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RICAM-Report 2019-34

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December 20, 2019

We propose local multigrid solvers for adaptively refined isogeometric discretization using HB- and THB-splines. We prove robust convergence of the proposed solvers with respect to the number of levels and the mesh sizes of the hierarchical discretization space and provide some numerical experiments. Smoothing is only performed in or near the refinement areas on each level, leading to a computationally efficient method.

The main analytical tools are quasi-interpolators for THB-spline basis functions and the abstract convergence theory of subspace correction methods.

1 Introduction

Elliptic partial differential equations with local features such as singularities are typically solved numerically through the use of adaptive refinement. Historically, much effort has been put into the development of a more unified approach to the combined process of adaptive refinement and multigrid solution, which in addition could also be used with high order methods, see [26, 36] and the references therein. In the context of Isogeometric Analysis (IgA; see [20]), the development of an adaptive isogeometric method (AIGM) for solving elliptic second-order partial differential equations with truncated hierarchical B-splines (THB-splines) of arbitrary degree, different order of continuity and any dimension has been addressed in [8, 10, 9]. AIGM based on local refinements can be written using the standard loop of the form

$$\textbf{SOLVE} \quad \rightarrow \quad \textbf{ESTIMATE} \quad \rightarrow \quad \textbf{MARK} \quad \rightarrow \quad \textbf{REFINE}.$$

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Apart from **SOLVE**, these steps have been thoroughly discussed in the aforementioned papers. The distinct feature of applying multigrid or multilevel solvers to adaptively refined spaces is that the number of new degrees of freedom may not grow exponentially with the number of refinement steps, which would be the case for global refinement. Thus, local smoothing strategies are required in order to achieve optimal computational complexity, leading to so-called local multigrid methods.

The literature on fast solvers for AIGM schemes is still quite sparse. To our knowledge, the first work on fast solvers for adaptive IgA was [19], where a multigrid solver for HB- and THB-spline discretizations was constructed. Here, smoothing was still performed globally, and no convergence analysis was given. Additive (BPX) solvers for analysis-suitable T-splines were described in [13], where also the robust convergence with respect to the mesh size was proved. The construction closely follows that of the corresponding finite element result in [12].

In the present work, we present the first local multigrid solver for linear systems associated to the HB- or THB-discretization of elliptic partial differential equations and present an h-robust convergence analysis. Our construction and analysis are significantly simpler than the corresponding one for T-splines [13] and directly exploit the inherent multilevel structure of hierarchical spline spaces. Surprisingly, the resulting theory is simpler than that of any analogous solvers in the finite element method (FEM) world [12, 18, 34]. There is a certain similarity to a multigrid scheme used in deal.II which is described in [21]; however, no convergence analysis is given therein. An important tool in our analysis is the existence of of a versatile quasi-interpolant for THB-splines [33, 32].

The theoretical framework for our convergence theory is provided by the theory of subspace correction methods [35]. This general approach involves the solving of subproblems on suitable chosen subspaces and combining these corrections either additively (*parallel subspace correction*, PSC) or multiplicatively (*successive subspace correction* SSC). When applied to a hierarchy of refinement levels, the former lead to additive multilevel (BPX-like) solvers, whereas the latter lead to multigrid methods. The interpretation of an abstract multilevel method as an SSC method was introduced in Bramble, Pasciak, Wang and Xu [6]. The equivalence of certain multigrid methods to SSC methods was also discussed in Xu [35]. In this setting, smoothing iterations take the role of approximate subspace solvers.

For adaptive refinement strategies, smoothing must be done locally in a certain sense in order to maintain optimal computational complexity. It has long been known that, in order not to degrade the convergence rate, smoothing has to be performed at least for all basis functions contained in the refined region; this usually means including degrees of freedom neighboring the newly added ones in the smoothing set [1, 5, 7, 24, 25, 29, 23]. An exception is the hierarchical basis method [2], where only newly added degrees of freedom are smoothed, but which is limited to two-dimensional problems.

In the present work, we show that smoothing only those fine B-spline functions supported within the refinement region is sufficient for h-robust convergence.

One can show that if the local smoothing procedure is computationally optimal, these so-called "local multilevel methods" are of optimal computational complexity in the sense that if N is the size of the linear system on the finest level, only $\mathcal{O}(N)$ operations have to be performed for one sweep of this method. Under the same conditions also the PSC-like BPX preconditioner exhibits optimal computational complexity [4, 15, 28, 27].

The paper is organized as follows. In Section 2 we review the basic features of HB- and THB-splines and quasi-interpolants (QIs); for the latter, we extend an existing approximation result. We also describe the isogeometric discretization to be solved and review the theory of subspace correction methods. In Section 3 we give a decomposition of hierarchical spline spaces obtained by adaptive refinement and prove its stability and strengthened Cauchy-Schwarz inequality under suitable smoothing properties. We also verify these smoothing properties for some standard smoothers. In Section 4 we combine the previous results in order to give a rigorous convergence analysis of a local multigrid solver for hierarchical spline spaces and discuss possible extensions.

2 Preliminaries

In this section we review some known results on THB-splines, quasi-interpolants and convergence theory for abstract subspace correction methods. In addition, in Section 2.2 we give new approximation and stability results for THB-spline quasi-interpolants in Sobolev space norms which generalize previous results from [33, 32].

2.1 THB-splines

Let $D \subseteq \mathbb{R}^d, d \in \mathbb{N}$ be a closed hyper-rectangle, $\mathbb{B}^0 \subseteq \mathbb{B}^1 \subseteq \ldots \subseteq \mathbb{B}^N, N \in \mathbb{N}$ be nested d-variate tensor product spline spaces on D spanned by the normalized tensor product B-spline basis for $l = 0, \ldots, N$,

$$\mathcal{B}^l := \{\beta^{(l,\boldsymbol{i})} : \boldsymbol{i} \in \mathcal{I}^l\}, \mathcal{I}^l := \{(i_1, \ldots, i_d) : i_k = 1, \ldots, n_k^l, k = 1, \ldots, d\},\ \mathbb{B}^l := \operatorname{span} \mathcal{B}^l$$

on corresponding *uniform open knot sequences* of degree $p_k \in \mathbb{N}, k = 1, \ldots, d$, i.e., the resulting mesh G^l consists of hyper-cubes (*cells*) with edge size $h_l = h_0 2^{-l}, l = 1, \ldots, N$ for some fixed $h_0 > 0$. The (non-empty) quadrilateral cells $\Upsilon^l \in G^l$ are the Cartesian product of d open intervals between adjacent grid values. For any coordinate direction $k = 1, \ldots, d$, each grid value appears in the knot vector as many times as specified by a certain multiplicity. At any level the multiplicity of each knot may vary between one (single knots) and p_k. To enforce nestedness of the spline spaces we assume the knot sequences to be also nested, i.e. we assume $\mathbb{B}^l$ to be obtained from $\mathbb{B}^{l-1}$ by *dyadic refinement*, hence $h_l = h_{l-1}/2, l = 1, \ldots, N$, where h_l denotes the uniform grid mesh size of level l and that every knot of level $l-1$ is also present at level l at least with the same multiplicity in the corresponding coordinate direction.

We also take as given a sequence of nested domains $D = \Omega^0 \supseteq \Omega^1 \supseteq \ldots \supseteq \Omega^N$ being closed subsets of D, which are defined as the union of the closure of cells that belong to the tensor product grid G^{l-1} of the previous level. By assumption $\partial\Omega^l$ is aligned with the knot lines of $\mathbb{B}^l, l = 0, \ldots, N$. The union of the associated grids is referred to as the *hierarchical mesh*

$$\mathcal{G} := \bigcup_{l=0}^{N} (G^l \cap \Omega^l \setminus \Omega^{l+1}), \qquad l = 0, \dots, N. \tag{1}$$

Let us denote by $\operatorname{supp} f$ the support of a function f intersected with D.

