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REGULARIZED RECONSTRUCTION OF THE ORDER IN SEMILINEAR SUBDIFFUSION WITH MEMORY

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Abstract. For $\nu\in(0,1)$, we analyze the semilinear integro-differential equation on the multidimensional space domain $\Omega\subset\mathbb{R}^n$ in the unknown $v=v(x,t)$:

$$\mathbf{D}_t^\nu v-\mathcal{L}_1 v-\int_0^t \mathcal{K}(t-s)\mathcal{L}_2 v(\cdot,s)ds=f(x,t,v)+g(x,t)$$

where $\mathbf{D}_t^\nu$ is the Caputo fractional derivative and $\mathcal{L}_1$ and $\mathcal{L}_2$ are uniform elliptic operators of the second order with time-dependent smooth coefficients. We obtain the explicit formula reconstructing the order of the fractional derivative ν for small time state measurements. The formula gives rise to a regularization algorithm for calculating ν from possibly noisy measurements. We present several numerical tests illustrating the algorithm when it is equipped with quasi-optimality criteria for choosing the regularization parameters.

1. Introduction

Fractional partial differential equations have applications in many fields, including mathematical modeling [24], electromagnetism [9], polymer science [3], viscoelasticity [28], hydrology [4,12], geophysics [34], biophysics [8], finance [25], and prediction of extreme events like earthquake [5]. Over the last two decades the standard diffusive transport has been found to be inadequate to explain a wide-range phenomena that are observed in experiments [10, 12, 13, 29]. The feature of these anomalies in diffusion/transport processes is that the mean square displacement of the diffusing species $\langle(\mathbf{\Delta x})^2\rangle$ scales as a nonlinear power law in time, i.e. $\langle(\mathbf{\Delta x})^2\rangle\sim t^\nu$, $\nu>0$ [24]. For a subdiffusive process, the value of ν is such that $0<\nu<1$, while for normal diffusion $\nu=1$, and for a superdiffusive process, we have $\nu>1$. However, something a value of the subdiffusion order is not given a priori.

In this paper, we discuss an approach to the reconstruction of a subdiffusion order ν from small time state measurements. To this end, we analyze an inverse problem of recovering the order ν.

Let Ω be a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, $\partial\Omega\in\mathcal{C}^{k+\alpha}$, $k\geq 2$. For an arbitrary given time $T>0$ we denote

$$\Omega_T=\Omega\times(0,T)\quad\text{and}\quad\partial\Omega_T=\partial\Omega\times[0,T].$$

For $\nu\in(0,1)$, we consider the nonlinear equation with the unknown function $v=v(x,t):\Omega_T\to\mathbb{R}$,

$$\mathbf{D}_t^\nu v-\mathcal{L}_1 v-\mathcal{K}\star\mathcal{L}_2 v=f(x,t,v)+g(x,t),\ (x,t)\in\Omega_T,\tag{1.1}$$

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supplemented with the initial condition

$$v(x,0)=v_0(x) \text{ in } \bar{\Omega}, \tag{1.2}$$

subject either to the Dirichlet boundary condition (**DBC**)

$$v(x,t)=0 \text{ on } \partial\Omega_T, \tag{1.3}$$

or to the condition of the third kind (**III BC**)

$$\mathcal{M}_1 v+\mathcal{K}_1\star\mathcal{M}_2 v+\sigma v=0 \quad \text{ on } \partial\Omega_T, \tag{1.4}$$

where a positive number σ, the functions v_0, g, f, the operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, $\mathcal{M}_2$, and the memory kernels $\mathcal{K}$, $\mathcal{K}_1$ are assumed to be given. Here, the *star* denotes the usual time-convolution product on $(0,t)$, namely

$$(\mathfrak{H}_1\star\mathfrak{H}_2)(t):=\int\limits_0^t\mathfrak{H}_1(t-s)\mathfrak{H}_2(s)ds,\ t>0,$$

while the symbol $\mathbf{D}_t^\nu$ stands the Caputo fractional derivative of order ν with respect to time t (see e.g. (2.4.1) in [18]), defined as

$$\mathbf{D}_t^\nu v(x,t)=\frac{1}{\Gamma(1-\nu)}\frac{\partial}{\partial t}\int\limits_0^t\frac{[v(x,\tau)-v(x,0)]}{(t-\tau)^\nu}d\tau,\ \nu\in(0,1)$$

with Γ being the Euler Gamma-function. An equivalent definition in the case of absolutely continous functions reads

$$\mathbf{D}_t^\nu v(x,t)=\frac{1}{\Gamma(1-\nu)}\int_0^t\frac{1}{(t-s)^\nu}\frac{\partial v}{\partial s}(x,s)ds.$$

