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Abstract

In this work, we further develop multigoal-oriented a posteriori error estimation for the nonlinear, stationary, incompressible Navier-Stokes equations. It is an extension of our previous work [B. Endtmayer, U. Langer, T. Wick: Two-side a posteriori error estimates for the DWR method, *SISC*, 2019, accepted]. We now focus on h mesh refinement and p enrichment for the error estimator. These advancements are demonstrated with the help of a numerical example.

1 Introduction

Multigoal-oriented error estimation offers the opportunity to control several quantities of interest simultaneously. In recent years, we have developed a version [3, 4] which relies on the dual-weighted residual method [2], and also balances the discretization error with the nonlinear iteration error [12]. The localization is based on the weak formulation proposed in [13]. Our method uses on hierarchical finite element spaces. Here, we investigate h refinement along with p refinement to generate enriched spaces. These ideas are applied to the stationary incompressible Navier-Stokes equations. It is well-known that the spaces for the velocities and the pressure must be balanced in order to satisfy an inf-sup condition [6]. These requirements must be reflected in the design of the adjoint problems in dual-weighted residual error estimation and our proposed p refinement. To demonstrate the performance of the error estimator, we adopt the 2D-1 fluid flow benchmark [14].

2 The Model Problem and Discretization

2.1 The Model Problem

We consider the stationary Navier Stokes 2D-1 benchmark problem [14] as our model problem. This configuration was also considered in [4]. The domain $\Omega \subset \mathbb{R}^2$ is given by $(0, 2.2) \times (0, H) \setminus B$, and B is the ball with the center $(0.2, 0.2)$ and the radius 0.05 as given in [14] and visualized in Figure 1. Find $\vec{u} = (u, p)$ such that

$$\begin{aligned} -\operatorname{div}(\nu(\nabla u + \nabla u^T)) + (u \cdot \nabla)u - \nabla p &= 0, & \text{in } \Omega, \\ -\operatorname{div}(u) &= 0, & \text{in } \Omega, \\ u &= u_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ u &= 0 & \text{on } \Gamma_{\text{no-slip}}, \\ \nu(\nabla u + \nabla u^T)\vec{n} + p\vec{n} &= 0 & \text{on } \Gamma_{\text{out}}, \end{aligned}$$

where $\Gamma_{\text{in}} := \{x = 0\} \cap \partial\Omega$, $\Gamma_{\text{no-slip}} := \overline{\partial\Omega \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}})}$ and $\Gamma_{\text{out}} := (\{x = 2.2\} \cap \partial\Omega) \setminus \partial(\{x = 2.2\} \cap \partial\Omega)$. Furthermore, the viscosity $\nu = 10^{-3}$ and $u_{\text{in}}(x, y) = (0.3w(y), 0)$ with $w(y) = 4y(H - y)/H^2$ and $H = 0.41$. The corresponding weak form reads as follows: Find $\vec{u} = (u, p) \in V_{BC} := [H^1(\Omega)]_{BC}^2 \times L^2(\Omega)$ such that

$$A(\vec{u})(\vec{v}) = 0 \quad \forall \vec{v} = (v_u, v_p) \in V_0 := [H_0^1(\Omega)]^2 \times L^2(\Omega) \quad (1)$$

with

$$\begin{aligned} A(\vec{u})(\vec{v}) := & (\nu(\nabla u + \nabla u^T), \nabla v_u)_{[L^2(\Omega)]^{2 \times 2}} + ((u \cdot \nabla)u, v_u)_{[L^2(\Omega)]^2} \\ & + (p, \operatorname{div}(v_u))_{L^2(\Omega)} - (\operatorname{div}(u), v_p)_{L^2(\Omega)}, \end{aligned}$$

where $[H^1(\Omega)]_{BC}^2 := \{u \in [H^1(\Omega)]^2 : u|_{\Gamma_{\text{in}}} = u_{\text{in}} \wedge u|_{\Gamma_{\text{no-slip}}} = 0\}$ and $[H_0^1(\Omega)]^2 := \{v \in [H^1(\Omega)]^2 : v|_{\Gamma_{\text{in}}} = 0 \wedge v|_{\Gamma_{\text{no-slip}}} = 0\}$.



Figure 1: The computational domain Ω (left) and the initial mesh (right).

