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A polynomial identity implying Schur's partition theorem

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A POLYNOMIAL IDENTITY IMPLYING SCHUR'S PARTITION THEOREM

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Abstract. We propose and prove a new polynomial identity that implies Schur's partition theorem. We give combinatorial interpretations of some of our expressions in the spirit of Kurşungöz. We also present some related polynomial and q-series identities.

1. Introduction and background

Since the 2018 Combinatory Analysis conference in honor of G. E. Andrews' birthday, in a series of papers Kurşungöz presented his technique of writing generating functions for the number of partitions with gap conditions on some classical partition theorems [16–18]. His approach is backed with a combinatorial construction. This construction can be used to find finite analogs of these generating functions. Berkovich and the author [9] have found finite analogs of the Capparelli's partition theorem related generating functions presented by Kanade–Russell and Kurşungöz [15, 16]. Comparing these polynomials with the earlier found finite analogs of Alladi–Andrews–Gordon and Berkovich and the author's [1, 8], they listed polynomial identities that directly imply Capparelli's partition theorems [9]. These polynomial identities led to many q-series relations involving the q-trinomial coefficients and, with the use of trinomial version of the Bailey lemma, proven infinite families of q-series identities in the spirit of the Andrews–Gordon Identities [10, 11]. Following the footsteps of [9] and using other combinatorial arguments, the author presented other polynomial and q-series identities that are related with the classical partition theorems: namely the Euler, the Rogers–Ramanujan, the Göllnitz–Gordon, and the little Göllnitz theorems [22]. It should be noted that Kurşungöz also approached the Göllnitz–Gordon theorem [16], and the comparison of his construction versus the author equivalent formulas are discussed in [22].

In this work, we will follow the footsteps of [9, 10, 18, 22] and present a new polynomial identity that directly implies Schur's partition theorem followed up with the study of some related q-series identities.

We define a *partition* $\pi = (\lambda_1, \lambda_2, \dots)$ as a *non-decreasing finite sequence of positive integers*, which are called parts of the partition π. We will use $\nu(\pi)$ and $|\pi|$ to denote the number of parts and the sum of all parts (size) of the partition π, respectively. The empty sequence $\emptyset$ is the only conventional partition with 0 parts and 0 size.

We start with an equivalent formulation of the Schur's partition theorem [21]:

Theorem 1.1 (Schur, 1926). *For any non-negative integer n, the number of partitions of n into distinct parts ± 1 modulo 3 is equal to the number of partitions of n, where the gap between parts is at least 3 with the gap at least 6 if the parts are multiples of 3.*

This classical example of congruence–gap partition theorem is well studied and there are many proofs [2–6, 12, 13]. Out of this long list of proofs, the first and the only polynomial identity that implies Theorem 1.1 should be credited to Alladi–Berkovich [2].

Here we prove a new polynomial identity in the spirit of the polynomial identities that yield Capparelli's partition theorems [9]. We will show that the following new polynomial identity implies Schur's partition theorem:

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Theorem 1.2. *For any fixed integer N, let $\mathcal{N} := \mathcal{N}(m,n_1,n_2) := N-m-n_1-n_2$, then we have*

$$\sum_{m,n_1,n_2\geq 0} q^{A(n_1,n_2,m)} {3\mathcal{N} \brack m}_q {\mathcal{N}+\lfloor \frac{n_1}{2}\rfloor \brack \lfloor \frac{n_1}{2}\rfloor}_{q^6} {\mathcal{N}+\lfloor \frac{n_2}{2}\rfloor \brack \lfloor \frac{n_2}{2}\rfloor}_{q^6} = \sum_{j=-N}^{N} q^{\frac{j(3j-1)}{2}} \binom{N;j;q^3}{j}_2, \tag{1.1}$$

where

$$A(n_1,n_2,m) := \frac{(2m+n_1+n_2+1)(2m+n_1+n_2)}{2} + m(n_1+n_2) + (n_1+n_2)^2 - n_1. \tag{1.2}$$

The rest of this paper is organized as follows. We start with the necessary definitions that appear in Theorem 1.2 and the rest of the paper in Section 2. A direct proof of Theorem 1.2 is given in Section 3. Section 4 has the combinatorial connection of Theorem 1.2 to Theorem 1.1 showing that Theorem 1.2 implies Schur's Theorem. Some q-series and combinatorial identities of this study is discussed in Section 5.

2. Necessary Definitions and Some Useful Formulae

In this work, we will use the standard notations [7,14,23]. For variables a and q with $|q|<1$, we define the q-Pochhammer symbols and a useful abbreviation as:

$$(a)_\infty := (a;q)_\infty := (1-a)(1-aq)(1-aq^2)(1-aq^3)\dots,$$
$$(a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n;q)_\infty},$$
$$(a_1,a_2,\dots,a_k;q)_n := (a_1)_n(a_2)_n\dots(a_k)_n.$$

We note two well known properties of q-Pochhammer symbols:

$$(a;q)_{n-k} = \frac{(a;q)_n}{(\frac{q^{1-n}}{a};q)_k}\left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}, \tag{2.1}$$

and

$$(q^{-1};q^{-1})_n = (-1)^n q^{-\binom{n+1}{2}}(q;q)_n. \tag{2.2}$$

Let $m,\ n,\ a,$ and $b\in\mathbb{Z}$, we define the q-binomial coefficients and the two types of q-trinomial coefficients as

$${n+m \brack m}_q := \begin{cases} \dfrac{(q)_{n+m}}{(q)_n(q)_m}, & \text{if } n+m\geq m\geq 0,\\ 0, & \text{otherwise,}\end{cases} \tag{2.3}$$

$$\binom{m;b;q}{a}_2 := \sum_{k\geq 0} q^{k(k+b)} {m \brack k}_q {m-k \brack k+a}_q = \sum_{k\geq 0} \frac{q^{k(k+b)}(q)_m}{(q)_k(q)_{k+a}(q)_{m-2k-a}}, \tag{2.4}$$

and

$$T_n\binom{m;q}{a} := q^{\frac{m(m-n)-a(a-n)}{2}} \binom{m;a-n;q^{-1}}{a}_2, \tag{2.5}$$

respectively. The following properties of q-binomial coefficients are well known:

$$\lim_{n\to\infty} {n \brack m}_q = \frac{1}{(q)_m}, \tag{2.6}$$

and

$${n+m \brack m}_{q^{-1}} = q^{-mn} {n+m \brack m}_q. \tag{2.7}$$

We would also like to take a second here to recall two classic generating function interpretations [7]. The expression

$$\frac{1}{(q;q)_n}$$

is the generating function for the number of partitions into $\leq n$ parts, and the q-binomial coefficients

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_q$$

is the generating function for the number of partitions into $\leq n$ parts each $\leq m$. We will use these interpretations in our constructions of the generating functions that count the number of partitions that satisfy the gap conditions of Theorem 1.1.

