

# **On quasi-Monte Carlo methods in weighted ANOVA spaces**

**P. Kritzer, F. Pillichshammer, G.W.  
Wasilkowski**

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# On Quasi-Monte Carlo Methods in Weighted ANOVA Spaces

P. Kritzer\*, F. Pillichshammer† and G. W. Wasilkowski

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## Abstract

In the present paper we study quasi-Monte Carlo rules for approximating integrals over the  $d$ -dimensional unit cube for functions from weighted Sobolev spaces of regularity one. While the properties of these rules are well understood for anchored Sobolev spaces, this is not the case for the ANOVA spaces, which are another very important type of reference spaces for quasi-Monte Carlo rules.

Using a direct approach we provide a formula for the worst case error of quasi-Monte Carlo rules for functions from weighted ANOVA spaces. As a consequence we bound the worst case error from above in terms of weighted discrepancy of the employed integration nodes. On the other hand we also obtain a general lower bound in terms of the number  $n$  of used integration nodes.

For the one-dimensional case our results lead to the optimal integration rule and also in the two-dimensional case we provide rules yielding optimal convergence rates.

*Keywords:* Quasi-Monte Carlo integration, ANOVA space, worst case error, weighted discrepancy

*MSC 2010:* 65D30, 65C05, 11K38

## 1 Introduction

In this paper we study numerical integration of functions  $f$  over the  $d$ -dimensional unit cube  $[0, 1]^d$ . A powerful method is the quasi-Monte Carlo (QMC) method which approximates the integral by

$$\int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x} \approx \frac{1}{n} \sum_{j=1}^n f(\mathbf{t}_j),$$

where the  $\mathbf{t}_j$  are given points in  $[0, 1]^d$ . The latter expression is called an ( $n$ -point) QMC rule and is denoted by  $\text{QMC}_{d,n}$ , with  $d$  indicating the dimension. We call the set  $\mathcal{P} = \{\mathbf{t}_1, \dots, \mathbf{t}_n\}$  the underlying node set of the QMC rule. General introductions to QMC methods can, e.g., be found in [5, 25, 28].

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The basis of the QMC method is the fact that the absolute error of a QMC rule

$$\left| \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x} - \text{QMC}_{d,n}(f) \right|$$

can be separated into properties of the integrand  $f$  on the one hand, and distribution properties of the node set underlying the QMC rule on the other hand. Such estimates are called Koksma-Hlawka type inequalities, which go back to Koksma [18] (for  $d = 1$ ) and Hlawka [17] (for arbitrary  $d \in \mathbb{N}$ ). In classical cases the absolute error is bounded by the product of the variation of the integrand  $f$  in the sense of Hardy and Krause, and the star-discrepancy of the node set  $\mathcal{P}$ . Until today, several variants of these classical results for various function classes have been developed, see, e.g., [1, 3, 13, 30, 31, 35] to mention just a few references.

Nowadays it is very convenient to introduce Koksma-Hlawka type inequalities as equalities for the worst case integration errors of QMC rules for functions from Banach spaces  $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$  which are defined as

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}) = \sup_{\substack{f \in \mathcal{F} \\ \|f\|_{\mathcal{F}} \leq 1}} \left| \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x} - \text{QMC}_{d,n}(f) \right|.$$

For example, the classical Koksma-Hlawka inequality can then be stated as

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}) = L_{p^*}(\mathcal{P}),$$

where  $L_{p^*}(\mathcal{P})$  is the so-called  $L_{p^*}$ -discrepancy of the node set  $\mathcal{P}$  (see Section 2.4 below for a precise definition), when considering the norm

$$\|f\|_{\mathcal{F}} = \left( \int_{[0,1]^d} \left| \frac{\partial^d f}{\partial \mathbf{x}}(\mathbf{x}) \right|^p \, d\mathbf{x} \right)^{1/p}$$

for the space of functions anchored at  $\mathbf{0}$ , see [29, Sec. 3.1.5], where  $p, p^* \in [1, \infty]$  are conjugate, i.e.,  $1/p + 1/p^* = 1$ . For further information we also refer to [5, 30].

The most prominent example of a function space in this context is the anchored Sobolev space of regularity one, and, even more general, the  $\gamma$ -weighted anchored Sobolev spaces of regularity one. The exact definitions of these spaces will be given in Section 2.3. For these Sobolev spaces the Koksma-Hlawka theory is very well understood: the worst case error is exactly the weighted  $L_{p^*}$ -discrepancy.

Related and also often considered as reference spaces for QMC rules are the ( $\gamma$ -weighted) ANOVA spaces which will be formally introduced in Section 2.2; see, e.g., [4, 5, 12, 14, 29, 36]. While for special choices of weights the anchored space and the ANOVA space can be related in terms of embeddings (see, e.g., [9, 11, 12, 16, 21]) and therefore the error analysis for these two spaces is (up to embedding constants) equivalent, this is not possible for general weights.

In the present paper we provide a direct approach for the error analysis of QMC rules in  $\gamma$ -weighted ANOVA spaces. It is an advantage that this approach will work for general choices of non-negative weights without any restriction (cf. Remark 2 in Section 3).

A further advantage appears when we restrict ourselves to the 1D case (i.e.,  $d = 1$ ). Recall that the optimality of the composite midpoint rule for the  $L_{p^*}$ -discrepancy has been known for quite some time but only for  $p^* \in \{2, \infty\}$ , see [27], and was very recently extended to arbitrary  $p^* \in [1, \infty]$  in [22]. Using our results for the 1D case, we are able to provide an elementary

proof of the optimality of the composite midpoint rule for arbitrary  $p^*$  for integration in the ANOVA space (cf. Theorem 13 in Section 4.1).

The paper is organized as follows. In Section 2 we introduce the basic notation, the weighted ANOVA and anchored spaces of regularity one, and several notions of discrepancy. Section 3 is devoted to the error analysis of QMC rules in weighted ANOVA spaces. The main results show how the respective worst case errors can be related to the weighted discrepancy of the underlying node sets (cf. Corollary 5). Furthermore, we provide a general lower bound on the worst case error. In Section 4 two subsections are devoted to the 1D and the 2D cases, respectively. In the 1D case we will show that the composite midpoint rule is the optimal QMC rule among all QMC rules (Theorem 13). In the 2D case we show that, for example, shifted Hammersley point sets achieve the optimal convergence rate of the error (Example 17).

## 2 Basic definitions and facts

We begin with the notation used in the paper.

### 2.1 Notation

For a positive integer  $d$ , we write  $[d]$  to denote

$$[d] = \{1, 2, \dots, d\}.$$

We use  $\mathbf{u}$  and  $\mathbf{v}$  for monotonically increasing sequences of numbers from  $[d]$ , e.g.,

$$\mathbf{u} = (u_1, \dots, u_k), \quad \text{where } 1 \leq u_1 < \dots < u_k \leq d, \quad \text{and } k = |\mathbf{u}|.$$

This includes the empty sequence  $\mathbf{u} = \emptyset$  with  $|\emptyset| = 0$ . Often, it is convenient to treat the  $\mathbf{u}$ 's as sets, since then we can write  $\mathbf{u} \subseteq [d]$ ,  $j \in \mathbf{u}$ ,  $\mathbf{u} \setminus \mathbf{v}$ , etc.

For  $\mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d$  and  $\mathbf{u} \subseteq [d]$ , by  $\mathbf{x}_{\mathbf{u}}$  we mean the point in  $[0, 1]^{|\mathbf{u}|}$  with the coordinates  $x_j$  for  $j \in \mathbf{u}$ . That is,

$$\mathbf{x}_{\mathbf{u}} = (x_{u_1}, \dots, x_{u_k}) \quad \text{for } \mathbf{u} = (u_1, \dots, u_k).$$

For  $\mathbf{u} \neq \emptyset$ , we write  $\partial^{(\mathbf{u})}$  to denote mixed first order partial derivatives,

$$\partial^{(\mathbf{u})} = \prod_{j \in \mathbf{u}} \frac{\partial}{\partial x_{u_j}}.$$

For  $\mathbf{u} = \emptyset$ ,  $\partial^{(\emptyset)}$  is the identity operator.

We consider weights  $\boldsymbol{\gamma} = (\gamma_{\mathbf{u}})_{\mathbf{u} \subseteq [d]}$ , where the  $\gamma_{\mathbf{u}}$ 's are nonnegative reals. Sometimes we will use  $\mathcal{U}_+$  to list the sequences corresponding to positive weights,

$$\mathcal{U}_+ = \{\mathbf{u} \subseteq [d] : \gamma_{\mathbf{u}} > 0\}.$$

### 2.2 $\boldsymbol{\gamma}$ -weighted ANOVA spaces

For given  $d$ , weights  $\boldsymbol{\gamma}$ , and  $p \in [1, \infty]$ , the corresponding space  $\mathcal{F}_d = \mathcal{F}_{d,p,\boldsymbol{\gamma}}$  is the Banach space of functions

$$f : [0, 1]^d \rightarrow \mathbb{R}$$

endowed with the norm

$$\|f\|_{\mathcal{F}_d} = \left[ \sum_{\mathbf{u} \subseteq [d]} \gamma_{\mathbf{u}}^{-p} \left\| \int_{[0,1]^{d-|\mathbf{u}|}} \partial^{(\mathbf{u})} f(\cdot; \mathbf{t}_{[d] \setminus \mathbf{u}}) d\mathbf{t}_{[d] \setminus \mathbf{u}} \right\|_{L_p([0,1]^{|\mathbf{u}|})}^p \right]^{1/p} \quad \text{if } p < \infty,$$

and

$$\|f\|_{\mathcal{F}_d} = \max_{\mathbf{u} \subseteq [d]} \gamma_{\mathbf{u}}^{-1} \left\| \int_{[0,1]^{d-|\mathbf{u}|}} \partial^{(\mathbf{u})} f(\cdot; \mathbf{t}_{[d] \setminus \mathbf{u}}) d\mathbf{t}_{[d] \setminus \mathbf{u}} \right\|_{L_{\infty}([0,1]^{|\mathbf{u}|})} \quad \text{if } p = \infty.$$

By a convention  $0/0 = 0$  so that for  $\gamma_{\mathbf{u}} = 0$  the corresponding integral part of the definition is also zero. Note also that for  $\mathbf{u} = \emptyset$ ,

$$\int_{[0,1]^{d-|\emptyset|}} \partial^{(\emptyset)} f(\cdot; \mathbf{t}_{[d] \setminus \emptyset}) d\mathbf{t}_{[d] \setminus \emptyset} = \int_{[0,1]^d} f(\mathbf{t}) d\mathbf{t},$$

and for  $\mathbf{u} = [d]$  the above  $L_p$ -norms equal

$$\left\| \frac{\partial^d}{\partial x_1 \cdots \partial x_d} f \right\|_{L_p([0,1]^d)} \quad \text{for all } p \in [1, \infty].$$

