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DETERMINATION OF THE FRACTIONAL ORDER IN QUASILINEAR SUBDIFFUSION EQUATIONS

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Abstract. We analyze the inverse boundary value-problem to determine the fractional order ν of nonautonomous quasilinear subdiffusion equations with memory terms from observations of their solutions during small time. We obtain an explicit formula reconstructing the order. Based on the Tikhonov regularization scheme and the quasi-optimality criterion, we construct the computational algorithm to find the order ν from noisy discrete measurements. We present several numerical tests illustrating the algorithm in action.

1. Introduction

Anomalous processes in porous, fractal, biological media are modeled by differential equation containing fractional derivatives in time or/and space (see e.g., [4, 5, 7, 8, 31]). In practice, the parameters of media or model are unknown or scarcely known, and can be reconstructed by solving inverse problems for governing differential equations.

Inverse problems in fractional diffusion models have been intensively studied during the present decade. Series of papers are related with a reconstruction of unknown source terms, boundary and initial conditions, coefficients and order derivatives from some extra measurement data [11–14, 20, 24, 28, 30, 32].

Physically, the time-fractional order diffusion process represents some slow diffusion with tail effect, which has been found in many modern industry and environmental areas [22, 23]. Such slow diffusion is called subdiffusion and described with time fractional diffusion equations. On the one hand, the order ν of the involved fractional derivative identifies a rate of diffusion process and, on the other hand, ν is a "memory" parameter of a diffusion system [2, 5, 10, 26].

In this paper, we discuss an approach to the reconstruction of a subdiffusion order ν, $\nu \in (0, 1)$, from small time state observation data. To this end, we analyze an inverse problem of recovering the order ν of a semilinear subdiffusion equation with memory terms with the unknown function $v = v(x, t) : \Omega \times (0, T) \to \mathbb{R}$

$$\mathbf{D}_t^{\nu} v - \mathcal{L}_1 v - \int_0^t \mathcal{K}(t-s)\mathcal{L}_2 v(\cdot, s)ds = f(x, t, v) + f_0(x, t), \quad \nu \in (0, 1), \tag{1.1}$$

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where the symbol $\mathbf{D}_t^\nu$ stands the Caputo fractional derivative of order ν with respect to time t (see e.g. (2.4.1) in [15]), defined as

$$\mathbf{D}_t^\nu v(x,t) = \frac{1}{\Gamma(1-\nu)} \frac{\partial}{\partial t} \int\limits_0^t \frac{[v(x,\tau) - v(x,0)]}{(t-\tau)^\nu} d\tau, \ \nu \in (0,1)$$

with Γ being the Euler Gamma-function. $\mathcal{L}_1$ and $\mathcal{L}_2$ are uniformly elliptic operators of the second order with time-dependent smooth coefficients.

A motivation for the study of equations (1.1) arises from theoretical and experimental investigations of materials with memory [2, 5, 10, 16, 26].

The problem of recovering the order ν for an equation like (1.1) has been studied in [12,13,19]. Namely, in [12], the following two explicit formulas for a homogenous linear autonomous equation without memory terms have been derived for $x_0 \in \Omega$:

$$\nu = \lim_{t\to 0} \frac{t v_t(x_{0,t})}{v(x_0,t) - v_0(x_0)}, \quad \nu = \lim_{t\to\infty} \frac{t v_t(x_{0,t})}{v(x_0,t)}.$$

Then the paper [13] has studied a linear equation like (1.1) with time-independent coefficients in a one-dimensional domain Ω, $\mathcal{L}_2 = \frac{\partial^2}{\partial x^2}$, and derived the formula

$$\nu = \lim_{t\to 0} \frac{\ln |v(x_0,t) - v(x_0,0)|}{\ln t}. \tag{1.2}$$

Moreover, in [13] it has been proposed to approximate the order ν by the pre-limit value of (1.2) at the smallest observation time $t = t_1$:

$$\nu = \nu^{t_1} \approx \frac{\ln |v(x_0,t_1) - v(x_0,0)|}{\ln t_1}.$$

In [19], the validity of formula (1.2) has been extended to the case of the nonlinear equation of more general form (1.1) in the multidimensional case of Ω. Moreover, using Tikhonov regularization scheme [29] and multiple quasi-optimality criterion [9, 21], the authors proposed techniques to reconstruct the order ν from noisy discrete observation data. Numerical tests in [19] have shown efficiency of both formula (1.2) and the regularization algorithm in the case of a low noise level.

In this paper incorporating non-vanishing memory kernels in (1.1) and analyzing this model in the fractional Hölder classes, we obtain a new explicit reconstruction formula for order ν:

$$\nu = \lim_{t\to 0} \frac{t[v(x_0,t) - v(x_0,0)]}{\int\limits_0^t [v(x_0,\tau) - v(x_0,0)] d\tau} - 1$$

where the only requirement for $x_0 \in \bar{\Omega}$ is that

$$\mathcal{L}_1 v(x_0,t)|_{t=0} + f(x_0,0,v(x_0,0)) + f_0(x_0,0) \neq 0.$$

Then we show how to use this formula in the case where we have only noisy observations $\psi_\delta(t_k) \approx v(x_0,t_k)$ at a finite number N of time moments $t = t_k$, $k = 1,2,...,N$, $0 < t_1 < t_2 < ... < t_N$. To this end, we at first propose to reconstruct $\psi(t) = \psi_{\delta,\lambda}(t)$ by means of the regularized regression from given noisy data ψ_δ, where the regularization is performed in the finite-dimensional space

$$\operatorname{span}\{t^{\nu_i}, \mathcal{P}_j^{(0,-\gamma)}, \quad i = 1,2,3, \quad j = 1,2,...,m(N)\}$$

according to the Tikhonov scheme with the penalty term $\lambda\|\cdot\|_{L^2_{t^{-\gamma}}(0,t_N)}$. Here λ is the regularization parameter, ν_i are our initial guesses about ν (in particular, $\nu_i = 0$), $L^2_{t^{-\gamma}}(0, t_N)$ is a weighted space L^2 with the weight $t^{-\gamma}$, $\gamma \in (0, 1)$; $\mathcal{P}_j^{(0,-\gamma)}$ are Jacobi polynomials shifted to $[0, t_N]$.

Then, according to our formula, we consider the quantities

$$\nu(\lambda, t) = \frac{t[\psi_{\delta,\lambda}(t) - v(x_0, 0)]}{\int_0^t [\psi_{\delta,\lambda}(\tau) - v(x_0, 0)]d\tau} - 1$$

calculated for the sequences of (regularization) parameters: $\lambda \in \{\lambda_p\}$, $t \in \{\tilde{t}_q\}$.

Finally, the regularized reconstructor

$$\nu := \nu_{reg} = \nu(\tilde{\lambda}, \tilde{t})$$

is chosen from the set of approximate values $\{\nu(\lambda_p, \tilde{t}_q)\}$ by applying two-parameter quasi-optimality criterion [9] selecting $\tilde{\lambda} \in \{\lambda_p\}$, $\tilde{t} \in \{\tilde{t}_q\}$.

As shown by the numerical tests (see Section 6.2), in the case of high noise level, the new recovering formula for the order ν gives more accurate outputs than the formula (1.2) and, hence, takes advantage over (1.2).

The paper is organized as follows. In the next section we state inverse problem and introduce the function spaces. The main theoretical results of the paper along with the general assumptions on the model are stated in Section 3. In Section 4, we discuss existence, uniqueness in (1.1), prove the explicit formula for ν. The influence of noise on the calculation of the order ν is discussed in Section 5. Finally, Section 6 is devoted to the description of the algorithm for regularized recovering of ν. The proposed method is illustrated by numerical examples and compared with the known approaches.

2. Statement of the Problem and Function Spaces

Let Ω be a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, $\partial\Omega \in \mathcal{C}^{k+\alpha}$, $k \geq 2$, $\alpha \in (0, 1)$. For an arbitrary given time $T > 0$ we denote

$$\Omega_T = \Omega \times (0, T) \quad \text{and} \quad \partial\Omega_T = \partial\Omega \times [0, T].$$

For the given functions $a_{ij}(x, t)$, $a_i(x, t)$, $b_i(x, t)$, $i, j = 1, 2, ..., n$, $a_0(x, t)$, $b_0(x, t)$, we define the operator $\mathcal{M}$ and the second order elliptic operators in the divergence form $\mathcal{L}_1$ and $\mathcal{L}_2$ as

$$\mathcal{L}_1 := \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} a_i(x, t)\frac{\partial}{\partial x_i} + a_0(x, t),$$

$$\mathcal{L}_2 := \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x, t)\frac{\partial}{\partial x_i} + b_0(x, t),$$

$$\mathcal{M} := \sum_{ij=1}^{n} a_{ij}(x, t)N_i(x)\frac{\partial}{\partial x_j},$$

where $N = \{N_1(x), ..., N_n(x)\}$ is the outward normal to $\partial\Omega$.