Definition 2.1 ([17]). Let $f \in \mathbb{B}^l, l \in \{0, \dots, N-1\}$ and let

$$f = \sum_{\beta \in \mathcal{B}^{l+1}} c_\beta^{l+1}(f)\beta, \qquad c_\beta^{l+1}(f) \in \mathbb{R}, \tag{2}$$

be its representation with respect to $\mathbb{B}^{l+1}$. The *truncation* of f with respect to $\mathcal{B}^{l+1}$ and Ω^{l+1} is defined as

$$\overset{l+1}{\operatorname{trunc}} f := \sum_{\substack{\beta \in \mathcal{B}^{l+1} \\ \operatorname{supp} \beta \not\subseteq \Omega^{l+1}}} c_\beta^{l+1}(f)\beta. \tag{3}$$

Now we can introduce the *truncated hierarchical B-spline (THB-spline) basis.*

Definition 2.2 ([17]). The *THB-spline basis* $\mathcal{T}$ is recursively defined as

$$\begin{aligned} \mathcal{T}^0 &:= \{\beta \in \mathcal{B}^0 : \operatorname{supp} \beta \neq \emptyset\}, \\ \mathcal{T}^{l+1} &:= \{\overset{l+1}{\operatorname{trunc}} f : f \in \mathcal{T}^l, \operatorname{supp} f \not\subseteq \Omega^{l+1}\} \\ &\quad \cup \{\beta \in \mathcal{B}^{l+1} : \operatorname{supp} \beta \subseteq \Omega^{l+1}\}, \qquad l = 0, \dots, N-1, \\ \mathcal{T} &:= \mathcal{T}^N. \end{aligned}$$

THB-splines are non-negative, linearly independent and form a partition of unity [17].

Lemma 2.1 ([17]). *For every $\tau \in \mathcal{T}^l, l \in \{0, \dots, N\}$ there exists a unique $\beta \in \mathcal{B}^l$ with*

$$\tau = \overset{l+1}{\operatorname{Trunc}} \beta := \overset{N}{\operatorname{trunc}} \overset{N-1}{\operatorname{trunc}} \cdots \overset{l+1}{\operatorname{trunc}} \beta, \qquad \tau|_{\Omega^l \setminus \Omega^{l+1}} = \beta|_{\Omega^l \setminus \Omega^{l+1}}.$$

With the sets

$$\mathcal{B}_*^l := \{\beta^{(l,\boldsymbol{i})} : \boldsymbol{i} \in \mathcal{I}_*^l\}, \quad \mathcal{I}_*^l := \{\boldsymbol{i} \in \mathcal{I}^l : \Omega^l \supseteq \operatorname{supp} \beta^{(l,\boldsymbol{i})} \not\subseteq \Omega^{l+1}\}, \quad \mathbb{B}_*^l := \operatorname{span} \mathcal{B}_*^l$$

of *active basis functions* [16] on level $l = 0, \dots, N$, the index set $\mathcal{I}$ of THB-splines basis functions can then be defined as

$$\mathcal{I} := \{(l, \boldsymbol{i}) : \boldsymbol{i} \in \mathcal{I}_*^l, l = 0, \dots, N\}, \tag{4}$$

and definition 2.2 directly implies the equivalent representation

$$\mathcal{T} = \{\tau^{(l,\boldsymbol{i})} : (l, \boldsymbol{i}) \in \mathcal{I}\}, \quad \mathbb{T} := \operatorname{span} \mathcal{T}, \quad \text{with} \quad \tau^{(l,\boldsymbol{i})} := \overset{l+1}{\operatorname{Trunc}} \beta^{(l,\boldsymbol{i})}. \tag{5}$$

We see from this construction that the support of any $\tau \in \mathcal{T}$ can be given as the union of closed cells in $\mathcal{G}$.

It is a well known fact [17] that without the truncation mechanism, the *hierarchical B-spline (HB-spline) basis* spans the same space, i.e.

$$\mathbb{T} = \operatorname{span}\left(\bigcup_{l=0}^{N} \mathcal{B}_*^l\right) = \operatorname{span}\{\beta^{(l,\boldsymbol{i})} : (l,\boldsymbol{i}) \in \mathcal{I}\}. \tag{6}$$

Therefore, we will refer to $\mathbb{T}$ as a *hierarchical spline space* when the particular choice of basis is of no interest.

Definition 2.3 ([32, 33])**.** For each cell $\Upsilon \in \mathcal{G}$ in the hierarchical mesh, let $\delta_{\mathcal{T},\Upsilon}$ be the largest difference between the levels of the THB-splines $\mathcal{T}$ supported on Υ. The *mesh level disparity* $\delta_{\mathcal{T}}$ is defined as the maximum of the values $\delta_{\mathcal{T},\Upsilon}$ related to all cells $\Upsilon \in \mathcal{G}$.

From this, one can bound the number of overlapping basis functions $\tau \in \mathcal{T}$ on a cell $\Upsilon \in \mathcal{G}$ by

$$c_{\mathcal{T}} := (\delta_{\mathcal{T}} + 1)\bar{c}_{\boldsymbol{p}}, \quad \text{with} \quad \bar{c}_{\boldsymbol{p}} := \prod_{k=1}^{d} (p_k + 1),$$

see [33, 8].

Corollary 2.1 ([8])**.** *Under the above assumptions, one has*

$$|\Upsilon| \eqsim |\operatorname{supp} \tau|, \quad \forall \Upsilon \in \mathcal{G}, \Upsilon \cap \operatorname{supp} \tau \neq \emptyset,$$

where $|\Upsilon|$ *denotes the* d*-dimensional measure of* $\Upsilon \in \mathcal{G}$ *and the hidden constants in the above inequalities depend on* $\delta_{\mathcal{T}}$*, but not on* $\mathcal{T}, \mathcal{G}$ *or* N*.*

Remark 2.1. All results in this paper can easily be generalized to quasi-uniform meshes [8], which would demand a more complex notation.

For the rest of this paper, we will always indicate any inequality which does not depend on the depth N (or, equivalently, on h_N or $\mathcal{G}$) of the spline hierarchy with $\lesssim, \gtrsim$. We write $\eqsim$, if the relation holds for both $\lesssim$ and $\gtrsim$.

Definition 2.4. The *support extension* $S(\Upsilon, k) \subseteq G^k$ of $\Upsilon \in G^l$ with respect to $k, 0 \leq k \leq l, l \in \{0, \dots, N\}$ is defined as

$$S(\Upsilon, k) := \{\Upsilon' \in G^k : \exists \beta \in \mathcal{B}^k, \operatorname{supp} \beta \cap \Upsilon' \neq \emptyset, \operatorname{supp} \beta \cap \Upsilon \neq \emptyset\}.$$

By a slight abuse of notation, we will also denote by $S(\Upsilon, k)$ the region occupied by the closure of elements in $S(\Upsilon, k)$.