Consider next the ANOVA decomposition of functions  $f \in \mathcal{F}_d$ ,

$$f = \sum_{\mathbf{u} \subseteq [d]} f_{\mathbf{u}},$$

where

$$f_{\emptyset} = \int_{[0,1]^d} f(\mathbf{x}) d\mathbf{x}$$

and, for nonempty  $\mathbf{u}$ ,  $f_{\mathbf{u}}$  depends only on  $\mathbf{x}_{\mathbf{u}}$ , and

$$\int_0^1 f_{\mathbf{u}}(\mathbf{t}) dt_j = 0 \quad \text{for any } j \in \mathbf{u}.$$

From [12] we know that  $f_{\mathbf{u}} \equiv 0$  for  $\mathbf{u} \notin \mathcal{U}_+$ , i.e.,

$$f = \sum_{\mathbf{u} \in \mathcal{U}_+} f_{\mathbf{u}}.$$

Moreover, for nonempty  $\mathbf{u} \in \mathcal{U}_+$

$$f_{\mathbf{u}}(\mathbf{x}) = \int_{[0,1]^{|\mathbf{u}|}} h_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) K_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) d\mathbf{t}_{\mathbf{u}},$$

where  $h_{\mathbf{u}} \in L_p([0,1]^{|\mathbf{u}|})$  and

$$K_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) = \prod_{\ell \in \mathbf{u}} K(x_{\ell}, t_{\ell}) \quad \text{with} \quad K(x, t) = \begin{cases} t & \text{if } x \geq t, \\ t - 1 & \text{if } x < t. \end{cases}$$

More precisely, let  $F_{\emptyset}$  be the space of constant functions with the absolute value as its norm. For nonempty  $\mathbf{u} \in \mathcal{U}_+$ , let

$$F_{\mathbf{u}} = \left\{ f_{\mathbf{u}} = \int_{[0,1]^{|\mathbf{u}|}} h_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) K_{\mathbf{u}}(\cdot, \mathbf{t}_{\mathbf{u}}) d\mathbf{t}_{\mathbf{u}} : h_{\mathbf{u}} \in L_p([0,1]^{|\mathbf{u}|}) \right\},$$

which is a Banach space with the norm

$$\|f_{\mathbf{u}}\|_{F_{\mathbf{u}}} = \|h_{\mathbf{u}}\|_{L_p([0,1]^{|\mathbf{u}|})}.$$

Then

$$\mathcal{F}_d = \bigoplus_{\mathbf{u} \in \mathcal{U}_+} F_{\mathbf{u}} \quad \text{and} \quad \|f\|_{\mathcal{F}_d} = \left[ \sum_{\mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{-p} \|f_{\mathbf{u}}\|_{F_{\mathbf{u}}}^p \right]^{1/p}$$

with the obvious modifications if  $p = \infty$ .

### 2.3 $\gamma$ -weighted anchored spaces

Consider the following Banach space  $\mathcal{F}_d^{\mathfrak{h}} = \mathcal{F}_{d,p,\gamma}^{\mathfrak{h}}$  of functions

$$f : [0, 1]^d \rightarrow \mathbb{R}$$

whose norm is given by

$$\|f\|_{\mathcal{F}_d^{\mathfrak{h}}} = \left[ \sum_{\mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{-p} \left\| \partial^{(\mathbf{u})} f(\cdot; \mathbf{0}_{[d] \setminus \mathbf{u}}) \right\|_{L_p([0,1]^{|\mathbf{u}|})}^p \right]^{1/p} \quad \text{if } p < \infty$$

and

$$\|f\|_{\mathcal{F}_d^{\mathfrak{h}}} = \max_{\mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{-1} \left\| \partial^{(\mathbf{u})} f(\cdot; \mathbf{0}_{[d] \setminus \mathbf{u}}) \right\|_{L_{\infty}([0,1]^{|\mathbf{u}|})} \quad \text{if } p = \infty.$$

Consider next the *anchored* decompositions of  $f \in \mathcal{F}_d^{\mathfrak{h}}$ ,

$$f = \sum_{\mathbf{u} \in \mathcal{U}_+} f_{\mathbf{u}}^{\mathfrak{h}},$$

where, for  $\mathbf{u} \neq \emptyset$ ,  $f_{\mathbf{u}}^{\mathfrak{h}}$  depends only on  $\mathbf{x}_{\mathbf{u}}$  and

$$f_{\mathbf{u}}^{\mathfrak{h}}(\mathbf{x}_{\mathbf{u}}) = 0 \quad \text{if } x_j = 0 \text{ for some } j \in \mathbf{u}.$$

We know from [12] that for nonempty  $\mathbf{u} \in \mathcal{U}_+$

$$f_{\mathbf{u}}^{\mathfrak{h}}(\mathbf{x}) = \int_{[0,1]^d} h_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) K_{\mathbf{u}}^{\mathfrak{h}}(\mathbf{x}_{\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) d\mathbf{t}_{\mathbf{u}},$$

where  $h_{\mathbf{u}} \in L_p([0, 1]^{|\mathbf{u}|})$  and

$$K_{\mathbf{u}}^{\mathfrak{h}}(\mathbf{x}_{\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) = \prod_{\ell \in \mathbf{u}} K^{\mathfrak{h}}(x_{\ell}, t_{\ell}) \quad \text{with} \quad K^{\mathfrak{h}}(x, t) = \begin{cases} 1 & \text{if } x \geq t, \\ 0 & \text{if } x < t. \end{cases}$$

As in the previous section, let  $F_{\emptyset}^{\mathfrak{h}} = F_{\emptyset}$  be the space of constant functions with the absolute value as its norm. For nonempty  $\mathbf{u}$ , let

$$F_{\mathbf{u}}^{\mathfrak{h}} = \left\{ f_{\mathbf{u}}^{\mathfrak{h}} = \int_{[0,1]^{|\mathbf{u}|}} h_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) K_{\mathbf{u}}^{\mathfrak{h}}(\cdot, \mathbf{t}_{\mathbf{u}}) d\mathbf{t}_{\mathbf{u}} : h_{\mathbf{u}} \in L_p([0, 1]^{|\mathbf{u}|}) \right\},$$

which is a Banach space with the norm

$$\|f_{\mathbf{u}}^{\mathfrak{h}}\|_{F_{\mathbf{u}}^{\mathfrak{h}}} = \|h_{\mathbf{u}}\|_{L_p([0,1]^{|\mathbf{u}|})}.$$

Then

$$\mathcal{F}_d^{\mathfrak{h}} = \mathcal{F}_{d,p,\gamma}^{\mathfrak{h}} = \bigoplus_{\mathbf{u} \in \mathcal{U}_+} F_{\mathbf{u}}^{\mathfrak{h}} \quad \text{and} \quad \|f\|_{\mathcal{F}_d^{\mathfrak{h}}} = \left[ \sum_{\mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{-p} \|f_{\mathbf{u}}^{\mathfrak{h}}\|_{F_{\mathbf{u}}^{\mathfrak{h}}}^p \right]^{1/p}$$

is the  $\gamma$ -weighted anchored space of functions with anchor  $\mathbf{0}$ .

We now recall the following relation between the ANOVA and the anchored spaces, see [12, Proposition 13].

**Proposition 1.** *For any  $p \in [1, \infty]$  and  $\gamma$  the following holds. The  $\gamma$ -weighted anchored and ANOVA spaces are equal (as sets of functions) if and only if*

$$\gamma_{\mathbf{u}} > 0 \quad \text{implies} \quad \gamma_{\mathbf{v}} > 0 \quad \text{for all} \quad \mathbf{v} \subseteq \mathbf{u}. \quad (1)$$

Moreover, if (1) does not hold, then

$$\mathcal{F}_{d,p,\gamma}^{\mathfrak{h}} \not\subseteq \mathcal{F}_{d,p,\gamma} \quad \text{and} \quad \mathcal{F}_{d,p,\gamma} \not\subseteq \mathcal{F}_{d,p,\gamma}^{\mathfrak{h}}.$$

## 2.4 Discrepancy and weighted discrepancy

We now recall the definition of (weighted) discrepancy, which is related to the errors of QMC methods studied in this paper.

For a point set  $\mathcal{P} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  in  $[0, 1]^d$  the local discrepancy function  $\Delta_{\mathcal{P}} : [0, 1]^d \rightarrow \mathbb{R}$  is defined as

$$\Delta_{\mathcal{P}}(\mathbf{t}) = \frac{|\{j \in \{1, \dots, n\} : \mathbf{x}_j \in [\mathbf{0}, \mathbf{t}]\}|}{n} - \lambda([\mathbf{0}, \mathbf{t}]),$$

where  $\mathbf{t} = (t_1, \dots, t_d) \in [0, 1]^d$ ,  $[\mathbf{0}, \mathbf{t}] = [0, t_1] \times \dots \times [0, t_d]$  and  $\lambda([\mathbf{0}, \mathbf{t}]) = t_1 \cdots t_d$ . The local discrepancy function can be expressed in terms of the indicator function, namely,

$$\Delta_{\mathcal{P}}(\mathbf{t}) = \frac{1}{n} \sum_{j=1}^n 1_{[\mathbf{0}, \mathbf{t}]}(\mathbf{x}_j) - \lambda([\mathbf{0}, \mathbf{t}]),$$

where  $1_{[\mathbf{0}, \mathbf{t}]}(\mathbf{x}_j) = 1$  if  $\mathbf{x}_j \in [\mathbf{0}, \mathbf{t}]$  and 0 otherwise. Note that

$$1_{[\mathbf{0}, \mathbf{t}]}(\mathbf{x}_j) = \prod_{i=1}^d 1_{[0, t_i]}(x_{j,i}),$$

where  $x_{j,i}$  is the  $i^{\text{th}}$  component of  $\mathbf{x}_j$ .

