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Asymptotically best possible Lebesque-type inequalities for the Fourier sums on sets of generalized Poisson integrals

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Abstract In this paper we establish Lebesgue-type inequalities for 2π-periodic functions f, which are defined by generalized Poisson integrals of the functions φ from L_p, $1 \le p < \infty$. In these inequalities uniform norms of deviations of Fourier sums $\|f - S_{n-1}\|_C$ are expressed via best approximations $E_n(\varphi)_{L_p}$ of functions φ by trigonometric polynomials in the metric of space L_p. We show that obtained estimates are asymptotically best possible.
Keywords Lebesgue-type inequalities, Fourier sums, generalized Poisson integrals, best approximations by trigonometric polynomials

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1 Introduction

Let L_p, $1 \le p < \infty$, be the space of 2π–periodic functions f summable to the power p on $[0, 2\pi)$, in which the norm is given by the formula $\|f\|_p = \Big(\int\limits_0^{2\pi} |f(t)|^p dt\Big)^{\frac{1}{p}}$; L_∞ be the space of measurable and essentially bounded 2π–periodic functions f with the norm $\|f\|_\infty = \operatorname*{ess\,sup}\limits_t |f(t)|$; C be the space of continuous 2π–periodic functions f, in which the norm is specified by the equality $\|f\|_C = \max\limits_t |f(t)|$.

Denote by $C^{\alpha,r}_\beta L_p$, $\alpha > 0$, $r > 0$, $\beta \in \mathbb{R}$, $1 \le p \le \infty$, the set of all 2π–periodic functions, representable for all $x \in \mathbb{R}$ as convolutions of the form (see, e.g., [1, p. 133])

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int\limits_{-\pi}^{\pi} P_{\alpha,r,\beta}(x-t)\varphi(t)dt, \ a_0 \in \mathbb{R}, \quad \varphi \perp 1, \tag{1}$$

where $\varphi \in L_p$ and $P_{\alpha,r,\beta}(t)$ are fixed generated kernels

$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\big(kt - \frac{\beta\pi}{2}\big), \quad \alpha, r > 0, \quad \beta \in \mathbb{R}. \tag{2}$$

The kernels $P_{\alpha,r,\beta}$ of the form (2) are called generalized Poisson kernels. For $r = 1$ and $\beta = 0$ the kernels $P_{\alpha,r,\beta}$ are usual Poisson kernels of harmonic functions.

If the functions f and φ are related by the equality (1), then function f in this equality is called generalized Poisson integral of the function φ and is denoted by $\mathcal{J}_\beta^{\alpha,r}(\varphi)(f(\cdot)) = \mathcal{J}_\beta^{\alpha,r}(\varphi,\cdot)$. The function φ in equality (1) is called as generalized derivative of the function f and is denoted by $f_\beta^{\alpha,r}$ $(\varphi(\cdot) = f_\beta^{\alpha,r}(\cdot))$.

The set of functions f from $C_\beta^{\alpha,r}L_p$, $1 \le p \le \infty$, such that $f_\beta^{\alpha,r} \in B_p^0$, where

$$B_p^0 = \{\varphi : \ ||\varphi||_p \le 1, \ \varphi \perp 1\},$$

we will denote by $C_{\beta,p}^{\alpha,r}$.

Let τ_{2n-1} be the space of all trigonometric polynomials of degree at most $n-1$ and let $E_n(f)_{L_p}$ be the best approximation of the function $f \in L_p$ in the metric of space L_p, $1 \le p \le \infty$, by the trigonometric polynomials t_{n-1} of degree $n-1$, i.e.,

$$E_n(f)_{L_p} = \inf_{t_{n-1}\in\tau_{2n-1}} \|f - t_{n-1}\|_p.$$

Analogously, by $E_n(f)_C$ we denote the best uniform approximation of the function f from C by trigonometric polynomials of order $n-1$, i.e.,

$$E_n(f)_C = \inf_{t_{n-1}\in\tau_{2n-1}} \|f - t_{n-1}\|_C.$$

Let $\rho_n(f;x)$ be the following quantity

$$\rho_n(f;x) := f(x) - S_{n-1}(f;x), \tag{3}$$

where $S_{n-1}(f;\cdot)$ are the partial Fourier sums of order $n-1$ of a function f.

Least upper bounds of the quantity $\|\rho_n(f;\cdot)\|_C$ over the classes $C_{\beta,p}^{\alpha,r}$, we denote by $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$, i.e.,

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C = \sup_{f\in C_{\beta,p}^{\alpha,r}} \|\rho_n(f;\cdot)\|_C, \quad r > 0, \ \alpha > 0, \ 1 \le p \le \infty. \tag{4}$$

Asymptotic behaviour of the quantities $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$ of the form (4) was studied in [1]–[10].