3. Proof of Theorem 1.2

We start by noting that the right-hand side of (1.1) is the $S_{3N-1}(1,q) := \mathcal{R}_N(q)$ function defined in [6] that Andrews originally used to prove Schur's theorem directly. In his proof, he shows that this object satisfies the recurrence relation

$$\mathcal{R}_N(q) = (1+q^{3N-2}+q^{3N-1})\mathcal{R}_{N-1}(q) + q^{3N-3}(1-q^{3N-3})\mathcal{R}_{N-2}(q). \tag{3.1}$$

We would like to note that this recurrence can directly be found and automatically proven using Symbolic Computation tools Sigma and qMultiSum [19, 20]. Same is true for the left-hand side sum of (1.1). Using the mentioned implementations, we first prove that the left-hand side summand, to be denoted by $\mathcal{F}_N(m,n_1,n_2)$, satisfies the recurrence

$$\begin{aligned} \mathcal{F}_N(m,n_1,n_2) = \mathcal{F}_{N-1}(m,n_1,n_2) &+ q^{6N-5}F_{N-2}(m,n_1,n_2-1) + q^{6N-7}F_{N-2}(m,n_1-1,n_2) \\ &+ q^{3N-3}(1+q+q^2)\mathcal{F}_{N-2}(m-1,n_1,n_2) + q^{6N-8}(1+q+q^2)F_{N-3}(m-2,n_1,n_2) \\ &- q^{12N-24}\mathcal{F}_{N-4}(m,n_1-1,n_2-1) + q^{9N-15}F_{N-4}(m-3,n_1,n_2). \end{aligned} \tag{3.2}$$

To do so, one needs to split the treathment of the $\mathcal{F}_N(m,n_1,n_2)$ into 4 cases, that correspond to the even and odd values of n_1 and n_2. After this manual adjustment one can directly see that formally the summand $\mathcal{F}_N(m,n_1,n_2)$ satisfies the (3.2) in each case. The symbolic recurrence rules needs to be checked for exceptions which yield initial conditions. This is especially important for the transaction from the recurrence of the summand to the recurrence of the sum. In this case, we require

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix}_q := 1$$

as our only exceptional condition to the definition (2.3). This is the analogous common convention of $\begin{bmatrix} -1 \\ 0 \end{bmatrix}_q = 1$ due to the floor function. This common convention and its analogue here appears due to q-binomial coefficient recurrences failing when $n = m = 0$.

By summing (3.2) over m, n_1, and n_2 from $-\infty$ to ∞, we see that the left-hand side sum $\mathcal{L}_N(q)$ satisfies the recurrence

$$\begin{aligned} \mathcal{L}_N(q) = \mathcal{L}_{N-1}(q) &+ q^{3N-3}(1+q+q^2+q^{3N-4}+q^{3N-2})\mathcal{L}_{N-2}(q) \\ &+ q^{6N-8}(1+q+q^2)\mathcal{L}_{N-3}(q) + q^{9N-15}(1-q^{3N-9})\mathcal{L}_{N-4}(q). \end{aligned} \tag{3.3}$$

Using the recurrence (3.1) in an iterative fashion on its own terms

$$(q^{3N-2}+q^{3N-1})\mathcal{R}_{N-1}(q) \qquad \text{and} \qquad q^{6N-6}\mathcal{R}_{N-2}(q)$$

is enough to show that $\mathcal{R}_N(q)$ also satisfies (3.3). Now that we established that both sides of (1.1) satisfy the same recurrence (3.3), the last task is to check confirm that the first four initial conditions of both sides are the same. For that we give the list:

$$\begin{aligned}
&\mathcal{L}_0(q)=\mathcal{R}_0(q)=1,\\
&\mathcal{L}_1(q)=\mathcal{R}_1(q)=1+q+q^2,\\
&\mathcal{L}_2(q)=\mathcal{R}_2(q)=1+q+q^2+q^3+q^4+2q^5+q^6+q^7,\\
&\mathcal{L}_3(q)=\mathcal{R}_3(q)\\
&\qquad\quad =1+q+q^2+q^3+q^4+2q^5+2q^6+3q^7+3q^8+2q^9+2q^{10}+2q^{11}+2q^{12}+2q^{13}+q^{14}+q^{15}.
\end{aligned}$$

This proves the identity (1.1) for any non-negative N. For negative values of N both sides of (1.1) is 0.

4. Combinatorics of Theorem 1.2

Let $n_1,\ n_2$, and m be non-negative integers and let the partition $\pi_{n_1,n_2,m}$, to be called *minimal configuration*, be defined as n_1 consecutive 1 modulo 3 parts followed by n_2 consecutive 2 modulo 3 parts followed by m parts that are exactly 4 apart from their neighboring parts. For positive $n_1,\ n_2$, and m, we have

$$\pi_{n_1,n_2,m} := (\underline{1,4,7,\dots,3(n_1-1)+1},\ \underline{3n_1+2,3n_1+5,\dots,3(n_1+n_2-1)+2},$$

$$3(n_1+n_2)+3, 3(n_1+n_2)+7, 3(n_1+n_2)+11,\dots,3(n_1+n_2)+3+4(m-1)), \tag{4.1}$$

where we underline the *initial chain* of the n_1 consecutive 1 mod 3 parts and also underline the following n_2 consecutive 2 mod 3 parts. We do not underline the m 4-apart parts and call these parts *singletons*. If $n_1,\ n_2$, or m is 0, in the (4.1) we ignore the related portion of the partition with these numbers. As an example, when $n_1=n_2=m=0$, we get an empty list (the unique partition of 0) as our minimal configuration.

It is easy to see that the minimal configuration $\pi_{n_1,n_2,m}$ satisfies the gap conditions of the Schur Theorem (Theorem 1.1). Moreover, this partition has n_1+n_2+m parts and its size is exactly $A(n_1,n_2,m)$ as in (1.2). The name minimal configuration comes from the fact that $\pi_{n_1,n_2,m}$ is the partition with the smallest size that satisfies the gap conditions of Theorem 1.1 that has n_1+n_2-2 gaps of size exactly 3 into n_1+n_2+m parts.