For  $p^* \in [1, \infty]$  the  $L_{p^*}$ -discrepancy of  $\mathcal{P}$  is defined as the  $L_{p^*}$ -norm of the local discrepancy function, i.e.,

$$L_{p^*}(\mathcal{P}) = \|\Delta_{\mathcal{P}}\|_{L_{p^*}([0,1]^d)}.$$

Furthermore, the  $\gamma$ -weighted  $L_{p^*}$ -discrepancy of  $\mathcal{P}$  is defined as

$$L_{p^*,\gamma}(\mathcal{P}) = \left[ \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{p^*} \|\Delta_{\mathcal{P}_{\mathbf{u}}}\|_{L_{p^*}([0,1]^{|\mathbf{u}|})}^{p^*} \right]^{1/p^*} \quad \text{for} \quad p^* < \infty$$

and

$$L_{\infty,\gamma}(\mathcal{P}) = \max_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}} \|\Delta_{\mathcal{P}_{\mathbf{u}}}\|_{L_{\infty}([0,1]^{|\mathbf{u}|})} \quad \text{for} \quad p^* < \infty,$$

where  $\Delta_{\mathcal{P}_{\mathbf{u}}}$  denotes the local discrepancy function of the set that consists of the projected points of  $\mathcal{P}$  to the coordinates with indices in  $\mathbf{u}$ . Weighted  $L_{p^*}$ -discrepancy was first introduced and studied by Sloan and Woźniakowski [35]. For further information on weighted discrepancy we also refer to [5, 30].

### 3 Quasi-Monte Carlo methods and their errors

In this main section we consider QMC methods of the form

$$\text{QMC}_{d,n}(f) = \frac{1}{n} \sum_{j=1}^n f(\mathbf{t}_j)$$

for some deterministically chosen points  $\mathbf{t}_j \in [0, 1]^d$ . We are interested in their worst case errors with respect to the unit ball of the space  $\mathcal{F}_d$  defined as

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) = \sup_{\substack{f \in \mathcal{F}_d \\ \|f\|_{\mathcal{F}_d} \leq 1}} \left| \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x} - \text{QMC}_{d,n}(f) \right|.$$

**Remark 2.** If Condition (1) is satisfied, then to study QMC in the ANOVA space one may consider the embedding operator  $\iota : \mathcal{F}_d \rightarrow \mathcal{F}_d^{\text{fn}}$ ,  $\iota(f) = f$ . Then

$$\frac{1}{\|\iota^{-1}\|} \text{error}(\text{QMC}_{d,n}; \mathcal{F}_d^{\text{fn}}) \leq \text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) \leq \|\iota\| \text{error}(\text{QMC}_{d,n}; \mathcal{F}_d^{\text{fn}}),$$

where  $\|\iota\|$  and  $\|\iota^{-1}\|$  are the operator norms of the embedding operator  $\iota$  and its inverse  $\iota^{-1}$ , respectively. It is well known (see, e.g., [35]) that

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d^{\text{fn}}) = L_{p^*, \gamma}(\overline{\mathcal{P}}),$$

where  $L_{p^*, \gamma}(\overline{\mathcal{P}})$  is the weighted  $L_{p^*}$ -discrepancy of the point set<sup>1</sup>

$$\overline{\mathcal{P}} = \{\mathbf{1} - \mathbf{t}_j : j = 1, \dots, n\},$$

where  $\mathbf{1} - \mathbf{t}_j$  is defined as the component-wise difference of the vector containing only ones and  $\mathbf{t}_j$ . Estimates or, in some cases, exact values of  $\max\{\|\iota\|, \|\iota^{-1}\|\}$  can be found in, e.g., [9, 11, 12, 14, 16, 20] (see also Remark 14). Results on weighted discrepancy can be found in, e.g., [5, 6, 24, 35].

However, in order to follow the approach as sketched in Remark 2 one requires the assumption that Condition (1) is satisfied. For example, this condition is not satisfied for weights of the form  $\gamma_{[d]} = 1$  and  $\gamma_{\mathbf{u}} = 0$  for all  $\mathbf{u} \subsetneq [d]$ . In the present paper we follow a direct approach of an error analysis for QMC rules in the  $\gamma$ -weighted ANOVA space that does not require Condition (1) and the embedding of the ANOVA space into the anchored space.

#### 3.1 A formula for the worst case error

The following theorem gives a formula for the worst case integration error.

**Theorem 3.** *For any QMC rule  $\text{QMC}_{d,n}$*

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) = \frac{1}{n} \left[ \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{p^*} \int_{[0,1]^{|\mathbf{u}|}} \left| \sum_{j=1}^n K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) \right|^{p^*} d\mathbf{t}_{\mathbf{u}} \right]^{1/p^*} \quad \text{for } p^* < \infty$$

<sup>1</sup>We remark that in [35] the anchored space with anchor  $\mathbf{1}$  is considered which results in a worst case error of exactly  $L_{p^*, \gamma}(\mathcal{P})$ , where  $\mathcal{P}$  is the node set of the QMC rule. Here we have chosen the anchor as  $\mathbf{0}$ , and therefore in the formula for the worst case error the point set  $\overline{\mathcal{P}}$  appears.



and

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) = \frac{1}{n} \max_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}} \left\| \sum_{j=1}^n K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \cdot_{\mathbf{u}}) \right\|_{L_{\infty}([0,1]^{|\mathbf{u}|})} \quad \text{for } p^* = \infty,$$

where we write  $\mathbf{x}_{j,\mathbf{u}}$  short hand for  $(\mathbf{x}_j)_{\mathbf{u}}$ .

*Proof.* We present the proof only for  $p^* < \infty$  since it is very similar for  $p^* = \infty$ .

In the following, let, for  $\mathbf{u} \subseteq [d]$ ,  $\text{QMC}_{\mathbf{u},n}$  denote the projection of the rule  $\text{QMC}_{d,n}$  onto those coordinates with indices in  $\mathbf{u}$  (i.e., the rule is based on  $|\mathbf{u}|$ -dimensional integration nodes obtained by projecting the nodes of  $\text{QMC}_{d,n}$  accordingly). For any  $f \in \mathcal{F}_d$ ,

$$\begin{aligned} \left| \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x} - \text{QMC}_{d,n}(f) \right| &= \left| \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \text{QMC}_{d,n}(f_{\mathbf{u}}) \right| \\ &= \frac{1}{n} \left| \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \sum_{j=1}^n \int_{[0,1]^{|\mathbf{u}|}} h_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) \, d\mathbf{t}_{\mathbf{u}} \right| \\ &= \frac{1}{n} \left| \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \int_{[0,1]^{|\mathbf{u}|}} h_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) \sum_{j=1}^n K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) \, d\mathbf{t}_{\mathbf{u}} \right| \\ &\leq \frac{1}{n} \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \|h_{\mathbf{u}}\|_{L_p} \left[ \int_{[0,1]^{|\mathbf{u}|}} \left| \sum_{j=1}^n K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) \right|^{p^*} \, d\mathbf{t}_{\mathbf{u}} \right]^{1/p^*}. \quad (2) \end{aligned}$$

Using Hölder's inequality one more time we get another upper bound,

$$\left| \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x} - \text{QMC}_{d,n}(f) \right| \leq \frac{1}{n} \|f\|_{\mathcal{F}_d} \left[ \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{p^*} \int_{[0,1]^{|\mathbf{u}|}} \left| \sum_{j=1}^n K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) \right|^{p^*} \, d\mathbf{t}_{\mathbf{u}} \right]^{1/p^*}. \quad (3)$$

To prove equality, we will use the fact that Hölder's inequality is sharp. For that purpose recall that for two functions  $f$  and  $g$

$$\int_D |g f| = \|g\|_{L_p(D)} \|f\|_{L_{p^*}(D)} \quad \text{if } |g| = c|f|^{p^*-1} \text{ a.e. on } D,$$

and for two sequences of numbers  $a_i$  and  $b_i$

$$\sum_i |a_i b_i| = \left[ \sum_i |a_i|^p \right]^{1/p} \left[ \sum_i |b_i|^{p^*} \right]^{1/p^*} \quad \text{if } |a_i| = c|b_i|^{p^*-1} \text{ for all } i.$$

Consider next the function  $f$  with  $h_{\emptyset} = 0$  and

$$h_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) = \left| \sum_{j=1}^n K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) \right|^{p^*-1}.$$

For such  $h_{\mathbf{u}}$ 's we have equality in (2). Moreover, for every  $\mathbf{u} \neq \emptyset$ ,

$$\begin{aligned} \|h_{\mathbf{u}}\|_{L_p([0,1]^{|\mathbf{u}|})} &= \left[ \int_{[0,1]^{|\mathbf{u}|}} \left| \sum_{j=1}^n K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) \right|^{p(p^*-1)} \, d\mathbf{t}_{\mathbf{u}} \right]^{1/p} \\ &= \left\| \sum_{j=1}^n K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \cdot_{\mathbf{u}}) \right\|_{L_{p^*}([0,1]^{|\mathbf{u}|})}^{p^*-1}, \end{aligned}$$

since  $p(p^* - 1) = p^*$  and  $1/p = (p^* - 1)/p^*$ . Therefore we have equality also in (3) which completes the proof.  $\square$

Next we relate the worst case error to the  $L_{p^*}$ -discrepancy of the point sets underlying the QMC rule.

**Lemma 4.** *For every nonempty  $\mathbf{u} \subseteq [d]$  we have*

$$\sum_{j=1}^n K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) = n \sum_{k=1}^{|\mathbf{u}|} (-1)^k \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}}) \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i,$$

where  $\Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}})$  denotes the local discrepancy function in  $\mathbf{t}_{\mathbf{v}}$  of the set that consists of the points of  $\mathcal{P}$  projected onto the coordinates with indices in  $\mathbf{v}$ .