In [11]–[15] the analogs of the Lebesque inequalities for functions $f \in C_\beta^{\alpha,r}L_p$ have been found in the case $r \in (0,1)$ and $p = \infty$, and also in the case $r \ge 1$

and $1 \le p \le \infty$, where the estimates for the deviations $\|f(\cdot) - S_{n-1}(f;\cdot)\|_C$ are expressed in terms of the best approximations $E_n(f_\beta^{\alpha,r})_{L_p}$. Namely, in [11] it was proved that for arbitrary $f \in C_\beta^{\alpha,r}$, $r \in (0,1)$, $\beta \in \mathbb{R}$, the following inequality holds

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \le \Big(\frac{4}{\pi^2}\ln n^{1-r} + \mathcal{O}(1)\Big)e^{-\alpha n^r} E_n(f_\beta^{\alpha,r})_C, \tag{5}$$

where $\mathcal{O}(1)$ is a quantity uniformly bounded with respect to n, β and $f \in C_\beta^{\alpha,r}C$. It was also shown that for any function $f \in C_\beta^{\alpha,r}C$ and for every $n \in \mathbb{N}$ one can find a function $\mathcal{F}\cdot) = \mathcal{F}f;n;\cdot)$ in the set $C_\beta^{\alpha,r}C$, such that $E_n(\mathcal{F}_\beta^{\alpha,r})_C = E_n(f_\beta^{\alpha,r})_C$ and for this function the relation (5) becomes an equality.

The present paper is a continuation of [11]–[15], and is devoted to obtain asymptotically best possible analogs of Lebesgue-type inequalities on the sets $C_\beta^{\alpha,r}L_p$, $r \in (0,1)$ and $p \in [1,\infty)$. This case was not considered yet.

It should be also noticed, that asymptotically best possible Lebesgue inequalities on classes of generalized Poisson integrals $C_\beta^{\alpha,r}L_p$ for $r \in (0,1)$, $p = \infty$ and $r \ge 1$, $1 \le p \le \infty$ also were established for approximations by Lagrange trigonometric interpolation polynomials with uniform distribution of interpolation nodes (see, e.g., [16]–[18]).

2 Main results

Let us formulate the results of the paper.

By $F(a,b;c;d)$ we denote Gauss hypergeometric function

$$F(a,b;c;z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k}\frac{z^k}{k!}, \tag{6}$$

$$(x)_k := x(x+1)(x+2)...(x+k-1).$$

For arbitrary $\alpha > 0$, $r \in (0,1)$ and $1 \le p < \infty$ we denote by $n_0 = n_0(\alpha,r,p)$ the smallest integer n such that

$$\frac{1}{\alpha r}\frac{1}{n^r} + \frac{\alpha r p}{n^{1-r}} \le \begin{cases} \frac{1}{14}, & p = 1, \\ \frac{1}{(3\pi)^3}\cdot\frac{p-1}{p}, & 1 < p < \infty. \end{cases} \tag{7}$$

The following theorem takes place.

Theorem 1. *Let $0 < r < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. Then in the case $1 < p < \infty$ for any function $f \in C^{\alpha,r}_{\beta} L_p$ and $n \geq n_0(\alpha, r, p)$, the following inequality holds*

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \leq e^{-\alpha n^r} n^{\frac{1-r}{p}} \Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}} (\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}} \Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big)$$

$$+\gamma_{n,p} \Big(\Big(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big) \frac{1}{n^{\frac{1-r}{p}}} + \frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}} \frac{1}{n^r} \Big) \Bigg) E_n(f^{\alpha,r}_{\beta})_{L_p}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{8}$$

where $F(a,b;c;d)$ is Gauss hypergeometric function.

Moreover, for any function $f \in C^{\alpha,r}_{\beta} L_p$ one can find a function mathcal$F(x) = \mathcal{F}f; n; x)$, such that $E_n(\mathcal{F}^{\alpha,r}_{\beta})_{L_p} = E_n(f^{\alpha,r}_{\beta})_{L_p}$ and the following equality holds

$$\|\mathcal{F}\cdot) - S_{n-1}(F;\cdot)\|_C$$

$$= e^{-\alpha n^r} n^{\frac{1-r}{p}} \Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}} (\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}} \Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big)$$

$$+ \gamma_{n,p} \Big(\Big(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big) \frac{1}{n^{\frac{1-r}{p}}} + \frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}} \frac{1}{n^r} \Big) \Bigg) E_n(f^{\alpha,r}_{\beta})_{L_p}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{9}$$

In (8) and (9) the quantity $\gamma_{n,p} = \gamma_{n,p}(\alpha, r, \beta)$ is such that $|\gamma_{n,p}| \leq (14\pi)^2$.

Proof of Theorem 1. Let us prove at the beginning the inequality (8) .