We would like to start with all such minimal configurations and build up all partitions that satisfy Schur's gap conditions, bijectively. For that we will define "the forward motions of the parts" of the minimal configurations first. This will be done in a similar fashion to [9, 16–18, 22], mostly resembling the lines of [22].

Before presenting the details, we would like to summarize the way we will approach the forward motions. First, we will move the singletons; starting from the largest singleton (greatest as an integer) to the smallest singleton. We will preserve the order of the singletons of $\pi_{n_1,n_2,m}$ by moving each part less than or equal to the amount of movement of the previous (greater) part. Then, we will define the motion of the 2 modulo 3 parts as pairs splitting from the end of the 2 modulo 3 initial chain of $\pi_{n_1,n_2,m}$. Once again, this motion will be done starting from the greatest pair (the order with respect to the sum of the pair's parts) to the smallest pair. We will maintain the ordering of the pairs by letting any pair to move at most the same amount as the previous pair that moved before it. We will define crossing over a singleton for these 2 mod 3 pairs, as these pairs may come close to a singleton that moved before any one of the pairs and may violate the Schur's gap conditions. Finally, we will define the motion of the 1 modulo 3 pairs in a similar fashion to the 2 modulo 3 pairs. In this case, we will need the additional treatment of a 1 modulo 3 pair crossing over consecutive 2 modulo 3 parts of the partition. All the defined motions will be bijective maps (meaning every step will be reversible) and at each step we will make sure the outcome partition satisfies the Theorem 1.1's gap conditions.

Starting from the largest part (the last part) we can move the m-singletons forwards by adding each element a non-negative value: r_m to the largest part, r_{m-1} to the second largest with $r_m \geq r_{m-1}$... r_1 to the smallest singleton $r_2 \geq r_1 \geq 0$. The order $0 \leq r_1 \leq r_2 \leq \cdots \leq r_m$ is enough to ensure that order of the singletons are preserved after the motions. Such a list $(r_1, r_2, \ldots, r_m)$ with $0 \leq r_1 \leq r_2 \leq \cdots \leq r_m$ may not be a partition itself; some r_i values might be 0. On the other hand, by ignoring the zero values, it is clear that every such list (used in the forward motions of the singletons) corresponds to a unique partition into $\leq m$ parts. Therefore, the generating function that is related with the forward motions of m singletons is the generating function for the number of partitions into $\leq m$ parts:

$$\frac{1}{(q)_m}. \tag{4.2}$$

It is clear that the motions of the singletons are reversible.

After moving the singletons, we start moving the initial chain of the n_2 2 modulo 3 parts (signified by the underlining of all the related parts). In this motion we first split the last two elements of the initial chain, making them a pair (signified by under-braces)

$$\underline{3n_1+2, 3n_1+5, \ldots, 3(n_1+n_2-2)+2, 3(n_1+n_2-1)+2}$$

$$\mapsto \underline{3n_1+2, 3n_1+5, \ldots, 3(n_1+n_2-3)+2}\, \underbrace{3(n_1+n_2-2)+2, 3(n_1+n_2-1)+2}.$$

Later we will start moving these pairs by moving one to the next possible location where the numbers again become a pair of consecutive 2 modulo 3 parts. Before doing so, note that we are splitting and moving two parts of an n_2 length initial chain together. Hence, we can at most split and move $\lfloor n_2/2 \rfloor$ pairs. In the motion of these pairs, similar to the singletons case, we will move the greatest pair (ordered with respect to sum of the parts in the pair) forwards the most, then the second largest pair less than the motion of the first pair etc.

For a given pair $\underbrace{x, y}$ of π that satisfies the gap conditions of Schur's theorem (Theorem 1.1), if π does not have a part z such that $3 \leq z - y < 6$, we define the motion of this pair as

$$\underbrace{x, y} \mapsto \underbrace{x+3, y+3}. \tag{4.3}$$

This forward motions adds a total of 6 to the size of the partition π, the greater part of the pair moves 3 steps forwards, and it does not change the residue class of x and y modulo 3. Moreover, it is clearly bijective (meaning every motion is reversible) and can be undone.

There might be a z value that is in 3, 4 or 5 distance to the larger part of the pair that we would like to move. This forward motions needs us to define particular reversible rules so that the outcome partition would still satisfy the gap conditions of Schur's theorem. Given a pair $\underbrace{3k+2, 3k+5}$, we define the following rules for crossing singletons. Similar to adjustments explained in [18], we need to handle different cases differently. These cases will depend on the number of singletons that one pair needs to cross in a given circumstance:

- Case 1: If the pair is crossing a single singleton, which is < 6 distant to the pair and the larger part of the pair is ≥ 9 distant to the following larger singleton (if any), we define the following forward motions. For $r = 0, 1, 2$, we have

$$\begin{aligned} &(\text{parts } \leq 3k-1), \underbrace{3k+2, 3k+5}, 3k+8+r, (\text{parts } \geq 3k+14) \\ &\qquad\qquad \mapsto \ \ldots, 3k+2+r, \underbrace{3k+8, 3k+11}, \ldots. \end{aligned} \tag{4.4}$$

  In any given partition, the outcome of this map can be identified easily as there is a singleton followed by a pair and this map can be reversed by using the map in the other direction. This is enough to say that this map that is defined on partitions with this property is bijective and the inverse map is also can be seen at (4.4) in the reverse order.

We can also use $r = 3$ for the reverse direction of this motion. This is analogous to the adjustment rule [18, (9), pg 6]. The outcomes of both approaches are the same and their inverse images are the same.

- Case 2: If the pair is crossing two close singletons (a singleton followed by another singleton that is ≤ 5 distant) where employing a case of the (4.4) would break the Schur's gap conditions, we use the following reversible motions. For $r, s \in \{0, 1\}$ with $r \leq s$, we define the motions:

$$\begin{aligned} &\dots, \underbrace{3k+2, 3k+5}, 3k+8+r, 3k+12+s, \text{(parts} \geq 3k+17) \\ &\qquad \mapsto \dots, 3k+2+r, 3k+6+s, \underbrace{3k+11, 3k+14}, \dots . \end{aligned} \tag{4.5}$$

Notice that, in the $r = 0$ cases one can use (4.4) with the pair $\underbrace{3k+5, 3k+8}$ instead of using (4.5). Once again, this is analogous to the rule [18, (9), pg 6]. The outcomes of both approaches are the same and their inverse images are the same.