*Proof.* Observe that

$$\sum_{j=1}^n K_{\mathbf{u}}(\mathbf{x}_{j,\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) = \sum_{j=1}^n \prod_{i \in \mathbf{u}} (t_i - 1_{[0,t_i]}(x_{j,i})). \quad (4)$$

Let us now rewrite the product above as

$$\begin{aligned} \prod_{i \in \mathbf{u}} (t_i - 1_{[0,t_i]}(x_{j,i})) &= \sum_{\mathbf{v} \subseteq \mathbf{u}} \prod_{i \in \mathbf{v}} (-1_{[0,t_i]}(x_{j,i})) \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i \\ &= \sum_{k=0}^{|\mathbf{u}|} \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \prod_{i \in \mathbf{v}} (-1_{[0,t_i]}(x_{j,i})) \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i. \end{aligned}$$

We have

$$\sum_{k=0}^{|\mathbf{u}|} \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} (-1)^{|\mathbf{v}|} \prod_{i \in \mathbf{u}} t_i = \left( \prod_{i \in \mathbf{u}} t_i \right) \sum_{k=0}^{|\mathbf{u}|} \binom{|\mathbf{u}|}{k} (-1)^k = \left( \prod_{i \in \mathbf{u}} t_i \right) (1 + (-1))^{|\mathbf{u}|} = 0.$$

Hence

$$\begin{aligned} \prod_{i \in \mathbf{u}} (t_i - 1_{[0,t_i]}(x_{j,i})) &= \sum_{k=0}^{|\mathbf{u}|} \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \left[ \prod_{i \in \mathbf{v}} (-1_{[0,t_i]}(x_{j,i})) \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i - (-1)^{|\mathbf{v}|} \prod_{i \in \mathbf{u}} t_i \right] \\ &= \sum_{k=0}^{|\mathbf{u}|} (-1)^k \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \left[ \prod_{i \in \mathbf{v}} 1_{[0,t_i]}(x_{j,i}) - \prod_{i \in \mathbf{v}} t_i \right] \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i. \end{aligned}$$

For  $k = 0$  we have  $\mathbf{v} = \emptyset$  and hence

$$\prod_{i \in \mathbf{v}} 1_{[0,t_i]}(x_{j,i}) - \prod_{i \in \mathbf{v}} t_i = 0.$$

This means that it suffices to start the summation with the index 1. Hence

$$\prod_{i \in \mathbf{u}} (t_i - 1_{[0,t_i]}(x_{j,i})) = \sum_{k=1}^{|\mathbf{u}|} (-1)^k \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} [1_{[0,\mathbf{t}_{\mathbf{v}}]}(\mathbf{x}_{j,\mathbf{v}}) - \lambda([0, \mathbf{t}_{\mathbf{v}}])] \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i.$$

Summation over all  $j = 1, \dots, n$  gives

$$\sum_{j=1}^n \prod_{i \in \mathbf{u}} (t_i - 1_{[0, t_i)}(x_{j,i})) = \sum_{k=1}^{|\mathbf{u}|} (-1)^k \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} n \Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}}) \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i.$$

□

The following corollary to Theorem 3 bounds the error in terms of the weighted  $L_{p^*}$ -discrepancy of the node set  $\mathcal{P}$  underlying the QMC rule for suitably modified weights  $\tilde{\gamma}$ .

**Corollary 5.** *For any QMC rule  $\text{QMC}_{d,n}$  and  $p \in [1, \infty]$  we have*

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) \leq L_{p^*, \tilde{\gamma}}(\mathcal{P}),$$

where the latter is the  $\tilde{\gamma}$ -weighted  $L_{p^*}$ -discrepancy of  $\mathcal{P}$  and  $\tilde{\gamma} = (\tilde{\gamma}_{\mathbf{u}})_{\mathbf{u} \subseteq [d]}$  with

$$\tilde{\gamma}_{\mathbf{u}} = \left( (p^* + 1)^{|\mathbf{u}|} \sum_{\substack{\mathbf{v} \in \mathcal{U}_+ \\ \mathbf{v} \supseteq \mathbf{u}}} \gamma_{\mathbf{v}}^{p^*} \left( \frac{2^{p^* - 1}}{p^* + 1} \right)^{|\mathbf{v}|} \right)^{1/p^*} \quad \text{for } p^* < \infty,$$

and

$$\tilde{\gamma}_{\mathbf{u}} = 2^{|\mathbf{u}|} \gamma_{\mathbf{u}} \quad \text{for } p^* = \infty.$$

For the proof of Corollary 5 we use the following simple lemma.

**Lemma 6.** *For  $p^* \in [1, \infty)$  and  $x_k \geq 0$  for  $k = 1, 2, \dots, \ell$  we have*

$$\left( \sum_{k=1}^{\ell} x_k \right)^{p^*} \leq \ell^{p^* - 1} \sum_{k=1}^{\ell} x_k^{p^*}$$

with equality if  $x_1 = \dots = x_{\ell}$ .

*Proof.* We use Hölder's inequality and the fact that  $p^*/p = p^* - 1$  to obtain

$$\begin{aligned} \left( \sum_{k=1}^{\ell} x_k \right)^{p^*} &= \left( \sum_{k=1}^{\ell} x_k \cdot 1 \right)^{p^*} \\ &\leq \left( \left( \sum_{k=1}^{\ell} x_k^{p^*} \right)^{1/p^*} \left( \sum_{k=1}^{\ell} 1^p \right)^{1/p} \right)^{p^*} \\ &= \ell^{p^* / p} \sum_{k=1}^{\ell} x_k^{p^*} \\ &= \ell^{p^* - 1} \sum_{k=1}^{\ell} x_k^{p^*}. \end{aligned}$$

□

*Proof of Corollary 5.* Consider first  $p > 1$  and hence  $p^* < \infty$ . According to Theorem 3 and Lemma 4 we have

$$\begin{aligned}
\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) &= \left[ \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{p^*} \int_{[0,1]^{|\mathbf{u}|}} \left| \sum_{k=1}^{|\mathbf{u}|} (-1)^k \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}}) \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i \right|^{p^*} d\mathbf{t}_{\mathbf{u}} \right]^{1/p^*} \\
&\leq \left[ \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{p^*} \int_{[0,1]^{|\mathbf{u}|}} \left( \sum_{\emptyset \neq \mathbf{v} \subseteq \mathbf{u}} |\Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}})| \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i \right)^{p^*} d\mathbf{t}_{\mathbf{u}} \right]^{1/p^*} \\
&\leq \left[ \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{p^*} 2^{|\mathbf{u}|(p^*-1)} \int_{[0,1]^{|\mathbf{u}|}} \sum_{\emptyset \neq \mathbf{v} \subseteq \mathbf{u}} |\Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}})|^{p^*} \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i^{p^*} d\mathbf{t}_{\mathbf{u}} \right]^{1/p^*},
\end{aligned} \tag{5}$$

where we applied Lemma 6 to the innermost sum. Interchanging the integral and the inner sum gives

$$\begin{aligned}
&\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) \\
&\leq \left[ \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{p^*} 2^{|\mathbf{u}|(p^*-1)} \sum_{\emptyset \neq \mathbf{v} \subseteq \mathbf{u}} \left( \int_{[0,1]^{|\mathbf{v}|}} |\Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}})|^{p^*} d\mathbf{t}_{\mathbf{v}} \right) \left( \int_0^1 t^{p^*} dt \right)^{|\mathbf{u}|-|\mathbf{v}|} \right]^{1/p^*} \\
&= \left[ \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{p^*} 2^{|\mathbf{u}|(p^*-1)} \sum_{\emptyset \neq \mathbf{v} \subseteq \mathbf{u}} (L_{p^*}(\mathcal{P}_{\mathbf{v}}))^{p^*} \left( \frac{1}{p^* + 1} \right)^{|\mathbf{u}|-|\mathbf{v}|} \right]^{1/p^*} \\
&= \left[ \sum_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}}^{p^*} \left( \frac{2^{p^*-1}}{p^* + 1} \right)^{|\mathbf{u}|} \sum_{\emptyset \neq \mathbf{v} \subseteq \mathbf{u}} (p^* + 1)^{|\mathbf{v}|} (L_{p^*}(\mathcal{P}_{\mathbf{v}}))^{p^*} \right]^{1/p^*}.
\end{aligned}$$

Now we interchange the order of summation and obtain in this way

$$\begin{aligned}
\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) &\leq \left[ \sum_{\emptyset \neq \mathbf{v} \in \mathcal{U}_+} \left( (p^* + 1)^{|\mathbf{v}|} \sum_{\substack{\mathbf{u} \in \mathcal{U}_+ \\ \mathbf{u} \supseteq \mathbf{v}}} \gamma_{\mathbf{u}}^{p^*} \left( \frac{2^{p^*-1}}{p^* + 1} \right)^{|\mathbf{u}|} \right) (L_{p^*}(\mathcal{P}_{\mathbf{v}}))^{p^*} \right]^{1/p^*} \\
&= \left[ \sum_{\emptyset \neq \mathbf{v} \in \mathcal{U}_+} \tilde{\gamma}_{\mathbf{v}}^{p^*} (L_{p^*}(\mathcal{P}_{\mathbf{v}}))^{p^*} \right]^{1/p^*} \\
&= L_{p^*, \tilde{\gamma}}(\mathcal{P}).
\end{aligned}$$

If  $p = 1$ , and hence  $p^* = \infty$ , we trivially have

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) \leq \max_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}} \sum_{\emptyset \neq \mathbf{v} \subseteq \mathbf{u}} L_{\infty}(\mathcal{P}_{\mathbf{v}}),$$

and from this we obtain

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) \leq \max_{\emptyset \neq \mathbf{u} \in \mathcal{U}_+} \gamma_{\mathbf{u}} 2^{|\mathbf{u}|} L_{\infty}(\mathcal{P}_{\mathbf{u}}) = L_{\infty, \tilde{\gamma}}(\mathcal{P}).$$

□

**Remark 7.** For product weights  $\gamma_{\mathbf{u}} = \prod_{j \in \mathbf{u}} \gamma_j$  with a sequence  $(\gamma_j)_{j \geq 1}$  of positive weights we have

$$\tilde{\gamma}_{\mathbf{u}} = \left( \prod_{j \in \mathbf{u}} \frac{2\gamma_j}{2^{1/p^*}} \right) \prod_{j \in [d] \setminus \mathbf{u}} \left( 1 + \frac{2^{p^*-1}}{p^*+1} \gamma_j^{p^*} \right)^{1/p^*} \quad \text{for } p^* < \infty,$$

and

$$\tilde{\gamma}_{\mathbf{u}} = \prod_{j \in \mathbf{u}} (2\gamma_j) \quad \text{for } p^* = \infty.$$

**Remark 8.** To have small worst case error, one should use node sets with low weighted  $L_{p^*}$ -discrepancy. These discrepancies have been well studied with respect to both, the order of magnitude in  $n$  as well as their dependence on the dimension  $d$ . There are constructions of  $n$ -element point sets in  $[0, 1]^d$  yielding a convergence rate of order  $O((\log n)^{(d-1)/2}/n)$  if  $p^* \in [1, \infty)$  and  $O((\log n)^{d-1}/n)$  if  $p^* = \infty$ . Furthermore, conditions on the weights are known which guarantee various kinds of tractability for the weighted discrepancy and hence for the corresponding integration problem in the ANOVA space. For information see, for example, [5, 6, 24, 30] and the references therein.

### 3.2 A general lower bound for the worst case error

We now provide the following general lower bound.