For $\nu \in (0, 1)$, we consider the nonlinear equation with the unknown function $v = v(x, t) : \Omega_T \to \mathbb{R}$,

$$\mathbf{D}_t^\nu v - \mathcal{L}_1 v - \mathcal{K} \star \mathcal{L}_2 v = f(x, t, v) + f_0(x, t) \quad \text{in} \quad \Omega_T, \tag{2.1}$$

supplemented with the initial condition

$$v(x,0) = v_0(x) \text{ in } \bar{\Omega}, \tag{2.2}$$

subject either to the Dirichlet boundary condition (**DBC**)

$$v(x,t) = 0 \text{ on } \partial\Omega_T, \tag{2.3}$$

or to the condition of the third kind (**III BC**)

$$\mathcal{M}v + \mathcal{K}_1 \star \mathcal{M}v + \sigma v = 0 \quad \text{ on } \partial\Omega_T, \tag{2.4}$$

where a positive number σ, the functions v_0, f_0, f and the memory kernels $\mathcal{K}$, $\mathcal{K}_1$ are assumed to be given. Here, $\star$ denotes the usual time-convolution product on $(0,t)$, namely

$$(\mathfrak{H}_1 \star \mathfrak{H}_2)(t) := \int_0^t \mathfrak{H}_1(t-s)\mathfrak{H}_2(s)ds, \; t > 0.$$

Finally, to state the inverse problem, we introduce the observation data $\psi(t)$ for small time $t \in [0, t^\star)$, $t^\star < \min(1, T)$,

$$v(x_0, t) = \psi(t), \tag{2.5}$$

where in the **DBC** case the point $x_0 \in \Omega$, and in the **III BC** case $x_0 \in \bar{\Omega}$.

Statement of the inverse problem: for the given functions f, f_0 ψ, v_0, $\mathcal{K}$, $\mathcal{K}_1$ and the given coefficients of the operators $\mathcal{L}_i$, $i = 1, 2$, and $\mathcal{M}$, the inverse problem (IP) is to look for the order $\nu \in (0,1)$, such that the solution $v = v_\nu(x,t)$ of the direct problem (2.1)-(2.4) satisfies observation data (2.5) for small time t.

We carry out our analysis of IP (2.1)-(2.5) in the framework of the fractional Hölder spaces. To this end, in what follows we take two arbitrary (but fixed) parameters

$$\alpha, \beta \in (0,1).$$

For any Banach space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$, we consider the usual spaces

$$\mathcal{C}([0,T], \mathbf{X}), \quad \mathcal{C}^\beta[0,T] \quad \text{and} \quad L^p(\Omega), \quad \text{for } p \in (1, +\infty).$$

Besides, denoting

$$\langle u \rangle^{(\beta)}_{x,\Omega_T} = \sup\Big\{ \frac{|u(x_1,t) - u(x_2,t)|}{|x_1 - x_2|^\beta} : x_1 \neq x_2;\, x_1, x_2 \in \bar{\Omega}, t \in [0,T] \Big\},$$

$$\langle u \rangle^{(\beta)}_{t,\Omega_T} = \sup\Big\{ \frac{|u(x,t_1) - u(x,t_2)|}{|t_1 - t_2|^\beta} : x \in \bar{\Omega}, t_1 \neq t_2; t_1, t_2 \in [0,T] \Big\},$$

we introduce the fractional Hölder spaces $\mathcal{C}^{l+\alpha, \frac{l+\alpha}{2}\beta}(\bar{\Omega}_T)$, according to the following definition: a function $u = u(x,t)$ belongs to the classes $\mathcal{C}^{l+\alpha, \frac{l+\alpha}{2}\beta}(\bar{\Omega}_T)$, for $l = 0, 1, 2$, if this function together with its corresponding derivatives is continuous and the norms below are finite

$$\|u\|_{\mathcal{C}^{l+\alpha, \frac{l+\alpha}{2}\beta}(\bar{\Omega}_T)} = \|u\|_{\mathcal{C}([0,T], \mathcal{C}^{l+\alpha}(\bar{\Omega}))} + \sum_{|j|=0}^{l} \langle D_x^j u \rangle^{(\frac{l+\alpha-|j|}{2}\beta)}_{t,\Omega_T}, \quad l = 0, 1,$$

$$\|u\|_{\mathcal{C}^{2+\alpha, \frac{2+\alpha}{2}\beta}(\bar{\Omega}_T)} = \|u\|_{\mathcal{C}([0,T], \mathcal{C}^{2+\alpha}(\bar{\Omega}))} + \|\mathbf{D}_t^\beta u\|_{\mathcal{C}^{\alpha, \frac{\alpha}{2}\beta}(\bar{\Omega}_T)} + \sum_{|j|=1}^{2} \langle D_x^j u \rangle^{(\frac{2+\alpha-|j|}{2}\beta)}_{t,\Omega_T}.$$

The main properties of these spaces have been described in Section 2 [17]. In a similar way, we introduce the space $\mathcal{C}^{l+\alpha, \frac{l+\alpha}{2}\beta}(\partial\Omega_T)$ for $l = 0, 1, 2$.

Moreover, we will use the Hilbert spaces $L^2_w(t_1,t_2)$ of real-valued functions that are square integrable with a positive weight $w(t)$ on (t_1,t_2). We remind that the inner product in the space $L^2_w(t_1,t_2)$ is defined as

$$\langle f,g\rangle_w := \int_{t_1}^{t_2} w(t)f(t)g(t)dt,$$

which induces the corresponding norm $\|f\|^2_{L^2_w(t_1,t_2)} = \langle f,f\rangle_w$.

3. The Main Result

First, we state our general hypotheses on the structural terms in the model (2.1)-(2.5).

H1 (Ellipticity conditions): There are positive constants μ_1, μ_2, $\mu_1<\mu_2$, such that for any $(x,t,\xi)\in\bar{\Omega}_T\times\mathbb{R}^n$

$$\mu_1|\xi|^2 \le \sum_{ij=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le \mu_2|\xi|^2.$$

H2 (Conditions on the coefficients): For $i,j=1,\dots,n$, $\alpha\in(0,1)$,

$$a_{ij}(x,t), a_i(x,t), b_i(x,t) \in \mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}(\bar{\Omega}_T),$$
$$a_0(x,t), b_0(x,t) \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T).$$

H3 (Conditions on the given functions): The following inclusions hold with $\alpha,\beta\in(0,1)$

$$\mathcal{K}_1(t), \mathcal{K}(t) \in L^1(0,T)$$
$$v_0(x) \in C^{2+\alpha}(\bar{\Omega}),$$
$$f_0(x,t) \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T),$$
$$\psi(t)\in\mathcal{C}^{\beta}[0,t^\star],\ \mathbf{D}_t^{\beta}\psi(t)\in\mathcal{C}^{\frac{\alpha\beta}{2}}[0,t^\star].$$

H4 (Compatibility conditions): Considering **DBC** (2.3), we assume that the following compatibility conditions hold at the initial time $t=0$ for every $x\in\partial\Omega$

$$v_0(x)=0 \quad \text{and} \quad 0=\mathcal{L}_1 v_0(x)\big|_{t=0}+f(x,0,v_0)+f_0(x,0),$$

while in the case of **III BC** (2.4) it is assumed that

$$\mathcal{M}v_0(x)\big|_{t=0}+\sigma v_0=0.$$

H5 (Conditions on the nonlinearity): There exists a positive constant L such that for $(x_i,t_i,v_i)\in\bar{\Omega}_T\times\mathbb{R}$

$$|f(x_1,t_1,v_1)-f(x_2,t_2,v_2)| \le L(|x_1-x_2|+|t_1-t_2|+|v_1-v_2|),$$

and

$$|f(x,t,0)|\le L$$

for $(x,t)\in\bar{\Omega}_T$.

We are now in the position to state our main result.

Theorem 3.1. *Let $T>0$ be arbitrary fixed and the assumptions **H1-H5** hold. We assume that*

$$\mathcal{L}_1 v_0(x_0)|_{t=0} + f(x_0,0,v_0(x_0)) + f_0(x_0,0) \neq 0. \tag{3.1}$$

Then the pair $(\nu, v_\nu(x,t))$ solves (2.1)-(2.5)*, where ν is defined as*

$$\nu = \lim_{t\to 0} \frac{t[\psi(t)-v_0(x_0)]}{\int\limits_0^t [\psi(\tau)-v_0(x_0)]d\tau} - 1, \tag{3.2}$$

and $v_\nu(x,t)$ is a unique solution of direct problem (2.1)-(2.4) *satisfying regularity condition $v_\nu \in \mathcal{C}^{2+\alpha,\frac{\alpha+2}{2}\nu}(\bar{\Omega}_T)$.*

Remark 3.1. *Actually, a slight modification of the proof below allows the same results to be obtained for* (2.1)-(2.5) *with inhomogeneous boundary conditions:*

$$\begin{aligned} &v(x,t) = \varphi_1(x,t) \quad \text{on} \quad \partial\Omega_T, \\ &\mathcal{M}v + \mathcal{K}_1 \star \mathcal{M}v + \sigma v = \varphi_2(x,t) \quad \text{on} \quad \partial\Omega_T, \end{aligned}$$

if $\varphi_1 \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}}(\partial\Omega_T)$, $\varphi_2 \in \mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}(\partial\Omega_T)$ and the corresponding compatibility conditions hold.