- Case 3: The only case that is not covered is the following. If the larger part of the pair to move is 3 distant to the smallest singleton and it needs to cross three singletons with gaps exactly 4. Here we define the motion:

$$\begin{aligned} &\dots, \underbrace{3k+2, 3k+5}, 3k+8, 3k+12, 3k+16, \text{(parts} \geq 3k+20) \\ &\qquad \mapsto \quad \dots, 3k+2, 3k+6, 3k+10, \underbrace{3k+14, 3k+17}, \dots . \end{aligned} \tag{4.6}$$

We would like to note that the image of this motion is identifiable in a partition. The image of this motion is the only case where three singletons with gap 4 each is followed by a pair, where the smaller part of the pair is also 4 distant to the largest singleton. This makes it possible to revert this motion.

Also observe that a possible part of the partition (if any) that follows the part $3k+16$ in (4.6) is at least of size $3k+20$. The gap between the largest part of our last motion (4.6) $3k+17$ has at least a gap of 3 with this possible part $3k+20$. Therefore, one can stop the crossing of the pairs over singletons here. This also means one can stop defining particular rules here as well. If they would like to move the pair $\underbrace{3k+14, 3k+17}$ once again, they can start with checking and employing the motion rules (4.3)-(4.6).

Hence, the list of motions (4.3), (4.4), (4.5), and (4.6) is the full reversible list of motions for the $\lfloor n_2/2 \rfloor$ 2 modulo 3 pairs. Furthermore, each of these motions add 6 to the total size of the partition once employed. Recalling that a pair can move at most the same amount as the previous pair is enough to see that the generating function related with the motions of the $\lfloor n_2/2 \rfloor$ 2 modulo 3 pairs is in bijection with the partitions into $\leq \lfloor n_2/2 \rfloor$ parts. The generating function for the forward motions of the 2 modulo 3 initial chain is

$$\frac{1}{(q^6; q^6)_{\lfloor n_2/2 \rfloor}}. \tag{4.7}$$

Also, observe that in all these motions the pairs move

$$3 + 3 \times \text{“the number of singletons crossed”} \tag{4.8}$$

steps forwards.

Finally, we move on to the motions starting from the initial chain of the n_1 1 modulo 3 parts. Similar to the previous case, we first split the last two elements of the initial chain, making them a pair (signified by under-braces)

$$\begin{aligned} &\underline{1, 4, 7, \dots, 3(n_1-3)+1, 3(n_1-2)+1, 3(n_1-1)+1} \\ &\qquad \mapsto \underline{1, 4, 7, \dots, 3(n_1-3)+1}, \underbrace{3(n_1-2)+1, 3(n_1-1)+1}. \end{aligned}$$

Similar to the previous (2 modulo 3 initial chain) case we can split and move at most $\lfloor n_1/2 \rfloor$. Moreover, (4.3) is still valid for this case, and for the rest of the crossing rules all one needs to do is to use the same cases related to (4.4)-(4.6) and subtract 1 from each and every term in these motions. All the size and number of forward motion observations that is made for the 2 modulo 3 pairs are still valid for the 1 modulo 3 pairs.

One new situation in this case appears if a 1 modulo 3 pair comes close to a group of consecutive 2 modulo 3 parts of the partition. In this situation, we define the following map. Let $l \geq 3$ be the number of consecutive parts with gap between them ≤ 4, then for $r, s \in \{0, 1\}$, we have

$$\begin{aligned}&\dots, \underbrace{3k+1, 3k+4}, 3k+7+r, 3k+11, \dots, 3k+3l+5+s, (\text{parts } \geq 3k+3l+10)\\ &\mapsto \dots, 3k+1+r, 3k+5, \dots, 3k+3l-1+s, \underbrace{3k+3l+4, 3k+3l+7}, \dots.\end{aligned} \tag{4.9}$$

In the 1 modulo 3 pair motions we will not be defining any 1 modulo 3 pair crossing another pair of the sort. This clearly determines as the possible beginning sequence of this consecutive numbers list as either $(3k+7, 3k+11)$ or $(3k+8, 3k+11)$. There can be many consecutive 3 modulo 3 parts in this list, and this rule covers 1 modulo 3 pairs crossing over 2 modulo 3 pairs. In any given Schur's partition if there ois a 2 modulo 3 pair that has parts less than any 1 modulo 3 pair, this is a clear indication that this rule has been employed. The reversal of this rule is also clear since the gaps between the list of values that a pair jumps over does not change.

Note that $l = 1$ and 2 cases are covered under the relative versions of (4.4) and (4.5) for the 1 modulo 3 pairs. Moreover, note that in this forward motions the pair makes l extra motions and again the size of the overall partition raises only by 6. By the same argument as the previous case now we can see that the generating function corresponding to the forward motions of the 1 modulo 3 initial chain is

$$\frac{1}{(q^6; q^6)_{\lfloor n_1/2 \rfloor}}. \tag{4.10}$$

We would like to give two examples of these rules.

Example 1: We would like to start with the minimal configuration $\pi_{2,3,3}$ and exemplify some forwards motions. Looking at (4.1), we have

$$\pi_{2,3,3} = (\underline{1,4}, \underline{8,11,14}, 18, 22, 26).$$

We would like to move all the singletons 2 step forwards. This yields

$$\pi_1 = (\underline{1,4}, \underline{8,11,14}, 20, 24, 28).$$

Next, we look at the 2 modulo 3 pairs. At the end of the 2 modulo 3 initial chain of 3 elements, we have the pair $\underbrace{11, 14}$ that we can split. We would like to move this pair 2 times. The singleton 20 is the closest larger singleton to the pair $\underbrace{11, 14}$ and it is ≥ 6 away from the larger part, 14, of the pair. Therefore, we can move this pair once by the rule (4.3) and get

$$\pi_2 = (\underline{1,4}, \underline{8}, \underbrace{14, 17}, 20, 24, 28).$$

We cannot repeat the same motion due to the close proximity of the singleton 20. For the second forwards movement of this pair we shall look at the applicable cases. The motion (4.4) does not apply since the second singleton 25 is ≤ 9 distance to the larger part of the pair 17. The motion (4.5) does not apply since the largest singleton 28 is ≤ 12 distance to the larger part of the singleton 17. Therefore, we employ the motion (4.6) and get

$$\pi_3 = (\underline{1,4}, \underline{8}, 14, 18, 22, \underbrace{26, 29}).$$

Since $\underline{8}$ is a single standing part of the 2 modulo 3 initial chain, we can not move it. We would like to move the pair $\underbrace{1, 4}$ three times. For the first motion, since the pair will go over the remaining portion of the 2 mod 3 initial chain, we need to use the motion (4.9), with $(k, l) = (0, 1)$. This yields

$$\pi_3 = (\underline{2}, \underbrace{7, 10}, 14, 18, 22, \underbrace{26, 29}).$$