2.2 Quasi-interpolants

The subsequent construction, following [32, 33], allows the simple design of *quasi-interpolants* (QIs) in $\mathbb{T}$, once a sequence of QIs in the spaces $\mathbb{B}^l, l = 0, \dots, N$ is given. Let f be a function on D. Consider a sequence of *one-level QIs*

$$\mathfrak{Q}^l f := \sum_{\boldsymbol{i} \in \mathcal{I}^l} \lambda^{(l,\boldsymbol{i})}(f) \beta^{(l,\boldsymbol{i})}, \qquad l = 0, \dots, N,$$

where $\lambda^{(l,\boldsymbol{i})}, \boldsymbol{i} \in \mathcal{I}^l, l = 0, \ldots, N$ are suitable *linear functionals*. We say that $\lambda^{(l,\boldsymbol{i})}$ is supported on $\Lambda^{(l,\boldsymbol{i})}$ iff

$$f|_{\Lambda^{(l,\boldsymbol{i})}} \equiv 0 \implies \lambda^{(l,\boldsymbol{i})}(f) = 0.$$

If $\Lambda^{(l,\boldsymbol{i})}$ with this property are choosen as small as possible, one refers to $\Lambda^{(l,\boldsymbol{i})}$ as the *support* of $\lambda^{(l,\boldsymbol{i})}$. We define our *hierarchical QI* in $\mathbb{T}$ as

$$\mathfrak{Q} : f \in L^2(D) \mapsto \sum_{(l,\boldsymbol{i})\in\mathcal{I}} \lambda^{(l,\boldsymbol{i})}(f)\tau^{(l,\boldsymbol{i})} \in \mathbb{T}. \tag{7}$$

Theorem 2.1 ([32, 33])**.** *If each $\lambda^{(l,\boldsymbol{i})}, \boldsymbol{i} \in \mathcal{I}^l, l = 0, \ldots, N$ is supported on $\Omega^l \setminus \Omega^{l+1}$ and $\mathfrak{Q}^l f = f$ for any $f \in \mathbb{B}^l, l = 0, \ldots, N$ then*

$$\mathfrak{Q}f = f, \qquad \forall f \in \mathbb{T}. \tag{8}$$

Consequently, the construction of a hierarchical QI for THB-splines has been reduced to the construction of appropriate one-level QIs as outlined in the above theorem. For the remainder of this paper we make the following assumptions.

QI1 The mesh level disparity $\delta_{\mathcal{T}}$ is bounded independently of the number N of levels in the hierarchy. This can be guaranteed by suitable refinement strategies, [8].

QI2 The linear functionals $\lambda^{(l,\boldsymbol{i})}, \boldsymbol{i} \in \mathcal{I}^l, l = 0, \ldots, N$ are locally supported,

$$\operatorname{diam} \Lambda^{(l,\boldsymbol{i})} \leq C_\Lambda h_l, \qquad \forall \boldsymbol{i} \in \mathcal{I}^l_*, \tag{9}$$

where C_Λ is a constant independent of h_l and $\operatorname{diam} \Lambda^{(l,\boldsymbol{i})}$ denotes the diameter of $\Lambda^{(l,\boldsymbol{i})}$.

QI3 The linear functionals $\lambda^{(l,\boldsymbol{i})}, (l, \boldsymbol{i}) \in \mathcal{I}$ are bounded in the L^q-norm, $1 \leq q \leq \infty$,

$$|\lambda^{(l,\boldsymbol{i})}(f)| \leq C_\lambda (h_l)^{-d/q} \|f\|_{L^q(\Lambda^{(l,\boldsymbol{i})})}, \qquad \forall (l, \boldsymbol{i}) \in \mathcal{I}, \tag{10}$$

where C_λ is a constant independent of h_l and $\|.\|_{L^q(\Lambda^{(l,\boldsymbol{i})})}$ denotes the usual q-Lebesgue norm on $\Lambda^{(l,\boldsymbol{i})}, (l, \boldsymbol{i}) \in \mathcal{I}$.

QI4 The linear functionals $\lambda^{(l,\boldsymbol{i})}, \boldsymbol{i} \in \mathcal{I}^l, l = 0, \ldots, N$ are choosen so that $\mathfrak{Q}^l, l = 0, \ldots, N$ reproduces the tensor product polynomial space $\mathbb{P}_{\boldsymbol{\rho}}$ of degree $\boldsymbol{\rho}$,

$$\mathfrak{Q}^l g = g, \qquad \forall g \in \mathbb{P}_{\boldsymbol{\rho}}, \tag{11}$$

for some $\boldsymbol{\rho} = (\rho, \ldots, \rho) \in \mathbb{N}_0^d$ with $0 \leq \rho \leq p$.

In the following two examples, we present one-level QIs that satisfy the conditions (9)-(11).

Example 2.1. For $d = 1$ we consider the QI developed in [22, Section 5.3.1] of the form (7) with

$$\lambda^{(l,\boldsymbol{i})}(f) := \frac{1}{h_l}\int_{\xi_n^l}^{\xi_{n+1}^l}\Big(\sum_{j=0}^{p} a_{i,j}\left(\frac{x-\xi_n^l}{h_l}\right)^j\Big) f(x)dx, \tag{12}$$

where $[\xi_n^l, \xi_{n+1}^l) \in G^l$ can be any knot interval in the support of the B-spline $\beta^{(l,\boldsymbol{i})}$ of degree p, and the coefficients $a_{i,j} \in \mathbb{R}, i, j = 0, \dots, p$ are chosen in a special way. These functionals clearly satisfy (9) with $C_\Lambda = 1$, and from [22, Lemma 3] we know that they also satisfy (10). Finally, from [22, Lemma 2] it follows that (11) is satisfied for each $0 \le \rho \le p$.

Example 2.2. For $d > 1$ we construct the QI by taking the tensor product of univariate schemes defined in the previous example 2.1. More precisely, given a d-variate function f we define the linear functionals as

$$\lambda^{(l,\boldsymbol{i})}(f) := (\lambda^{(l,i_1)} \cdots \lambda^{(l,i_d)})(f), \qquad \forall (l, \boldsymbol{i}) \in \mathcal{I},$$

assuming that $\lambda^{(l,i_k)}, k = 1, \dots, d$ are the linear functionals in (12) operating on functions of the k-th variable. From the properties of the univariate scheme it follows that also this multivariate scheme satisfies the conditions (9)-(11).

Suppose $\Upsilon^l \in G^l$ is a cell of a given level $l = 0, \dots, N$ in $\Omega^l \setminus \Omega^{l+1}$. One can check that $\operatorname{diam} \Upsilon^l = h_l\sqrt{d}$. We define

$$\Lambda_{\Upsilon^l} := \operatorname{conv}\Big(\bigcup_{(k,\boldsymbol{i})\in\mathcal{I}:\operatorname{supp}\tau^{(k,\boldsymbol{i})}\cap\Upsilon^l\neq\emptyset} \Lambda^{(k,\boldsymbol{i})} \cup \Upsilon^l\Big) \subseteq S(\Upsilon^l, l-\delta_{\mathcal{T}}), \tag{13}$$

where $\operatorname{conv}\Lambda$ denotes the convex hull of a set $\Lambda \subseteq \mathbb{R}^d$. Taking into account the bounded mesh level disparity, we have

$$\operatorname{diam}\Lambda_{\Upsilon^l} \le C_{\Lambda_\Upsilon} h_l,$$

where C_{Λ_Υ} is a constant independent of h_l. Let us denote by $|.|_{W_q^k(\Lambda)}$ the usual seminorm on the Sobolev space $W_q^k(\Lambda)$. We are now ready to extend the L^q-norm approximation estimate provided in [32, Corollary 1] as follows.

Theorem 2.2. *Under the above assumptions, let $\Upsilon^l \in G^l$ be a cell of level l in $\Omega^l \setminus \Omega^{l+1}$, and let Λ_{Υ^l} be the corresponding set as defined in (13). Let $\mathfrak{Q}$ satisfy (9)-(11) (e.g., be constructed according to example 2.1 and example 2.2). If $f \in W_q^{\rho+1}(\Lambda_{\Upsilon^l}), 1 \le q \le \infty, 0 \le \rho \le \min_{j=1,\dots,d} p_j$, then for any $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d), 0 \le \alpha_j \le p_j, j = 1, \dots, d$ one has*

$$\|D^{\boldsymbol{\alpha}}(f - \mathfrak{Q}f)\|_{L^q(\Upsilon^l)} \le C h_l^{\rho+1-|\boldsymbol{\alpha}|}|f|_{W_q^{\rho+1}(\Lambda_{\Upsilon^l})}, \tag{14}$$

where the constant C is independent of f and h_l and $|\boldsymbol{\alpha}| = \sum_{j=1}^d \alpha_j$.