**Theorem 9.** *Assume that  $\gamma_{\mathbf{u}} > 0$  for every  $\emptyset \neq \mathbf{u} \subseteq [d]$ . Then for every  $p \in [1, \infty)$  there exists a positive constant  $c = c(p^*, d, \boldsymbol{\gamma})$  such that any QMC rule based on an  $n$ -element point set in  $[0, 1]^d$  has the worst case error bounded from below by*

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) \geq c \frac{(\ln n)^{\frac{d-1}{2}}}{n}. \quad (6)$$

For  $p = \infty$  we have

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) \geq c \frac{(\ln n)^{\frac{1}{2}}}{n}. \quad (7)$$

For  $d = 2$  and  $p = 1$  the lower bound (6) can be improved to

$$\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) \geq c \frac{\ln n}{n}. \quad (8)$$

For the proof we need the following technical lemma.

**Lemma 10.** *Let  $d \in \mathbb{N}$ ,  $d \geq 2$ , and let  $T_d(\ell)$  for  $\ell \in [d]$  be defined by*

$$T_d(\ell) := \frac{(\ell!)^2}{(d!)^2}.$$

*Then it is true for every  $\ell \in \{2, \dots, d\}$  that*

$$T_d(\ell) > \sum_{k=1}^{\ell-1} \binom{\ell}{k} T_d(k).$$

*Proof.* Let, for  $\ell \in [d]$ ,

$$s(\ell) := \sum_{k=1}^{\ell-1} \binom{\ell}{k} (k!)^2.$$

Showing the desired inequality is equivalent to showing that  $s(\ell) < (\ell!)^2$  for all  $\ell \in \{2, \dots, d\}$ . This is done by induction on  $\ell$ . It is easily checked that the assertion holds for  $\ell = 1, 2$ . Assume that we have  $s(\ell) < (\ell!)^2$ . Now we consider  $s(\ell + 1)$ . We have

$$\begin{aligned} s(\ell + 1) &= \sum_{k=1}^{\ell} \binom{\ell + 1}{k} (k!)^2 \\ &= \sum_{k=1}^{\ell} \frac{\ell + 1}{k} \binom{\ell}{k-1} (k!)^2 \\ &= \sum_{k=0}^{\ell-1} \frac{\ell + 1}{k+1} \binom{\ell}{k} ((k+1)!)^2 \\ &= \ell + 1 + \sum_{k=1}^{\ell-1} \binom{\ell}{k} (k!)^2 (\ell + 1)(k + 1) \\ &\leq \ell + 1 + (\ell + 1) \ell s(\ell) \\ &< \ell + 1 + (\ell + 1) \ell (\ell!)^2 \\ &\leq (\ell + 1)^2 (\ell!)^2 \\ &= ((\ell + 1)!)^2. \end{aligned}$$

This completes the proof.  $\square$

*Proof of Theorem 9.* Note that for every  $p^* \in (1, \infty]$  there exists a  $C = C(d, p^*) > 0$  such that for every  $n$ -element point set  $\mathcal{P}$  in  $[0, 1]^d$  we have

$$L_{p^*}(\mathcal{P}) \geq C(d, p^*) \frac{(\ln n)^{\frac{d-1}{2}}}{n}. \quad (9)$$

For  $p^* \geq 2$  this is a famous result by Roth [32] that was extended later by Schmid [34] to the case  $p^* \in (1, 2)$ . For  $p^* = 1$  we always have

$$L_1(\mathcal{P}) \geq C(d, p^*) \frac{(\ln n)^{\frac{1}{2}}}{n}, \quad (10)$$

as shown by Halász [10] for  $d = 2$ , but the result holds for all  $d \in \mathbb{N}$  (cf. [2]). For  $d = 2$  and  $p^* = \infty$  the lower bound (9) can, according to Schmidt [33], be tightened to

$$L_{\infty}(\mathcal{P}) \geq C(2, \infty) \frac{\ln n}{n}. \quad (11)$$

We now use the sequence  $T_d(\ell)$  for  $\ell \in [d]$  from Lemma 10,  $T_d(\ell) := (\ell!)^2 / (d!)^2$ .

According to (5), we have for  $p > 1$ , and hence  $p^* < \infty$ ,

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) = \left[ \sum_{\emptyset \neq \mathbf{u} \subseteq [d]} \gamma_{\mathbf{u}}^{p^*} \int_{[0,1]^{|\mathbf{u}|}} \left| \sum_{k=1}^{|\mathbf{u}|} (-1)^k \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}}) \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i \right|^{p^*} d\mathbf{t}_{\mathbf{u}} \right]^{1/p^*}. \quad (12)$$

Assume that  $p < \infty$  and hence  $p^* > 1$ . Suppose first that

$$L_{p^*}(\mathcal{P}_{\{i\}}) \geq T_d(1)C(d, p^*) \frac{(\ln n)^{\frac{d-1}{2}}}{n} \quad (13)$$

for some  $i \in [d]$ . Then we obtain from (5) or (12), respectively, that

$$\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) \geq \gamma_{\{i\}} L_{p^*}(\mathcal{P}_{\{i\}}) \geq \gamma_{\{i\}} T_d(1)C(d, p^*) \frac{(\ln n)^{\frac{d-1}{2}}}{n}.$$

If (13) does not hold for any  $i \in [d]$ , let  $\ell \in \{2, \dots, d\}$  be minimal such that the following two conditions hold:

(i) There exists  $\mathbf{u} \subseteq [d]$ ,  $\mathbf{u} \neq \emptyset$ , with  $|\mathbf{u}| = \ell$  such that

$$L_{p^*}(\mathcal{P}_{\mathbf{u}}) \geq T_d(\ell)C(d, p^*) \frac{(\ln n)^{\frac{d-1}{2}}}{n},$$

(ii) but

$$L_{p^*}(\mathcal{P}_{\mathbf{v}}) < T_d(|\mathbf{v}|)C(d, p^*) \frac{(\ln n)^{\frac{d-1}{2}}}{n}$$

for all  $\mathbf{v} \subsetneq \mathbf{u}$ ,  $\mathbf{v} \neq \emptyset$ .

Note that  $\ell \geq 2$  since we assumed that (13) does not hold for any  $i \in [d]$ , and  $\ell \leq d$  due to the fact that (9) holds and  $T_d(d) = 1$ .

Let now  $\ell$  be defined as above, and let  $\mathbf{u} \subseteq [d]$  be such that Condition (i) holds. Then it follows from (12) that

$$\begin{aligned} \text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) &\geq \left[ \gamma_{\mathbf{u}}^{p^*} \int_{[0,1]^\ell} \left| (-1)^\ell \Delta_{\mathcal{P}_{\mathbf{u}}}(\mathbf{t}_{\mathbf{u}}) + \sum_{k=1}^{\ell-1} (-1)^k \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}}) \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i \right|^{p^*} d\mathbf{t}_{\mathbf{u}} \right]^{1/p^*} \\ &= \left[ \gamma_{\mathbf{u}}^{p^*} \int_{[0,1]^\ell} \left| \Delta_{\mathcal{P}_{\mathbf{u}}}(\mathbf{t}_{\mathbf{u}}) - (-1)^{\ell-1} \sum_{k=1}^{\ell-1} (-1)^k \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}}) \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i \right|^{p^*} d\mathbf{t}_{\mathbf{u}} \right]^{1/p^*}. \end{aligned}$$

Let now

$$g_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) := (-1)^{\ell-1} \sum_{k=1}^{\ell-1} (-1)^k \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}}) \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i \quad \text{for } \mathbf{t}_{\mathbf{u}} \in [0, 1]^\ell.$$

Then,

$$\begin{aligned} \text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) &\geq \gamma_{\mathbf{u}} \left[ \int_{[0,1]^d} |\Delta_{\mathcal{P}_{\mathbf{u}}}(\mathbf{t}_{\mathbf{u}}) - g_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}})|^{p^*} d\mathbf{t}_{\mathbf{u}} \right]^{1/p^*} \\ &= \gamma_{\mathbf{u}} \|\Delta_{\mathcal{P}_{\mathbf{u}}} - g_{\mathbf{u}}\|_{L_{p^*}} \\ &\geq \gamma_{\mathbf{u}} \left| \|\Delta_{\mathcal{P}_{\mathbf{u}}}\|_{L_{p^*}} - \|g_{\mathbf{u}}\|_{L_{p^*}} \right| \\ &= \gamma_{\mathbf{u}} \left| L_{p^*}(\mathcal{P}_{\mathbf{u}}) - \|g_{\mathbf{u}}\|_{L_{p^*}} \right|. \end{aligned}$$

However,

$$\begin{aligned}
\|g_{\mathbf{u}}\|_{L_{p^*}} &\leq \sum_{k=1}^{\ell-1} \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \left\| \left( \prod_{i \in \mathbf{u} \setminus \mathbf{v}} t_i \right) \Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}}) \right\|_{L_{p^*}} \\
&\leq \sum_{k=1}^{\ell-1} \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} \left( \frac{1}{(p^* + 1)^{1/p^*}} \right)^{\ell-k} \|\Delta_{\mathcal{P}_{\mathbf{v}}}(\mathbf{t}_{\mathbf{v}})\|_{L_{p^*}} \\
&\leq \sum_{k=1}^{\ell-1} \sum_{\substack{\mathbf{v} \subseteq \mathbf{u} \\ |\mathbf{v}|=k}} L_{p^*}(\mathcal{P}_{\mathbf{v}}) \\
&\leq \sum_{k=1}^{\ell-1} \binom{\ell}{k} T_d(k) C(d, p^*) \frac{(\ln n)^{\frac{d-1}{2}}}{n} \\
&= C(d, p^*) \frac{(\ln n)^{\frac{d-1}{2}}}{n} \sum_{k=1}^{\ell-1} \binom{\ell}{k} T_d(k).
\end{aligned}$$

This yields

$$\begin{aligned}
\text{error}(\text{QMC}_{d,n}; \mathcal{F}_d) &\geq \gamma_{\mathbf{u}} |L_{p^*}(\mathcal{P}_{\mathbf{u}}) - \|g_{\mathbf{u}}\|_{L_{p^*}}| \\
&\geq C(d, p^*) \frac{(\ln n)^{\frac{d-1}{2}}}{n} \left( T_d(\ell) - \sum_{k=1}^{\ell-1} \binom{\ell}{k} T_d(k) \right) \\
&= C(d, p^*) \frac{(\ln n)^{\frac{d-1}{2}}}{n} c_{\ell},
\end{aligned}$$

where  $c_{\ell} > 0$  by Lemma 10. The same proof idea with some obvious modifications also works for  $p = 1$  and hence  $p^* = \infty$ . Therefore (6) is completely proven. Furthermore, the same proof but with all terms  $(\ln n)^{\frac{d-1}{2}}$  replaced by  $(\ln n)^{\frac{1}{2}}$  works for (7) ( $p = \infty$  and  $p^* = 1$ ).