4. Proof of the Main Results

4.1. **Solvability of problem (2.1)-(2.5), explicit formula for ν.** We start our analysis with considering properties of the pair $(\nu, v_\nu(x,t))$ that solves the problem (2.1)-(2.5). The first step on this route is related with obtaining formula (3.2).

Lemma 4.1. *Let T be arbitrarily positive fixed and $\beta\in(0,1)$. Let a function $\Phi(t)$ and its fractional derivative $\mathbf{D}_t^\beta\Phi(t)$ be continuous on $[0,T]$. Then for each $t\in[0,T]$ the function $\Phi(t)$ allows the following representation*

$$\Phi(t) = \Phi(0) + t^\beta \frac{\mathbf{D}_t^\beta \Phi(0)}{\Gamma(1+\beta)} + \frac{1}{\Gamma(\beta)} \int\limits_0^t (t-\tau)^{\beta-1}[\mathbf{D}_\tau^\beta\Phi(\tau) - \mathbf{D}_t^\beta\Phi(0)]d\tau. \tag{4.1}$$

Assume in addition that $\mathbf{D}_t^\beta\Phi(0)\neq 0$ and

$$\omega(t) := \sup_{\tau\in[0,t]} |\mathbf{D}_\tau^\beta\Phi(\tau) - \mathbf{D}_t^\beta\Phi(0)| \to 0 \quad \text{as} \quad t\to 0,$$

then

$$\beta + 1 = \lim_{t\to 0} \frac{t[\Phi(t)-\Phi(0)]}{\int\limits_0^t [\Phi(\tau)-\Phi(0)]d\tau}. \tag{4.2}$$

Proof. The straightforward calculations based on the smoothness of the function $\Phi(t)$ provide representation (4.1). Further, equality (4.1) ensures the relations

$$\Phi(t)-\Phi(0) = t^\beta \Big[\frac{\mathbf{D}_t^\beta\Phi(0)}{\Gamma(1+\beta)} + \frac{t^{-\beta}}{\Gamma(\beta)} \int\limits_0^t (t-\tau)^{\beta-1}[\mathbf{D}_\tau^\beta\Phi(\tau) - \mathbf{D}_t^\beta\Phi(0)]d\tau\Big],$$

$$\int\limits_0^t [\Phi(\tau)-\Phi(0)]d\tau = t^{\beta+1}\Big[\frac{\mathbf{D}_t^\beta\Phi(0)}{\Gamma(2+\beta)} + \frac{t^{-1-\beta}}{\Gamma(\beta)} \int\limits_0^t d\tau \int\limits_0^\tau (\tau-s)^{\beta-1}[\mathbf{D}_s^\beta\Phi(s) - \mathbf{D}_t^\beta\Phi(0)]ds\Big],$$

where

$$\left|\frac{t^{-\beta}}{\Gamma(\beta)}\int\limits_0^t (t-\tau)^{\beta-1}[\mathbf{D}_t^{\beta}\Phi(\tau)-\mathbf{D}_t^{\beta}\Phi(0)]d\tau\right| \leq \frac{\omega(t)}{\Gamma(1+\beta)} \underset{t\to 0}{\rightarrow} 0,$$

$$\left|\frac{t^{-1-\beta}}{\Gamma(\beta)}\int\limits_0^t d\tau\int\limits_0^{\tau} (\tau-s)^{\beta-1}[\mathbf{D}_s^{\beta}\Phi(s)-\mathbf{D}_t^{\beta}\Phi(0)]ds\Big]\right| \leq \frac{\omega(t)}{\Gamma(2+\beta)} \underset{t\to 0}{\rightarrow} 0.$$

Taking into account these inequalities and the fact that $\mathbf{D}_t^{\beta}\Phi(0)\neq 0$, we arrive at (4.2)

$$\lim_{t\to 0}\frac{t[\Phi(t)-\Phi(0)]}{\int_0^t[\Phi(\tau)-\Phi(0)]d\tau} = \frac{\frac{\mathbf{D}_t^{\beta}\Phi(0)}{\Gamma(1+\beta)}}{\frac{\mathbf{D}_t^{\beta}\Phi(0)}{\Gamma(2+\beta)}} = \beta+1,$$

that completes the proof of the lemma. □

Note that the right-hand side of (4.2) exists and is bounded under weaker conditions on the function $\Phi(t)$.

Remark 4.1. *It is apparent that if the function $\Phi(t)\in\mathcal{C}^{\beta}[0,T]$, $\beta\in(0,1)$, then*

$$\lim_{t\to 0}\frac{t[\Phi(t)-\Phi(0)]}{\int_0^t[\Phi(\tau)-\Phi(0)]d\tau} \leq 1+\beta.$$

Returning to the observation $\psi(t)$ and taking into account the condition **H3**, one can easily obtain

$$\sup_{\tau\in[0,t]}|\mathbf{D}_t^{\beta}\psi(\tau)-\mathbf{D}_t^{\beta}\psi(0)| \leq t^{\frac{\alpha\beta}{2}}\langle\mathbf{D}_t^{\beta}\psi\rangle_{t,[0,T]}^{(\frac{\alpha\beta}{2})} \underset{t\to 0}{\rightarrow} 0.$$

Thus, we can apply Lemma 4.1 to the measurements $\psi(t)$ and conclude that

$$\beta = \lim_{t\to 0}\frac{t[\psi(t)-\psi(0)]}{\int_0^t[\psi(\tau)-\psi(0)]d\tau} - 1. \tag{4.3}$$

Then, the following result is a direct consequence of Theorem 6.2 [17] and condition (2.5).

Proposition 4.1. *Let $\nu\in(0,1)$, and let the assumptions of Theorem 3.1 hold. If the pair (ν, v_{ν}) satisfies (2.1)-(2.5), then*

$$\psi(t) = v_{\nu}(x_0,t)\in\mathcal{C}^{\nu}[0,t^{\star}],$$

$$\mathbf{D}_t^{\nu}\psi(t) = \mathbf{D}_t^{\nu}v_{\nu}(x_0,t)\in\mathcal{C}^{\frac{\alpha\nu}{2}}[0,t^{\star}],$$

$$\mathbf{D}_t^{\nu}\psi(0) = \mathcal{L}_1 v_0(x_0)|_{t=0} + f(x_0,0,v_0(x_0)) + f_0(x_0,0)\neq 0.$$

After that, relation (4.3) together with Proposition 4.1 and Theorem 6.2 from [17] ensure the existence of the pair (ν, v_{ν}) which solves (2.1)-(2.5), where

$$\nu = \beta = \lim_{t\to 0}\frac{t[\psi(t)-v_0(x_0)]}{\int_0^t[\psi(\tau)-v_0(x_0)]d\tau} - 1.$$

4.2. **Uniqueness of the solution** (ν, v_ν) **of problem** (2.1)**-**(2.5)**.** To finish the proof of Theorem 3.1, we are left to check the uniqueness of the pair (ν, v_ν) which satisfies relations (2.1)-(2.5).

Lemma 4.2. *Under the assumptions **H1-H5** and restriction* (3.1)*, the order* ν *can be identified uniquely by the measurements* $\psi(t)$ *using formula* (3.2)*, and there exists a unique pair* (ν, v_ν) *satisfying relations* (2.1)*-*(2.5)*.*

Proof. For the same observation data (2.5), the same right-hand sides and coefficients in model (2.1)-(2.4), we suppose the existence of two solutions (ν_1, v_{ν_1}) and (ν_2, v_{ν_2}) of problem (2.1)-(2.5), where the quantities ν_1 and ν_2 defined with formula (3.2). As for the functions v_{ν_1} and v_{ν_2}, they solve the corresponding direct problems (2.1)-(2.4).

For simplicity of consideration, we assume that $0 < \nu_1 < \nu_2 < 1$. Proposition 4.1 and restriction (3.1) ensure

$$\begin{gathered}\psi(t) \in \mathcal{C}^{\nu_2}[0, t^*], \quad \mathbf{D}_t^{\nu_i}\psi(t) \in \mathcal{C}^{\alpha\nu_i/2}[0, t^*], \\ \psi(0) = v_0(x_0) \quad \text{and} \quad \mathbf{D}_t^{\nu_i}\psi(0) \neq 0. \end{gathered} \tag{4.4}$$

Following Lemma 4.1, we can represent the function $\psi(t)$ as

$$\psi(t) = v_0(x_0) + \frac{t^{\nu_1}}{\Gamma(1+\nu_1)}\mathbf{D}_t^{\nu_1}\psi(0) + \frac{1}{\Gamma(\nu_1)}\int_0^t \frac{[\mathbf{D}_\tau^{\nu_1}\psi(\tau) - \mathbf{D}_\tau^{\nu_1}\psi(0)]}{(t-\tau)^{1-\nu_1}}d\tau,$$

$$\psi(t) = v_0(x_0) + \frac{t^{\nu_2}}{\Gamma(1+\nu_2)}\mathbf{D}_t^{\nu_2}\psi(0) + \frac{1}{\Gamma(\nu_2)}\int_0^t \frac{[\mathbf{D}_\tau^{\nu_2}\psi(\tau) - \mathbf{D}_\tau^{\nu_2}\psi(0)]}{(t-\tau)^{1-\nu_2}}d\tau.$$

For sufficiently small t these relations together with properties (4.4) of the function $\psi(t)$ provide the equality

$$\frac{\mathbf{D}_t^{\nu_1}\psi(0)}{\Gamma(1+\nu_1)} + O(t^{\alpha\nu_1/2}) = \frac{t^{\nu_2-\nu_1}}{\Gamma(1+\nu_2)}\mathbf{D}_t^{\nu_2}\psi(0) + O(t^{\nu_2-\nu_1+\alpha\nu_1/2}).$$

Then, passing to the limit in the last equality as $t \to 0$, we conclude that

$$\frac{\mathbf{D}_t^{\nu_1}\psi(0)}{\Gamma(1+\nu_1)} = 0,$$

that contradicts relation (4.4) above.