Finally, coming to the involved operators $\mathcal{L}_i$ and $\mathcal{M}_i$, we consider two different cases. In the case of nonlinear sources (NS), i.e. $f(x,t,v)\neq 0$, and a multidimensional domain Ω $(n\geq 2)$, $\mathcal{L}_i$ and $\mathcal{M}_i$ read as

$$\begin{aligned}
\mathcal{L}_1&:=\sum_{ij=1}^n\frac{\partial}{\partial x_i}\bigg(A_{ij}(x,t)\frac{\partial}{\partial x_j}\bigg)+\sum_{i=1}^n A_i(x,t)\frac{\partial}{\partial x_i}+A_0(x,t),\\
\mathcal{L}_2&:=\sum_{ij=1}^n\frac{\partial}{\partial x_i}\bigg(A_{ij}(x,t)\frac{\partial}{\partial x_j}\bigg)+\sum_{i=1}^n B_i(x,t)\frac{\partial}{\partial x_i}+B_0(x,t),\\
\mathcal{M}_1&=\mathcal{M}_2:=\sum_{ij=1}^n A_{ij}(x,t)N_i(x)\frac{\partial}{\partial x_j},
\end{aligned}\tag{1.5}$$

where $N=\{N_1(x),...,N_n(x)\}$ is the outward normal to $\partial\Omega$. In the case of a linear source (LS), i.e. $f\equiv 0$, or in the 1-dimensional case and NS, the operators have the form

$$\begin{aligned}
\mathcal{L}_1&:=\sum_{ij=1}^n a_{ij}(x,t)\frac{\partial^2}{\partial x_j\partial x_i}+\sum_{i=1}^n a_i(x,t)\frac{\partial}{\partial x_i}+a_0(x,t),\\
\mathcal{L}_2&:=\sum_{ij=1}^n b_{ij}(x,t)\frac{\partial^2}{\partial x_j\partial x_i}+\sum_{i=1}^n b_i(x,t)\frac{\partial}{\partial x_i}+b_0(x,t),\\
\mathcal{M}_1&=\sum_{i=1}^n c_i(x,t)\frac{\partial}{\partial x_i}+c_0(x,t),
\end{aligned}\tag{1.6}$$

$$\mathcal{M}_2 = \sum_{i=1}^{n} d_i(x,t)\frac{\partial}{\partial x_i} + d_0(x,t).$$

To formulate the inverse problem, we introduce the observation data $\psi(t)$ for small time $t \in [0, t^\star)$, $t^\star < \min(1, T)$,

$$v(x_0, t) = \psi(t), \tag{1.7}$$

where the point $x_0 \in \Omega$ in the **DBC** case, and $x_0 \in \bar{\Omega}$ in the **III BC** case.

For the given functions f, g ψ and the given coefficients of the operators $\mathcal{L}_i$ and $\mathcal{M}_i$, $i = 1, 2$, the inverse problem (IP) is to find the order $\nu \in (0, 1)$, such that the solution of the direct problems (1.1)-(1.6) satisfies observation data (1.7) for small time t.

A motivation for the study of equations (1.1) arises from theoretical and experimental investigations of materials with memory [3, 4, 6, 12, 28]. Indeed, in the mentioned papers, the authors stated that the passage of the fluid through the porous matrix may cause a local variation of the permeability. That implies the presence of "memory" in the matrix or in the fluid. In practice [6, 12], the flow may perturb the porous formation by causing particle migration resulting in pore clogging, or chemical reacting with the medium so to enlarge the pores or diminish their size. Moreover, in biological systems, the filtering properties of membranes may become saturated.

To adequately represent the memory, the derivative of fractional order which weighs the 'past' of the solution is introduced in the constitutive equations. More discussion about physical phenomena that can be modeled by (1.1)-(1.6) with $\mathcal{K} \neq 0$ can be found in [3, 4, 6, 12, 19, 28].