Proof. By the partition of unity property of the basis, definition (13) and condition (10) we have

$$|\mathfrak{Q}f(\boldsymbol{x})| \le C_\lambda h_l^{-d/q}\|f\|_{L^q(\Lambda_{\Upsilon^l})}, \qquad \forall \boldsymbol{x} = (x_1, \dots, x_d) \in \Upsilon^l,$$

where we used the fact that

$$h_l = \min_{(k,\boldsymbol{i})\in\mathcal{I}:\operatorname{supp}\tau^{(k,\boldsymbol{i})}\cap\Upsilon^l\neq\emptyset} h_k.$$

Then, taking the L^q-norm we get

$$\|\mathfrak{Q}f\|_{L^q(\Upsilon^l)} \le C_1\|f\|_{L^q(\Lambda_{\Upsilon^l})}. \tag{15}$$

Now, consider the averaged Taylor polynomial $\mathfrak{F}_{\rho,B_{\Lambda_{\Upsilon^l}}}$ defined in [32, Definition 1] with $p=\rho$ and $B=B_{\Lambda_{\Upsilon^l}}$, the ball with largest radius contained in Λ_{Υ^l}. Using the bound (15) and the error estimate from [32, Lemma 2] we arrive at

$$\|\mathfrak{Q}(f-\mathfrak{F}_{\rho,B_{\Lambda_{\Upsilon^l}}}f)\|_{L^q(\Upsilon^l)} \le C_1\|f-\mathfrak{F}_{\rho,B_{\Lambda_{\Upsilon^l}}}f\|_{L^q(\Lambda_{\Upsilon^l}} \le C_2\Big(\operatorname{diam}\Lambda_{\Upsilon^l}\Big)^{\rho+1}|f|_{W_q^{\rho+1}(\Lambda_{\Upsilon^l})}. \tag{16}$$

The inequality (16) is similar to the inequality [32, (18)] but it has no restrictions on the relations between the degree ρ, the dimension d and the L^q-norm.

Finally, we follow the same line of arguments as in the proof of [32, Theorem 4]. In particular, we combine the inequalities in [32, (15)-(17)] with (16), and we get the error estimate (14). We note that $D^{\boldsymbol{\alpha}}g=0$ for any polynomial $g\in\mathbb{P}_{\boldsymbol{\rho}}$ whenever $\alpha_j>\rho_j$ for some $j=1,\dots,d$. □

2.3 Adaptive isogeometric discretizations

We assume that a computational domain $G(D)\subset\mathbb{R}^d$ is given via a bijective and sufficiently smooth *geometry mapping* $G: D\to G(D)\subset\mathbb{R}^d$, which maps the parameter domain $D=(0,1)^d$ to the physical domain $G(D)$. In Isogeometric Analysis (IgA) [20], this mapping is typically given in terms of spline basis functions, $G\in(\mathbb{B}^0)^d$, but its concrete form is irrelevant to our discussion. The standard IgA approach is then to transform the variational problem of interest back to the parameter domain. Choosing a suitable Hilbert space $\mathcal{V}$ over D, we obtain a variational problem of the form: for $f\in\mathcal{V}'$, find $u\in\mathcal{V}$ such that

$$a(u,v)=\langle f,v\rangle,\qquad \forall v\in\mathcal{V}.$$

Here we assume that the bilinear form on the physical domain $G(D)$ is symmetric, positive definite, and bounded, which then also holds for $a(\cdot,\cdot):\mathcal{V}\times\mathcal{V}\to\mathbb{R}$. The concrete coefficient within the bilinear form $a(\cdot,\cdot)$ is modified by this transformation.

Our case of interest are adaptive discretizations of such variational problems using HB- or THB-splines. Thus, let $\mathbb{T}$ be a hierarchical spline space as defined in Section 2.1 (e.g., obtained by an AIGM scheme as outlined in the introduction) and let $\mathcal{V}:=\mathbb{T}\cap H_0^1(D)$ denote our finite dimensional trial space, where $H_0^1(D)$ is the Sobolev space $H^1(D)=W_2^1(D)$ with zero trace at ∂D. We assume here pure Dirichlet boundary conditions for simplicity, but other boundary conditions pose no problem.

We pass to the operator notation $A:\mathcal{V}\to\mathcal{V}'$ by setting $\langle Au,v\rangle=a(u,v)\ \forall u,v\in\mathcal{V}$, resulting in the operator equation $Au=f$. The main object of this paper is to construct fast multigrid solvers for this discretized problem.

2.4 Subspace correction methods

We describe here subspace correction methods and their convergence theory in the abstract setting, which will provide the basis for the convergence analysis of our multigrid method.

Let $(\mathcal{V}, (.,.))$ be a finite dimensional Hilbert space and choose a *space decomposition* of $\mathcal{V}$ of the form

$$\mathcal{V} = \sum_{i=0}^{N} \mathcal{V}_i$$

with subspaces $\mathcal{V}_i \subseteq \mathcal{V}$, $i = 0, \dots, N \in \mathbb{N}$. For a given $u \in \mathcal{V}$, the decomposition $u = \sum_{i=0}^{N} u_i$, $u_i \in \mathcal{V}_i$ is in general not unique.

Let $A : \mathcal{V} \to \mathcal{V}'$ be linear, bounded, symmetric and positive definite (s.p.d.), i.e., A is isomorphic and $\|.\|_A^2 := \langle A.,.\rangle$ constitutes a norm, where $\mathcal{V}', \mathcal{V}_i'$ denote the dual spaces of $\mathcal{V}, \mathcal{V}_i$, $i = 0, \dots, N$, respectively. Throughout this paper, we use the following notation for $i = 0, \dots, N$:

- $P_i : \mathcal{V} \to \mathcal{V}_i$ the (*energy*) projection with respect to $(.,.)_A := \langle A.,.\rangle$, i.e.

$$(P_i v, v_i)_A = (v, v_i)_A, \qquad \forall v \in \mathcal{V}, v_i \in \mathcal{V}_i, i = 0, \dots, N,$$

- $A_i : \mathcal{V}_i \to \mathcal{V}_i'$ the restriction of A to the subspace $\mathcal{V}_i$, i.e.

$$\langle A_i u_i, v_i\rangle = \langle A u_i, v_i\rangle, \qquad \forall u_i, v_i \in \mathcal{V}_i, i = 0, \dots, N,$$

- $R_i : \mathcal{V}_i' \to \mathcal{V}_i$ a s.p.d. approximation of A_i^{-1},
- $T_i : \mathcal{V} \to \mathcal{V}_i$ the auxiliary operator $T_i = R_i A_i P_i = R_i A$, $i = 0, \dots, N$.

With a slight abuse of notation we still use T_i to denote the restriction $T_i|_{\mathcal{V}_i} : \mathcal{V}_i \to \mathcal{V}_i$ and $T_i^{-1} = (T_i|_{\mathcal{V}_i})^{-1} : \mathcal{V}_i \to \mathcal{V}_i$.

For a given right-hand side $f \in \mathcal{V}'$, our goal is to solve the operator equation

$$Au = f$$

for $u \in \mathcal{V}$ by means of the above space decomposition. There are two common ways to achieve this:

PSC (Parallel subspace correction method) This method performs corrections on each subspace in parallel for a given $u^0 \in \mathcal{V}$,

$$u^{k+1} = u^k + B(f - Au^k), \quad B = \sum_{i=0}^{N} R_i,$$

for $k = 0, 1, 2, \dots$ The error equation reads

$$u - u^{k+1} = (I - T)(u - u^k), \quad T = \sum_{i=0}^{N} T_i = BA, \quad \text{for} \quad k = 0, 1, 2, \dots$$

SSC (Successive subspace correction method) This method performs the corrections in a successive way for a given $u^0 \in \mathcal{V}$,

$$v^0 = u^k, \quad v^{i+1} = v^i + R_i(f - Av^i), i = 0, \ldots, N, \quad u^{k+1} = v^{N+1},$$

for $k = 0, 1, 2, \ldots$ The error equation reads

$$u - u^{k+1} = \Big(\prod_{i=0}^{N}(I - T_i)\Big)(u - u^k), \quad \text{for} \quad k = 0, 1, 2, \ldots,$$

where $\prod_{i=0}^{N}(I - T_i) := (I - T_0)(I - T_1) \cdots (I - T_N)$.