For the second motion, we need to employ the 1 modulo 3 analog (where we subtract 1 from each part) of (4.5) where $(k, r, s) = (2, 1, 1)$, since the largest singleton 22 is ≥ 12 distant to the larger part of the pair $\underbrace{7, 10}$ and get

$$\pi_3 = (\underline{2}, 8, 12, \underbrace{16, 19}, 22, \underbrace{26, 29}).$$

The last movement of the 1 modulo 3 pair is done by the rule (4.9) with $(k, l, r, s) = (5, 3, 0, 0)$ and get the final partition

$$\pi_4 = (\underline{2}, 8, 12, 16, \underbrace{20, 23}, \underbrace{28, 31}).$$

In Kurşungöz's notations [18], one can see this example as the quadruple partition vector

$$(([1, 4], 8, [11, 14], 18, 22, 26), (0, 2, 2, 2), (12), (18)).$$

This is to say that we start with the minimal configuration $([1, 4], 8, [11, 14], 18, 22, 26)$, term-by-term add $(2, 2, 2)$ to the singletons $(18, 22, 26)$, evenly distribute 12 to the pair $[11, 14]$, while we do the proper adjustment rules, and finally evenly distribute 18 to the pair $[1, 4]$ while we do the proper adjustment rules. Our set up and minimal configurations does not require us to keep track of a intermediate singleton between pairs such as the part 8 in this example.

Example 2: We would like to start with the partition

$$\pi^* = (4, 7, 10, 16, 20, 25, 28, 31),$$

which satisfies the Schur's theorem's gap conditions. We first look if there is a clear piece of the 1 modulo 3 initial chain. In this example we do not see such a piece. Then we identify the pairs. We do this identification by looking from smallest part to the largest, and pairing up parts that are exactly 3 distant. Once a part is paired it will not be used in any other pairing. We have

$$\pi^* = (\underbrace{4, 7}, 10, 16, 20, \underbrace{25, 28}, 31).$$

Then, we start by moving the smallest 1 modulo 3 pair $\underbrace{4, 7}$ using the list of rules: (4.3), (1 modulo 3 analogues of) (4.4), (4.5), (4.6), and the rule (4.9) in the reverse direction.

Here we can move the pair $\underbrace{4, 7}$ using (4.3) in reverse one time and get

$$\pi_1^* = (\underbrace{1, 4}, 10, 16, 20, \underbrace{25, 28}, 31).$$

Since this pair cannot move anymore, we change our focus to the next 1 modulo 3 pair $\underbrace{25, 28}$. In the backwards motion, this pair needs to cross the singletons 16 and 20, this is 1 modulo 3 analog (when we subtract 1 from each part) of (4.5) in reverse with $(k, r, s) = (5, 0, 0)$. Hence, we get

$$\pi_2^* = (\underbrace{1, 4}, 10, \underbrace{16, 19}, 22, 26, 31).$$

Then (4.3) can be used in reverse once followed by (4.4) with $(k, r) = (2, 3)$ again in reverse direction to yield

$$\pi_3^* = (\underbrace{1, 4}, \underbrace{7, 10}, 16, 22, 26, 31).$$

There are no more 1 modulo 3 pairs or any 2 modulo 3 pairs. So now it is only a matter of moving back the singletons starting from the smallest one. This way we get to the minimal configuration

$$\pi_{4,0,4} = (\underline{1, 4, 7, 10}, 14, 18, 22, 26).$$

In Kurşungöz's notations [18], π^* has the following vector partition decomposition

$$(([1, 4], [7, 10], 14, 18, 22, 26), (2, 4, 4, 5), (6, 18), \emptyset).$$

Now we would like to combine our observations about the generating functions to give a new formula for the generating function of partitions that satisy the Schur's theorem's gap conditions. Putting together

(4.2), (4.7) and (4.10), it is easy to see that

$$\frac{q^{A(n_1,n_2,m)}}{(q^6;q^6)_{\lfloor n_1/2\rfloor}(q^6;q^6)_{\lfloor n_2/2\rfloor}(q)_m} \tag{4.11}$$

is the generating function for all the partitions that satisfies the gap conditions of Theorem 1.1 that can be constructed from the minimal configuration $\pi_{n_1,n_2,m}$ defined in (4.1), where $A(n_1, n_2, m)$ is as defined in (1.2). By summing over all possible n_1, n_2, and m we get the following theorem.

Theorem 4.1. *Let $A(n_1, n_2, m)$ be as defined in* (1.2)*, then*

$$\sum_{n_1,n_2,m\geq 0} \frac{x^{n_1+n_2+m}q^{A(n_1,n_2,m)}}{(q^6;q^6)_{\lfloor n_1/2\rfloor}(q^6;q^6)_{\lfloor n_2/2\rfloor}(q)_m} \tag{4.12}$$

is the generating function for the number of partitions that satisfy the gap conditions of Schur's theorem (Theorem 1.1), where the exponent of x counts the number of parts of the counted partitions.

The triple series (4.12) is the analogue of the double sums presented for the Göllnitz–Gordon and little Göllnitz theorems in [22]. This series (as well as the ones in [22]) are inspired by Kurşungöz's recent works [16–18]. Due to the difference in the minimal configuration setups and some of the motions, the author and Kurşungöz gets equivalent but different representations for the same generating functions. Here we present Kurşungöz's version [18, Theorem 2, p 3] of the generating function represented in Theorem 4.1.

Theorem 4.2 (Kurşungöz, 2018)**.** *Let*

$$K(n_1, n_2, m) := 6(n_1 + n_2)^2 + 2m^2 + 6m(n_1 + n_2) - n_1 + n_2 + m, \tag{4.13}$$

then

$$\sum_{n_1,n_2,m\geq 0} \frac{x^{2n_1+2n_2+m}q^{K(n_1,n_2,m)}}{(q^6;q^6)_{n_1}(q^6;q^6)_{n_2}(q)_m} \tag{4.14}$$

is the generating function for the number of partitions that satisfy the gap conditions of Schur's theorem (Theorem 1.1), where the exponent of x counts the number of parts of the counted partitions.