2.2 Discretization

Let \mathcal{T}_h be a decomposition of $\Omega \subset \mathbb{R}^2$ into quadrilateral elements. Furthermore, we assume that $\mathcal{T}_{\frac{h}{2}}$ is the uniform refinement of \mathcal{T}_h . We discretize our problem using $[Q_c^2]^2$, i.e. piecewise bi-quadratic elements for the velocity u , and Q_c^1 , i.e. piecewise bi-linear elements for the pressure p . The resulting space using the mesh \mathcal{T}_h will be denoted by V_h . For a more detailed explanation of the discretization, we refer to [4]. The resulting space using the mesh $\mathcal{T}_{\frac{h}{2}}$ will be denoted by $V_{\frac{h}{2}}$. We say $V_{\frac{h}{2}}$ is the (hierarchical) h -refined finite element space of V_h . Furthermore we consider using $[Q_c^4]^2$, i.e. piecewise bi-quartic elements for the velocity u , and Q_c^2 , i.e. piecewise bi-quadratic elements

for the pressure p . The resulting finite element space using the mesh \mathcal{T}_h will be denoted by $V_h^{(2)}$. Here we have the property that $V_h \subset V_h^{(2)}$. We say $V_h^{(2)}$ is the (hierarchical) p -refined finite element space of V_h . The corresponding discretized problems read as: Find $\vec{u}_h \in V_h \cap V_{BC}$, $\vec{u}_{\frac{h}{2}} \in V_{\frac{h}{2}} \cap V_{BC}$ and $\vec{u}_h^{(2)} \in V_h^{(2)} \cap V_{BC}$ such that

$$\begin{aligned} A(\vec{u}_h)(\vec{v}_h) &= 0 & \forall \vec{v}_h \in V_h \cap V_0, \\ A(\vec{u}_{\frac{h}{2}})(\vec{v}_{\frac{h}{2}}) &= 0 & \forall \vec{v}_{\frac{h}{2}} \in V_{\frac{h}{2}} \cap V_0, \\ A(\vec{u}_h^{(2)})(\vec{v}_h^{(2)}) &= 0 & \forall \vec{v}_h^{(2)} \in V_h^{(2)} \cap V_0. \end{aligned}$$

Remark 2.1. *We would like to mention that the domain Ω is not of polygonal shape. Therefore, a decomposition into quadrilateral elements is not possible. However, we approximate the ball B by a polygonal domain, which is adapted after every refinement process by describing it as a spherical manifold in deal.II [1] using the command `Triangulation::set_manifold`.*

3 Dual Weighted Residual Method and Error Representation

We are primarily interested in one or more particular quantities of interest. We employ the dual weighted residual (DWR) method [2] for estimating the error in these quantities. To connect the quantity of interest J with the model problem, we consider the adjoint problem.

3.1 The Adjoint Problem

The adjoint problem reads as follows: Find $\vec{z} \in V_0$ such that

$$A'(\vec{u})(\vec{v}, \vec{z}) = J'(\vec{u})(\vec{v}) \quad \forall \vec{v} \in V_0, \quad (2)$$

where A' and J' denote the Frechet derivative of A and J , respectively, and \vec{u} is the solution of the model problem (1).

Theorem 3.1. *Let us assume that $J \in \mathcal{C}^3(V_{BC}, \mathbb{R})$. If \vec{u} solves the model problem (1) and \vec{z} solves the adjoint problem (2), then, for arbitrary fixed $\vec{u} \in V_{BC}$ and $\vec{z} \in V_0$, the following error representation formula holds:*

$$\begin{aligned} J(\vec{u}) - J(\vec{u}) &= \frac{1}{2}\rho(\vec{u})(\vec{z} - \vec{z}) + \frac{1}{2}\rho^*(\vec{u}, \vec{z})(\vec{u} - \vec{u}) + \rho(\vec{u})(\vec{z}) + \mathcal{R}^{(3)}, \\ \text{where } \rho(\vec{u})(\cdot) &:= -A(\vec{u})(\cdot), \quad \rho^*(\vec{u}, \vec{z})(\cdot) := J'(\vec{u})(\cdot) - A'(\vec{u})(\cdot, \vec{z}), \text{ and} \\ \mathcal{R}^{(3)} &:= \frac{1}{2} \int_0^1 [J'''(\vec{u} + s\vec{e})(\vec{e}, \vec{e}, \vec{e}) - A'''(\vec{u} + s\vec{e})(\vec{e}, \vec{e}, \vec{e}, \vec{z} + s\vec{e}^*) - 3A''(\vec{u} + s\vec{e})(\vec{e}, \vec{e}, \vec{e}^*)]s(s-1) ds, \\ &\text{with } \vec{e} = \vec{u} - \vec{u} \text{ and } \vec{e}^* = \vec{z} - \vec{z}. \end{aligned} \quad (3)$$