Let $f \in C^{\alpha,r}_{\beta} L_p$, $1 \leq p \leq \infty$. Then, at every point $x \in \mathbb{R}$ the following integral representation is true:

$$\rho_n(f;x) = f(x) - S_{n-1}(f;x) = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f^{\alpha,r}_{\beta}(t) P^{(n)}_{\alpha,r,\beta}(x-t) dt, \tag{10}$$

where

$$P^{(n)}_{\alpha,r,\beta}(t) := \sum_{k=n}^{\infty} e^{-\alpha k^r} \cos\Big(kt - \frac{\beta\pi}{2}\Big), \ 0 < r < 1, \ \alpha > 0, \ \beta \in \mathbb{R}. \tag{11}$$

The function $P^{(n)}_{\alpha,r,\beta}(t)$ is orthogonal to any trigonometric polynomial t_{n-1} of degree not greater than $n-1$. Hence, for any polynomial $t_{n-1} \in \tau_{2n-1}$ we obtain

$$\rho_n(f;x) = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \delta_n(t) P^{(n)}_{\alpha,r,\beta}(x-t) dt, \tag{12}$$

where

$$\delta_n(x) = \delta_n(\alpha, r, \beta; x) := f_\beta^{\alpha,r}(x) - t_{n-1}(x). \tag{13}$$

Further we choose the polynomial t_{n-1}^* of the best approximation of the function $f_\beta^{\alpha,r}$ in the space L_p, i.e., such that

$$\|f_\beta^{\alpha,r} - t_{n-1}^*\|_p = E_n(f_\beta^{\alpha,r})_{L_p}, \quad 1 \le p \le \infty,$$

to play the role of t_{n-1} in (12). Thus, by using the inequality

$$\left\| \int_{-\pi}^{\pi} K(t-u)\varphi(u)du \right\|_C \le \|K\|_{p'} \|\varphi\|_p, \tag{14}$$

$$\varphi \in L_p, \quad K \in L_{p'}, \quad 1 \le p \le \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

(see, e.g., [19, p. 43]), we get

$$\|f(\cdot) - S_{n-1}(f; \cdot)\|_C \le \frac{1}{\pi} \|P_{\alpha,r,\beta}^{(n)}\|_{p'} E_n(f_\beta^{\alpha,r})_{L_p}. \tag{15}$$

It follows from the paper [9] (see, e.g., also [8] and [10]) for arbitrary $r \in (0,1)$, $\alpha > 0$, $\beta \in \mathbb{R}$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $n \in \mathbb{N}$ and $n \ge n_0(\alpha, r, p)$ the following estimate holds

$$\frac{1}{\pi}\|P_{\alpha,r,\beta}^{(n)}\|_{p'} = e^{-\alpha n^r} n^{\frac{1-r}{p}} \left(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} \left(\int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2+1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} \right.$$

$$\left. + \gamma_{n,p}^{(1)} \left(\frac{1}{(\alpha r)^{1+\frac{1}{p}}} \left(\int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2+1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}} \right) \right), \tag{16}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and the quantity $\gamma_{n,p}^{(1)} = \gamma_{n,p}^{(1)}(\alpha, r, \beta)$ satisfies the inequality $|\gamma_{n,p}^{(1)}| \le (14\pi)^2$.

In [8] and [9] it was mentioned that formula (16) also holds, if in its second part instead $\frac{1}{\pi}\|P_{\alpha,r,\beta}^{(n)}\|_{p'}$ to put $\frac{1}{\pi} \inf\limits_{\lambda \in \mathbb{R}} \|P_{\alpha,r,\beta}^{(n)} - \lambda\|_{p'}$ or $\sup\limits_{h \in \mathbb{R}} \frac{1}{2\pi} \|P_{\alpha,r,\beta}^{(n)}(t+h) - P_{\alpha,r,\beta}^{(n)}(t)\|_{p'}$

Formula (106) from [10] gives the following estimate

$$\left(\int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\right)^{\frac{1}{p'}} = \left(\int\limits_0^{\infty} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\right)^{\frac{1}{p'}} + \frac{\Theta^{(1)}_{\alpha,r,p,n}}{p'-1}\Big(\frac{\alpha r}{\pi n^{1-r}}\Big)^{p'-1}, \quad |\Theta^{(1)}_{\alpha,r,p,n}|<2. \tag{17}$$

In the work [9] (see formula (27)) it was shown, that for arbitrary $1<p'<\infty$ the following equality takes place

$$\left(\int\limits_0^{\infty} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\right)^{\frac{1}{p'}} = F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big). \tag{18}$$

Taking into account the following estimate

$$\left(\int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\right)^{\frac{1}{p'}} \le \left(\int\limits_0^{\infty} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\right)^{\frac{1}{p'}} < \left(1+\int\limits_1^{\infty}\frac{dt}{t^{p'}}\right)^{\frac{1}{p'}} < (p)^{\frac{1}{p'}}, \tag{19}$$

formulas (16)–(19) imply that for $n \ge n_0(\alpha, r, p)$, $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$,

$$\begin{aligned}\frac{1}{\pi}\big\|P^{(n)}_{\alpha,r,\beta}\big\|_{p'} &= e^{-\alpha n^r} n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) \\ &\quad + \gamma^{(1)}_{n,p}\Big(\frac{1}{p'-1}\frac{(\alpha r)^{\frac{p'-1}{p}}}{n^{(1-r)(p'-1)}} + \frac{p^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}}\Big)\Bigg) \\ &= e^{-\alpha n^r} n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) \\ &\quad + \gamma^{(2)}_{n,p}\Big(\Big(1+\frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}} + \frac{p^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Big)\Bigg),\end{aligned} \tag{20}$$

where the quantities $\gamma^{(i)}_{n,p} = \gamma^{(i)}_{n,p}(\alpha, r, \beta)$, satisfy the inequality $|\gamma^{(i)}_{n,p}| \le (14\pi)^2$, $i=1,2$. Formula (8) follows from (15) and (20).