It remains to prove (8). For  $d = 2$  and  $p = 1$  (i.e.,  $p^* = \infty$ ) we have

$$\begin{aligned}
&\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) \\
&= \max \left\{ \gamma_{\{1\}} L_{\infty}(\mathcal{P}_{\{1\}}), \gamma_{\{2\}} L_{\infty}(\mathcal{P}_{\{2\}}), \gamma_{\{1,2\}} \sup_{(t_1, t_2) \in [0,1]^2} |\Delta_{\mathcal{P}}(t_1, t_2) - g(t_1, t_2)| \right\}, \quad (14)
\end{aligned}$$

where

$$g(t_1, t_2) := t_1 \Delta_{\mathcal{P}_{\{2\}}}(t_2) + t_2 \Delta_{\mathcal{P}_{\{1\}}}(t_1).$$

Let  $C = C(2, \infty)$  from (11). Now we consider two cases:

**Case 1:**  $L_{\infty}(\mathcal{P}_{\{i\}}) \geq \frac{C \ln n}{4n}$  for at least one  $i \in [2]$ , say for  $i = 1$ . Then we obtain from (14) that

$$\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) \geq \gamma_{\{1\}} L_{\infty}(\mathcal{P}_{\{1\}}) \geq \gamma_{\{1\}} \frac{C \ln n}{4n}.$$

**Case 2:**  $L_{p^*}(\mathcal{P}_{\{i\}}) < \frac{C \ln n}{4n}$  for all  $i \in [2]$ . Then we obtain from (14) that

$$\begin{aligned}
\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) &\geq \gamma_{\{1,2\}} \sup_{(t_1, t_2) \in [0,1]^2} |\Delta_{\mathcal{P}}(t_1, t_2) - g(t_1, t_2)| \\
&= \gamma_{\{1,2\}} \|\Delta_{\mathcal{P}} - g\|_{L_{\infty}} \\
&\geq \gamma_{\{1,2\}} \left| \|\Delta_{\mathcal{P}}\|_{L_{\infty}} - \|g\|_{L_{\infty}} \right| \\
&= \gamma_{\{1,2\}} |L_{\infty}(\mathcal{P}) - \|g\|_{L_{\infty}}|.
\end{aligned}$$



We have

$$\begin{aligned} \|g\|_{L_\infty} &\leq \|t_1 \Delta_{\mathcal{P}_{\{2\}}}(t_2)\|_{L_\infty} + \|t_2 \Delta_{\mathcal{P}_{\{1\}}}(t_1)\|_{L_\infty} \\ &= (L_\infty(\mathcal{P}_{\{1\}}) + L_\infty(\mathcal{P}_{\{2\}})) < \frac{C \ln n}{2n}. \end{aligned}$$

Hence, together with (11), we get

$$\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) > \gamma_{\{1,2\}} \frac{C \ln n}{2n}.$$

In any case we have

$$\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) > c(p^*, \gamma) \frac{\ln n}{n},$$

where we can choose

$$c(p^*, \gamma) = C \min_{u \neq 0} \frac{\gamma_u}{2^{|u|}},$$

with  $C$  taken from (11). □

## 4 Low-dimensional cases

In this section we study the special low-dimensional cases  $d \in \{1, 2\}$  in greater detail and provide optimal QMC rules.

### 4.1 The case 1D

From (5) we have:

**Corollary 11.** *For any  $n$ -element point set  $\mathcal{P}$  in  $[0, 1)$ , the error of the corresponding QMC method is (modulo  $\gamma_{\{1\}}$ ) the  $L_{p^*}$ -discrepancy of  $\mathcal{P}$ ,*

$$\text{error}(\text{QMC}_{1,n}, \mathcal{F}_1) = \gamma_{\{1\}} L_{p^*}(\mathcal{P}).$$

Next we show that the composite midpoint rule

$$\text{QMC}_{1,n}^{\text{MP}}(f) = \frac{1}{n} \sum_{j=1}^n f(x_j) \quad \text{with} \quad x_j = \frac{2j-1}{2n}$$

is optimal among all QMC rules based on  $n$  nodes in  $[0, 1]$ . This is equivalent to the fact that the point set formed by

$$y_j = \frac{2j-1}{2n} \quad \text{for } j = 1, \dots, n \tag{15}$$

has optimal  $L_{p^*}$ -discrepancy among all  $n$ -element point sets in  $[0, 1)$ . The latter is well known for  $p^* \in \{2, \infty\}$ , see [27], and has been shown recently in [22] for arbitrary  $p^* \in [1, \infty]$ . Here we give an elementary proof for the case of general  $p^* \in [1, \infty]$ .

We begin with the following lemma.

**Lemma 12.** *For any  $p^* \in [1, \infty]$ , if an  $n$ -element point set  $\mathcal{P} = \{x_1, \dots, x_n\}$  has the least  $L_{p^*}$ -discrepancy, then each subinterval  $[\frac{j-1}{n}, \frac{j}{n})$ ,  $j = 1, \dots, n$ , contains exactly one point from  $\mathcal{P}$ .*

*Proof.* We provide a proof for  $p^* < \infty$  only. The case for  $p^* = \infty$  is addressed at the end of the proof of the next theorem.

Let  $\mathcal{P} = \{x_1, \dots, x_n\}$  with  $0 \leq x_1 \leq \dots \leq x_n \leq 1$ . We have

$$[L_{p^*}(\mathcal{P})]^{p^*} = \sum_{j=1}^n e_j(\mathcal{P}), \quad \text{where} \quad e_j(\mathcal{P}) = \int_{(j-1)/n}^{j/n} |\Delta_{\mathcal{P}}(t)|^{p^*} dt.$$

To simplify the notation in this proof, we introduce

$$s_j = s_j(\mathcal{P}) := |\mathcal{P} \cap [0, \frac{j-1}{n})| \quad \text{for} \quad j = 1, \dots, n.$$

Then the  $e_j(\mathcal{P})$ 's satisfy the following properties:

$$e_j(\mathcal{P}) = \int_{(j-1)/n}^{j/n} \left| \frac{s_j}{n} - t \right|^{p^*} dt$$

if  $\mathcal{P} \cap [\frac{j-1}{n}, \frac{j}{n}) = \emptyset$ , and

$$e_j(\mathcal{P}) = \int_{(j-1)/n}^{x_{j,1}} \left| \frac{s_j}{n} - t \right|^{p^*} dt + \int_{x_{j,1}}^{x_{j,2}} \left| \frac{s_j + 1}{n} - t \right|^{p^*} dt + \dots + \int_{x_{j,k_j}}^{j/n} \left| \frac{s_j + k_j}{n} - t \right|^{p^*} dt,$$

if  $\mathcal{P} \cap [\frac{j-1}{n}, \frac{j}{n}) = \{x_{j,1}, \dots, x_{j,k_j}\}$ .

Suppose, by a contradiction, that for some  $\ell$ , the subinterval  $[(\ell-1)/n, \ell/n)$  does not contain any point from  $\mathcal{P}$ . If there are more such subintervals, then we choose  $\ell$  to be the smallest index. We now consider two cases.

**Case 1:** Suppose that there is  $m < \ell$  such that  $[(m-1)/n, m/n)$  contains more than one point from  $\mathcal{P}$ . Choose the largest such  $m$  if there are more of such subintervals. Then

$$\left[ \frac{m-1}{n}, \frac{m}{n} \right) \cap \mathcal{P} = \{x_{m,1}, \dots, x_{m,k_m}\} \quad \text{with} \quad k_m \geq 2,$$

$$\left[ \frac{j-1}{n}, \frac{j}{n} \right) \cap \mathcal{P} = \{x_{j,1}\} \quad \text{for} \quad j = m+1, \dots, \ell-1, \quad \text{and} \quad \left[ \frac{\ell-1}{n}, \frac{\ell}{n} \right) \cap \mathcal{P} = \emptyset.$$

Note that, due to  $k_m \geq 2$ , we have

$$s_{m+1} \geq m-1 + k_m \geq m+1, \quad \text{and} \quad s_{m+j} = s_{m+1} + j-1 \geq m+j \quad \text{for} \quad j = 2, \dots, \ell-m. \quad (16)$$

Consider next  $\tilde{\mathcal{P}}$  which is obtained from  $\mathcal{P}$  by removing the point  $x_{m,k_m}$  and adding  $y_m$  inside  $((\ell-1)/n, \ell/n]$ . Clearly  $e_i(\mathcal{P}) = e_i(\tilde{\mathcal{P}})$  for  $i < m$  and  $i > \ell$ . Note that

$$s_{m+j}(\tilde{\mathcal{P}}) = s_{m+j}(\mathcal{P}) - 1 \quad \text{for} \quad j = 1, \dots, \ell-m.$$

Therefore,

$$\begin{aligned} |L_{p^*}(\mathcal{P})|^{p^*} - |L_{p^*}(\tilde{\mathcal{P}})|^{p^*} &= \int_{x_{m,k_m}}^{x_{m+1,1}} \left[ \left| \frac{s_{m+1}}{n} - t \right|^{p^*} - \left| \frac{s_{m+1}-1}{n} - t \right|^{p^*} \right] dt \\ &+ \sum_{j=1}^{\ell-m-2} \int_{x_{m+j,1}}^{x_{m+j+1,1}} \left[ \left| \frac{s_{m+j+1}}{n} - t \right|^{p^*} - \left| \frac{s_{m+j+1}-1}{n} - t \right|^{p^*} \right] dt \\ &+ \int_{x_{\ell-1,1}}^{y_m} \left[ \left| \frac{s_{\ell}}{n} - t \right|^{p^*} - \left| \frac{s_{\ell}-1}{n} - t \right|^{p^*} \right] dt. \end{aligned}$$

Due to (16), all integrals in the sum above are positive. Indeed, if  $s_{m+j+1} \geq m + j + 2$  then  $s_{m+j+1} - 1 \geq m + j + 1 > nx_{m+j+1,1}$  and

$$\frac{s_{m+j+1}}{n} - t > \frac{s_{m+j+1} - 1}{n} - t > 0 \quad \text{for any } t \in [x_{m+j,1}, x_{m+j+1,1}].$$

If  $s_{m+j+1} = m + j + 1$  then  $\frac{s_{m+j+1}-1}{n} = \frac{m+j}{n} \in [x_{m+j,1}, x_{m+j+1,1}]$  and, therefore

$$\int_{x_{m+j,1}}^{x_{m+j+1,1}} \left( \frac{m+j+1}{n} - t \right)^{p^*} dt > \int_{x_{m+j,1}}^{x_{m+j+1,1}} \left| \frac{m+j}{n} - t \right|^{p^*} dt.$$

Hence  $L_{p^*}(\mathcal{P}) > L_{p^*}(\tilde{\mathcal{P}})$ .