This contradiction is resolved if we admit that $\nu_1 = \nu_2$. Then, Theorem 6.2 in [17] guarantees the equality $v_{\nu_1} = v_{\nu_2}$, that finishes the proof. $\square$

Summing up, we conclude that the arguments of Subsections 4.1 and 4.2 completes the proof of Theorem 3.1. It is worth mentioning, the results of Theorem 3.1 can be extended to equations in non-divergent form. To this end, it is enough to recast the arguments above and apply the results from [16] and [18].

Moreover, the condition on the nonlinearity f can be relaxed if the kernel $\mathcal{K}$ fulfills a stronger requirement.

Remark 4.2. *Let* $\mathcal{K} \in \mathcal{C}^1[0, T]$ *and for every* $\varrho > 0$ *and for any* $(x_i, t_i, v_i) \in \bar{\Omega}_T \times [-\varrho, \varrho]$*, there exists a constant* $C_\varrho > 0$ *such that*

$$|f(x_1, t_1, v_1) - f(x_2, t_2, v_2)| \leq C_\varrho\big(|x_1 - x_2| + |t_1 - t_2| + |v_1 - v_2|\big).$$

Moreover, there is a constant $L > 0$ such that the inequality

$$|f(x,t,v)| \leq L[1+|v|]$$

*holds for any $(x,t,v) \in \bar{\Omega}_T \times \mathbb{R}$. Then, under the assumptions **H1-H3**, the results of Theorem 3.1 hold.*

5. Influence of noisy data on the calculation of the order ν

In this section we analyze formula (3.2) where the observation data $\psi(t)$ is replaced by noisy data $\psi_\delta(t)$.

We suppose that

$$|\psi(t) - \psi_\delta(t)| \leq \delta \Xi(t), \quad 0 \leq t \leq t^\star, \tag{5.1}$$

where $\Xi(t)$ is a nonnegative function and δ denotes the noise level. Since initial condition (2.2) is supposed to be known, it is natural to assume that $\psi(0) = \psi_\delta(0) = v_0(x_0)$, and therefore $\Xi(0) = 0$.

In the paper, we analyze three noise models corresponding to different behavior patterns of $\Xi(t)$ as $t \to 0$. The first-type noise (**FTN**) model corresponds to the case of

$$\Xi(t) = o(t^\nu) \quad \text{as} \quad t \to 0, \tag{5.2}$$

while the second type noise (**STN**) model is characterized by

$$\Xi(t) = O(t^\nu) \quad \text{as} \quad t \to 0, \tag{5.3}$$

and finally, the third type noise (**TTN**) is identified as

$$\Xi(t) t^{-\nu} \to \infty \quad \text{if} \quad t \to 0. \tag{5.4}$$

Note that condition (5.3) can be rewritten as

$$\Xi(t) = C_1 t^\nu + o(t^\nu) \tag{5.5}$$

where C_1 is some positive constant.

Remark 5.1. *It is easy to examine that the function*

$$\Xi(t) = \begin{cases} C_1 t|\ln t| & \text{in } \textbf{\textit{FTN}} \text{ case}, \\ C_1 t^\nu & \text{in } \textbf{\textit{STN}} \text{ case}, \\ C_2 + C_3 t^\nu |\ln t| & \text{in } \textbf{\textit{TTN}} \text{ case} \end{cases}$$

with a positive constant C_1 and nonnegative C_2 and C_3, $C_2 + C_3 > 0$, satisfies conditions (5.2)-(5.5).

Note that the considered noise models (5.1)-(5.4) are similar to ones proposed in Section 4 [12].

In the sequel we will deal with

$$C_0 := \mathcal{L}_1 v_0(x_0)|_{t=0} + f(x_0, 0, v_0(x_0)) + f_0(x_0, 0),$$

$$\nu_\delta = \lim_{t\to 0} \frac{t[\psi_\delta(t) - v_0(x_0)]}{\int\limits_0^t [\psi_\delta(\tau) - v_0(x_0)]d\tau} - 1.$$

Lemma 5.1. *Let the assumptions of Theorems 3.1 hold, and let relations* (5.1)-(5.5) *be satisfied with* $\delta \in (0,1)$ *and* C_1 *such that*

$$\frac{C_1}{||C_0| - C_1\delta|} < 1.$$

Then there are the following absolute error estimates:

$$|\nu - \nu_\delta| = 0 \quad \textit{in } \boldsymbol{FTN} \textit{ case,}$$

and

$$|\nu - \nu_\delta| \leq \frac{4C_1\delta}{||C_0| - C_1\delta|} \quad \textit{in } \boldsymbol{STN} \textit{ case,}$$

while in the ***TTN*** *case we have*

$$|\nu - \nu_\delta| \leq 1 + \nu + \lim_{t\to 0} \frac{t\Xi(t)}{\int\limits_0^t \Xi(\tau)d\tau}.$$

Proof. Let us denote

$$\mathcal{I}(\psi) = \int\limits_0^t [\psi(\tau) - v_0(x_0)]d\tau$$

and consider the difference

$$\begin{aligned}
\Delta :&= \frac{t[\psi(t) - v_0(x_0)]}{\mathcal{I}(\psi)} - \frac{t[\psi_\delta(t) - v_0(x_0)]}{\mathcal{I}(\psi_\delta)} \\
&= \frac{t[\psi(t) - v_0(x_0)][\mathcal{I}(\psi_\delta) - \mathcal{I}(\psi)]}{\mathcal{I}(\psi)\mathcal{I}(\psi_\delta)} + \frac{t[\psi(t) - \psi_\delta(t)]}{\mathcal{I}(\psi_\delta)} \\
&\equiv \Delta_1 + \Delta_2.
\end{aligned} \tag{5.6}$$

Then we evaluate separately each term Δ_i in the right-hand side of (5.6).

Based on representation (4.1) and estimate (5.1), we obtain the following relations for sufficiently small time $t \in [0, t^\star]$:

$$\begin{aligned}
\psi(t) - v_0(x_0) &= \frac{t^\nu \mathbf{D}_t^\nu \psi(0)}{\Gamma(1+\nu)} + O(t^{\nu + \alpha\nu/2}), \\
\mathcal{I}(\psi) &= \frac{t^{\nu+1} \mathbf{D}_t^\nu \psi(0)}{\Gamma(2+\nu)} + O(t^{\nu+1+\alpha\nu/2}), \\
|\mathcal{I}(\psi_\delta)| &\geq \bigg| \Big| \frac{t^{\nu+1} \mathbf{D}_t^\nu \psi(0)}{\Gamma(2+\nu)} + O(t^{\nu+1+\alpha\nu/2}) \Big| - \delta \int\limits_0^t \Xi(\tau)d\tau \bigg|,
\end{aligned} \tag{5.7}$$

telling us, that the following inequality holds

$$|\Delta_1| \leq \frac{\Big| \frac{\mathbf{D}_t^\nu \psi(0)}{\Gamma(1+\nu)} + O(t^{\alpha\nu/2}) \Big| \delta \int\limits_0^t \Xi(\tau)d\tau}{\Big| \frac{\mathbf{D}_t^\nu \psi(0)}{\Gamma(2+\nu)} + O(t^{\alpha\nu/2}) \Big| \bigg| \Big| \frac{t^{\nu+1}\mathbf{D}_t^\nu \psi(0)}{\Gamma(2+\nu)} + O(t^{\nu+1+\alpha\nu/2}) \Big| - \delta \int\limits_0^t \Xi(\tau)d\tau \bigg|}, \quad t \in [0, t^\star].$$

After that assumptions (5.2)-(5.5) together with direct calculations provide estimates

$$\lim_{t\to 0}|\Delta_1| \leq \begin{cases} 0 & \text{in } \mathbf{FTN} \text{ case}, \\ \frac{2\delta C_1}{||\mathbf{D}_t^\nu \psi(0)|-\delta C_1|} & \text{in } \mathbf{STN} \text{ case}, \\ 1+\nu & \text{in } \mathbf{TTN} \text{ case}. \end{cases} \tag{5.8}$$