Note that in the case of nested spaces $\mathcal{V}_i \subseteq \mathcal{V}_{i+1}, i = 0, \ldots, N-1$, **SSC** is nothing else but the multigrid V-cycle [35, 36]. The convergence analysis of **PSC** and **SSC** according to [35] rests upon the following two assumptions.

A1 *Stable decomposition*: For any $v \in \mathcal{V}$ there exists a decomposition $v = \sum_{k=0}^{N} v_k, v_k \in \mathcal{V}_k, k = 0, \ldots, N$ such that

$$\sum_{k=0}^{N} \|v_k\|^2_{R_k^{-1}} \leq K_0 \|v\|^2_A.$$

A2 *Strengthened Cauchy-Schwarz (SCS) inequality*: For any $u_k, v_k \in \mathcal{V}_k, k = 0, \ldots, N$ one has

$$|\sum_{i=0}^{N} \sum_{j=i+1}^{N} (T_i u_i, T_j v_j)_A| \leq K_1 \Big(\sum_{i=0}^{N} (T_i u_i, u_i)_A\Big)^{1/2} \Big(\sum_{j=0}^{N} (T_j v_j, v_j)_A\Big)^{1/2}.$$

Theorem 2.3 ([35])**.** *Let **A1** and **A2** be satisfied. Then **PSC** and **SSC** satisfy the convergence results*

$$\kappa(BA) \leq K_0 K_1,$$

$$\|\prod_{i=0}^{N}(I - T_i)\|^2_A \leq 1 - \frac{2 - \omega}{K_0(1 + K_1)^2} \qquad \text{with} \qquad \omega := \max_{k=0,\ldots,N} \rho(R_k A_k),$$

where κ denotes the condition number *and ρ the* spectral radius.

3 Space decomposition of hierarchical spline spaces

The hierarchical spline space $\mathbb{T}$ as defined in Section 2.1 is induced by the domain hierarchy $\Omega^0 \supseteq \Omega^1 \supseteq \ldots \supseteq \Omega^N$. As a consequence of $\mathbb{B}^l(\Omega^l) \supseteq \mathbb{B}^l_*$, the nestedness of $\mathbb{B}^l, l = 0, \ldots, N$ and (6) we have the representation

$$\mathbb{T} = \sum_{l=0}^{N} \mathbb{B}^l(\Omega^l), \tag{17}$$

where $\mathbb{B}^l(\Omega) := \operatorname{span}\{\beta \in \mathcal{B}^l : \operatorname{supp}\beta \subseteq \Omega\}$ denotes the space spanned by the tensor product B-splines of level $l = 0, \dots, N$ whose support is entirely contained in a given set $\Omega \subseteq D$. Note that these spaces also contain B-spline basis functions which are eliminated by the Kraft selection mechanism and thus not contained in the basis of the hierarchical spline space.

The relation (17) provides us with the main space decomposition upon which we will build a subspace correction method as described in Section 2.4, meaning that smoothing will only be performed in subspaces of tensor product splines on a single level.

Let $\widetilde{\mathbb{T}}_l := \mathbb{B}^l(\Omega^l)$ and introduce the auxiliary hierarchical spline spaces

$$\mathbb{T}_k := \sum_{l=0}^{k} \widetilde{\mathbb{T}}_l, \qquad k = 0, \dots, N,$$

such that $\mathbb{T}_0 \subseteq \mathbb{T}_1 \subseteq \dots \subseteq \mathbb{T}_N = \mathbb{T}$, with $\mathcal{I}_k, \widetilde{\mathcal{I}}_k, \mathcal{T}_k, \widetilde{\mathcal{T}}_k$ denoting the active index sets and bases of $\mathbb{T}_k, \widetilde{\mathbb{T}}_k, k = 0, \dots, N$, respectively. Note that $\widetilde{\mathcal{I}}_k = \mathcal{I}_k \setminus \mathcal{I}_{k-1}, k = 1, \dots, N$. We also denote the hierarchical meshes as in (1) corresponding to these spaces by $\mathcal{G}_k$.

We have the standard inverse inequality ([3])

$$\|v_k\|_A^2 \eqsim |v_k|_1^2 \lesssim h_k^{-2} \|v_k\|_0^2 \qquad \forall v_k \in \widetilde{\mathbb{T}}_k,\ k = 0, \dots, N, \tag{18}$$

where $\|.\|_0 := \|.\|_{L^2(D)}$, $|.|_1 := |.|_{W_2^1(D)}$, $\|.\|_1 := \|.\|_{W_2^1(D)}$ are the usual Sobolev norms.

In Sections 3.1 and 3.2 we will verify **A1** and **A2** for the space decomposition (17) on the basis of a *smoothing property* on the subspace solvers $R_k, k = 0, \dots, N$: we assume that

$$\left.\begin{aligned} \langle A_k v_k, v_k \rangle &\le \omega \langle R_k^{-1} v_k, v_k \rangle \\ h_k^{-2} \|v_k\|_0^2 &\eqsim \|v_k\|_{R_k^{-1}}^2 \end{aligned}\right\} \quad \forall v_k \in \widetilde{\mathbb{T}}_k,\ k = 0, \dots, N \tag{\textbf{SP}}$$

with some constant $\omega \in (0, 2)$.

In Section 3.3 we will prove this smoothing property for some standard smoothing iterations.

3.1 Stability of the decomposition

Let $\mathfrak{Q}_{-1} := 0$ and $\mathfrak{Q}_k : L^2(D) \to \mathbb{T}_k,\ k = 0, \dots, N$, be THB-spline QIs constructed according to Theorem 2.1, for instance those given in Example 2.1 and Example 2.2. As a direct consequence of the construction of $\mathfrak{Q}_k$ one obtains the following property.

Corollary 3.1. *Let $f \in L^2(D)$. Then we have for any $k = 0, \dots, N$*

$$(\mathfrak{Q}_k - \mathfrak{Q}_{k-1}) f \in \widetilde{\mathbb{T}}_k, \qquad (\mathfrak{Q}_k - \mathfrak{Q}_{k-1}) f \equiv 0 \text{ on } D \setminus \widetilde{\Omega}_k, \qquad \widetilde{\Omega}_k := \bigcup_{\tau \in \widetilde{\mathcal{T}}_k} \operatorname{supp} \tau.$$

We define the auxiliary sets

$$\widetilde{\Lambda}_k := \bigcup_{\substack{\Upsilon \in \mathcal{G}_k \\ \Upsilon \cap \widetilde{\Omega}_k \neq \emptyset}} \Lambda_\Upsilon, \qquad \widetilde{\Lambda}_k \supseteq \widetilde{\Omega}_k, \qquad k = 0, \dots, N.$$

Remark 3.1. Note that from (13) one has $\widetilde{\Lambda}_k \subseteq \bigcup_{\substack{\Upsilon\in\mathcal{G}_k \\ \Upsilon\cap\widetilde{\Omega}_k\neq\emptyset}} S(\Upsilon, k-\delta)$, that is, $\widetilde{\Lambda}_k$ can be seen as the support extension with respect to the level $k-\delta$ of $\widetilde{\Omega}_k, k = 0,\dots,N$ with equality possible in the tensor product setting.