**Case 2:** Suppose that Case 1 is not applicable. Let  $\ell$  be as before and let  $m > \ell$  be the smallest index such that  $[(m-1)/n, m/n)$  contains more than one point of  $\mathcal{P}$ . Let  $\mathcal{P} \cap [(m-1)/n, m/n) = \{x_{m,1}, \dots, x_{m,k_m}\}$  with  $k_m \geq 2$ . Using similar arguments as in Case 1, one can verify that the  $L_{q^*}$ -discrepancy of  $\mathcal{P}$  is larger than the discrepancy of  $\tilde{\mathcal{P}}$ , where now  $\tilde{\mathcal{P}}$  has  $x_{m,1}$  replaced by  $y_{\ell,1} \in [(\ell-1)/n, \ell/n)$ .  $\square$

**Theorem 13.** *We have*

$$\text{error}(\text{QMC}_{1,n}^{\text{MP}}; \mathcal{F}_1) = \frac{\gamma_{\{1\}}}{2(p^* + 1)^{1/p^*} n} \quad (17)$$

with the convention that  $(p^* + 1)^{1/p^*} = 1$  for  $p^* = \infty$ . Moreover,

$$\text{error}(\text{QMC}_{1,n}^{\text{MP}}; \mathcal{F}_1) = \min\{\text{error}(\text{QMC}_{1,k}; \mathcal{F}_1) : k \leq n\}. \quad (18)$$

*Proof.* We begin with  $p^* < \infty$ . Let  $y_1, \dots, y_n$  be the nodes of the midpoint rule given by (15). From (5) we have

$$\begin{aligned} \text{error}(\text{QMC}_{1,n}^{\text{MP}}; \mathcal{F}_1) &= \gamma_{\{1\}} \left[ \int_0^1 \left| \frac{|\{k \in \{1, \dots, n\} : y_k < t\}|}{n} - t \right|^{p^*} dt \right]^{1/p^*} \\ &= \gamma_{\{1\}} \left[ \sum_{j=0}^n \int_{y_j}^{y_{j+1}} \left| \frac{|\{k \in \{1, \dots, n\} : y_k < t\}|}{n} - t \right|^{p^*} dt \right]^{1/p^*}, \end{aligned}$$

where we put  $y_0 = 0$  and  $y_{n+1} = 1$ . The first and last integrals in the sum above are equal to

$$\int_0^{y_1} |0 - t|^{p^*} dt = \int_{y_n}^1 |1 - t|^{p^*} dt = \frac{1}{(p^* + 1)(2n)^{p^*+1}}.$$

The other integrals are equal to

$$\int_{y_j}^{y_{j+1}} \left| \frac{j}{n} - t \right|^{p^*} dt = 2 \int_{j/n}^{(2j+1)/(2n)} \left( t - \frac{j}{n} \right)^{p^*} dt = \frac{2}{(p^* + 1)(2n)^{p^*+1}}.$$

This proves that the sum of the integrals is equal to

$$\sum_{j=0}^n \int_{y_j}^{y_{j+1}} \left| \frac{|\{k \in \{1, \dots, n\} : y_k < t\}|}{n} - t \right|^{p^*} dt = \frac{1}{(p^* + 1)(2n)^{p^*}},$$

which completes the proof for  $p^* < \infty$ .

For  $p^* = \infty$ , we have

$$\begin{aligned} \text{error}(\text{QMC}_{1,n}^{\text{MP}}; \mathcal{F}_1) &= \gamma_{\{1\}} \sup_{t \in [0,1]} \left| \frac{|\{j \in \{1, \dots, n\} : y_j < t\}|}{n} - t \right| \\ &= \gamma_{\{1\}} \max_{j=0, \dots, n} \max_{t \in [y_j, y_{j+1}]} \left| \frac{j}{n} - t \right| = \frac{\gamma_{\{1\}}}{2n}. \end{aligned}$$

This completes the proof of (17).

To show (18), consider a general point set  $\mathcal{P} = \{x_1, \dots, x_n\}$ ,

$$0 =: x_0 \leq x_1 \leq \dots \leq x_n \leq x_{n+1} := 1.$$

For  $p^* < \infty$ , the worst case error of the corresponding QMC rule  $\text{QMC}_{1,n}$  raised to the power  $p^*$  is equal to

$$E(x_1, \dots, x_n) := \gamma_{\{1\}}^{p^*} \sum_{\ell=0}^n \int_{x_\ell}^{x_{\ell+1}} \left| \frac{\ell}{n} - t \right|^{p^*} dt. \quad (19)$$

Its partial derivative with respect to  $x_k$  is

$$\gamma_{\{1\}}^{p^*} \left( \left| \frac{k-1}{n} - x_k \right|^{p^*} - \left| \frac{k}{n} - x_k \right|^{p^*} \right),$$

which is zero if and only if

$$\frac{k-1}{n} - x_k = - \left( \frac{k}{n} - x_k \right), \quad \text{i.e., iff } x_k = \frac{2k-1}{2n}.$$

This means that the only possible extremal point is given by  $x_k = \frac{2k-1}{2n}$  for  $1 \leq k \leq n$ . It is easy to see that the minimum of  $E$  is not attained if  $x_1 = 0$  and/or  $x_n = 1$ . Due to Lemma 12 it is not attained if  $x_i = x_{i+1}$  for some  $i$ , which completes the proof of (18) for  $p^* < \infty$ .

For  $p^* = \infty$ , we need to show

$$\gamma_{\{1\}} \max_{\ell=0, \dots, n} \max_{t \in [x_\ell, x_{\ell+1}]} \left| \frac{\ell}{n} - t \right| \geq \frac{\gamma_{\{1\}}}{2n}. \quad (20)$$

To prove (20), suppose by contrary that for some point set  $\mathcal{P} = \{x_1, \dots, x_n\}$  it holds that

$$\max_{\ell=0, \dots, n} \max_{t \in [x_\ell, x_{\ell+1}]} \left| \frac{\ell}{n} - t \right| < \frac{1}{2n}.$$

Then  $|0 - x_1| < 1/(2n)$  and  $|1 - x_n| < 1/(2n)$ . Moreover,

$$\left| \frac{\ell}{n} - x_\ell \right| < \frac{1}{2n} \quad \text{and} \quad \left| x_{\ell+1} - \frac{\ell}{n} \right| < \frac{1}{2n},$$

which implies that

$$x_{\ell+1} - x_\ell \leq |\ell/n - x_\ell| + |x_{\ell+1} - \ell/n| < 1/n \quad \text{for } \ell = 1, \dots, n-1.$$

Therefore

$$1 = x_1 + (x_2 - x_1) + \dots + (x_n - x_{n-1}) + 1 - x_n < \frac{1}{2n} + \frac{n-1}{n} + \frac{1}{2n} = 1,$$

which is a contradiction.

This completes the proof of (20) and of the theorem.  $\square$

**Remark 14.** From [20, Theorem 8] it follows that for  $p = 2$  and  $\gamma_\emptyset = 1$  the norms of the corresponding embeddings are equal and

$$\|i\| = \|i^{-1}\| = \left(1 + \frac{\gamma_{\{1\}}}{\sqrt{3}} \left(\sqrt{1 + \frac{\gamma_{\{1\}}^2}{12}} + \frac{\gamma_{\{1\}}}{\sqrt{12}}\right)\right)^{1/2},$$

which could be large. For instance for  $\gamma_{\{1\}} = 1, 2, 3$ , these norms are equal to  $1.329\dots, 1.732\dots$ , and  $2.188\dots$ , respectively. Hence, using the embedding approach we would get  $\|i\|\gamma_{\{1\}}/(2\sqrt{3}n)$  as an upper bound for the error of  $\text{QMC}_{1,n}^{\text{MP}}$ . Proceeding directly as in Theorem 13, however, we get the exact value of the error of the midpoint rule which is  $\gamma_{\{1\}}/(2\sqrt{3}n)$ .

## 4.2 The case 2D

Now we consider the two-dimensional case and show that here the lower bound in Theorem 9 is best possible with respect to the order of magnitude in  $n$ . In the following we assume that the two-dimensional point sets  $\mathcal{P} = \{(x_j, y_j) : j = 1, \dots, n\}$  under consideration are projection regular in the sense that

$$\{x_j : j = 1, \dots, n\} = \{y_j : j = 1, \dots, n\} = \{j/n : j = 0, \dots, n-1\}. \quad (21)$$

We will prove the following result:

**Theorem 15.** *Let  $\text{QMC}_{2,n}$  be the QMC rule based on a two-dimensional point set  $\mathcal{P}$  that satisfies projection regularity (21). Then we have*

$$\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) \leq \frac{1}{n} \left[ \frac{\gamma_{\{1\}}^{p^*} + \gamma_{\{2\}}^{p^*}}{p^* + 1} + 3^{p^*-1} \gamma_{\{1,2\}}^{p^*} \left( \frac{2}{(p^* + 1)^2} + (nL_{p^*}(\mathcal{P}))^{p^*} \right) \right]^{1/p^*}.$$

On the other hand, there exists a positive number  $C = C(p^*)$  such that

$$\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) \geq C \gamma_{\{1,2\}} L_{p^*}(\mathcal{P}).$$

For the proof we need the following easy lemma:

**Lemma 16.** *If  $\mathcal{P} = \{(x_j, y_j) : j = 1, \dots, n\}$  satisfies (21), we have for  $t \in [0, 1]$  that*

$$\sum_{j=1}^n 1_{[0,t)}(x_j) = \sum_{j=1}^n 1_{[0,t)}(y_j) = \lceil nt \rceil.$$

*Proof.* Since  $\mathcal{P}$  satisfies (21) we obtain

$$\sum_{j=1}^n 1_{[0,t)}(x_j) = \sum_{\substack{j=0 \\ j < nt}}^{n-1} 1 = \sum_{j=0}^{\lceil nt \rceil - 1} 1 = \lceil nt \rceil.$$

□

*Proof of Theorem 15.* First we show the upper bound. We need to study

$$\sum_{j=1}^n \prod_{i \in \mathbf{u}} (t_i - 1_{[0, t_i]}(x_{j,i})) \quad (22)$$

for  $\mathbf{u} = \{1\}$ ,  $\mathbf{u} = \{2\}$ , and  $\mathbf{u} = \{1, 2\}$ , where  $x_{j,1} = x_j$  and  $x_{j,2} = y_j$ .

