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Uniform approximations by Fourier sums on classes of convolutions of periodic functions

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Abstract We establish asymptotic estimates for exact upper bounds of uniform approximations by Fourier sums on the classes of 2π–periodic functions, which are represented by convolutions of functions $\varphi(\varphi\perp 1)$ from unit ball of the space L_1 with fixed kernels Ψ_β of the form $\Psi_\beta(t)=\sum\limits_{k=1}^{\infty}\psi(k)\cos\left(kt-\frac{\beta\pi}{2}\right)$, $\sum\limits_{k=1}^{\infty}k\psi(k)<\infty$, $\psi(k)\geq 0$, $\beta\in\mathbb{R}$.

1 Introduction

Let L_1 be the space of 2π–periodic functions f summable on $[0,2\pi)$, in which the norm is given by the formula $\|f\|_1=\int\limits_0^{2\pi}|f(t)|dt$; L_∞ be the space of measurable and essentially bounded 2π–periodic functions f with the norm $\|f\|_\infty=\operatorname{ess\,sup}\limits_t|f(t)|$; C be the space of continuous 2π–periodic functions f, in which the norm is specified by the equality $\|f\|_C=\max\limits_t|f(t)|$.

Let $\psi(k)$ be an arbitrary fixed sequence of real, nonnegative numbers and let β be a fixed real number.

We set

$$B_1^0:=\{\varphi:\ ||\varphi||_1\leq 1,\ \varphi\perp 1\}. \tag{1}$$

Further let $C^\psi_{\beta,1}$ be the set of all functions f, which are represented for all x as convolutions of the form

$$f(x)=\frac{a_0}{2}+\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\varphi(t)\Psi_\beta(x-t)dt,\ a_0\in\mathbb{R},\ \varphi\in B_1^0, \tag{2}$$

where Ψ_β is a fixed kernel of the form

$$\Psi_\beta(t) = \sum_{k=1}^{\infty} \psi(k) \cos\big(kt - \frac{\beta\pi}{2}\big), \ \psi(k) \geq 0, \ \beta \in \mathbb{R}, \tag{3}$$

and the following condition holds:

$$\sum_{k=1}^{\infty} \psi(k) < \infty. \tag{4}$$

Condition (4) provides an embedding $C^{\psi}_{\beta,1} \subset C$.

For $\psi(k) = e^{-\alpha k^r}$, $\alpha, r > 0$, the kernels $\Psi_\beta(t)$ of the form (3) are called generalized Poisson kernels $P_{\alpha,r,\beta}(t) = \sum\limits_{k=1}^{\infty} e^{-\alpha k^r} \cos\big(kt - \frac{\beta\pi}{2}\big)$ and the classes of functions f, generated by these kernels are the classes of generalized Poisson integrals $C^{\alpha,r}_{\beta,1}$.

For the classes $C^{\psi}_{\beta,1}$ we consider the quantities

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \sup_{f \in C^{\psi}_{\beta,1}} \|f(\cdot) - S_{n-1}(f;\cdot)\|_C, \tag{5}$$

where $S_{n-1}(f;\cdot)$ are the partial Fourier sums of order $n-1$ for a function f.

Approximations by Fourier sums on other classes of differentiable functions in uniform metric were investigated in works [1]–[10].

In this paper we consider Kolmogorov–Nikolsky problem about finding of asymptotic equalities of the quantity (5) as $n \to \infty$.

2 Main result

The following statement holds.

Theorem 2.1. *Let $\sum\limits_{k=1}^{\infty} k\psi(k) < \infty$, $\psi(k) \geq 0$, $k = 1, 2, ...$ and $\beta \in \mathbb{R}$. Then as $n \to \infty$ the following asymptotic equality holds*

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \frac{1}{\pi} \sum_{k=n}^{\infty} \psi(k) + \frac{\mathcal{O}(1)}{n} \sum_{k=1}^{\infty} k\psi(k+n), \tag{6}$$

where $O(1)$ is a quantity uniformly bounded in all parameters.