To prove the second part of Theorem 1, according to the equality (12), for arbitrary $\varphi \in L_p$ we should find the function $\Phi(\cdot) = \Phi(\varphi, n; \cdot) \in L_p$, such that

$E_n(\Phi)_{L_p} = E_n(\varphi)_{L_p}$ and for all $n \geq n_0(\alpha, r, p)$ the following equality holds

$$\frac{1}{\pi}\left|\int\limits_{-\pi}^{\pi}(\Phi(t) - t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(-t)dt\right| = e^{-\alpha n^r} n^{\frac{1-r}{p}}\left(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\left(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\right)\right.$$

$$\left. +\gamma_{n,p}\left(\left(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\right)\frac{1}{n^{\frac{1-r}{p}}} + \frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\right)\right) E_n(\varphi)_{L_p}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{21}$$

where t^*_{n-1} is the polynomial of the best approximation of the order $n-1$ of the function Φ in the space L_p, $|\gamma_{n,p}| \leq (14\pi)^2$.

In this case for an arbitrary function $f \in C^{\alpha,r}_{\beta} L_p$, $1 < p < \infty$, there exists a function $\Phi(\cdot) = \Phi(f^{\alpha,r}_{\beta}; \cdot)$, such that $E_n(\Phi)_{L_p} = E_n(f^{\alpha,r}_{\beta})_{L_p}$, and for $n \geq n_0(\alpha, r, p)$ the formula (21) holds, where as function φ we take the function $f^{\alpha,r}_{\beta}$.

Let us assume

$$\mathcal{F}\cdot) = \mathcal{J}^{\alpha,r}_{\beta}\left(\Phi(\cdot) - \frac{a_0}{2}\right),$$

where

$$a_0 = a_0(\Phi) := \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\Phi(t)dt.$$

The function F is the function, which we have looked for, because $F \in C^{\alpha,r}_{\beta} L_p$ and

$$E_n(\mathcal{F}^{\alpha,r}_{\beta})_{L_p} = E_n(\Phi - \frac{a_0}{2})_{L_p} = E_n(\Phi)_{L_p} = E_n(f^{\alpha,r}_{\beta})_{L_p},$$

so (10), (12), (8) and (21) imply (9).

At last let us prove (21). Let $\varphi \in L_p$, $1 < p < \infty$. Then as a function $\Phi(t)$ we consider the function

$$\Phi(t) = \|P^{(n)}_{\alpha,r,-\beta}\|^{1-p'}_{p'}|P^{(n)}_{\alpha,r,-\beta}(t)|^{p'-1}\mathrm{sign}(P^{(n)}_{\alpha,r,-\beta}(t))E_n(\varphi)_{L_p} \tag{22}$$

For this function

$$\begin{aligned}\|\Phi\|_p &= \|P^{(n)}_{\alpha,r,-\beta}\|^{1-p'}_{p'}\||P^{(n)}_{\alpha,r,-\beta}|^{p'-1}\|_p E_n(\varphi)_{L_p} \\ &= \|P^{(n)}_{\alpha,r,-\beta}\|^{1-p'}_{p'}\|P^{(n)}_{\alpha,r,-\beta}\|^{p'-1}_{p'} E_n(\varphi)_{L_p} = E_n(\varphi)_{L_p}.\end{aligned}$$

Now we show that the polynomial t^*_{n-1} of best approximation of order $n-1$ in the space L_p of the function $\Phi(t)$ equals identically to zero: $t^*_{n-1} \equiv 0$.

For any $t_{n-1} \in \tau_{2n-1}$

$$\int\limits_0^{2\pi} t_{n-1}(t)|\Phi(t)|^{p-1}\mathrm{sign}(\Phi(t))dt = \|P^{(n)}_{\alpha,r,-\beta}\|^{-1}_{p'}(E_n(\varphi)_{L_p})^{p-1}\int\limits_{-\pi}^{\pi} t_{n-1}(t)P^{(n)}_{\alpha,r,-\beta}(t)dt = 0.$$

Then, according to Proposition 1.4.12 of the work [19, p. 29] we can make conclusion, that the polynomial $t^*_{n-1}\equiv 0$ is the polynomial of the best approximation of the function $\Phi(t)$ in the space L_p, $1<p<\infty$.