To avoid any speculative trivial transformation between (4.12) and (4.14) please note that

$$A(2n_1, 2n_2, m) - K(n_1, n_2, m) = 2m. \tag{4.15}$$

We would also like to present the equality of the series (4.12) and (4.14) after doing even-odd splits for the variables n_1 and n_2 and regrouping in (4.12). We will also be using (4.15) to write the q-factors in the summands using the same quadratic $K(n_1, n_2, m)$.

Theorem 4.3. *We have*

$$\sum_{n_1,n_2,m\geq 0} \frac{x^{2n_1+2n_2+m}q^{K(n_1,n_2,m)+2m}}{(q^6;q^6)_{n_1}(q^6;q^6)_{n_2}(q)_m}(1 + xq^{6n_1+6n_2+3m+1} + xq^{6n_1+6n_2+3m+2} + x^2q^{12n_1+12n_2+6m+6})$$

$$= \sum_{n_1,n_2,m\geq 0} \frac{x^{2n_1+2n_2+m}q^{K(n_1,n_2,m)}}{(q^6;q^6)_{n_1}(q^6;q^6)_{n_2}(q)_m}, \tag{4.16}$$

where $K(n_1, n_2, m)$ is as in (4.13)*.*

Now we start finding a finite analogue of (4.12). Let N be a non-negative integer. We would like to find all the partitions with the largest part $\leq N$ that are counted by (4.12). For that we need to count how many times a singleton, a 2 modulo 3 pair and a 1 modulo 3 pair can move forward before exceeding N and change our generating functions from reciprocal of a q-factorials to the necessary q-binomials.

The largest singleton of the minimal configuration $\pi_{n_1,n_2,m}$, $3(n_1 + n_2) + 3 + 4(m - 1)$, can only move $N - [3(n_1 + n_2) + 3 + 4(m - 1)]$ steps forward before exceeding the imposed bound. Therefore, with the new bound, the motions for the singletons is related with the partitions into $\leq m$ parts, where each part is $\leq N-[3(n_1+n_2)+3+4(m-1)]$. The generating function for all such partitions is

$$\begin{bmatrix} N-[3(n_1+n_2)+3+4(m-1)]+m \\ m \end{bmatrix}_q \tag{4.17}$$

Each forward movement of a 2 modulo 3 pair gets it 3 units closer to the bound N. Then, ignoring the singletons for a second, the largest 2 modulo 3 pair $\underbrace{3(n_1+n_2-2)+2, 3(n_1+n_2-1)+2}$ can move at most

$$\left\lfloor \frac{N-[3(n_1+n_2-1)+2]}{3} \right\rfloor$$

steps forwards before the larger part, $3(n_1+n_2-1)+2$, of the pair goes over the bound on the largest part N. Recall (4.8): crossing over singletons make these pairs move extra steps forwards. There are m singletons that are greater than the largest pair $\underbrace{3(n_1+n_2-2)+2, 3(n_1+n_2-1)+2}$. Hence before reaching the bound this pair would need to cross all of those m singletons, and move an extra 3 steps forwards each time. Therefore, the actual number of steps this pair can take forwards before passing the bound N is

$$\left\lfloor \frac{N-[3(n_1+n_2-1)+2]}{3} \right\rfloor - m.$$

This shows us that the bounded forward motions of the 2 modulo 3 pairs is related with partitions into $\leq \lfloor n_2/2 \rfloor$ parts each $\leq \lfloor (N-[3(n_1+n_2-1)+2])/3 \rfloor - m$. This implies that the related generating function for this motion (that changes the size by 6 each time) is

$$\begin{bmatrix} \left\lfloor \frac{N-[3(n_1+n_2-1)+2]}{3} \right\rfloor - m + \left\lfloor \frac{n_2}{2} \right\rfloor \\ \left\lfloor \frac{n_2}{2} \right\rfloor \end{bmatrix}_{q^6}. \tag{4.18}$$

Finally, Similar to the previous case, forgetting about the the n_2 2 modulo 3 parts and the m singletons, the largest 1 modulo 3 pair, $\underbrace{3(n_1-2)+1, 3(n_1-1)+1}$, can move

$$\left\lfloor \frac{N-[3(n_1-1)+1]}{3} \right\rfloor$$

forwards before $3(n_1-1)+1$ goes over N. Including our observations about the extra steps one pair takes while crossing over parts, we see that the actual number of steps forwards that the largest pair can take is

$$\left\lfloor \frac{N-[3(n_1-1)+1]}{3} \right\rfloor - m - n_2.$$

With that, similar to the previous case, we see that the generating function related to the forward motions of the $\lfloor n_1/2 \rfloor$ 1 modulo 3 pairs is

$$\begin{bmatrix} \left\lfloor \frac{N-[3(n_1-1)+1]}{3} \right\rfloor - m - n_2 + \left\lfloor \frac{n_1}{2} \right\rfloor \\ \left\lfloor \frac{n_1}{2} \right\rfloor \end{bmatrix}_{q^6}. \tag{4.19}$$

Putting (4.17), (4.18), and (4.19) together, we get that

$$\begin{aligned} q^{A(n_1,n_2,m)} &\begin{bmatrix} N-3(n_1+n_2+m)+1 \\ m \end{bmatrix}_q \\ &\times \begin{bmatrix} \left\lfloor \frac{N-[3(n_1-1)+1]}{3} \right\rfloor - m - n_2 + \left\lfloor \frac{n_1}{2} \right\rfloor \\ \left\lfloor \frac{n_1}{2} \right\rfloor \end{bmatrix}_{q^6} \begin{bmatrix} \left\lfloor \frac{N-[3(n_1+n_2-1)+2]}{3} \right\rfloor - m + \left\lfloor \frac{n_2}{2} \right\rfloor \\ \left\lfloor \frac{n_2}{2} \right\rfloor \end{bmatrix}_{q^6} \end{aligned} \tag{4.20}$$

is the generating function for the number of partitions that satisfies the gap conditions of Theorem 1.1 that can be constructed from the minimal configuration $\pi_{n_1,n_2,m}$ with the extra bound on the largest part $\leq N$. Summing (4.20) over n_1, n_2, and m yields the following theorem.