Proof. According to (2) and (5) we have that

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \frac{1}{\pi} \sup_{\varphi \in B^0_1} \Bigg\| \int\limits_{-\pi}^{\pi} \varphi(t) \Psi_{\beta,n}(x-t) dt \Bigg\|_C, \tag{7}$$

where

$$(8)\qquad \Psi_{\beta,n}(t) := \sum_{k=n}^{\infty} \psi(k) \cos\Big(kt - \frac{\beta\pi}{2}\Big), \quad \beta \in \mathbb{R}.$$

Taking into account the invariance of the sets B_1^0 under shifts of the argument, from (7) we conclude that

$$(9)\qquad \mathcal{E}_n(C^{\psi}_{\beta,1})_C = \frac{1}{\pi} \sup_{\varphi\in B_1^0} \int_{-\pi}^{\pi} \varphi(t)\Psi_{\beta,n}(t)dt.$$

On the basis of the duality relation (see, e.g., [2, Chapter 1, Section 1.4]) we obtain

$$(10)\qquad \sup_{\varphi\in B_1^0} \int_{-\pi}^{\pi} \Psi_{\beta,n}(t)\varphi(t)dt = \inf_{\lambda\in\mathbb{R}} \|\Psi_{\beta,n}(t) - \lambda\|_C,$$

We represent the function $\Psi_{\beta,n}(t)$, which is defined by formula (8), in the form

$$(11)\qquad \Psi_{\beta,n}(t) = g_{\psi,n}(t)\cos\Big(nt - \frac{\beta\pi}{2}\Big) + h_{\psi,n}(t)\sin\Big(nt - \frac{\beta\pi}{2}\Big),$$

where

$$(12)\qquad g_{\psi,n}(t) := \sum_{k=0}^{\infty} \psi(k+n)\cos kt,$$

$$(13)\qquad h_{\psi,n}(t) := -\sum_{k=0}^{\infty} \psi(k+n)\sin kt.$$

It is obvious that

$$(14)\qquad \inf_{\lambda\in\mathbb{R}} \|\Psi_{\beta,n}(t) - \lambda\|_C \le \|\Psi_{\beta,n}\|_C$$

and

$$(15)\qquad \frac{1}{2}\Big\|\Psi_{\beta,n}\Big(t+\frac{\pi}{n}\Big) - \Psi_{\beta,n}(t)\Big\|_C \le \inf_{\lambda\in\mathbb{R}} \|\Psi_{\beta,n}(t) - \lambda\|_C.$$

In view of representation (11) and applying mean value theorem, we obtain that

$$\begin{aligned}
&\frac{1}{2}\left\|\Psi_{\beta,n}\left(t+\frac{\pi}{n}\right)-\Psi_{\beta,n}(t)\right\|_C\\
=&\frac{1}{2}\Big\|2\Psi_{\beta,n}(t)+g_{\psi,n}\Big(t+\frac{\pi}{n}\Big)\cos\left(nt-\frac{\beta\pi}{2}\right)+h_{\psi,n}\Big(t+\frac{\pi}{n}\Big)\sin\left(nt-\frac{\beta\pi}{2}\right)\\
-&\left(g_{\psi,n}(t)\cos\left(nt-\frac{\beta\pi}{2}\right)+h_{\psi,n}(t)\sin\left(nt-\frac{\beta\pi}{2}\right)\right)\Big\|_C\\
=&\|\Psi_{\beta,n}\|_C+\mathcal{O}(1)\left(\left\|g_{\psi,n}\Big(t+\frac{\pi}{n}\Big)-g_{\psi,n}(t)\right\|_C+\left\|h_{\psi,n}\Big(t+\frac{\pi}{n}\Big)-h_{\psi,n}(t)\right\|_C\right)\\
=&\|\Psi_{\beta,n}\|_C+\mathcal{O}(1)\left(\frac{1}{n}\left\|g'_{\psi,n}\right\|_C+\frac{1}{n}\left\|h'_{\psi,n}\right\|_C\right)\\
=&\|\Psi_{\beta,n}\|_C+\frac{\mathcal{O}(1)}{n}\sum_{k=1}^{\infty}k\psi(k+n). \qquad (16)
\end{aligned}$$

So, formulas (9), (10) and (14)– (16) imply

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C=\frac{1}{\pi}\|\Psi_{\beta,n}\|_C+\frac{\mathcal{O}(1)}{n}\sum_{k=1}^{\infty}k\psi(k+n). \qquad (17)$$