Concerning the difference Δ_2, we apply again (5.1)-(5.5) and (5.7) and arrive at the inequality

$$\lim_{t\to 0}|\Delta_2| \leq \lim_{t\to 0} \frac{t\delta\Xi(t)}{\left|\left|\frac{t^{\nu+1}\mathbf{D}_t^\nu\psi(0)}{\Gamma(2+\nu)} + O(t^{\nu+1+\alpha\nu/2})\right| - \delta\int\limits_0^t \Xi(\tau)d\tau\right|}$$

$$\leq \begin{cases} 0 & \text{in } \mathbf{FTN} \text{ case}, \\ \frac{2\delta C_1}{||\mathbf{D}_t^\nu \psi(0)|-\delta C_1|} & \text{in } \mathbf{STN} \text{ case}, \\ \lim\limits_{t\to 0} \frac{t\Xi(t)}{\int\limits_0^t \Xi(\tau)d\tau} & \text{in } \mathbf{TTN} \text{ case}. \end{cases}$$

Combining these estimates with inequality (5.8) and representation (5.6), we obtain

$$|\nu-\nu_\delta| \leq \begin{cases} 0 & \text{in } \mathbf{FTN} \text{ case}, \\ \frac{4\delta C_1}{||\mathbf{D}_t^\nu \psi(0)|-\delta C_1|} & \text{in } \mathbf{STN} \text{ case}, \\ 1+\nu+\lim\limits_{t\to 0} \frac{t\Xi(t)}{\int\limits_0^t \Xi(\tau)d\tau} & \text{in } \mathbf{TTN} \text{ case}. \end{cases}$$

Finally, taking into account Theorem 3.1 and Proposition 4.1, we conclude that $\mathbf{D}_t^\nu\psi(0) = C_0$, and complete the proof of this lemma. □

It is worth mentioning, that due to $C_0 \neq 0$, we have

$$|\nu-\nu_\delta| \leq O(\delta)$$

in the **STN** case. Moreover, in the case $\Xi(t) = C_2 + C_3 t^\nu |\ln t|$, the quantity $\lim\limits_{t\to 0} \frac{t\Xi(t)}{\int_0^t \Xi(\tau)d\tau}$ can be calculated explicitly

$$\lim_{t\to 0} \frac{t\Xi(t)}{\int\limits_0^t \Xi(\tau)d\tau} = \lim_{t\to 0} \frac{C_2 + C_3 t^\nu |\ln t|}{C_2 + \frac{C_3 t^\nu}{(\nu+1)^2} + \frac{C_3 t^\nu |\ln t|}{\nu+1}}$$

$$= \begin{cases} 1 & \text{if} \quad C_2 \neq 0, \\ 1+\nu & \text{if} \quad C_2 = 0. \end{cases}$$

Thus, taking into account this relation and the estimate of $|\nu-\nu_\delta|$ in Lemma 5.1 for **TTN** case, we obtain the inequality

$$|\nu-\nu_\delta| \leq \begin{cases} 2+\nu & \text{for} \quad \Xi(t) = C_2 + C_3 t^\nu |\ln t|, \\ 2+2\nu & \text{for} \quad \Xi(t) = C_3 t^\nu |\ln t|. \end{cases}$$

6. Regularized reconstruction of the order ν

In this section, we discuss a regularization of the reconstruction formula (3.2). It is worth mentioning, that this formula has been obtained assuming enough smoothness of the measurements $\psi(t)$, while in practice only noisy discrete observation data are mostly available. Therefore, the obtained formula should be regularized to deal with such data. Here we propose an approach which combines formula (3.2) with a regularization procedure.

We remark that the availability of only discrete observations is more problematic issue than the presence of noise in continuous data $\psi_\delta(t)$, because in virtue of Lemma 5.1, such continuous noisy data in principle allow us to evaluate the order ν rather accurately.

6.1. **Algorithm of reconstruction.** Let us assume that we afford to observe the solution $v(x,t)$ of (2.1)-(2.5) at the point $x = x_0 \in \bar{\Omega}$ and at time moments t_k, $k = 1, 2, ..., N$, $0 < t_1 < t_2 < ... < t_N \leq t^\star$. However, these measurements are spoiled by an additive noise so, that what we really have is

$$\psi_{\delta,k} = v(x_0, t_k) + \delta_k, \quad k = 1, 2, ..., N.$$

Moreover, initial condition (2.2) allows us to know the value $\psi_0 = v(x_0, 0) = v_0(x_0)$.

In order to use such discrete noisy data in formula (3.2), a reconstruction algorithm should at first approximately recover the function $\psi(t) = v(x_0, t)$ from the values $\psi_{\delta,k}$, $k = 0, 1, ..., N$, where with a bit abuse of symbols we set

$$\psi_{\delta,0} = \psi_0 = v_0(x_0).$$

It is important to note that Lemma 4.1 and Proposition 4.1 provide the following asymptotic for $t \leq t^\star$:

$$\psi(t) = v_0(x_0) + O(t^\nu),$$

telling us that our desired function should be square integrable on $(0, t_N)$, $t_N \leq t^\star$, with unbounded weight $w(t) = t^{-\gamma}$, $\gamma \in (0,1)$. Hence, the elements of the space $L^2_{t^{-\gamma}}(0, t_N)$ are admissible approximations for the function $\psi(t)$.

According to the Tikhonov regularization scheme [29], the above mentioned approximate recovery of $\psi(t)$ from noisy values $\{\psi_{\delta,k}\}_{k=0}^N$ can be performed by minimizing a penalized least squares functional

$$\sum_{k=0}^{N} (\psi(t_k) - \psi_{\delta,k})^2 + \lambda \|\psi\|^2_{L^2_{t^{-\gamma}}(0,t_N)} \to \min, \tag{6.1}$$

where λ is a regularization parameter.

Next important issue is related with the choice of a suitable basis in $L^2_{t^{-\gamma}}(0, t_N)$. As well known, the Jacobi polynomials (see, e.g. [1, 27])

$$\mathcal{P}_m^{(0,-\gamma)}(t/t_N) = \sum_{m=0}^{n} p_{n,m} (2t/t_N - 2)^{n-m} (2t/t_N)^m, \quad t \in (0, t_N),$$

where

$$p_{n,m} = \frac{1}{2^n} \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} n-\gamma \\ n-m \end{pmatrix},$$

constitute an orthogonal system in $L^2_{t^{-\gamma}}(0, t_N)$. Thus, it is natural to search the minimizer of (6.1) as a linear combination of $\mathcal{P}_m^{(0,-\gamma)}(t/t_N)$. Moreover, having a series of initial guesses $\nu_1, \nu_2, ..., \nu_{\mathcal{J}}$ for the value of ν, one may incorporate the functions t^{ν_j}, $j=1,2,...,\mathcal{J}$, into the basis in which minimization problem (6.1) has to be solved.

Then an approximate minimizer of (6.1) can be represented as

$$\psi_{\delta,\lambda}(t)=\sum_{j=1}^{\mathcal{J}} c_j t^{\nu_j}+\sum_{j=\mathcal{J}+1}^{P} c_j \mathcal{P}^{(0,-\gamma)}_{j-\mathcal{J}-1}(t/t_N), \tag{6.2}$$

where the coefficients c_j solve the corresponding system of linear algebraic equations written in the matrix form as follows:

$$(B^T B+\lambda\mathcal{E})\vec{c}=B^T\vec{\psi}_\delta,$$

where we put
$\vec{c}=(c_1,c_2,...,c_P)^T$, $\vec{\psi}_\delta=(\psi_{\delta,0},\psi_{\delta,1},...,\psi_{\delta,N})^T$, $B=\{B_{i,j}\}_{i=0,j=1}^{N\ \ P}$, $B_{ij}=e_j(t_i)$, $\mathcal{E}=\{\mathcal{E}\}_{l,m=1}^{P}$, $\mathcal{E}_{l,m}=\int\limits_0^{t_N} t^{-\gamma}e_l(t)e_m(t)dt$, and

$$e_l(t)=\begin{cases} t^{\nu_l}, & l=1,2,...,\mathcal{J},\\ \mathcal{P}^{(0,-\gamma)}_{l-\mathcal{J}-1}(t/t_N), & l=1+\mathcal{J},...,P.\end{cases}$$

To calculate the elements $\mathcal{E}_{l,m}$ and the quantity $\int\limits_0^{\tilde t}[\psi_{\delta,\lambda}(\tau)-\psi_0]d\tau$ below, we apply the following relations

$$\int\limits_0^{t_N}\mathcal{P}^{(0,-\gamma)}_n(t/t_N)\mathcal{P}^{(0,-\gamma)}_k(t/t_N)t^{-\gamma}dt=\begin{cases}\frac{t_N^{1-\gamma}}{2n-\gamma+1}, & n=k,\\ \\ 0, & n\neq k,\end{cases}$$

$$\int\limits_0^{t_N} t^{-\gamma}\mathcal{P}^{(0,-\gamma)}_n(t/t_N)t^{\bar\nu}dt=\frac{t_N^{\bar\nu-\gamma+1}}{\bar\nu-\gamma+1}\,{}_2F_1(-n,-\gamma+n+1;2+\bar\nu-\gamma;1),$$

where $\bar\nu\in(0,1)$, and ${}_2F_1(\cdot)$ is the hypergeometric function [1].

It is worth mentioning that even after the approximate reconstruction of measurements $\psi(t)$ in the form of $\psi_{\delta,\lambda}(t)$, the problem of calculating the limit in reconstruction formula (3.2) is an ill-posed and needs to be regularized.