Theorem 4.4. *For any non-negative integer N, the expression*

(4.21)

$$\sum_{n_1,n_2,m\geq 0} x^{n_1+n_2+m} q^{A(n_1,n_2,m)} \begin{bmatrix} N-3(n_1+n_2+m)+1 \\ m \end{bmatrix}_q \times \begin{bmatrix} \left\lfloor \frac{N-[3(n_1-1)+1]}{3} \right\rfloor - m - n_2 + \left\lfloor \frac{n_1}{2} \right\rfloor \\ \left\lfloor \frac{n_1}{2} \right\rfloor \end{bmatrix}_{q^6} \begin{bmatrix} \left\lfloor \frac{N-[3(n_1+n_2-1)+2]}{3} \right\rfloor - m + \left\lfloor \frac{n_2}{2} \right\rfloor \\ \left\lfloor \frac{n_2}{2} \right\rfloor \end{bmatrix}_{q^6}$$

where $A(n_1,n_2,m)$ is defined as in Theorem 1.2, is the generating function for the number of partitions that satisfy the gap conditions of Theorem 1.1 with the extra condition that each part is $\leq N$, where the exponent of x counts the number of parts.

One direct corollary of Theorem 4.4 is the interpretation of the left-hand side of (1.1) when $N \mapsto 3N-1$.

Corollary 4.5. *For any positive integer N, and $\mathcal{N} := N - n_1 - n_2 - m$, the expression*

$$\sum_{m,n_1,n_2\geq 0} q^{A(n_1,n_2,m)} \begin{bmatrix} 3\mathcal{N} \\ m \end{bmatrix}_q \begin{bmatrix} \mathcal{N} + \left\lfloor \frac{n_1}{2} \right\rfloor \\ \left\lfloor \frac{n_1}{2} \right\rfloor \end{bmatrix}_{q^6} \begin{bmatrix} \mathcal{N} + \left\lfloor \frac{n_2}{2} \right\rfloor \\ \left\lfloor \frac{n_2}{2} \right\rfloor \end{bmatrix}_{q^6},$$

where $A(n_1,n_2,m)$ is defined as in Theorem 1.2, is the generating function for the number of partitions that satisfy the gap conditions of Theorem 1.1 with the extra condition that each part is $\leq 3N-1$.

On the other hand, Andrews [6] interpreted the right-hand side of (1.1) as the same generating function in the interpretation of Corollary 4.5. This is also proves the validity of Theorem 1.2 for positive values of N, this time using only the combinatorial constructions. In [6, (3.9), pg. 147], Andrews also shows that the right-hand side sum converges to the generating function for the number of partitions into distinct parts ± 1 mod 3:

$$(-q,-q^2;q^3)_\infty.$$

This shows that after taking limits $N \to \infty$ of (1.1), and using (2.6) as needed, we have

$$\sum_{n_1,n_2,m\geq 0} \frac{q^{A(n_1,n_2,m)}}{(q^6;q^6)_{\lfloor n_1/2 \rfloor}(q^6;q^6)_{\lfloor n_2/2 \rfloor}(q)_m} = (-q,-q^2;q^3)_\infty,$$

which is the analytic version of the Schur's theorem (Theorem 1.1). This shows that the polynomial identity (1.1) (keeping the interpretation, Theorem 4.4 in mind) implies Theorem 1.1.

5. Some Implications of Theorem 1.2

We start by sending $q \mapsto 1/q$ in (1.1) followed by the use of (2.7) and multiplying both sides with $q^{3N^2/2}$. This yields the equivalent formula

$$\sum_{m,n_1,n_2\geq 0} q^{B(n_1,n_2,m,N)-A(n_1,n_2,m)} \begin{bmatrix} 3\mathcal{N} \\ m \end{bmatrix}_q \begin{bmatrix} \mathcal{N} + \left\lfloor \frac{n_1}{2} \right\rfloor \\ \left\lfloor \frac{n_1}{2} \right\rfloor \end{bmatrix}_{q^6} \begin{bmatrix} \mathcal{N} + \left\lfloor \frac{n_2}{2} \right\rfloor \\ \left\lfloor \frac{n_2}{2} \right\rfloor \end{bmatrix}_{q^6} = \sum_{j=-N}^{N} q^{\frac{j}{2}} T_0\binom{N;q^3}{j}, \tag{5.1}$$

where $A(n_1,n_2,m)$ is as in (1.2), $\mathcal{N} = N - n_1 - n_2 - m$, and

$$B(n_1,n_2,m,N) = \frac{3N^2}{2} - (3\mathcal{N}-m)m - 6\mathcal{N}\left(\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor\right). \tag{5.2}$$

Note that the sides in (5.1) are not polynomials but multiplying both sides with $q^{N/2}$ is enough to make them polynomials. After multiplying both sides of (5.1) by $q^{N/2}$, writing the definition of (2.4) in for the right-hand side of (5.1) and using (2.2) multiple times we see that

$$\sum_{j=-\infty}^{\infty} q^{\frac{N+j}{2}} T_0\binom{N;q^3}{j} = \sum_{k,l\geq 0} \frac{q^{k+\frac{l(3l+1)}{2}}(q^3;q^3)_N}{(q^3;q^3)_{N-k-l}(q^3;q^3)_k(q^3;q^3)_l}, \tag{5.3}$$

after simple changes of variables. We use (2.1) for the term $(q^3;q^3)_{(N-k)-l}$ to separate the difference of the variable l. This way we end up with the expression

$$\sum_{k\geq 0} q^k \begin{bmatrix} N \\ k \end{bmatrix}_{q^3} \sum_{l\geq 0} \frac{(q^{-3(N-k)};q^3)_l}{(q^3;q^3)_l} \left(-q^{3(N-k)+2}\right)^l.$$

The inner sum can be summed using the q-binomial theorem [14, II.4, p 354], and we get

$$\sum_{j=-\infty}^{\infty} q^{\frac{N+j}{2}} T_0\binom{N;q^3}{j} = \sum_{k\geq 0} q^k \begin{bmatrix} N \\ k \end{bmatrix}_{q^3} (-q^2;q^3)_{N-k}. \tag{5.4}$$

Not only that, (5.4) with the use of [14, II.1, p 354] on the right-hand side, yields

$$\lim_{N\to\infty} \sum_{j=-\infty}^{\infty} q^{\frac{N+j}{2}} T_0\binom{N;q^3}{j} = \frac{1}{(q^2;q^3)_\infty (q;q^6)_\infty}. \tag{5.5}$$