The kernel $\Psi_{\beta,n}$ can be written in the form

$$\begin{aligned}
\Psi_{\beta,n}(t)=&\sqrt{g^2_{\psi,n}(t)+h^2_{\psi,n}(t)}\times\\
&\times\left(\frac{g_{\psi,n}(t)}{\sqrt{g^2_{\psi,n}(t)+h^2_{\psi,n}(t)}}\cos\left(nt-\frac{\beta\pi}{2}\right)+\frac{h_{\psi,n}(t)}{\sqrt{g^2_{\psi,n}(t)+h^2_{\psi,n}(t)}}\sin\left(nt-\frac{\beta\pi}{2}\right)\right)\\
=&\sqrt{g^2_{\psi,n}(t)+h^2_{\psi,n}(t)}\cos\left(nt-\frac{\beta\pi}{2}-arg(g_{\psi,n}(t)+ih_{\psi,n}(t))\right). \qquad (18)
\end{aligned}$$

Let

$$t_0:=\frac{1}{n}\left(\frac{\beta\pi}{2}+arg(g_{\psi,n}(t)+ih_{\psi,n}(t))\right). \qquad (19)$$

Then

$$\|\Psi_{\beta,n}\|_C\ge\Psi_{\beta,n}(t_0)=\sqrt{g^2_{\psi,n}(t_0)+h^2_{\psi,n}(t_0)}\ge|g_{\psi,n}(t_0)|. \qquad (20)$$

Using mean value theorem we have that

$$\begin{aligned}
|g_{\psi,n}(t_0)|=g_{\psi,n}(0)+|g_{\psi,n}(t_0)-g_{\psi,n}(0)|=g_{\psi,n}(0)+\frac{\mathcal{O}(1)}{n}\|g'_{\psi,n}\|_C\\
=\sum_{k=n}^{\infty}\psi(k)+\frac{\mathcal{O}(1)}{n}\sum_{k=1}^{\infty}k\psi(k+n). \qquad (21)
\end{aligned}$$

On the other hand it is clear that

$$\|\Psi_{\beta,n}\|_C \leq \sum_{k=n}^{\infty} \psi(k). \tag{22}$$

Thus,

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \frac{1}{\pi}\sum_{k=n}^{\infty} \psi(k) + \frac{\mathcal{O}(1)}{n}\sum_{k=1}^{\infty} k\psi(k+n). \tag{23}$$

Theorem 2.1 is proved. □

Corollary 2.2. *Let the sequence $\psi(k)$, which generates the classes $C^{\psi}_{\beta,1}$, satisfies the condition D_0, i.e., $\psi(k) > 0$ and*

$$\lim_{k\to 0} \frac{\psi(k+1)}{\psi(k)} = 0.$$

Then, the following asymptotic equality holds as $n \to \infty$

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \frac{1}{\pi}\psi(n) + \frac{\mathcal{O}(1)}{n} \sum_{k=n+1}^{\infty} k\psi(k), \tag{24}$$

where $O(1)$ is a quantity uniformly bounded in all parameters.

Proof. Indeed, let $\psi \in D_0$, then the right hand of (6) can be written in the form

$$\begin{aligned}\mathcal{E}_n(C^{\psi}_{\beta,1})_C &= \frac{1}{\pi}\psi(n) + \mathcal{O}(1)\left(\sum_{k=n+1}^{\infty} \psi(k) + \frac{1}{n}\sum_{k=0}^{\infty} k\psi(k+n)\right)\\ &= \frac{1}{\pi}\psi(n) + \frac{\mathcal{O}(1)}{n}\sum_{k=1}^{\infty}(k+n)\psi(k+n)\\ &= \frac{1}{\pi}\psi(n) + \frac{\mathcal{O}(1)}{n}\sum_{k=n+1}^{\infty} k\psi(k).\end{aligned}$$

Corollary 2.2 is proved. □

Asymptotic equality (24) with written in another form reminder was obtained earlier in [4] and [5].

Corollary 2.3. *Let $\psi(k) = e^{-\alpha k^{-r}}$, $r > 1$, $\alpha > 0$ and $\beta \in \mathbb{R}$. Then as $n \to \infty$ the following asymptotic equality holds*

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,1})_C = e^{-\alpha n^r}\Big(\frac{1}{\pi} + \mathcal{O}(1)e^{-\alpha r n^{r-1}}\Big(1 + \frac{1}{\alpha r(n+1)^{r-1}}\Big)\Big), \tag{25}$$

where $O(1)$ is a quantity uniformly bounded in n and β.