As an approximate value of that limit one can chose the quantity

$$\nu_\delta(\lambda,\tilde t)=\frac{\tilde t[\psi_{\delta,\lambda}(\tilde t)-\psi_0]}{\int\limits_0^{\tilde t}[\psi_{\delta,\lambda}(\tau)-\psi_0]d\tau}-1 \tag{6.3}$$

computed at a point $t=\tilde t$ which is located sufficiently close to zero and playing the role of a regularization parameter.

Thus, the regularized approximate value $\nu_\delta(\lambda,\tilde t)$ of the order ν suggested by the proposed algorithm depends on two regularization parameters λ and $\tilde t$ that have to be properly selected. Since in practice the amplitudes δ_k of the noise perturbations are usually unknown, one should rely on the so-called noise level-free regularization parameter choice rules.

We recall that the quasi-optimality criterion [29] is one of the simplest and the oldest, however still quite efficient techniques among such strategies. Its version for the choice of multiple regularization parameters, such as λ and $\tilde{t}$, has been discussed in [9, 21].

In order to apply the quasi-optimality criterion in the present context one ought to consider two geometric sequences of regularization parameters values

$$\lambda = \lambda_i = \lambda_1 q_1^{i-1}, \quad i = 1, 2, ..., M_1,$$

$$\tilde{t} = \tilde{t}_j = \tilde{t}_1 q_2^{j-1}, \quad j = 1, 2, ..., M_2,$$

$$0 < q_1, \ q_2 < 1,$$

and the values $\nu_\delta(\lambda_i, \tilde{t}_j)$ should be calculated for all considered i, j.
Then, for each $\tilde{t}_j$ one needs to find $\lambda_{i_j} \in \{\lambda_i\}_{i=1}^{M_1}$ such that

$$\left|\nu_\delta(\lambda_{i_j}, \tilde{t}_j) - \nu_\delta(\lambda_{i_j-1}, \tilde{t}_j)\right| = \min\{|\nu_\delta(\lambda_i, \tilde{t}_j) - \nu_\delta(\lambda_{i-1}, \tilde{t}_j)|, \quad i = 2, 3, ..., M_1\}.$$

Next, $\tilde{t}_{j_0}$ is taken from $\{\tilde{t}_j\}_{j=1}^{M_2}$ such that

$$\left|\nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0}) - \nu_\delta(\lambda_{i_{j_0-1}}, \tilde{t}_{j_0-1})\right| = \min\{|\nu_\delta(\lambda_{i_j}, \tilde{t}_j) - \nu_\delta(\lambda_{i_{j-1}}, \tilde{t}_{j-1})|, \ j = 2, 3, ..., M_2\}.$$

Finally, the value $\nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0})$ is selected as the output of the proposed algorithm.

In the next subsection we illustrate the performance of the algorithm calculating $\nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0})$ by a series of numerical tests. In particular, we compare the proposed approximations $\nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0})$ with the ones constructed according to [19], where multiple quasi-optimality criterion described above has been applied to the sequence

$$\nu_\delta^{\ln}(\lambda_i, \tilde{t}_j) = \frac{\ln |\psi_{\delta,\lambda_i}(\tilde{t}_j) - \psi_0|}{\ln \tilde{t}_j} \tag{6.4}$$

defined by employing formula (1.2) instead of (6.3).

6.2. **Numerical experiments.** Three different numerical tests corresponding to the final moment $T = 0.1$ and the time to measurements $t^\star = 0.003$ are examined below to test the proposed algorithm.

At first, we consider problem (2.1)-(2.5) in the one-dimensional domain $\Omega := (0, L)$:

$$\begin{cases} \mathbf{D}_t^\nu v - a(x,t)v_{xx} + \tilde{a}(x,t)v_x - \int\limits_0^t \mathcal{K}(t-s)b(x,s)v_{xx}(x,s)ds \\ = f(x,t,v) + f_0(x,t) \qquad \text{in} \quad (0,L)\times(0,T), \\ v(x,0) = v_0(x), \qquad x \in [0,L], \\ v_x(0,t) = v_x(L,t) = 0, \qquad t \in [0,T]. \end{cases} \tag{6.5}$$

In general, it is problematic to find the solution of (6.5) in the analytical form. Therefore, to generate the synthetic test data, we have to solve problem (6.5) numerically using the computational scheme described briefly below.

Introducing the space-time mesh with nodes

$$x_k = kh, \quad \tau_j = j\tau, \quad k = 0, 1, \dots, \tilde{N}, \quad j = 0, 1, \dots, \tilde{M}, \quad h = L/\tilde{N}, \quad \tau = T/\tilde{M},$$

and approximating the differential equation from (6.5) at each level τ_{j+1}, we derive the following finite-difference scheme:

$$\tau^{-\nu}\sum_{m=0}^{j+1}(v_k^{j+1-m}-v_0(x_k))\rho_m-\frac{a_k^{j+1}}{h^2}(v_{k-1}^{j+1}-2v_k^{j+1}+v_{k+1}^{j+1})+\frac{\tilde{a}_k^{j+1}}{2h}(v_{k+1}^{j+1}-v_{k-1}^{j+1})$$

$$=\sum_{m=0}^{j}\left(b_k^m\frac{v_{k-1}^m-2v_k^m+v_{k+1}^m}{h^2}+b_k^{m+1}\frac{v_{k-1}^{m+1}-2v_k^{m+1}+v_{k+1}^{m+1}}{h^2}\right)\frac{\tilde{\mathcal{K}}_{m,j}}{2} \tag{6.6}$$

$$+f(x_k,\tau_j,v_k^j)+f_0(x_k,\tau_{j+1}),\quad k=1,\dots,\tilde{N}-1,\quad j=0,1,\dots,\tilde{M}-1,$$

where we denote the finite-difference approximation of the function v at the point (x_k,τ_j) by v_k^j and put

$$a_k^{j+1}=a(x_k,\tau_{j+1}),\quad \tilde{a}_k^{j+1}=\tilde{a}(x_k,\tau_{j+1}),\quad b_k^j=b(x_k,\tau_j),$$
$$\rho_m=(-1)^m\binom{\nu}{m},\quad \tilde{\mathcal{K}}_{m,j}=\int_{\tau_m}^{\tau_{m+1}}\mathcal{K}(\tau_{j+1}-s)ds.$$

Here we use the second-order finite-difference formulas to approximate the derivatives v_x and v_{xx} and Grünwald-Letnikov formula [6, 15, 30] to approximate the derivative $\mathbf{D}_t^\nu v$. Moreover, the trapezoid-rule is used to approximate the integrals in the sum

$$\sum_{m=0}^{j}\int_{\tau_m}^{\tau_{m+1}}\mathcal{K}(\tau_{j+1}-s)b(x,s)v_{xx}(x,s)ds.$$

It is worth noting that an improvement in the accuracy of calculations can be achieved by Richardson extrapolation and finite element method with mass lumping (see [6], [25]). We also use two fictitious mesh points outside the spatial domain to approximate the Neumann boundary conditions with the second order of accuracy (see, e.g. [11]).

In all our tests the noisy measurements are simulated according to (5.1), i.e.,

$$\psi_{\delta,k}=v(x_0,t_k)+\delta\Xi(t_k),\quad k=1,2,...,21,$$

and we consider three different types of the function Ξ mentioned in Remark 5.1

$$\textbf{N1:}\quad \Xi(t)=C_1t|\ln t|,$$
$$\textbf{N2:}\quad \Xi(t)=C_1t^\nu,$$
$$\textbf{N3:}\quad \Xi(t)=C_1t^\nu|\ln t|.$$

In our numeric experiments we test noise levels $C_1\delta=0.03,0.2,0.3$. Note that the application of formula (1.2) to noisy data with the noise level $C_1\delta=0.03$ was examined in [19], but here we are interested to compare the impact of noise on the application of formulas (1.2) and (3.2).

The solution $v(x,t)$ and the space location x_0 are changing from example to example. Moreover, we consider the following distribution of the observation time moment t_k:

$$t_1=5\tau,\quad t_2=6\tau,\quad t_k=(9+k)\tau,\quad k=3,4,...,21,$$

were $\tau=10^{-4}$ that corresponds to $\tilde{M}=10^3$ in scheme (6.6).

The sequences of the regularization parameters in our reconstruction algorithm are chosen as follows:

$$\lambda_i = 2^{1-i}, \quad i = 1, 2, \dots, 60, \quad \tilde{t}_j = 2^{1-j} t_N, \quad j = 1, 2, \dots, 10.$$

The approximate minimizer $\psi_{\delta,\lambda}(t)$ has the form (6.2) with $\mathcal{J} = 3$, $\nu_1 = 0.1$, $\nu_2 = 0.4$, $\nu_3 = 0.7$, $\gamma = 0.99$, $P = 9$ and $N = 21$, i.e., $t_N = t_{21}$.