To evaluate the $N\to\infty$ limit on the left-hand side of (5.1) with the extra $q^{N/2}$, one first needs to make a change of summation variables and rewrite the q-factor. We would like to use $y=\mathcal{N}$ as our summation variable instead of n_2, but the parity of N must be kept in check to correctly identify the exponent of the q-factor in this case. Let $r(a,b)$ be the remainder of the division $a\div b$, for $a,b\in\mathbb{N}$. After the change of variables, the left-hand side of (5.1) multiplied with an extra $q^{N/2}$ becomes

$$\sum_{m,n_1,y\geq 0} q^{Q(m,n_1,y,N)} \begin{bmatrix} 3y \\ m \end{bmatrix}_q \begin{bmatrix} y+\lfloor\frac{n_1}{2}\rfloor \\ y \end{bmatrix}_{q^6} \begin{bmatrix} y+\left\lfloor\frac{N-m-n_1-y}{2}\right\rfloor \\ y \end{bmatrix}_{q^6},$$

where

$$Q(m,n_1,y,N) = \binom{m}{2} + \frac{y(3y+1)}{2} + n_1 + 3y\,r(N+m+y,2) + 6y\,r(n_1,2)\,r(N+m+y+1,2).$$

Then, by taking the limit $N\to\infty$ for odd and even N and using (5.5) we get the following theorem.

Theorem 5.1. *Let $t=1,\ 2$, then*

$$\sum_{m,n_1,y\geq 0} \frac{q^{Q_t(m,n_1,y)}}{(q^6;q^6)_y} \begin{bmatrix} 3y \\ m \end{bmatrix}_q \begin{bmatrix} y+\lfloor\frac{n_1}{2}\rfloor \\ y \end{bmatrix}_{q^6} = \frac{1}{(q^2;q^3)_\infty (q;q^6)_\infty}, \tag{5.6}$$

where

$$Q_t(m,n_1,y) = \binom{m}{2} + \frac{y(3y+1)}{2} + n_1 + 3y\,r(m+y+t,2) + 6y\,r(n_1,2)\,r(m+y+1+t,2) \tag{5.7}$$

Recall that Warnaar [23, (10), pg 2516] proved the following summation formula.

$$\sum_{i\geq 0} q^{\frac{i^2}{2}} \begin{bmatrix} L \\ i \end{bmatrix}_q T_0\binom{i;q}{a} = q^{\frac{a^2}{2}} \begin{bmatrix} 2L \\ L-a \end{bmatrix}_q. \tag{5.8}$$

This can be applied to the right-side of (5.1) to get the following theorem.

Theorem 5.2. *Let $\mathcal{N}=N-m-n_1-n_2$, for any non-negative integer M we have*

$$\sum_{N,m,n_1,n_2\geq 0} q^{\frac{3N^2}{2}+B(n_1,n_2,m,N)-A(n_1,n_2,m)} \begin{bmatrix} 3\mathcal{N} \\ m \end{bmatrix}_q \begin{bmatrix} M \\ N \end{bmatrix}_{q^3} \begin{bmatrix} \mathcal{N}+\lfloor\frac{n_1}{2}\rfloor \\ \lfloor\frac{n_1}{2}\rfloor \end{bmatrix}_{q^6} \begin{bmatrix} \mathcal{N}+\lfloor\frac{n_2}{2}\rfloor \\ \lfloor\frac{n_2}{2}\rfloor \end{bmatrix}_{q^6} = (-q,-q^2;q^3)_M, \tag{5.9}$$

where $A(n_1,n_2,m)$ and $B(n_1,n_2,m,N)$ are defined as in (1.2) and (5.2), respectively.

Proof. We sum both sides of (5.1) over N from 0 to M after multiplying the summand with

$$q^{\frac{3N^2}{2}} \begin{bmatrix} M \\ N \end{bmatrix}_{q^3}.$$

This gives the left-hand side of (5.9). For the right-hand side of the formula, we interchange the order of summations, use (5.8) followed by the summation formula [7, (3.3.6). p. 36]. This yields

$$q^{\frac{(3M+1)M}{2}}(-q^{1-3M};q^3)_{2M},$$

which after basic simplifications is equal to the right-hand side of the equation (5.9). □

The limit $M\to\infty$ of (5.9) is much more straightforward than the limit $n\to\infty$. By employing (2.6), we get the following corollary of Theorem 5.2.

Corollary 5.3.

$$\sum_{N,m,n_1,n_2\geq 0}\frac{q^{\frac{3N^2}{2}+B(n_1,n_2,m,N)-A(n_1,n_2,m)}}{(q^3;q^3)_N}\begin{bmatrix}3\mathcal{N}\\ m\end{bmatrix}_q\begin{bmatrix}\mathcal{N}+\lfloor\frac{n_1}{2}\rfloor\\ \lfloor\frac{n_1}{2}\rfloor\end{bmatrix}_{q^6}\begin{bmatrix}\mathcal{N}+\lfloor\frac{n_2}{2}\rfloor\\ \lfloor\frac{n_2}{2}\rfloor\end{bmatrix}_{q^6}=(-q,-q^2;q^3)_\infty$$

where $A(n_1,n_2,m)$ and $B(n_1,n_2,m,N)$ are defined as in (1.2) *and* (5.2)*, respectively.*

Theorem 1.2 (and the equation (5.1)) and Theorem 5.2 also yield some intriguing combinatorial corollaries at the $q=1$ level.

Corollary 5.4. *For some non-negative integer M, $\mathcal{N}:=N-n_1+n_2-m$ and $\mathcal{M}:=M-n_1-n_2-m$, we have*

$$\sum_{m,n_1,n_2\geq 0}\binom{3\mathcal{M}}{m}\binom{\mathcal{M}+\lfloor\frac{n_1}{2}\rfloor}{\mathcal{M}}\binom{\mathcal{M}+\lfloor\frac{n_2}{2}\rfloor}{\mathcal{M}}=3^M, \tag{5.10}$$

and

$$\sum_{N,m,n_1,n_2\geq 0}\binom{M}{N}\binom{3\mathcal{N}}{m}\binom{\mathcal{N}+\lfloor\frac{n_1}{2}\rfloor}{\mathcal{N}}\binom{\mathcal{N}+\lfloor\frac{n_2}{2}\rfloor}{\mathcal{N}}=4^M. \tag{5.11}$$

Proof. The equation (5.11) is a clear consequence of (5.9), or one can get it from (5.10) as it is the classical binomial theorem. For the equation (5.10), one only needs to recall that

$$\sum_{j=-N}^{N}x^j\binom{N;j;1}{j}_2=(x^{-1}+1+x)^N,$$

and set x to 1. □

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