Proof. Formula (24) implies that as $n \to \infty$

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,1})_C = \frac{1}{\pi}e^{-\alpha n^r} + \frac{\mathcal{O}(1)}{n}\sum_{k=n+1}^{\infty} ke^{-\alpha k^r}. \tag{26}$$

It is easy to see that

$$\frac{1}{n}\sum_{k=n+1}^{\infty} ke^{-\alpha k^r} < \frac{1}{n}\left((n+1)e^{-\alpha(n+1)^r} + \int_{n+1}^{\infty} te^{-\alpha t^r}dt\right). \tag{27}$$

Integrating by parts, we get

$$\int_{n+1}^{\infty} te^{-\alpha t^r}dt = \int_{n+1}^{\infty} t^2\frac{1}{\alpha r t^r}\left(-e^{-\alpha t^r}\right)' dt \leq \frac{1}{\alpha r(n+1)^r}\int_{n+1}^{\infty} t^2\left(-e^{-\alpha t^r}\right)' dt$$

$$= \frac{1}{\alpha r(n+1)^r}\left((n+1)^2 e^{-\alpha(n+1)^r} + 2\int_{n+1}^{\infty} te^{-\alpha t^r}dt\right). \tag{28}$$

From the last equality we have

$$\left(1 - \frac{2}{\alpha r(n+1)^r}\right)\int_{n+1}^{\infty} te^{-\alpha t^r}dt \leq \frac{(n+1)^2 e^{-\alpha(n+1)^r}}{\alpha r(n+1)^r}, \tag{29}$$

which is equivalent to

$$\int_{n+1}^{\infty} te^{-\alpha t^r}dt \leq \frac{e^{-\alpha(n+1)^r}}{\alpha r(n+1)^{r-2}}\frac{\alpha r(n+1)^r}{\alpha r(n+1)^r - 2}$$

$$= \frac{e^{-\alpha(n+1)^r}}{\alpha r(n+1)^{r-2}}\left(1 + \frac{2}{\alpha r(n+1)^r - 2}\right). \tag{30}$$

Relations (27) and (30) yield that

$$\frac{1}{n}\sum_{k=n+1}^{\infty} ke^{-\alpha k^r} = \mathcal{O}\left(e^{-\alpha(n+1)^r} + \frac{e^{-\alpha(n+1)^r}}{\alpha r(n+1)^{r-2}}\left(1 + \frac{2}{\alpha r(n+1)^r - 2}\right)\right). \tag{31}$$

Combining (26) and (31) we obtain

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,1})_C = \frac{1}{\pi}e^{-\alpha n^r} + \mathcal{O}\left(e^{-\alpha(n+1)^r} + \frac{e^{-\alpha(n+1)^r}}{\alpha r(n+1)^{r-2}}\left(1 + \frac{2}{\alpha r(n+1)^r - 2}\right)\right)$$

$$= e^{-\alpha n^r}\left(\frac{1}{\pi} + \mathcal{O}\left(e^{-\alpha r n^{r-1}} + \frac{e^{-\alpha r n^{r-1}}}{\alpha r(n+1)^{r-2}}\right)\right).$$

Corollary 2.3 is proved. □

Formula (25) was obtained in [4] and [5].

For classes $C^{\alpha,1}_{\beta,1}$, generated by classes of Poisson kernels

$$P_{\alpha,1,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k} \cos\big(kt - \frac{\beta\pi}{2}\big),\ \alpha > 0,\ \beta \in \mathbb{R}, \tag{32}$$

the following statement holds.

Corollary 2.4. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the following asymptotic equality holds as $n \to \infty$*

$$\mathcal{E}_n(C^{\alpha,1}_{\beta,1})_C = e^{-\alpha n}\left(\frac{1}{\pi}\frac{1}{1-e^{-\alpha}} + \frac{\mathcal{O}(1)}{n}\frac{e^{-\alpha}}{(1-e^{-\alpha})^2}\right), \tag{33}$$

where $O(1)$ is a quantity uniformly bounded in all parameters.