For the function $\Phi(t)$ of the form (22) we can write

$$\frac{1}{\pi}\int\limits_{-\pi}^{\pi}(\Phi(t)-t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(-t)dt$$
$$=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\Phi(t)P^{(n)}_{\alpha,r,\beta}(-t)dt=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\Phi(t)P^{(n)}_{\alpha,r,-\beta}(t)dt$$
$$=\frac{1}{\pi}\|P^{(n)}_{\alpha,r,-\beta}\|^{1-p'}_{p'}E_n(\varphi)_{L_p}\int\limits_{-\pi}^{\pi}|P^{(n)}_{\alpha,r,-\beta}(t)|^{p'}dt=\frac{1}{\pi}\|P^{(n)}_{\alpha,r,-\beta}\|_{p'}E_n(\varphi)_{L_p}. \quad (23)$$

Thus from (20) and (23) we get (9). Theorem 1 is proved. □

Theorem 2. *Let* $0<r<1$, $\alpha>0$, $\beta\in\mathbb{R}$, $n\in\mathbb{N}$. *Then, for any* $f\in C^{\alpha,r}_{\beta}L_1$ *and* $n\geq n_0(\alpha,r,1)$ *the following inequality holds:*

$$\|f(\cdot)-S_{n-1}(f;\cdot)\|_C\leq e^{-\alpha n^r}n^{1-r}\Big(\frac{1}{\pi\alpha r}+\gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r}+\frac{1}{n^{1-r}}\Big)\Big)E_n(f^{\alpha,r}_{\beta})_{L_1}. \quad (24)$$

Moreover, for any function $f\in C^{\alpha,r}_{\beta}L_1$ *one can find a function* $\mathcal{F}x)=\mathcal{F}f;n,x)$ *in the set* $C^{\alpha,r}_{\beta}L_1$, *such that* $E_n(\mathcal{F}^{\alpha,r}_{\beta})_{L_1}=E_n(f^{\alpha,r}_{\beta})_{L_1}$ *and for* $n>n_0(\alpha,r,1)$ *the following equality holds*

$$\|\mathcal{F}\cdot)-S_{n-1}(F;\cdot)\|_C=e^{-\alpha n^r}n^{1-r}\Big(\frac{1}{\pi\alpha r}+\gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r}+\frac{1}{n^{1-r}}\Big)\Big)E_n(f^{\alpha,r}_{\beta})_{L_1}. \quad (25)$$

In (24) and (25) the quantity $\gamma_{n,1}=\gamma_{n,1}(\alpha,r,\beta)$ *is such that* $|\gamma_{n,1}|\leq(14\pi)^2$.

Proof of Theorem 2. At the beginning let us show that (24) holds. Let $f\in C^{\alpha,r}_{\beta}L_1$. Then, according to (12) and (14)

$$\|f(\cdot)-S_{n-1}(f;\cdot)\|_C=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}(f^{\alpha,r}_{\beta}(t)-t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(x-t)dt\leq\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_{\infty}E_n(f^{\alpha,r}_{\beta})_{L_1}, \quad (26)$$

where $t^*_{n-1}\in\tau_{2n-1}$ is the polynomial of the best approximation of the function $f^{\alpha,r}_{\beta}$ in the space L_1.

From formula (20) of the work [9] (see also [8] and [10]) for arbitrary $r \in (0,1)$, $\alpha > 0$, $\beta \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq n_0(\alpha, r, 1)$ it follows that

$$\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_\infty = e^{-\alpha n^r} n^{1-r}\Big(\frac{1}{\alpha r\pi} + \gamma_{n,1}\Big(\frac{1}{(\alpha r)^2 n^r} + \frac{1}{n^{1-r}}\Big)\Big), \tag{27}$$

where the quantity $\gamma_{n,1} = \gamma_{n,1}(\alpha, r, \beta)$ is such that $|\gamma_{n,1}| \leq (14\pi)^2$.

It is clear, that from $P^{(n)}_{\alpha,r,\beta} \in C$ it follows that the norm $\|P^{(n)}_{\alpha,r,\beta}\|_\infty$ in (26) and (27) can be substituted by $\|P^{(n)}_{\alpha,r,\beta}\|_C$.

Combining formulas (26) and (27), we get (24).

To prove the second part of Theorem 2 we need for any function $\varphi \in L_1$ to find the function $\Phi(\cdot) = \Phi(\varphi, \cdot) \in L_1$, such that $E_n(\Phi)_{L_1} = E_n(\varphi)_{L_1}$ and for all $n \geq n_0(\alpha, r, 1)$ the following equality holds

$$\frac{1}{\pi}\left|\int_{-\pi}^{\pi} (\Phi(t) - t^*_{n-1}(t))\, P^{(n)}_{\alpha,r,\beta}(-t)dt\right|$$
$$= e^{-\alpha n^r} n^{1-r}\Big(\frac{1}{\pi\alpha r} + \gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r} + \frac{1}{n^{1-r}}\Big)\Big)E_n(\varphi)_{L_1}, \tag{28}$$

where t^*_{n-1} is the polynomial of the best approximation of order $n-1$ of the function Φ in the space L_1 and $|\gamma_{n,1}| \leq (14\pi)^2$.

In this case for any function $f \in C^{\alpha,r}_\beta L_1$ there exists a function $\Phi(\cdot) = \Phi(f^{\alpha,r}_\beta; \cdot)$, such that $E_n(\Phi)_{L_1} = E_n(f^{\alpha,r}_\beta)$, and for $n \geq n_0(\alpha, r, 1)$ the formula (28) holds, where as function φ we will take the function $f^{\alpha,r}_\beta$.