We have seen in Section 2.1 that at most $c_{\mathcal{T}}$ basis functions overlap some $\Upsilon^k \in \mathcal{G}_k, k \in \{0,\dots,N\}$. By virtue of Corollary 2.1 we obtain the estimate

$$\sum_{\substack{\Upsilon\in\mathcal{G}_k \\ \Upsilon\cap\widetilde{\Omega}_k\neq\emptyset}} \|f\|^2_{0,\Lambda_\Upsilon} \lesssim \|f\|^2_{0,\widetilde{\Lambda}_k}.$$

Hence, (15) can be used to obtain for $k = 0,\dots,N$, $k' \in \{k-1, k\}$,

$$\|\mathfrak{Q}_{k'} f\|^2_{0,\widetilde{\Omega}_k} \lesssim \|f\|^2_{0,\widetilde{\Lambda}_k}, \qquad f \in L^2(D), \tag{19}$$

where we set $\|.\|_{0,\Omega} := \|.\|_{L^2(\Omega)}, |.|_{1,\Omega} := |.|_{W_1^2(\Omega)}$ for arbitrary $\Omega \subseteq \mathbb{R}^d$. Following [14, 11], we have

$$\forall \overline{v} \in \mathbb{B}^N: \quad \sum_{l=0}^{N} h_l^{-2}\|\overline{v}_l\|_0^2 \lesssim |\overline{v}|_1^2, \quad \overline{v}_l := (\mathfrak{P}^l - \mathfrak{P}^{l-1})\overline{v}, \tag{20}$$

where $\mathfrak{P}^l : L^2(D) \to \mathbb{B}^l$ denotes the L^2-projector into the tensor product spline space $\mathbb{B}^l, l = 0,\dots,N$ and $\mathfrak{P}^{-1} = 0$. We now prove an analogous result for the decomposition (17) of the hierarchical spline space.

Theorem 3.1. *For any $v \in \mathbb{T}$ there exist $v_k \in \widetilde{\mathbb{T}}_k, k = 0,\dots,N$ such that $v = \sum_{k=0}^N v_k$ and*

$$\sum_{k=0}^{N} h_k^{-2}\|v_k\|_0^2 \lesssim \|v\|_A^2.$$

Thus, assuming (**SP**), ***A1*** *holds for the space decomposition* (17).

Proof. Let $v \in \mathbb{T}$ be arbitrary and let

$$\begin{aligned} \overline{v}_l &:= (\mathfrak{P}^l - \mathfrak{P}^{l-1})v \in \mathbb{B}^l, \qquad l = 0,\dots,N, \\ v_k &:= (\mathfrak{Q}_k - \mathfrak{Q}_{k-1})v \in \widetilde{\mathbb{T}}_k, \qquad k = 0,\dots,N. \end{aligned}$$

It follows that

$$\sum_{k=0}^{N} v_k = \sum_{l=0}^{N} \overline{v}_l = v, \qquad v_k = (\mathfrak{Q}_k - \mathfrak{Q}_{k-1})\sum_{l=0}^{N}\overline{v}_l = (\mathfrak{Q}_k - \mathfrak{Q}_{k-1})\sum_{l=k}^{N}\overline{v}_l,$$

where the last equality follows by the fact that $\mathfrak{Q}_k \equiv \mathfrak{Q}_{k-1}$ on $\mathbb{B}^{k-1}, k = 1,\dots,N$. Hence, we have from (19)

$$\|v_k\|^2_{0,\widetilde{\Omega}_k} = \|(\mathfrak{Q}_k - \mathfrak{Q}_{k-1})\sum_{l=k}^{N}\overline{v}_l\|^2_{0,\widetilde{\Omega}_k} \lesssim \|\sum_{l=k}^{N}\overline{v}_l\|^2_{0,\widetilde{\Lambda}_k}, \qquad k = 0,\dots,N.$$

Since $\widetilde{\Lambda}_k \subseteq D, k = 0, \dots, N$ we have

$$\|v_k\|_0^2 = \|v_k\|_{0,\widetilde{\Omega}_k}^2 \lesssim \|\sum_{l=k}^{N} \overline{v}_l\|_{0,\widetilde{\Lambda}_k}^2 \lesssim \|\sum_{l=k}^{N} \overline{v}_l\|_0^2 \lesssim \sum_{l=k}^{N} \|\overline{v}_l\|_0^2, \qquad k = 0, \dots, N.$$

We now employ the discrete Hardy inequality (see [12, Lemma 4.3]): if $a_k, b_k \geq 0, k = 0, \dots, N$ satisfy $b_k \leq \sum_{l=k}^{N} a_l$ for all $k = 0, \dots, N$, then for any $s \in (0, 1)$ we have

$$\sum_{k=0}^{N} s^{-k} b_k \leq \frac{1}{1-s} \sum_{k=0}^{N} s^{-k} a_k.$$

Applying this result with $s = 1/4$ to $a_l = h_0^2 \|\overline{v}_l\|_0^2$ and $b_l = h_0^2 \|v_l\|_0^2$, we obtain

$$\sum_{l=0}^{N} h_l^{-2} \|v_l\|_0^2 \lesssim \sum_{l=0}^{N} h_l^{-2} \|\overline{v}_l\|_0^2,$$

and the desired result follows with (20). □

3.2 Strengthened Cauchy-Schwarz inequality

The hierarchy of tensor product B-spline functions satisfies the following form of a strengthened Cauchy-Schwarz inequality.

Lemma 3.1 ([13]). *Let $u_i \in \mathbb{B}^i, v_j \in \mathbb{B}^j, i, j \in \{0, \dots, N\}, j \geq i$. Then we have for $\gamma = 1/2$*

$$(u_i, v_j)_A \lesssim \gamma^{(j-i)} |u_i|_1 h_j^{-1} \|v_j\|_0.$$

In order to prove assumption **A2** for our space decomposition, we first note that (**SP**) implies, for all $v \in \mathcal{V}_k$,

$$\|R_k A_k v\|_0^2 \eqsim h_k^2 \|R_k A_k v\|_{R_k^{-1}}^2 = h_k^2 \langle A_k v, R_k A_k v \rangle \leq \omega h_k^2 \|v\|_A^2,$$

where the last inequality stems from the spectral equivalence (see, e.g., [35, Lemma 2.1])

$$\langle A_k v, v \rangle \leq \omega \langle R_k^{-1} v, v \rangle \ \forall v \quad \Longleftrightarrow \quad \langle A_k R_k A_k v, v \rangle \leq \omega \langle A_k v, v \rangle \ \forall v.$$

This provides us with the assumption for the following result.

Lemma 3.2. *Let $R_k : \mathcal{V}_k' \to \mathcal{V}_k$ satisfy*

$$\|R_k A_k v\|_0 \lesssim h_k \|v\|_{A_k} \qquad \forall v \in \mathcal{V}_k, k = 0, \dots, N.$$

Then one has for any $u, v \in \mathcal{V}$, $i, j = 0, \dots, N$,

$$(T_i u, T_j v)_A \lesssim \gamma^{|j-i|/2} (T_i u, u)_A (T_j v, v)_A.$$

Proof. Follows exactly the arguments of the proof of [35, Lemma 6.3], using the statement of Lemma 3.1 (since $\widetilde{\mathbb{T}}_k \subseteq \mathbb{B}^k$) in place of the corresponding FEM estimate used therein. □

We can now prove the desired assumption.