The algorithm outputs for the analyzed examples are shown in Tables 1-10, where the values $\nu_\delta^I := \nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0})$ and $\nu_\delta^{ln} := \nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0})$ are chosen by multiple quasi-optimality criterion described above from the sequences $\nu_\delta(\lambda_i, \tilde{t}_j)$, $\nu_\delta^{\ln}(\lambda_i, \tilde{t}_j)$ calculated according to formulas (6.3) and (6.4). Moreover, we also consider the approximation

$$\nu_\delta^{t_1} := \frac{\ln |\psi_{\delta,1} - v_0(x_0)|}{\ln t_1}$$

calculated according to the approach from [13].

Example 1. *Consider problem* (6.5) *with* $L = 1$ *and*

$$a(x,t) = \cos \pi x/2 + t, \quad \tilde{a}(x,t) = x + t, \quad b(x,t) = t^{1/3} + \sin \pi x,$$

$$\mathcal{K}(t) = t^{-1/3}, \quad v_0(x) = \cos \pi x, \quad f(x,t,v) = xt \sin(v^2),$$

$$g(x,t) = 1 + \pi^2 \Big(\cos \frac{\pi x}{2} + t + \frac{3t^{2/3} \sin(\pi x)}{2} + \frac{t\pi}{3 \sin \pi/3} \Big) \cos \pi x$$
$$-(x+t)\pi \sin \pi x - xt \sin \Big(\cos \pi x + \frac{t^\nu}{\Gamma(1+\nu)} \Big)^2.$$

In this example, we test the considered reconstruction algorithm for different values of the memory order $\nu = 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$. It is easy to verify that the function $v(x,t) = \cos \pi x + \frac{t^\nu}{\Gamma(1+\nu)}$ solves direct problem (6.5) with the parameters specified above. The solution is observed at the space location $x_0 = 0.70$. Then, simple calculations show that for this example the error bound in **STN** case (**N2** in our tests) provided by Lemma 5.1 for ν_δ^I and Proposition 3.4 [19] for $\nu_\delta^{\ln}$ has the value 4ϵ and ϵ, correspondingly, where

$$\epsilon := \frac{C_1 \delta}{||C_0| - C_1 \delta|} = \begin{cases} 0.0309 & \text{if} \quad C_1 \delta = 0.03, \\ 0.2500 & \text{if} \quad C_1 \delta = 0.2, \\ 0.4286 & \text{if} \quad C_1 \delta = 0.3. \end{cases} \tag{6.7}$$

Moreover, in contrast to ν_δ^I, the error bound with ε holds for the noise **N3** in the case of $\nu_\delta^{\ln}$ (see Proposition 3.4 [19]).

Although in Example 1 the analytic form of the solution $v(x,t)$ is known, we generate synthetic noisy data by using numerical scheme (6.6). Of course, this increases the noise level and makes the test even harder. Nevertheless, as it can be seen from Tables 1-3, the accuracy of our reconstruction algorithm has the order predicted by Lemma 5.1 and bound (6.7). This is remarkable, because Lemma 5.1 presupposes the availability of continuous data, while our algorithm operates only with discrete data and does not use any information about the noise level. Moreover, as numerical results below show, formula (6.3) gives more accurate results than formula (6.4). This is especially transparent for $\nu < 0.5$, that is, in the case of very slow diffusion. Finally, the reconstruction based on formula (6.4) provides better results than the approximation $\nu_\delta^{t_1}$ suggested in [13].

$C_1\delta = 0.03$			$C_1\delta = 0.2$			$C_1\delta = 0.3$			
ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν
0.0506	0.0469	0.0464	0.0516	0.0468	0.0463	0.0520	0.0468	0.0462	0.05
0.0987	0.0956	0.0934	0.0988	0.0955	0.0932	0.0988	0.0955	0.0931	0.1
0.1999	0.1908	0.1887	0.2006	0.1908	0.1883	0.2015	0.1908	0.1881	0.2
0.3005	0.2890	0.2856	0.3028	0.2890	0.2849	0.3021	0.2890	0.2845	0.3
0.4015	0.3894	0.3840	0.4058	0.3915	0.3824	0.4074	0.3914	0.3815	0.4
0.5025	0.4872	0.4835	0.5116	0.4877	0.4802	0.5154	0.4871	0.4783	0.5
0.6035	0.5878	0.5839	0.6179	0.5841	0.5769	0.6241	0.5852	0.5730	0.6
0.7040	0.6885	0.6847	0.7215	0.6855	0.6700	0.7300	0.6881	0.6621	0.7
0.8032	0.7890	0.7847	0.8177	0.7714	0.7552	0.8235	0.7645	0.7405	0.8
0.8964	0.8845	0.8820	0.8891	0.8417	0.8263	0.8870	0.8239	0.8020	0.9

Table 1: The quantities ν_δ^I, ν_δ^{ln}, $\nu_\delta^{t_1}$ in Example 1 for the noise **N1**

$C_1\delta = 0.03$			$C_1\delta = 0.2$			$C_1\delta = 0.3$			
ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν
0.0504	0.0435	0.	0.0504	0.0298	0.0231	0.0505	0.0266	0.0127	0.05
0.0988	0.0932	0.	0.0989	0.0811	0.0705	0.0990	0.0747	0.0604	0.1
0.1995	0.1883	0.	0.1995	0.1757	0.1666	0.1995	0.1695	0.1567	0.2
0.2998	0.2867	0.	0.2998	0.2748	0.2640	0.2998	0.2687	0.2544	0.3
0.4003	0.3882	0.	0.4002	0.3779	0.3628	0.4002	0.3718	0.3532	0.4
0.5006	0.4846	0.	0.5005	0.4725	0.4627	0.5005	0.4652	0.4531	0.5
0.6006	0.5840	0.	0.6006	0.5706	0.5636	0.6005	0.5661	0.5540	0.6
0.7005	0.6859	0.	0.7004	0.6721	0.6655	0.7004	0.6673	0.6557	0.7
0.8003	0.7927	0.	0.8002	0.7776	0.7682	0.8001	0.7689	0.7583	0.8
0.9001	0.8935	0.	0.9000	0.8782	0.8717	0.9000	0.8706	0.8615	0.9

Table 2: The quantities ν_δ^I, ν_δ^{ln}, $\nu_\delta^{t_1}$ in Example 1 for the noise **N2**

$C_1\delta = 0.03$			$C_1\delta = 0.2$			$C_1\delta = 0.3$			
ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν
0.0268	0.0208	0.0201	-0.0278	-0.0637	-0.0730	-0.0401	-0.0938	-0.1074	0.05
0.0765	0.0699	0.0676	0.0263	-0.0005	-0.0243	0.0152	-0.0250	-0.0583	0.1
0.1766	0.1667	0.1638	0.1292	0.0938	0.0738	0.1210	0.0694	0.0402	0.2
0.2772	0.2653	0.2613	0.2280	0.1919	0.1726	0.2173	0.1673	0.1392	0.3
0.3810	0.3675	0.3600	0.3317	0.2936	0.2719	0.3222	0.2689	0.2387	0.4
0.4782	0.4636	0.4599	0.4266	0.3901	0.3719	0.4202	0.3683	0.3387	0.5
0.5781	0.5629	0.5608	0.5266	0.4828	0.4723	0.5145	0.4590	0.4390	0.6
0.6776	0.6645	0.6627	0.6256	0.5875	0.5733	0.6135	0.5594	0.5398	0.7
0.7807	0.7701	0.7653	0.7374	0.6964	0.6746	0.7268	0.6711	0.6408	0.8
0.8760	0.8699	0.8688	0.8238	0.7882	0.7763	0.8145	0.7623	0.7421	0.9

Table 3: The quantities ν_δ^I, ν_δ^{ln}, $\nu_\delta^{t_1}$ in Example 1 for the noise **N3**

Example 2. *Consider problem* (6.5) *with* $L = 1$ *and* $\nu = 0.6$, 0.7, 0.8, 0.9,

$$a(x,t) = 1, \quad \tilde{a}(x,t) = 0,$$

$$\mathcal{K} = 0, \quad b(x,t) = 1,$$

$$f(x,t,v) = 0 \quad g(x,t) = 100t(-2x^3 + 3x^2) \quad \textit{and} \quad v_0(x) = -\frac{2}{3}x^3 + x^2 + 1.$$

In this example, an analytic form of the solution is unknown and we use the numerical scheme (6.6) to generate measurements $v(x_0, t)$ at the point $x_0 = 0.20$, for which

$$\varepsilon = \frac{C_1\delta}{||C_0| - C_1\delta|} = \begin{cases} 0.0256 & \text{if} \quad C_1\delta = 0.03, \\ 0.2000 & \text{if} \quad C_1\delta = 0.2, \\ 0.3333 & \text{if} \quad C_1\delta = 0.3. \end{cases}$$

and, hence, condition (3.1) holds.

For this example the corresponding numerical results are listed in Tables 4-6.