In the literature on the inverse problems of reconstructing the order ν in (1.1) one can find two approaches mainly associated with the case $\mathcal{K} = 0$.

The first approach is connected with finding an explicit formula for ν in terms of small or large time measurements [15, 16], such as

$$\nu = \lim_{t\to 0} \frac{t v_t(x_0,t)}{v(x_0,t) - v_0(x_0)} \quad \text{and} \quad \nu = \lim_{t\to\infty} \frac{t v_t(x_0,t)}{v(x_0,t)}, \tag{1.8}$$
$$\nu = \lim_{t\to 0} \frac{\ln|v(x_0,t) - v(x_0,0)|}{\ln t}.$$

The second technique is started in [12] and related with the minimization of a certain functional depending on the solution of the corresponding direct problem and on given measurements either on the whole time interval $[0, T]$ or for the final time $t = T$ [17, 22, 26, 31, 35].

Both of these approaches have certain advantages and disadvantages. Indeed, the second approach [17, 22, 26, 31, 35] needs not only the measurements but also all information on the coefficients and the right-hand sides in the direct problem, while a calculation by the explicit formula requires only the knowledge of the measurements. However, the first two explicit formulas in (1.8) have been proved only for a linear autonomous equation like (1.1) with $\mathcal{K} \equiv 0$, $f, g = 0$ and in the case of multidimensional domains of the special geometry. Moreover, these formulas involve the derivative, existence of which is not guaranteed in general by models (1.1)-(1.3). The last formula in (1.8) has been obtained for a linear equation with time independent coefficients, with the memory term $\mathcal{K}\star v_{xx}$, and only in the one-dimensional domain Ω. Furthermore, problems considered in [15, 16] do not cover the third kind boundary conditions, such as (1.4).

All this narrows the scope of the application. Another important question that is understudied in the literature is what happens if known measurements are noisy and given for a finite number of time moments.

In this paper, we aim at resolving the above-mentioned issues. Namely, introducing non-vanishing memory kernels in (1.1)-(1.7) and analyzing this model in the fractional Hölder classes, we prove the explicit reconstruction formula for order ν:

$$\nu=\lim_{t\to 0}\frac{\ln|\psi(t)-v(x_0,0)|}{\ln t}$$

where the only requirement for $x_0\in\bar{\Omega}$ is

$$\mathcal{L}_1 v(x_0,t)|_{t=0}+f(x_0,0,v(x_0,0))+g(x_0,0)\neq 0.$$

As we mentioned above, this formula for ν was previously obtained in [16]. However, unlike [16], we establish this formula in the case of nonlinear nonautonomous equation (1.1) with more general representations of memory terms. Moreover, we study here not only Dirichlet boundary conditions but also the **III BC** (1.4).

Then we show how to use this formula in the case where we have only noisy observations $\psi_\delta(t_k)\approx v(x_0,t_k)$ at a finite number N of time moments $t=t_k$, $k=1,2,...,N$. To overcome this difficulty, we at first propose to reconstruct $\psi(t)=\psi_{\delta,\lambda}(t)$ by means of the regularized regression from the given noisy data ψ_δ, where the regularization is performed in the finite-dimensional space

$$\text{span}\{t^{\nu_i},\mathcal{P}_j^{(0,-\gamma)},\quad i=1,2,3,\quad j=1,2,...,m(N)\}$$

according to the Tikhonov scheme with the penalty term $\lambda\|\cdot\|_{L^2_{t^{-\gamma}}(0,t_N)}$. Here λ is the regularization parameter, ν_i are our initial guesses about ν (in particular, $\nu_i=0$), $L^2_{t^{-\gamma}}(0,t_N)$ is a weighted space L^2 with the weight $t^{-\gamma}$, $\gamma\in(0,1)$; $\mathcal{P}_j^{(0,-\gamma)}$ are Jacobi polynomials shifted to $[0,t_N]$.

Then, according to our formula, we consider the quantities

$$\nu(\lambda,t)=\frac{\ln|\psi_{\delta,\lambda}(t)-v(x_0,0)|}{\ln t}$$

calculated for the sequences of (regularization) parameters: $\lambda\in\{\lambda_p\}$, $t\in\{\tilde{t}_q\}$.