Theorem 3.2. *Let the smoothers R_k, $k = 0, \dots, N$, satisfy* (**SP**)*. Then we have for all $u_k, v_k \in \widetilde{\mathbb{T}}_k$, $k = 0, \dots, N$,*

$$\left| \sum_{i=0}^{N} \sum_{j=i+1}^{N} (T_i u_i, T_j v_j)_A \right| \lesssim \Big(\sum_{i=0}^{N} (T_i u_i, u_i)_A \Big)^{1/2} \Big(\sum_{j=0}^{N} (T_j v_j, v_j)_A \Big)^{1/2},$$

*that is, **A2** holds for the space decomposition* (17)*.*

Proof. As described above, (**SP**) guarantees the assumptions of Lemma 3.2. The desired estimate then follows with the elementary inequality (see, e.g., [12, 35])

$$\sum_{l=0}^{N} \sum_{k=0}^{N} \gamma^{|l-k|} x_l y_k \lesssim \frac{2}{1-\gamma} \Big(\sum_{l=0}^{N} x_j^2 \Big)^{1/2} \Big(\sum_{k=0}^{N} y_k^2 \Big)^{1/2} \qquad \forall x_l, y_l \in \mathbb{R},\ l = 0, \dots, N. \quad \square$$

3.3 Smoothing property

We first state the well-known L_2-stability of B-splines.

Theorem 3.3 ([31, 30])**.** *Let $l \in \{0, \dots, N\}$ and $c^{(l,\boldsymbol{j})} \in \mathbb{R}, \boldsymbol{j} \in \mathcal{I}^l$. Then we have*

$$\| \sum_{\boldsymbol{j} \in \mathcal{I}^l} c^{(l,\boldsymbol{j})} \beta^{(l,\boldsymbol{j})} \|_0^2 \approx h_l^d \sum_{\boldsymbol{j} \in \mathcal{I}^l} |c^{(l,\boldsymbol{j})}|^2.$$

We need the following auxiliary result on the norms of B-spline basis functions.

Lemma 3.3. *The tensor product B-spline basis functions satisfy*

$$|\beta^{(l,\boldsymbol{i})}|_1^2 \approx h_l^{-2} \|\beta^{(l,\boldsymbol{i})}\|_0^2 \approx h_l^{d-2} \qquad \forall (l, \boldsymbol{i}) \in \mathcal{I}.$$

Proof. The bound $|\beta^{(l,\boldsymbol{i})}|_1 \lesssim h_l^{-1} \|\beta^{(l,\boldsymbol{i})}\|_0$ follows from the inverse inequality (18). The bound $\|\beta^{(l,\boldsymbol{i})}\|_0 \lesssim h_l |\beta^{(l,\boldsymbol{i})}|_1$ follows from Poincaré's inequality since $\operatorname{diam} \operatorname{supp} \beta^{(l,\boldsymbol{i})} \approx h$ and $\beta^{(l,\boldsymbol{i})}$ is zero on at least some part of the boundary of its support. The final equivalence follows from the B-spline stability result of Theorem 3.3. □

Our smoothers R_k are to be specified on the subspaces $\widetilde{\mathbb{T}}_k := \mathbb{B}^k(\Omega^k) \subset \mathbb{B}^k$, $k = 0, \dots, N$. Therefore it is natural to use the canonical B-spline basis for their representation, i.e., fix some k and let

$$v = \sum_{\boldsymbol{j} \in \widetilde{\mathcal{I}}_k} c^{\boldsymbol{j}} \beta^{(k,\boldsymbol{j})} \in \widetilde{\mathbb{T}}_k \quad \text{with } c^{\boldsymbol{j}} \in \mathbb{R}.$$

In the remainder of this section, we identify A_k with the local stiffness matrix with respect to this basis and also interpret R_k as a square matrix of the same size. We will also make use of the splitting $A_k = D - L - U$, where D, L, U denote the *diagonal*, *lower left triangular* and *upper right triangular* components of A_k.

3.3.1 The Richardson smoother

Let

$$R_k = \mu_k I,$$

where I is the identity matrix of the same size as A_k and the damping parameter $\mu_k > 0$ is chosen such that $\langle A_k v, v\rangle \leq \langle R_k^{-1} v, v\rangle$ for all v, i.e., $\mu_k \eqsim h_k^2$. Thus, the first equation of (**SP**) is immediately satisfied with $\omega = 1$. On the other hand, by Theorem 3.3 and Lemma 3.3 we have

$$h_k^{-2}\|v\|_0^2 \eqsim h_k^{d-2} \sum_{\boldsymbol{j}\in\widetilde{\mathcal{I}}_k} |c^{\boldsymbol{j}}|^2 \eqsim h_k^{-2} \sum_{\boldsymbol{j}\in\widetilde{\mathcal{I}}_k} \|\beta^{(k,\boldsymbol{j})}\|_0^2 |c^{\boldsymbol{j}}|^2 \eqsim \mu_k^{-1}\|v\|_I^2 = \|v\|_{R_k^{-1}}^2.$$

Thus, (**SP**) holds for the Richardson smoother.

3.3.2 The Jacobi smoother

Let

$$R_k = \mu_k D^{-1},$$

where the damping parameter $\mu_k > 0$ is chosen such that $\langle A_k v, v\rangle \leq \langle R_k^{-1} v, v\rangle$ for all v. Thus, the first equation of (**SP**) is immediately satisfied with $\omega = 1$. On the other hand, by Theorem 3.3 and Lemma 3.3 we have

$$h_k^{-2}\|v\|_0^2 \eqsim h_k^{d-2} \sum_{\boldsymbol{j}\in\widetilde{\mathcal{I}}_k} |c^{\boldsymbol{j}}|^2 \eqsim \sum_{\boldsymbol{j}\in\widetilde{\mathcal{I}}_k} |\beta^{(k,\boldsymbol{j})}|_1^2 |c^{\boldsymbol{j}}|^2 \eqsim \sum_{\boldsymbol{j}\in\widetilde{\mathcal{I}}_k} \|\beta^{(k,\boldsymbol{j})}\|_A^2 |c^{\boldsymbol{j}}|^2 = \mu_k \|v\|_{R_k^{-1}}^2.$$

A standard argument using the Cauchy-Schwarz inequality shows that μ_k can be chosen depending only on the number of nonzeros per row of A_k and thus on $\bar{c}_{\boldsymbol{p}}$, but not on h_k; hence, $\mu_k \eqsim 1$. This shows that (**SP**) is satisfied for the Jacobi smoother.

4 Convergence of local multigrid methods with (T)HB-splines

4.1 Robust convergence of the local multigrid method

The assumptions of the abstract convergence result Theorem 2.3 are now satisfied due to Theorem 3.1 and Theorem 3.2 assuming that the chosen smoothers satisfy (**SP**). We have verified that standard Richardson or Jacobi smoothers satisfy these assumptions in Section 3.3. Thus, we have shown the following main result.

Theorem 4.1. *Under the assumption that the mesh level disparity δ is uniformly bounded (which can be guaranteed by suitable refinement strategies [8, 9, 10]), the proposed local multigrid method for HB- and THB-spline spaces based on the space decomposition* (17) *and using Richardson or Jacobi smoothers converges uniformly with respect to the number of levels and the mesh sizes.*

Remark 4.1. Note that our theory covers HB- and THB-splines in the same framework. In fact, the choice of basis enters only in the smoothers, and since smoothing is only done in the B-spline spaces $\mathbb{B}^k(\Omega^k)$, our method even produces identical results independent of the choice of HB- or THB-spline bases.

The theory is significantly simpler than that for previously introduced local multigrid methods for FEM [12, 18, 34], where the specifics of the element refinement strategy have to be taken into account.