- $\mathbf{u} = \{1\}$ : According to Lemma 16, Eq. (22) is

$$\sum_{j=1}^n (t_1 - 1_{[0, t_1]}(x_j)) = nt_1 - \lceil nt_1 \rceil$$

and hence

$$\left| \sum_{j=1}^n (t_1 - 1_{[0, t_1]}(x_j)) \right| = \lceil nt_1 \rceil - nt_1.$$

This implies that

$$\begin{aligned} \int_0^1 \left| \sum_{j=1}^n (t_1 - 1_{[0, t_1]}(x_j)) \right|^{p^*} dt_1 &= \int_0^1 (\lceil nt_1 \rceil - nt_1)^{p^*} dt_1 \\ &= \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} (k - nt_1)^{p^*} dt_1 \\ &= \sum_{k=1}^n \frac{1}{n} \int_0^1 y^{p^*} dy \\ &= \frac{1}{p^* + 1}, \end{aligned}$$

where we used the substitution  $y = k - nt_1$ .

- $\mathbf{u} = \{2\}$ : In this case, according to Lemma 16, Eq. (22) is

$$\sum_{j=1}^n (t_2 - 1_{[0, t_2]}(y_j)) = nt_2 - \lceil nt_2 \rceil$$

and hence, as above,

$$\int_0^1 \left| \sum_{j=1}^n (t_2 - 1_{[0, t_2]}(y_j)) \right|^{p^*} dt_2 = \frac{1}{p^* + 1}.$$

- $\mathbf{u} = \{1, 2\}$ : Here, again according to Lemma 16, Eq. (22) is

$$\begin{aligned} &\sum_{j=1}^n (t_1 - 1_{[0, t_1]}(x_j))(t_2 - 1_{[0, t_2]}(y_j)) \\ &= nt_1 t_2 - t_2 \lceil nt_1 \rceil - t_1 \lceil nt_2 \rceil + \sum_{j=1}^n 1_{[0, t_1]}(x_j) 1_{[0, t_2]}(y_j) \\ &= 2nt_1 t_2 - t_2 \lceil nt_1 \rceil - t_1 \lceil nt_2 \rceil + \left( \sum_{j=1}^n 1_{[0, t_1]}(x_j) 1_{[0, t_2]}(y_j) - nt_1 t_2 \right) \\ &= 2nt_1 t_2 - t_2 \lceil nt_1 \rceil - t_1 \lceil nt_2 \rceil + n\Delta_P(t_1, t_2). \end{aligned}$$

Taking the absolute value and the  $p^*$ -th power we obtain with Lemma 6 that

$$\begin{aligned} & \left| \sum_{j=1}^n (t_1 - 1_{[0,t_1)}(x_j))(t_2 - 1_{[0,t_2)}(y_j)) \right|^{p^*} \\ & \leq 3^{p^*-1} \left( t_1^{p^*} (\lceil nt_2 \rceil - nt_2)^{p^*} + t_2^{p^*} (\lceil nt_1 \rceil - nt_1)^{p^*} + (n|\Delta_{\mathcal{P}}(t_1, t_2)|)^{p^*} \right). \end{aligned}$$

Now we integrate with respect to  $(t_1, t_2) \in [0, 1]^2$  and obtain

$$\int_{[0,1]^2} \left| \sum_{j=1}^n (t_1 - 1_{[0,t_1)}(x_j))(t_2 - 1_{[0,t_2)}(y_j)) \right|^{p^*} d(t_1, t_2) \leq 3^{p^*-1} \left( \frac{2}{(p^* + 1)^2} + (nL_{p^*}(\mathcal{P}))^{p^*} \right).$$

From the error formula in Theorem 3 we obtain

$$\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) \leq \frac{1}{n} \left[ \frac{\gamma_{\{1\}}^{p^*} + \gamma_{\{2\}}^{p^*}}{p^* + 1} + 3^{p^*-1} \gamma_{\{1,2\}}^{p^*} \left( \frac{2}{(p^* + 1)^2} + (nL_{p^*}(\mathcal{P}))^{p^*} \right) \right]^{1/p^*}.$$

This proves the upper bound.

Now we turn our attention to the lower bound. From the proof of Theorem 9 we know that

$$\sum_{j=1}^n (t_1 - 1_{[0,t_1)}(x_j))(t_2 - 1_{[0,t_2)}(y_j)) = n\Delta_{\mathcal{P}}(t_1, t_2) - g(t_1, t_2),$$

where

$$g(t_1, t_2) := -2nt_1t_2 + t_2\lceil nt_1 \rceil + t_1\lceil nt_2 \rceil.$$

This yields, using again Theorem 3,

$$\begin{aligned} \text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) & \geq \frac{1}{n} \gamma_{\{1,2\}} \left[ \int_{[0,1]^2} \left| n\Delta_{\mathcal{P}}(t_1, t_2) - g(t_1, t_2) \right|^{p^*} d(t_1, t_2) \right]^{1/p^*} \\ & = \frac{1}{n} \gamma_{\{1,2\}} \|n\Delta_{\mathcal{P}} - g\|_{L_{p^*}} \\ & \geq \frac{1}{n} \gamma_{\{1,2\}} \left| \|n\Delta_{\mathcal{P}}\|_{L_{p^*}} - \|g\|_{L_{p^*}} \right| \\ & = \frac{1}{n} \gamma_{\{1,2\}} \left| nL_{p^*}(\mathcal{P}) - \|g\|_{L_{p^*}} \right| \\ & = \gamma_{\{1,2\}} \left| L_{p^*}(\mathcal{P}) - \frac{1}{n} \|g\|_{L_{p^*}} \right|. \end{aligned}$$

With the same methods as in the proof of the upper bound we can show that

$$\|g\|_{L_{p^*}} \leq \frac{2^{p^*+1}}{(p^* + 1)^2}.$$

On the other hand, we know from (9) that there exists an absolute constant  $C > 0$  such that

$$L_{p^*}(\mathcal{P}) \geq C \frac{\sqrt{\ln n}}{n}.$$

Hence we have

$$\begin{aligned} \text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) &\geq \gamma_{\{1,2\}} \left( L_{p^*}(\mathcal{P}) - \frac{2^{p^*+1}}{(p^*+1)^2} \frac{1}{n} \right) \\ &\geq \gamma_{\{1,2\}} L_{p^*}(\mathcal{P}) \left( 1 - \frac{2^{p^*+1}}{(p^*+1)^2} \frac{1}{C \cdot \sqrt{\ln n}} \right). \end{aligned}$$

Now, for  $n$  large enough we have

$$1 - \frac{2^{p^*+1}}{(p^*+1)^2} \frac{1}{C \cdot \sqrt{\ln n}} > 0$$

and hence the result follows.  $\square$

Several constructions of two-dimensional projection regular point sets with best possible order of  $L_{p^*}$ -discrepancy for all  $p^* \in [1, \infty]$  are known, e.g., generalized Hammersley point sets [8], shifted Hammersley point sets [15, 26] or digital NUT nets [23]. As an example we would like to present the digitally shifted Hammersley point sets from [15] in greater detail:

**Example 17.** Let  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_m) \in \{0, 1\}^m$ . The two-dimensional digitally shifted Hammersley point set is given by

$$\mathcal{R}_{m,\boldsymbol{\sigma}} = \left\{ \left( \frac{t_m}{2} + \frac{t_{m-1}}{2^2} + \dots + \frac{t_1}{2^m}, \frac{t_1 \oplus \sigma_1}{2} + \frac{t_2 \oplus \sigma_2}{2^2} + \dots + \frac{t_m \oplus \sigma_m}{2^m} \right) : t_1, \dots, t_m \in \{0, 1\} \right\},$$

where  $t \oplus \sigma = t + \sigma \pmod{2}$  for  $t, \sigma \in \{0, 1\}$ . This point set contains  $n = 2^m$  elements. If  $\boldsymbol{\sigma} = \mathbf{0} = (0, 0, \dots, 0)$ , we obtain the classical two-dimensional Hammersley point set. From [15, Theorem 1] we obtain that if  $|\{j : \sigma_j = 0\}| = \lfloor m/2 \rfloor$ , then for  $p^* \in [1, \infty)$  we have

$$L_{p^*}(\mathcal{R}_{m,\boldsymbol{\sigma}}) \asymp \frac{\sqrt{m}}{2^m} \asymp \frac{\sqrt{\ln n}}{n}.$$

Furthermore, according to [7, 19],

$$L_\infty(\mathcal{R}_{m,\boldsymbol{\sigma}}) \asymp \frac{m}{2^m} \asymp \frac{\ln n}{n}.$$

Since the point sets  $\mathcal{R}_{m,\boldsymbol{\sigma}}$  are projection regular we obtain

$$\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) \asymp \begin{cases} \frac{\sqrt{\ln n}}{n} & \text{if } p > 1, \\ \frac{\ln n}{n} & \text{if } p = 1, \end{cases}$$

where  $n = 2^m$ , and these orders of magnitude are optimal according to Theorem 9. We remark that for the classical two-dimensional Hammersley point set (where  $\boldsymbol{\sigma} = \mathbf{0}$ ) we only get

$$\text{error}(\text{QMC}_{2,n}; \mathcal{F}_2) \asymp \frac{\ln n}{n}$$

for all  $p \in [1, \infty]$ . This is optimal only for  $p = 1$  (i.e.,  $p^* = \infty$ ).

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**Authors' addresses:**

Peter Kritzer  
Johann Radon Institute for Computational and Applied Mathematics (RICAM)  
Austrian Academy of Sciences  
Altenbergerstr. 69, 4040 Linz, Austria  
E-mail: [peter.kritzer@oeaw.ac.at](mailto:peter.kritzer@oeaw.ac.at)

Friedrich Pillichshammer  
Institut für Finanzmathematik und Angewandte Zahlentheorie  
Johannes Kepler Universität Linz  
Altenbergerstr. 69, 4040 Linz, Austria  
E-mail: [friedrich.pillichshammer@jku.at](mailto:friedrich.pillichshammer@jku.at)

G. W. Wasilkowski  
Computer Science Department, University of Kentucky  
301 David Marksbury Building  
329 Rose Street  
Lexington, KY 40506, USA  
E-mail: [greg@cs.uky.edu](mailto:greg@cs.uky.edu)