$C_1\delta = 0.03$			$C_1\delta = 0.2$			$C_1\delta = 0.3$			
ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν
0.6143	0.5645	0.5641	0.6179	0.5555	0.5580	0.6206	0.5501	0.5546	0.6
0.7009	0.6723	0.6615	0.7148	0.6668	0.6491	0.7222	0.6646	0.6423	0.7
0.8057	0.7697	0.7614	0.8205	0.7661	0.7362	0.8695	0.7542	0.7234	0.8
0.9007	0.8722	0.8598	0.8891	0.8293	0.8114	0.8888	0.8163	0.7895	0.9

Table 4: The quantities ν_δ^I, ν_δ^{ln}, $\nu_\delta^{t_1}$ in Example 2 for the noise **N1**

$C_1\delta = 0.03$			$C_1\delta = 0.2$			$C_1\delta = 0.3$			
ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν
0.6131	0.5626	0.5622	0.6017	0.5440	0.5464	0.6012	0.5337	0.5379	0.6
0.6974	0.6705	0.6608	0.6986	0.6602	0.6452	0.6992	0.6558	0.6368	0.7
0.8030	0.7708	0.7634	0.8042	0.7614	0.7474	0.8024	0.7550	0.7389	0.8
0.9040	0.8735	0.8675	0.9037	0.8623	0.8511	0.9033	0.8540	0.8423	0.9

Table 5: The quantities ν_δ^I, ν_δ^{ln}, $\nu_\delta^{t_1}$ in Example 2 for the noise **N2**

$C_1\delta = 0.03$			$C_1\delta = 0.2$			$C_1\delta = 0.3$			
ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν
0.5931	0.5445	0.5440	0.5403	0.4567	0.4635	0.5307	0.4537	0.4321	0.6
0.6809	0.6538	0.6428	0.6389	0.5887	0.5628	0.6294	0.5594	0.5316	0.7
0.7829	0.7521	0.7450	0.7352	0.6815	0.6640	0.7230	0.6655	0.6325	0.8
0.8815	0.8555	0.8486	0.8300	0.7826	0.7659	0.8161	0.7559	0.7304	0.9

Table 6: The quantities ν_δ^I, ν_δ^{ln}, $\nu_\delta^{t_1}$ in Example 2 for the noise **N3**

Example 3. *In this example we consider problem* (2.1)-(2.4) *in two-dimensional domain* $\Omega = (0,1) \times (0,1)$ *and for the memory order* $\nu = 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$*:*

$$\begin{cases} \mathbf{D}_t^\nu v - v_{xx} - v_{yy} - \frac{t^{-\nu}}{\Gamma(1-\nu)} \star [v_{xx} + v_{yy}] = [\cos \pi x + \cos \pi y][\Gamma(1+\nu) + \pi^2(1+t^\nu) \\ +\pi^2 t(1+\Gamma(1+\nu)) + \frac{(1+\pi^2)t^{1-\nu}}{\Gamma(2-\nu)} + \frac{\pi^2 t^{2-\nu}}{\Gamma(3-\nu)}] \qquad in \quad \Omega \times (0,T), \\ v(x,y,0) = \cos \pi x + \cos \pi y, \qquad (x,y) \in [0,1] \times [0,1], \\ v_x(0,y,t) = v_x(1,y,t) = 0, \qquad t \in [0,T],\ y \in [0,1], \\ v_y(x,0,t) = v_y(x,1,t) = 0, \qquad t \in [0,T],\ x \in [0,1]. \end{cases}$$

In this case, the exact solution is represented as $v(x,y,t) = (\cos \pi x + \cos \pi y)(1 + t + t^\nu)$, and this solution will be observed in the point $(x_0, y_0) = (0.65, 0.65)$.

ε			
$C_1\delta = 0.03$	$C_1\delta = 0.2$	$C_1\delta = 0.3$	ν
0.0351	0.2924	0.5138	0.05
0.0360	0.3013	0.5321	0.1
0.0373	0.3156	0.5621	0.2
0.0382	0.3253	0.5827	0.3
0.0387	0.3302	0.5933	0.4
0.0387	0.3308	0.5944	0.5
0.0384	0.3272	0.5867	0.6
0.0377	0.3200	0.5714	0.7
0.0368	0.3098	0.5498	0.8
0.0356	0.2971	0.5233	0.9

Table 7: The error bound ε in Example 3 for the noise levels $C_1\delta = 0.03, 0.2, 0.3$

The outputs for this example are listed in Tables 8-10.

$C_1\delta = 0.03$			$C_1\delta = 0.2$			$C_1\delta = 0.3$			
ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν
0.0950	0.0995	0.0084	0.0122	0.0272	0.1147	0.0231	-0.0241	0.0084	0.05
0.1453	0.1497	0.0585	0.0276	0.0724	0.1648	0.0398	0.0225	0.0585	0.1
0.2460	0.2496	0.1586	0.0677	0.1652	0.2650	0.0832	0.1160	0.1586	0.2
0.3477	0.3500	0.2588	0.1267	0.2614	0.3655	0.1497	0.2128	0.2588	0.3
0.4516	0.4493	0.3593	0.2217	0.3615	0.4666	0.2526	0.3134	0.3593	0.4
0.5564	0.5485	0.4603	0.3732	0.4660	0.5690	0.4022	0.4176	0.4603	0.5
0.6633	0.6474	0.5626	0.4567	0.5737	0.6742	0.4627	0.5241	0.5626	0.6
0.7706	0.7435	0.6676	0.5133	0.6812	0.7862	0.5601	0.6252	0.6676	0.7
0.8787	0.8314	0.7789	0.5613	0.7813	0.9161	0.6639	0.7240	0.7789	0.8
0.9652	0.9009	0.9071	0.6323	0.8871	1.1202	0.6652	0.8329	0.9071	0.9

Table 10: The quantities ν_δ^I, ν_δ^{ln}, $\nu_\delta^{t_1}$ in Example 3 for the noise **N3**

$C_1\delta = 0.03$			$C_1\delta = 0.2$			$C_1\delta = 0.3$			
ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν
0.0516	0.0600	0.0626	0.0503	0.0600	0.0628	0.0496	0.0600	0.0628	0.05
0.1000	0.1080	0.1126	0.1000	0.1080	0.1128	0.1000	0.1080	0.1129	0.1
0.2003	0.2102	0.2125	0.1989	0.2103	0.2129	0.1974	0.2104	0.2132	0.2
0.2999	0.3136	0.3122	0.2983	0.3105	0.3131	0.2961	0.3096	0.3137	0.3
0.3997	0.4124	0.4117	0.3982	0.4081	0.4136	0.3934	0.4084	0.4148	0.4
0.5069	0.5096	0.5105	0.4983	0.5106	0.5147	0.4903	0.5116	0.5172	0.5
0.6115	0.6087	0.6081	0.5976	0.6123	0.6170	0.5868	0.6140	0.6226	0.6
0.7105	0.7069	0.7030	0.6981	0.7158	0.7221	0.6821	0.7191	0.7348	0.7
0.8331	0.8001	0.7927	0.7956	0.8342	0.8337	0.7947	0.8477	0.8655	0.8
0.9347	0.8905	0.8723	1.0034	0.9690	0.9625	1.1844	1.0870	1.0741	0.9

Table 8: The quantities ν_δ^I, ν_δ^{ln}, $\nu_\delta^{t_1}$ in Example 3 for the noise **N1**

$C_1\delta = 0.03$			$C_1\delta = 0.2$			$C_1\delta = 0.3$			
ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν_δ^I	ν_δ^{ln}	$\nu_\delta^{t_1}$	ν
0.0517	0.0632	0.0670	0.0519	0.0813	0.0953	0.0520	0.0939	0.1153	0.05
0.1000	0.1108	0.1170	0.1000	0.1287	0.1453	0.1000	0.1413	0.1653	0.1
0.2007	0.2138	0.2168	0.2011	0.2314	0.2450	0.1993	0.2384	0.2650	0.2
0.3004	0.3183	0.3165	0.3008	0.3315	0.3446	0.3012	0.3422	0.3645	0.3
0.4012	0.4168	0.4157	0.4015	0.4287	0.4437	0.3990	0.4411	0.4634	0.4
0.5087	0.5127	0.5141	0.5096	0.5321	0.5417	0.5109	0.5450	0.5611	0.5
0.6148	0.6115	0.6108	0.6162	0.6391	0.6376	0.6193	0.6455	0.6564	0.6
0.7223	0.7083	0.7039	0.7264	0.7293	0.7292	0.7198	0.7470	0.7468	0.7
0.8355	0.7988	0.7903	0.8426	0.8175	0.8129	0.8534	0.8304	0.8283	0.8
0.9338	0.8812	0.8652	0.9388	0.8967	0.8836	0.9413	0.9069	0.8958	0.9

Table 9: The quantities ν_δ^I, ν_δ^{ln}, $\nu_\delta^{t_1}$ in Example 3 for the noise **N2**

7. Discussion and conclusion

In this paper, we propose an approach to reconstruct the order of semilinear subdiffusion. To this end, analyzing boundary value problems for the nonautonomous semilinear subdiffusion equations with memory terms in the fractional Hölder spaces, we obtain an explicit reconstruction formula for the order ν in terms of the smooth observation data for small time. Then, based on the Tikhonov regularization scheme and the quasi-optimality criterion, we construct the computational algorithm to find the order ν from noisy discrete measurements.

The computational results demonstrate that the proposed method effectively determines the unknown memory order ν. Moreover, the obtained formula (3.2) gives more accurate results than the formula (1.2) [13, 19].

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