4.2 Possible enlargements of the subspaces

In Section 3 we have presented a space decomposition $(\widetilde{\mathbb{T}}_k)_{k=0}^N$ which, together with suitable smoothing operators (R_k), satisfies the assumptions **A1**, **A2**. In a sense, the subspaces $\widetilde{\mathbb{T}}_k, k = 0, \dots, N$ to be smoothed over were chosen in a *minimal* way, since $\mathfrak{Q}_k - \mathfrak{Q}_{k-1} : L^2(D) \to \widetilde{\mathbb{T}}_k, k = 0, \dots, N$ are supported on $\widetilde{\Omega}_k$ only. As we have shown, this is sufficient in order to obtain h-robustness. However, in practice it may be desirable to enlarge the subspaces, in particular to improve the behavior of the solver for higher spline degrees. Therefore we now discuss the possibility of choosing larger subspaces, $\widetilde{\mathbb{T}}_k \subseteq \widetilde{\mathbb{T}}_k^+ \subseteq \mathbb{T}$, along with corresponding subspace solvers R_k^+ which we again assume to satisfy (**SP**).

Assumption **A1** can be proved in a completely analogous way using (**SP**).

With regards to **A2**, we note that adding only HB-basis functions of levels $0, \dots, k-1$, preserves the inclusion $\widetilde{\mathbb{T}}_k^+ \subseteq \mathbb{B}^k$, and Lemma 3.1 remains valid for the enlarged spaces. Adding only THB-basis functions of levels $0, \dots, k-1$ leads to a slightly weakened inclusion $\widetilde{\mathbb{T}}_k^+ \subseteq \mathbb{B}^{k+\delta-1}$; thus, an estimate analogous to Lemma 3.1 remains valid with slightly larger constants, depending only on δ. Hence the proof of Theorem 3.2 and thus **A2** remain intact provided that is is possible to verify (**SP**) for the enlarged subspaces.

Regarding the optimal computational complexity of the resulting methods, it is merely required that $\dim \widetilde{\mathbb{T}}_k^+ \eqsim \dim \widetilde{\mathbb{T}}_k$ in order to retain optimality.

We now present two practical enlargements of space decompositions which still satisfy **A1**, **A2**. The following definition resembles a scheme presented in [34]:

$$\begin{aligned}
\widetilde{\mathcal{I}}_k^{\mathcal{T}} &:= (\mathcal{I}_k \setminus \mathcal{I}_{k-1}) \cup \{(l, \boldsymbol{i}) \in \mathcal{I}_k \cap \mathcal{I}_{k-1} : \tau_k^{(l,\boldsymbol{i})} \neq \tau_{k-1}^{(l,\boldsymbol{i})}\}, \\
\widetilde{\mathbb{T}}_k^{\mathcal{T}} &:= \operatorname{span}\{\tau_k^{(l,\boldsymbol{i})} : (l, \boldsymbol{i}) \in \widetilde{\mathcal{I}}_k^{\mathcal{T}}\}.
\end{aligned}$$

In other words, $\widetilde{\mathbb{T}}_k^{\mathcal{T}}$ consists of the newly added THB-spline basis functions which are in $\mathcal{T}_k$, but not in $\mathcal{T}_{k-1}$, as well as "neighboring" THB-spline basis functions (of at most δ levels) which have been modified due to truncation.

Another enlargement, potentially easier to implement, is given by

$$\begin{aligned}
\widetilde{\mathcal{I}}_k^{\mathcal{B}} &:= (\mathcal{I}_k \setminus \mathcal{I}_{k-1}) \cup \{(l, \boldsymbol{i}) \in \mathcal{I}_k \cap \mathcal{I}_{k-1} : \operatorname{supp} \beta^{(l,\boldsymbol{i})} \cap \widetilde{\Omega}_k\}, \\
\widetilde{\mathbb{T}}_k^{\mathcal{B}} &:= \operatorname{span}\{\tau_k^{(l,\boldsymbol{i})} : (l, \boldsymbol{i}) \in \widetilde{\mathcal{I}}_k^{\mathcal{B}}\}.
\end{aligned}$$

In other words, $\widetilde{\mathbb{T}}_k^{\mathcal{B}}$ consists of the newly added THB-spline basis functions which are in $\mathcal{T}_k$, but not in $\mathcal{T}_{k-1}$, as well as THB-spline basis functions (of at most δ levels) which intersect $\widetilde{\Omega}_k$.

We point out that $\widetilde{\mathbb{T}}_k \subseteq \widetilde{\mathbb{T}}_k^{\mathcal{T}} \subseteq \widetilde{\mathbb{T}}_k^{\mathcal{B}}$ for all $k = 0, \ldots, N$.

5 Numerical results

We present some preliminary numerical tests for a one-dimensional problem. We solve the differential equation

$$-u'' + u = 1 \qquad \text{in } (0, 1)$$

and fix the refinement hierarchy

$$\Omega^l = (1 - 0.5^l, 1), \qquad l = 0, \ldots, L.$$

We discretize using HB-splines and then apply our proposed local multigrid solver for the decomposition (17) using two steps of forward Gauss-Seidel smoothing per level.

For demonstrating the robustness in the number of levels and mesh size, we fix the spline degree and vary the number L of refinement steps and the base mesh size h_0. The results for spline degrees $p = 1, 2, 3, 4$ shown in Table 5. We observe that the convergence rates are robust with respect to both h_0 and L.

In Table 2 we present convergence rates comparing exact subspace solvers to Gauss-Seidel smoothing, as well as the minimal (h-robust) subspace decomposition (17) to the enlarged decomposition based on the truncation criterion $\widetilde{\mathbb{T}}_k^{\mathcal{T}}$ from Section 4.2. We note that the solver degrades for higher spline degrees, however increasing the smoothing area slightly improves the convergence rates significantly when exact subspace solvers are used. The design of improved smoothers which approach this almost p-robust behavior is an important future task.

Acknowledgments

The authors are thankful for many productive discussions with Walter Zulehner (JKU Linz) as well as Stefan Takacs (RICAM).

The financial support by the project LIT-2016-2-YOU-006 funded by the Linz Institute of Technology (LIT) is gratefully acknowledged.

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h_0^{-1}	10	20	40
$L = 1$	0.06	0.10	0.07
$L = 2$	0.10	0.11	0.11
$L = 3$	0.07	0.11	0.12
$L = 4$	0.12	0.10	0.11
$L = 5$	0.11	0.12	0.11

h_0^{-1}	10	20	40
$L = 1$	0.31	0.31	0.31
$L = 2$	0.31	0.32	0.32
$L = 3$	0.31	0.32	0.32
$L = 4$	0.31	0.32	0.32
$L = 5$	0.31	0.32	0.32

h_0^{-1}	10	20	40
$L = 1$	0.74	0.74	0.73
$L = 2$	0.76	0.74	0.73
$L = 3$	0.76	0.74	0.73
$L = 4$	0.77	0.74	0.73
$L = 5$	0.77	0.74	0.73

h_0^{-1}	10	20	40
$L = 1$	0.94	0.94	0.94
$L = 2$	0.95	0.94	0.94
$L = 3$	0.95	0.94	0.94
$L = 4$	0.95	0.94	0.94
$L = 5$	0.96	0.94	0.94

Table 1: Convergence rates for varying mesh sizes and number of levels with $p = 1$ (top left), 2 (top right), 3 (bottom left), 4 (bottom right).

p	1	2	3	4	5	6
exact solvers in $\widetilde{\mathbb{T}}_k$	8e-5	0.25	0.68	0.93	0.99	0.999
Gauss-Seidel in $\widetilde{\mathbb{T}}_k$	0.09	0.32	0.74	0.94	0.99	0.999
exact solvers in $\widetilde{\mathbb{T}}_k^{\mathcal{T}}$	8e-5	0.02	0.05	0.08	0.12	0.16
Gauss-Seidel in $\widetilde{\mathbb{T}}_k^{\mathcal{T}}$	0.09	0.13	0.58	0.91	0.99	0.999

Table 2: Convergence rates with $L = 3$ for $h_0 = 1/20$ for exact subspace solvers vs Gauss-Seidel smoothing and the minimal decomposition (17) vs the enlarged decomposition $\widetilde{\mathbb{T}}_k^{\mathcal{T}}$ from Section 4.2.

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