Proof. We refer the reader to [3] and [12]. \square

Remark 3.2. *In practice, the arbitrary elements $\vec{u} \in V_{BC}$ and $\vec{z} \in V_0$ will be replaced by approximations \vec{u}_h and \vec{z}_h to the corresponding finite element solutions.*

Remark 3.3. *The error representation formula in Theorem 3.1 is exact but not computable, because \vec{u} and \vec{z} are not known.*

3.2 Error Estimation and Adaptive Algorithm

The different error estimator parts are discussed in [4]. In particular, it turns out that $\eta_h := \frac{1}{2}\rho(\vec{u})(\vec{z} - \vec{z}) + \frac{1}{2}\rho^*(\vec{u}, \vec{z})(\vec{u} - \vec{u})$ is related to the discretization error [12, 3, 4]. The idea is to replace the quantities $\vec{u} - \vec{u}$ and $\vec{z} - \vec{z}$ by some computable quantities. This can be done via higher order interpolation [2, 12] or hierarchically (via an additional solve on an enriched space) [2, 3, 10]. If u_h^\dagger, z_h^\dagger are the solution, then we approximate $\vec{u} - \vec{u}$ and $\vec{z} - \vec{z}$ by $u_h^\dagger - \vec{u}$ and $z_h^\dagger - \vec{z}$, respectively. The new computable error estimator then reads as

$$\eta_h^\dagger := \frac{1}{2}\rho(\vec{u})(z_h^\dagger - \vec{z}) + \frac{1}{2}\rho^*(\vec{u}, \vec{z})(u_h^\dagger - \vec{u}).$$

Under some saturation assumption, it was shown in [4] that the resulting error estimator is efficient and reliable. We consider the two different error estimators

$$\begin{aligned} \eta_h^{(2)} &:= \frac{1}{2}\rho(\vec{u})(z_h^{(2)} - \vec{z}) + \frac{1}{2}\rho^*(\vec{u}, \vec{z})(u_h^{(2)} - \vec{u}), \\ \eta_{\frac{h}{2}} &:= \frac{1}{2}\rho(\vec{u})(z_{\frac{h}{2}} - \vec{z}) + \frac{1}{2}\rho^*(\vec{u}, \vec{z})(u_{\frac{h}{2}} - \vec{u}). \end{aligned}$$

We call $\eta_h^{(2)}$ and $\eta_{\frac{h}{2}}$ the p -enriched and h -enriched error estimators, respectively. The error estimators are localized using the partition of unity technique proposed in [13]. The marking strategy and algorithms are the same as in [4].

Remark 3.4. *The efficiency and reliability are not guaranteed under the corresponding saturation assumption in [4] for $\eta_{\frac{h}{2}}$, since the boundary is adapted in every refinement step.*

Remark 3.5. *We use the algorithm presented in [4]. The algorithm using p enrichment coincides with Algorithm 3 in [4]. In the algorithm, where we use h enrichment, we replace $V_h^{(2)}$ by $V_{\frac{h}{2}}$.*

4 Numerical Experiment

We compare the two error estimators introduced in Section 3.2. In the p enriched case, we use uniform p refinement for the hierarchical approximation. The results for p enrichment have already been computed in [4]. In the h enriched case, we use uniform h refinement. The configuration of the problem is given in Section 2.1.

4.1 Quantities of Interest

We use the quantities of interest defined in [14, 4]:

$$\begin{aligned}\Delta p(\vec{u}) &:= p(X_1) - p(X_2), \\ c_{\text{drag}}(\vec{u}) &:= C \int_{\partial B} [\nu(\nabla u + \nabla u^T)\vec{n} - p\vec{n}] \cdot \vec{e}_1 \, ds_{(x,y)}, \\ c_{\text{lift}}(\vec{u}) &:= C \int_{\partial B} [\nu(\nabla u + \nabla u^T)\vec{n} - p\vec{n}] \cdot \vec{e}_2 \, ds_{(x,y)},\end{aligned}$$

where $C = 500$, $X_1 = (0.15, 0.2)$, $X_2 = (0.25, 0.2)$, $\vec{e}_1 := (1, 0)$, $\vec{e}_2 := (0, 1)$, and \vec{n} denotes the outer normal vector. To do adaptivity for all of them at once we combine them to one functional

$$J_{\mathfrak{E}}(\vec{v}_h) := \frac{|\Delta p(\vec{u}_h^+ - \vec{v}_h)|}{|\Delta p(\vec{u}_h)|} + \frac{|c_{\text{drag}}(\vec{u}_h^+ - \vec{v}_h)|}{|c_{\text{drag}}(\vec{u}_h)|} + \frac{|c_{\text{lift}}(\vec{u}_h^+ - \vec{v}_h)|}{|c_{\text{lift}}(\vec{u}_h)|}.$$