Proof. Denote $q = e^{-\alpha}$. Then, from Theorem 2.1 it follows

$$\begin{aligned}
\mathcal{E}_n(C^{\alpha,1}_{\beta,1})_C =& \frac{1}{\pi}\sum_{k=n}^{\infty} q^k + \frac{\mathcal{O}(1)}{n}\sum_{k=0}^{\infty} kq^{k+n} \\
=& \frac{1}{\pi}\frac{q^n}{1-q} + \frac{\mathcal{O}(1)}{n}\left(\sum_{k=n}^{\infty} kq^k - n\sum_{k=n}^{\infty} q^k\right) \\
=& \frac{1}{\pi}\frac{q^n}{1-q} + \frac{\mathcal{O}(1)}{n}\left(\frac{nq^n(1-q)+q^{n+1}}{(1-q)^2} - \frac{nq^n}{1-q}\right) \\
=& \frac{1}{\pi}\frac{q^n}{1-q} + \frac{\mathcal{O}(1)}{n}\frac{q^{n+1}}{(1-q)^2},
\end{aligned} \tag{34}$$

where we have used that

$$\sum_{k=n}^{\infty} kq^k = \frac{nq^n(1-q)+q^{n+1}}{(1-q)^2}.$$

Corollary 2.4 is proved. □

The asymptotic equality (33) was proved in [4] and [5].

Corollary 2.5. *Let $\psi(k) = e^{-\alpha k^{-r}}$, $0 < r < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$. Then as $n \to \infty$ the following asymptotic equality holds*

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,1})_C = \frac{e^{-\alpha n^r}}{\pi\alpha r} n^{1-r}\Big(1 + \mathcal{O}(1)\Big(\frac{1}{n^r} + \frac{1}{n^{1-r}}\Big)\Big), \tag{35}$$

where $O(1)$ is a quantity uniformly bounded in n and β.

Proof. Theorem 2.1 allows to write

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \frac{1}{\pi}\sum_{k=n}^{\infty} e^{-\alpha k^r} + \frac{\mathcal{O}(1)}{n}\sum_{k=0}^{\infty} k e^{-\alpha(k+n)^r}. \tag{36}$$

Formulas (90) and (91) of the work [7] imply that

$$\sum_{k=0}^{\infty} e^{-\alpha(k+n)^r} = \frac{e^{-\alpha n^r}}{\alpha r} n^{1-r}\Big(1+\mathcal{O}\Big(\frac{1}{n^r}+\frac{1}{n^{1-r}}\Big)\Big). \tag{37}$$

From formulas (94) and (97) of the work [7] it follows that

$$\frac{1}{n}\sum_{k=1}^{\infty} k e^{-\alpha(k+n)^r} = \mathcal{O}(1)\frac{1}{n}e^{-\alpha n^r}(n^{2-2r}+n) = \mathcal{O}(1)e^{-\alpha n^r} n^{1-r}\Big(\frac{1}{n^r}+\frac{1}{n^{1-r}}\Big). \tag{38}$$

Combining (36)–(38) we obtain (24). Corollary 2.5 is proved. □

Asymptotic equality (24) was proved in [7].

By $\mathfrak{M}$ we denote the set of all convex (downward) continuous functions $\psi(t)$, $t \geq 1$, such that $\lim\limits_{t\to\infty} \psi(t) = 0$.

Assume that the sequence $\psi(k)$, $k \in \mathbb{N}$, specifying the class $C^{\psi}_{\beta,1}$ is the restriction of the functions $\psi(t)$ from $\mathfrak{M}$ to the set of natural numbers.

We also consider the following characteristics of functions $\psi \in \mathfrak{M}$:

$$\alpha(t) := \frac{\psi(t)}{t|\psi'(t)|}$$

and

$$\lambda(t) := \frac{\psi(t)}{|\psi'(t)|}.$$

Theorem 2.6. *Let $\psi \in \mathfrak{M}$, $\alpha(t) \downarrow 0$, $\lambda(t) \to \infty$, $\lambda'(t) \to 0$ as $t \to \infty$ and $\beta \in \mathbb{R}$. Then as $n \to \infty$ the following asymptotic equality holds*

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \psi(n)\lambda(n)\left(\frac{1}{\pi} + \mathcal{O}\left(\frac{1}{\lambda(n)} + \alpha(n) + \varepsilon_n\right)\right), \tag{39}$$

where $\varepsilon_n := \sup\limits_{t\geq n} |\lambda'(t)|$ and $O(1)$ is a quantity uniformly bounded in n and β.