Let us consider the function

$$\mathcal{F}\cdot) = \mathcal{J}^{\alpha,r}_\beta(\Phi(\cdot) - \frac{a_0}{2}),$$

where

$$a_0 = a_0(\Phi) := \frac{1}{\pi}\int_{-\pi}^{\pi} \Phi(t)dt.$$

The function F is the function, which we look for, because $F \in C^{\alpha,r}_\beta L_1$ and

$$E_n(\mathcal{F}^{\alpha,r}_\beta)_{L_1} = E_n(\Phi - \frac{a_0}{2})_{L_1} = E_n(\Phi)_{L_1} = E_n(f^{\alpha,r}_\beta)_{L_1},$$

and on the basis (10), (12), (24) and (28) the formula (25) holds.

Let us prove (28). Let t^* be the point from the interval $T = \Big[\frac{\pi(1-\beta)}{2n},\ 2\pi + \frac{\pi(1-\beta)}{2n}\Big)$, where the function $|P^{(n)}_{\alpha,r,-\beta}|$ attains its largest value, i.e.,

$$|P^{(n)}_{\alpha,r,-\beta}(t^*)| = \|P^{(n)}_{\alpha,r,-\beta}\|_C = \|P^{(n)}_{\alpha,r,\beta}\|_C.$$

Let put $\Delta_k^n := \Big[\frac{(k-1)\pi}{n} + \frac{\pi(1-\beta)}{2n}, \frac{k\pi}{n} + \frac{\pi(1-\beta)}{2n}\Big)$, $k = 1,...,2n$. By k^* we denote the number, such that $t^* \in \Delta_{k^*}^n$. Taking into account, that function $P_{\alpha,r,-\beta}^{(n)}$ is absolutely continuous, so for arbitrary $\varepsilon > 0$ there exists a segment $\ell^* = [\xi^*, \xi^* + \delta] \subset \Delta_{k^*}^n$, such that for arbitrary $t \in \ell^*$ the following inequality holds $|P_{\alpha,r,-\beta}^{(n)}(t)| > \|P_{\alpha,r,\beta}^{(n)}\|_C - \varepsilon$. It is clear that $\operatorname{mes} \ell^* = |\ell^*| = \delta < \frac{\pi}{n}$.

For arbitrary $\varphi \in L_1$ and $\varepsilon > 0$ we consider the function $\Phi_\varepsilon(t)$, which on the segment T is defined with a help of equalities

$$\Phi_\varepsilon(t) = \begin{cases} E_n(\varphi)_{L_1} \frac{1-\varepsilon(2\pi-\delta)}{\delta} \operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big), & t \in \ell^*, \\ E_n(\varphi)_{L_1} \varepsilon \operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big), & t \in \mathrm{T} \setminus \ell^*. \end{cases}$$

For the function $\Phi_\varepsilon(t)$ for arbitrary small values of $\varepsilon > 0$ ($\varepsilon \in (0, \frac{1}{2\pi})$) the following equality holds

$$\|\Phi_\varepsilon\|_1 = E_n(\varphi)_{L_1} \frac{1-\varepsilon(2\pi-\delta)}{\delta} \int\limits_{\ell^*} \Big|\operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big)\Big| dt$$
$$+ E_n(\varphi)_{L_1} \varepsilon \int\limits_{\mathrm{T}\setminus\ell^*} \Big|\operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big)\Big| dt$$
$$= E_n(\varphi)_{L_1} \left(\frac{1-\varepsilon(2\pi-\delta)}{\delta}\delta + \varepsilon(2\pi-\delta) \right) = E_n(\varphi)_{L_1}. \qquad (29)$$

It should be noticed, that

$$\operatorname{sign}\Phi_\varepsilon(t) = \operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big). \qquad (30)$$

Since for arbitrary trigonometric polynomial $t_{n-1} \in \tau_{2n-1}$

$$\int\limits_0^{2\pi} t_{n-1}(t) \operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big) dt = 0,$$

so, taking into account (30)

$$\int\limits_0^{2\pi} t_{n-1}(t) \operatorname{sign}\Big(\Phi_\varepsilon(t) - 0\Big) dt = 0, \quad t_{n-1} \in \tau_{2n-1}.$$

According to Proposition 1.4.12 of the work [19, p.29] the polynomial $t_{n-1}^* \equiv 0$ is a polynomial of the best approximation of the function Φ_ε in the metric of the space L_1, i.e., $E_n(\Phi_\varepsilon)_{L_1} = \|\Phi_\varepsilon\|_1$, so (29) yields $E_n(\Phi_\varepsilon)_{L_1} = E_n(\varphi)_{L_1}$.