Finally, the regularized reconstructor

$$\nu:=\nu_{reg}=\nu(\tilde{\lambda},\tilde{t})$$

is chosen from the set of approximate values $\{\nu(\lambda_p,\tilde{t}_q)\}$ by applying two-parameter quasi-optimality criterion [11] selecting $\tilde{\lambda}\in\{\lambda_p\}$, $\tilde{t}\in\{\tilde{t}_q\}$.

The paper is organized as follows. In the next section we introduce the function spaces and state some auxiliary results, which will play a key role in our analysis. The main theoretical results of the paper along with the general assumptions on the model are stated in Section 3. In Section 4, we discuss existence, uniqueness in (1.1)-(1.7), prove the explicit formula for ν and analyze the influence of noise. Finally, Section 5 is devoted to the description of the algorithm for regularized recovering of ν. The proposed method is illustrated by numerical examples.

2. Function Spaces and Preliminaries

We carry out our analysis in the framework of the fractional Hölder spaces. For any Banach space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$, we consider the usual spaces

$$\mathcal{C}([0,T],\mathbf{X}), \quad \mathcal{C}^{\beta}[0,T] \quad \text{and} \quad W^{1,p}(\Omega), \quad L^{p}(\Omega), \quad \beta\in(0,1),\, p\in(1,+\infty).$$

Let

$$\langle u\rangle^{(\beta)}_{x,\Omega_T} = \sup\Big\{\frac{|u(x_1,t)-u(x_2,t)|}{|x_1-x_2|^{\beta}} : x_1\neq x_2;\, x_1,x_2\in\bar{\Omega}, t\in[0,T]\Big\},$$
$$\langle u\rangle^{(\beta)}_{t,\Omega_T} = \sup\Big\{\frac{|u(x,t_1)-u(x,t_2)|}{|t_1-t_2|^{\beta}} : x\in\bar{\Omega}, t_1\neq t_2; t_1,t_2\in[0,T]\Big\}.$$

We say that a function $u=u(x,t)$ belongs to the fractional Hölder spaces $\mathcal{C}^{l+\alpha,\frac{l+\alpha}{2}\beta}(\bar{\Omega}_T)$, $\alpha\in(0,1)$, $l=0,1,2$, if the function u together with its derivatives are continuous and the norms below are finite

$$\|u\|_{\mathcal{C}^{l+\alpha,\frac{l+\alpha}{2}\beta}(\bar{\Omega}_T)} = \|u\|_{\mathcal{C}([0,T],\mathcal{C}^{l+\alpha}(\bar{\Omega}))} + \sum_{|j|=0}^{l}\langle D^j_x u\rangle^{(\frac{l+\alpha-|j|}{2}\beta)}_{t,\Omega_T}, \quad l=0,1,$$

$$\|u\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\beta}(\bar{\Omega}_T)} = \|u\|_{\mathcal{C}([0,T],\mathcal{C}^{2+\alpha}(\bar{\Omega}))} + \|\mathbf{D}^{\beta}_t u\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta}(\bar{\Omega}_T)} + \sum_{|j|=1}^{2}\langle D^j_x u\rangle^{(\frac{2+\alpha-|j|}{2}\beta)}_{t,\Omega_T}.$$

The properties of these spaces have been discussed in Section 2 [20]. In a similar way, for $l=0,1,2$, we introduce the space $\mathcal{C}^{l+\alpha,\frac{l+\alpha}{2}\beta}(\partial\Omega_T)$.

In the paper, we also use the spaces $L^2_w(t_1,t_2)$ of real-valued functions that a square integrable with a positive weight $w(t)$. Note that $L^2_w(t_1,t_2)$ is a Hilbert space with the inner product

$$\langle f,g\rangle_w := \int_{t_1}^{t_2} w(t)f(t)g(t)dt$$

and the corresponding norm $\|\cdot\|_{L^2_w(t_1,t_2)}$.

We conclude this section with briefly review of the properties of Jacobi polynomials [1,30]. Note that, these polynomials play a key role in our numerical schemes presented in Section 5.