By $J_{\mathfrak{E}}^p$ or $J_{\mathfrak{E}}^h$, we denote the functionals where we replace \vec{u}_h^+ with $\vec{u}_h^{(2)}$ or $\vec{u}_{\frac{h}{2}}$, respectively. More information on how to treat multiple functionals at once can be found in [8, 7, 15, 11, 9, 5, 3, 4]. The implementation is done in the finite element library `deal.II` [1], and follows the code in [4]. In this section, we compare two different sequences of meshes. The sequences are generated by the error estimators $\eta_h^{(2)}$ and $\eta_{\frac{h}{2}}$. First of all, let us define the effectivity indices by

$$I_{eff}^p := \frac{\eta_h^{(2)}}{|J_{\mathfrak{E}}^p(u) - J_{\mathfrak{E}}^p(u_h)|} \quad \text{and} \quad I_{eff}^h := \frac{\eta_{\frac{h}{2}}}{|J_{\mathfrak{E}}^h(u) - J_{\mathfrak{E}}^h(u_h)|}.$$

The p enriched discrete remainder part of the error estimator $\eta_{\mathcal{R}}^{(2)}$ is defined as the quantity (3), where we replace $\vec{u}, \vec{z}, \vec{u}, \vec{z}$ by $\vec{u}_h, \vec{z}_h, \vec{u}_h^{(2)}, \vec{z}_h^{(2)}$, respectively. The h enriched discrete remainder part of the error estimator $\eta_{\mathcal{R}, \frac{h}{2}}$ is defined as the quantity (3), where we replace $\vec{u}, \vec{z}, \vec{u}, \vec{z}$ by $\vec{u}_h, \vec{z}_h, \vec{u}_{\frac{h}{2}}, \vec{z}_{\frac{h}{2}}$, respectively. Finally, we define the gaps between the theoretical findings in [4] by

$$\eta_{\mathbb{E}}^{(2)} := \left| |J(\vec{u}_h^{(2)}) - J(\vec{u}_h)| - |\eta_h^{(2)} + \rho(\vec{u}_h, \vec{z}_h) + \eta_{\mathcal{R}}^{(2)}| \right|,$$

and

$$\eta_{\mathbb{E}, \frac{h}{2}} := \left| |J(\vec{u}_{\frac{h}{2}}) - J(\vec{u}_h)| - |\eta_{\frac{h}{2}} + \rho(\vec{u}_h, \vec{z}_h) + \eta_{\mathcal{R}, \frac{h}{2}}| \right|.$$

4.2 Discussion of the Results

In Figure 2, the effectivity indices for the two different types of error estimators are shown on their respective grids. We see that h -enrichment delivers effectivity indices which are very close to one, whereas for p -enrichment we have effectivity indices in the range of 0.2 – 8.1. This was also observed in [4]. In the case of p -enrichment, the saturation assumption is violated multiple times, as we observe in Figure 3. The saturation assumption is violated if the error $|J_{\mathfrak{E}}^p(\vec{u}_h^{(2)}) - J_{\mathfrak{E}}^p(\vec{u})|$ in the

enriched solution is larger than $|J_{\mathfrak{E}}^p(\vec{u}_h) - J_{\mathfrak{E}}^p(\vec{u})|$. In the case of h -enrichment, this always happens. If we compare the errors of the single functionals, which are monitored in Figure 4, Figure 5 and Figure 6, we conclude that the meshes generated by the p -enriched error estimator lead to smaller errors in the single functionals. If all the conditions in [4] are fulfilled, then $\eta_{\mathbb{E}}^{(2)}$ and $\eta_{\mathbb{E}, \frac{h}{2}}$ are zero. However, in the computation of the error estimators, our overall round-off error is in the order of $\varepsilon(\text{double}) \times \text{DOFs}$, where $\varepsilon(\text{double}) = 2^{-52}$ is the machine precision for double floating point numbers¹. In the case of p enrichment, we observe in Figure 7 that $\eta_{\mathbb{E}}^{(2)}$ indeed is in the order or even better than the round off errors when summing up the different error contributions. In this case, all requirements are fulfilled. For h enrichment, we do not have the inclusion $V_h \subset V_{\frac{h}{2}}$ due to the geometrical approximation. Therefore, these conditions are violated. The effects are monitored in Figure 7 as well. The quantity $\eta_{\mathbb{E}, \frac{h}{2}}$ does not only contain numerical round off errors, but also errors coming from the geometrical approximation. However, this is a non-local quantity, and the localization is not straightforward.