Moreover, for the function Φ_ε

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(\Phi_\varepsilon(t)-t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(-t)dt=\frac{1}{\pi}\int_{-\pi}^{\pi}\Phi_\varepsilon(t)P^{(n)}_{\alpha,r,-\beta}(t)dt$$
$$=\frac{1-\varepsilon(2\pi-\delta)}{\pi\delta}E_n(\varphi)_{L_1}\int_{\ell^*}\operatorname{sign}\cos\Big(nt+\frac{\beta\pi}{2}\Big)P^{(n)}_{\alpha,r,-\beta}(t)dt$$
$$+\frac{\varepsilon}{\pi}E_n(\varphi)_{L_1}\int_{\mathrm{T}\setminus\ell^*}\operatorname{sign}\cos\Big(nt+\frac{\beta\pi}{2}\Big)P^{(n)}_{\alpha,r,-\beta}(t)dt. \tag{31}$$

Taking into account, that $\operatorname{sign}\Phi_\varepsilon(t)=(-1)^k$, $t\in\Delta^{(n)}_k$, $k=1,...,2n$, and also the embedding $\ell^*\subset\Delta^{(n)}_{k^*}$, we get

$$\left|\frac{1-\varepsilon(2\pi-\delta)}{\pi\delta}E_n(\varphi)_{L_1}\int_{\ell^*}\operatorname{sign}\cos\Big(nt+\frac{\beta\pi}{2}\Big)P^{(n)}_{\alpha,r,-\beta}(t)dt\right|$$
$$=\left|(-1)^{k^*}\frac{1-\varepsilon(2\pi-\delta)}{\pi\delta}E_n(\varphi)_{L_1}\int_{\ell^*}P^{(n)}_{\alpha,r,-\beta}(t)dt\right|$$
$$\geq\frac{1-\varepsilon(2\pi-\delta)}{\pi}E_n(\varphi)_{L_1}\Big(\|P^{(n)}_{\alpha,r,\beta}\|_C-\varepsilon\Big)$$
$$>\frac{1-2\pi\varepsilon}{\pi}E_n(\varphi)_{L_1}\Big(\|P^{(n)}_{\alpha,r,\beta}\|_C-\varepsilon\Big)$$
$$=\frac{1}{\pi}E_n(\varphi)_{L_1}\Big(\|P^{(n)}_{\alpha,r,\beta}\|_C-2\pi\varepsilon\|P^{(n)}_{\alpha,r,\beta}\|_C-\varepsilon+2\pi\varepsilon^2\Big)$$
$$>E_n(\varphi)_{L_1}\left(\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_C-\varepsilon\Big(2\|P^{(n)}_{\alpha,r,\beta}\|_C+\frac{1}{\pi}\Big)\right). \tag{32}$$

Also, it is not hard to see that

$$\left|\frac{\varepsilon}{\pi}E_n(\varphi)_{L_1}\int_{\mathrm{T}\setminus\ell^*}\operatorname{sign}\cos\Big(nt+\frac{\beta\pi}{2}\Big)P^{(n)}_{\alpha,r,-\beta}(t)dt\right|\leq\frac{\varepsilon}{\pi}E_n(\varphi)_{L_1}\|P^{(n)}_{\alpha,r,\beta}\|_C. \tag{33}$$

Formulas (31)–(33) yield the following inequality

$$\left|\int_{-\pi}^{\pi}\frac{1}{\pi}(\Phi_\varepsilon(t)-t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(-t)dt\right|$$
$$>E_n(\varphi)_{L_1}\left(\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_C-\varepsilon\Big(\Big(2+\frac{1}{\pi}\Big)\|P^{(n)}_{\alpha,r,\beta}\|_C+\frac{1}{\pi}\Big)\right). \tag{34}$$

Let us show, that on basis of the results of the work [10], the estimate (27) can be improved, if we decrease the diapason for $|\gamma_{n,1}|$.

Formulas (34), (50)–(52) of the work [10], and also Remark 1 from [10] allow us to write that for any $n \in \mathbb{N}$

$$\|P^{(n)}_{\alpha,r,\beta}\|_\infty = \|P_{\alpha,r,n}\|_\infty \Big(1 + \delta_n^{(1)} \frac{M_n}{n}\Big), \tag{35}$$

where

$$P_{\alpha,r,n}(t) := \sum_{k=0}^{\infty} e^{-\alpha(k+n)^r} e^{ikt},$$

$$M_n := \sup_{t\in\mathbb{R}} \frac{|P'_{\alpha,r,n}(t)|}{|P_{\alpha,r,n}(t)|},$$

and for $\delta_n^{(1)} = \delta_n^{(1)}(\alpha, r, \beta)$ the following estimate takes place $|\delta_n^{(1)}| \leq 5\sqrt{2}\pi$.