Usual Jacobi polynomials $P^{a,b}_n(t)$ are defined on $(-1,1)$ as

$$P^{a,b}_n(t) = \sum_{m=0}^{n} p_{n,m}(t-1)^{n-m}(t+1)^m,$$

where

$$p_{n,m} = \frac{1}{2^n}\begin{pmatrix} n+a \\ m\end{pmatrix}\begin{pmatrix} n+b \\ n-m\end{pmatrix}.$$

After shifting them to $(0,1)$, we obtain the Jacobi polynomials defined on $(0,1)$ as

$$\mathcal{P}^{(a,b)}_n(t) = P^{a,b}_n(2t-1).$$

For $a, b > -1$ these polynomials are orthogonal with respect to the weight function $\bar{w}(t) := (1-t)^a t^b$,

$$\int_0^1 \mathcal{P}_n^{(a,b)}(t)\mathcal{P}_k^{(a,b)}(t)\bar{w}(t)dt = \begin{cases} \frac{\Gamma(n+a+1)\Gamma(n+b+1)}{(2n+a+b+1)n!\Gamma(n+a+b+1)}, & n=k, \\ 0, & n \neq k. \end{cases}$$

Moreover, the straightforward calculations provide the identity

$$\int_0^T t^b \mathcal{P}_n^{(0,b)}(t/T) t^{\bar{\nu}} dt = \frac{T^{\bar{\nu}+b+1}}{\bar{\nu}+b+1}\, {}_2F_1(-n, b+n+1; 2+\bar{\nu}+b; 1),$$

where T is positive, $\bar{\nu} \in (0,1)$ and ${}_2F_1(\cdot)$ is the hypergeometric function [1].

3. The Main Result

In the sequel we rely on the following assumptions concerning model (1.1)-(1.7).

h1 (Ellipticity conditions): There are positive constants μ_1, μ_2, $\mu_1 < \mu_2$, such that

$$\mu_1|\xi|^2 \leq \sum_{ij=1}^n A_{ij}(x,t)\xi_i\xi_j \leq \mu_2|\xi|^2,$$

$$\mu_1|\xi|^2 \leq \sum_{ij=1}^n a_{ij}(x,t)\xi_i\xi_j \leq \mu_2|\xi|^2,$$

for any $(x,t,\xi) \in \bar{\Omega}_T \times \mathbb{R}^n$. There exists a positive constant μ_3 such that

$$\left| \sum_{i=1}^n c_i(x,t)N_i(x) \right| \geq \mu_3.$$

h2 (Smoothness of the coefficients): For $i, j = 1, \dots, n$, $\alpha \in (0,1)$,

$$\begin{gathered} A_{ij}(x,t), A_i(x,t), B_i(x,t) \in \mathcal{C}^{1+\alpha, \frac{1+\alpha}{2}}(\bar{\Omega}_T), \\ A_0(x,t), B_0(x,t) \in \mathcal{C}^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_T); \end{gathered} \tag{3.1}$$

$$\begin{gathered} a_{ij}(x,t), b_{ij}(x,t), a_i(x,t), b_i(x,t), a_0(x,t), b_0(x,t) \in \mathcal{C}^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_T), \\ c_i(x,t), d_i(x,t), c_0(x,t), d_0(x,t) \in \mathcal{C}^{1+\alpha, \frac{1+\alpha}{2}}(\bar{\Omega}_T). \end{gathered} \tag{3.2}$$

Moreover, in the 1-dimensional case and $f \neq 0$, we additionally assume that

$$\frac{\partial a_{11}}{\partial x}, \quad \frac{\partial b_{11}}{\partial x} \in \mathcal{C}(\bar{\Omega}_T). \tag{3.3}$$

h3 (Conditions on the kernel): Depending on the considered case, we assume the following:

$$\begin{cases} \mathcal{K}(t) \in L^1(0,T) & \text{in the LS case,} \\ \mathcal{K}(t) \in \mathcal{C}^1[0,T] & \text{in the NS case.} \end{cases} \tag{3.4}$$

h4 (Conditions on the given functions): The following inclusions hold with $\alpha, \beta \in (0,1)$

$$\mathcal{K}^1(t) \in L_1(0,T)$$
$$v_0(x) \in C^{2+\alpha}(\bar{\Omega}),$$
$$g(x,t) \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T),$$
$$\psi(t) \in \mathcal{C}^{\beta}[0,t^\star],\ \mathbf{D}_t^{\beta}\psi(t) \in \mathcal{C}^{\frac{\alpha\beta}{2}}[0,t^\star].$$