¹https://en.wikipedia.org/wiki/Machine_epsilon

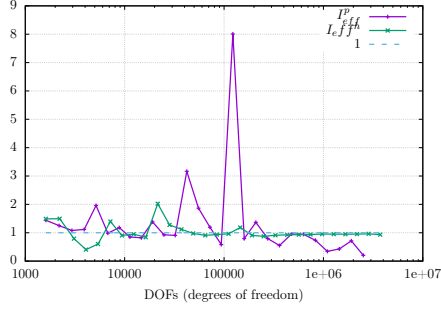


Figure 2: The two effectiveness indices on the corresponding meshes.

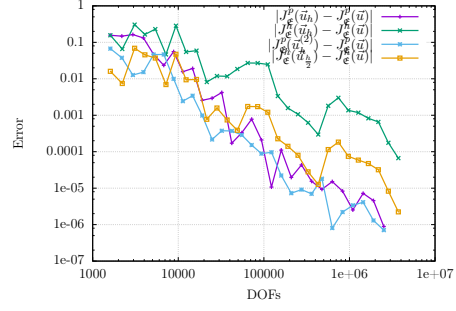


Figure 3: Errors in $J_{\mathcal{E}}^p$ and $J_{\mathcal{E}}^h$ in the solution and enriched solution.

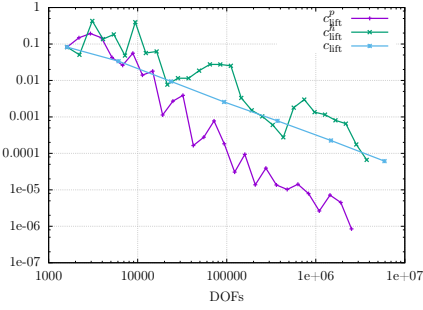


Figure 4: The errors in c_{lift} for refinement with p -enriched error estimation (c_{lift}^p), refinement with h -enriched error estimation (c_{lift}^h), and uniform refinement (c_{lift}).

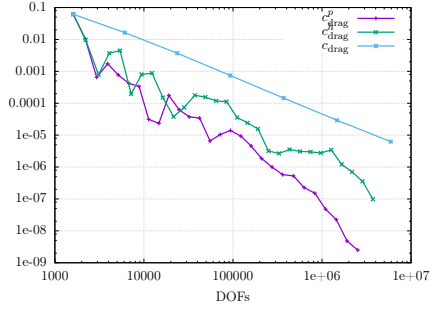


Figure 5: The errors in c_{drag} for refinement with p -enriched error estimation (c_{drag}^p), refinement with h -enriched error estimation (c_{drag}^h), and uniform refinement (c_{drag}).

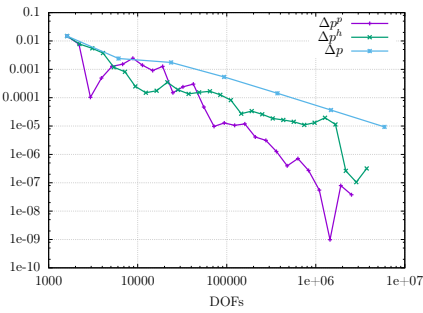


Figure 6: The errors in Δp for refinement with p -enriched error estimation (Δp^p), refinement with h -enriched error estimation (Δp^h), and uniform refinement (Δp).

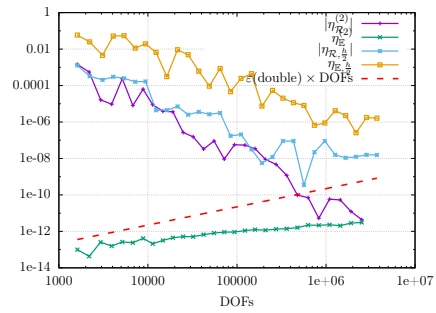


Figure 7: The remainder parts $\eta_{\mathcal{R}}^{(2)}$, $\eta_{\mathcal{R}, \frac{h}{2}}^{(2)}$ and gap parts $\eta_{\mathcal{E}}^{(2)}$, $\eta_{\mathcal{E}, \frac{h}{2}}^{(2)}$ for p and h enrichment. The remainder parts $\eta_{\mathcal{R}}^{(2)}$, $\eta_{\mathcal{R}, \frac{h}{2}}^{(2)}$ are indeed neglectable as usually done in literature.

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