Then, as it follows from the estimates (87) and (99) of the work [10] for $n \geq n_0(\alpha, r, 1)$

$$\|P_{\alpha,r,n}\|_\infty = \frac{e^{-\alpha n^r}}{\alpha r} n^{1-r} \left(1 + \theta_{\alpha,r,n}\Big(\frac{1-r}{\alpha r n^r} + \frac{\alpha r}{n^{1-r}}\Big)\right), \quad |\theta_{\alpha,r,n}| \leq \frac{14}{13} \tag{36}$$

and

$$M_n \leq \frac{784\pi^2}{117}\Big(\frac{n^{1-r}}{\alpha r} + \alpha r n^r\Big). \tag{37}$$

Combining formulas (35)–(37) we obtain that for $n \geq n_0(\alpha, r, 1)$

$$\begin{aligned}\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_\infty &= \frac{e^{-\alpha n^r}}{\alpha r \pi} n^{1-r} \left(1 + \theta_{\alpha,r,n}\Big(\frac{1-r}{\alpha r n^r} + \frac{\alpha r}{n^{1-r}}\Big)\right) \left(1 + \delta_n^{(1)} \frac{M_n}{n}\right) \\ &= e^{-\alpha n^r} n^{1-r}\Big(\frac{1}{\alpha r \pi} + \gamma_{n,1}\Big(\frac{1}{\alpha r n^{1-r}} + \frac{1}{n^{1-r}}\Big)\Big),\end{aligned} \tag{38}$$

where

$$|\gamma_{n,1}| \leq \frac{1}{\pi}\left(\frac{14}{13} + \frac{784\pi^2 5\sqrt{2}\pi}{117} + \frac{14\cdot 5\sqrt{2}\pi \cdot 784\pi^2}{13\cdot 117\cdot 14}\right) = \frac{14}{13\pi}\Big(1 + \frac{3920\sqrt{2}\pi^3}{117}\Big). \tag{39}$$

Let us choose ε small enough, that

$$\varepsilon < \frac{\left((14\pi)^2 - \frac{14}{13\pi}\left(1 + \frac{3920\sqrt{2}\pi^3}{117}\right)\right) e^{-\alpha n^r} n^{1-r}\big(\frac{1}{\alpha r n^r} + \frac{\alpha r}{n^{1-r}}\big)}{(2+\frac{1}{\pi})\|P^{(n)}_{\alpha,r,\beta}\|_\infty + \frac{1}{\pi}} \tag{40}$$

and for this ε we put

$$\Phi(t) = \Phi_\varepsilon(t). \tag{41}$$

The function $\Phi(t)$ is the function, which we looked for, because $E_n(\Phi)_{L_1} = E_n(\varphi)_{L_1}$ and according to (34), (38)–(40) for $n \geq n_0(\alpha, r, 1)$

$$\left|\frac{1}{\pi}(\Phi(t) - t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(-t)dt\right|$$
$$> E_n(\varphi)_{L_1}\left(\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_C - \left((14\pi)^2 - \frac{14}{13\pi}\Big(1+\frac{3920\sqrt{2}\pi^3}{117}\Big)\right)e^{-\alpha n^r}n^{1-r}\Big(\frac{1}{\alpha r n^r}+\frac{\alpha r}{n^{1-r}}\Big)\right)$$
$$\geq e^{-\alpha n^r}n^{1-r}\left(\frac{1}{\alpha r\pi} - (14\pi)^2\Big(\frac{1}{\alpha r n^r} + \frac{\alpha r}{n^{1-r}}\Big)\right)E_n(\varphi)_{L_1}. \tag{42}$$

Formulas (42), (26) and (27) imply (28). Theorem 2 is proved. □

It should be noticed, that inequalities (8) and (24) were announced in the work [15]. There it was also mentioned that that estimates (8) and (24) are asymptotically best possible on the classes $C^{\alpha,r}_{\beta,p}$, $1 \leq p < \infty$.

If $f \in C^{\alpha,r}_{\beta,p}$, then $\|f^{\alpha,r}_\beta\|_p \leq 1$, and $E_n(f^{\alpha,r}_\beta)_{L_p} \leq 1$, $1 \leq p < \infty$. Considering the least upper bounds of both sides of inequality (8) over the classes $C^{\alpha,r}_{\beta,p}$, $1 < p < \infty$, we arrive at the inequality

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \leq e^{-\alpha n^r}n^{\frac{1-r}{p}}\left(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}}F^{\frac{1}{p'}}\Big(\frac{1}{2},\frac{3-p'}{2};\frac{3}{2};1\Big)\right.$$
$$\left.+\gamma_{n,p}\left(\Big(1+\frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}} + \frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\right)\right)E_n(f^{\alpha,r}_\beta)_{L_p}, \quad \frac{1}{p}+\frac{1}{p'} = 1. \tag{43}$$

Comparing this relation with the estimate of Theorem 4 from [9] (see also [10]), we conclude that inequality (8) on the classes $C^{\alpha,r}_{\beta,p}$, $1 < p < \infty$, is asymptotically best possible.

In the same way, the asymptotic sharpness of the estimate (24) on the classes $C^{\alpha,r}_{\beta,1}$ follows from comparing inequality

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \leq e^{-\alpha n^r}n^{1-r}\Big(\frac{1}{\pi\alpha r} + \gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r} + \frac{1}{n^{1-r}}\Big)\Big)E_n(f^{\alpha,r}_\beta)_{L_1} \tag{44}$$

and formula (18) from [10].

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