Proof. From Theorem 2.1 we have that the following asymptotic equality holds as $n \to \infty$

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \frac{1}{\pi}\sum_{k=n}^{\infty} \psi(k) + \frac{\mathcal{O}(1)}{n}\sum_{k=1}^{\infty} k\psi(k+n). \tag{40}$$

Notice that

$$(41)\quad \sum_{k=0}^{\infty} k\psi(k+n) = \sum_{k=n}^{\infty} k\psi(k) - n\sum_{k=n}^{\infty}\psi(k) < \psi(n)n + \int_n^{\infty} t\psi(t)dt - n\int_n^{\infty}\psi(t)dt.$$

Let $\lambda'(t) \to 0$ as $t \to \infty$. Then integrating by parts, we get

$$\begin{aligned} I_1 :=& \int_n^{\infty}\psi(t)dt = \int_n^{\infty} -\psi'(t)\lambda(t)dt = \psi(n)\lambda(n) + \int_n^{\infty}\psi(t)\lambda'(t)dt \\ =&\psi(n)\lambda(n) + \lambda'(\theta_1)I_1, \end{aligned}$$

where θ_1 is a some point from the interval $[n, \infty)$.

Then

$$I_1\left(1 - \lambda'(\theta_1)\right) = \psi(n)\lambda(n)$$

and

$$I_1 = \psi(n)\lambda(n)\left(1 + \frac{\lambda'(\theta)}{1-\lambda'(\theta)}\right) = \psi(n)\lambda(n)\left(1 + \mathcal{O}(1)\varepsilon_n\right).$$

Again integrating by parts

$$\begin{aligned} I_2 :=& \int_n^{\infty} t\psi(t)dt = \int_n^{\infty} t^2\frac{\psi(t)}{-t\psi'(t)}(-\psi'(t))dt = \alpha(\theta_2)\int_n^{\infty} t^2(-\psi'(t))dt \\ =&\alpha(\theta_2)\left(n^2\psi(n) + 2\int_n^{\infty} t\psi(t)\right), \end{aligned}$$

where θ_2 is a some point from the interval $[n, \infty)$.

Assume that $\alpha(t)$ monotonically decreases. Then

$$I_2 \le \alpha(n)n^2\psi(n) + 2\alpha(n)I_2,$$

which is equivalent to

$$I_2\left(1 - 2\alpha(n)\right) \le \alpha(n)n^2\psi(n) = \psi(n)n\frac{\psi(n)}{|\psi'(n)|}$$

and

$$\frac{1}{n}I_2 \le \psi(n)\frac{\psi(n)}{|\psi'(n)|}\frac{1}{1-2\alpha(n)}.$$

Hence, if $\alpha(t) \downarrow 0$, then

$$(42)\quad \frac{1}{n}I_2 \le \psi(n)\frac{\psi(n)}{|\psi'(n)|}\left(1 + \frac{2\alpha(n)}{1-2\alpha(n)}\right)$$

and

$$I_1 = \psi(n)\frac{\psi(n)}{|\psi'(n)|} + \psi(n)\frac{\psi(n)}{|\psi'(n)|}\mathcal{O}(\varepsilon_n). \tag{43}$$

Combining (42) and (43), we obtain

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{\infty} k\psi(k+n) \leq& \psi(n) + \psi(n)\frac{\psi(n)}{|\psi'(n)|}\left(1 + \frac{2\alpha(n)}{1-2\alpha(n)}\right)\\ &- \left(\psi(n)\frac{\psi(n)}{|\psi'(n)|} + \psi(n)\frac{\psi(n)}{|\psi'(n)|}\mathcal{O}(\varepsilon_n)\right)\\ =&\psi(n) + \psi(n)\frac{\psi(n)}{|\psi'(n)|}\left(\frac{2\alpha(n)}{1-2\alpha(n)} + \mathcal{O}(\varepsilon_n)\right)\\ =&\psi(n)\frac{\psi(n)}{|\psi'(n)|}\mathcal{O}\left(\frac{1}{\lambda(n)} + \alpha(n) + \varepsilon_n\right). \end{aligned} \tag{44}$$

Moreover, taking into account (43),

$$\sum_{k=n}^{\infty}\psi(k) = I_1 + \mathcal{O}(1)\psi(n) = \psi(n)\frac{\psi(n)}{|\psi'(n)|}\left(1 + \mathcal{O}(1)\left(\varepsilon_n + \frac{1}{\lambda(n)}\right)\right). \tag{45}$$

Formulas (40), (44) and (45) imply (39). Theorem 2.6 is proved. □

Corollary 2.7. *Let $\psi(k) = (k+2)^{-\ln\ln(k+2)}$, $\beta \in \mathbb{R}$ and $k \in \mathbb{N}$. Then as $n \to \infty$ the following asymptotic equality holds*

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \frac{1}{\pi}\psi(n)\frac{n}{\ln\ln(n+2)} + \mathcal{O}(\psi(n)). \tag{46}$$

Proof. If $\psi(k) = e^{-\ln(k+2)\ln\ln(k+2)}$, then

$$\begin{aligned}\psi'(t) &= -e^{-\ln(t+2)\ln\ln(t+2)}\left(\frac{\ln\ln(t+2)}{t+2} + \frac{\ln(t+2)}{(t+2)\ln(t+2)}\right)\\ &= -e^{-\ln(t+2)(\ln\ln(t+2)}\frac{\ln\ln(t+2)+1}{t+2}.\end{aligned}$$

Doing elementary calculations, we get

$$\lambda(t) = \frac{t+2}{\ln\ln(t+2)+1} = \frac{t}{\ln\ln(t+2)} + \mathcal{O}(1), \tag{47}$$

$$\alpha(t) = \frac{t+2}{t}\frac{1}{\ln\ln(t+2)+1} \tag{48}$$

and

$$\lambda'(t) = \frac{\ln\ln(t+2) + 1 - \frac{1}{\ln(t+2)}}{(\ln\ln(t+2)+1)^2} \leq \frac{1}{\ln\ln(t+2)}. \tag{49}$$

Substituting (47)–(49) in (39) we obtain (46). Corollary 2.7 is proved. □

Corollary 2.8. *Let $\psi(k) = e^{-\ln^2 k}$, $\beta \in \mathbb{R}$ and $k \in \mathbb{N}$. Then as $n \to \infty$ the following asymptotic equality holds*

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \frac{1}{2\pi}\frac{\psi(n)n}{\ln n} + \mathcal{O}(\psi(n)). \tag{50}$$

Proof. It is easy to see

$$\psi'(t) = -2\frac{1}{t}e^{-\ln^2 t}\ln t. \tag{51}$$

Formula (51) yields

$$\lambda(t) = \frac{t}{2\ln t}, \quad \alpha(t) = \frac{1}{2\ln t} \tag{52}$$

and

$$\lambda'(t) = \frac{\ln t - 1}{2(\ln t)^2} \leq \frac{1}{2\ln t}. \tag{53}$$

Formulas (51)–(53) and (39) imply (50). Corollary 2.8 is proved. □

Corollary 2.9. *Let $\psi(k) = e^{-\frac{k+1}{\ln(k+1)}}$, $\beta \in \mathbb{R}$ and $k \in \mathbb{N}$. Then as $n \to \infty$ the following asymptotic equality holds*

$$\mathcal{E}_n(C^{\psi}_{\beta,1})_C = \psi(n)\ln(n+1)\left(\frac{1}{\pi} + \mathcal{O}\left(\frac{1}{\ln(n+1)}\right)\right). \tag{54}$$

Proof. Doing elementary calculations, we get

$$\psi'(t) = -e^{-\frac{t+1}{\ln(t+1)}}\frac{\ln(t+1)-1}{\ln^2(t+1)},$$

$$\lambda(t) = \frac{\ln^2(t+1)}{\ln(t+1)-1}, \quad \alpha(t) = \frac{\ln^2(t+1)}{t\ln(t+1)-t} = \mathcal{O}\left(\frac{1}{t\ln(t+1)}\right) \tag{55}$$

and

$$\lambda'(t) = \frac{1}{t+1} - \frac{1}{(t+1)(\ln(t+1)-1)^2} \leq \frac{1}{t+1}. \tag{56}$$

Formulas (55), (56) and (39) imply (54). Corollary 2.9 is proved. □

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