h5 (Compatibility conditions): In the case of **DBC** (1.3), the following compatibility conditions hold for every $x \in \partial\Omega$ at the initial time $t=0$

$$v_0(x) = 0 \quad \text{and} \quad 0 = \mathcal{L}_1 v_0(x)\big|_{t=0} + f(x,0,v_0) + g(x,0),$$

while in the case of **III BC** (1.4) it is assumed that

$$\mathcal{M}_1 v_0(x)\big|_{t=0} + \sigma v_0 = 0.$$

h6 (Conditions on the nonlinearity): For every $\varrho > 0$ and for any $(x_i, t_i, v_i) \in \bar{\Omega}_T \times [-\varrho, \varrho]$, there exists a constant $C_\varrho > 0$ such that

$$|f(x_1,t_1,v_1) - f(x_2,t_2,v_2)| \leq C_\varrho\big(|x_1 - x_2| + |t_1 - t_2| + |v_1 - v_2|\big).$$

Moreover, there is a constant $L > 0$ such that the inequality

$$|f(x,t,v)| \leq L[1 + |v|]$$

holds for any $(x,t,v) \in \bar{\Omega}_T \times \mathbb{R}$.

We are ready now to state our main results.

Theorem 3.1. *Let $T > 0$ be arbitrary fixed, $f \neq 0$, and let assumptions **h1**, **h3-h6** and* (3.1) *hold. We assume that*

$$\mathcal{L}_1 v_0(x_0)\Bigg|_{t=0} + f(x_0, 0, v_0(x_0)) + g(x_0, 0) \neq 0. \tag{3.5}$$

Then the pair $(\nu, v_\nu(x,t))$ solves (1.1)-(1.5), (1.7)*, where ν is defined as*

$$\nu = \lim_{t\to 0} \frac{\ln|\psi(t) - v_0(x_0)|}{\ln t}, \tag{3.6}$$

and $v_\nu(x,t)$ is a unique solution of direct problem (1.1)-(1.5) *satisfying regularity $v_\nu \in \mathcal{C}^{2+\alpha,\frac{\alpha+2}{2}\nu}(\bar{\Omega}_T)$.*

Theorem 3.2. *Let $T > 0$ be arbitrary fixed, and $f \equiv 0$. Then under assumptions **h1**, **h3-h5**,* (3.2)*, and* (3.5)*, the relations* (1.1)-(1.4), (1.6), (1.7) *are satisfied by the pair $(\nu, v_\nu(x,t))$, where ν is defined by* (3.6) *and v_ν is a unique solution of the direct problem* (1.1)-(1.4), (1.6)*, satisfying regularity $v_\nu \in \mathcal{C}^{2+\alpha,\frac{\alpha+2}{2}\nu}(\bar{\Omega}_T)$.*

Remark 3.1. *In the case $\Omega \subset \mathbb{R}$, results of Theorems 3.1 and 3.2 hold if assumptions **h1**, **h4-h6**,* (3.2), (3.3) *and* (3.5) *are fulfilled, and the kernel $\mathcal{K}$ meets the requirement*

$$|\mathcal{K}| \leq Ct^{-\theta} \quad \textit{for any} \quad t \in [0,T] \quad \textit{and} \quad \theta \in (0,1).$$

Remark 3.2. *Note that in NS case the assumptions on the kernel $\mathcal{K}$ in Theorem 3.1 and Remark 3.1 can be relaxed, provided that the nonlinearity f fulfills a stronger requirement. Namely, f has to satisfy global Lipschitz continuity, and $\mathcal{K} \in L^1(0,T)$.*

Remark 3.3. *Actually, a slight modification of the proof allows the same results to be obtained for* (1.1)-(1.7) *with inhomogeneous boundary conditions:*

$$v(x,t) = \psi_1(x,t) \quad \text{on} \quad \partial\Omega_T,$$
$$\mathcal{M}_1 v + \mathcal{K}_1 \star \mathcal{M}_2 v + \sigma v = \psi_2(x,t) \quad \text{on} \quad \partial\Omega_T,$$

if $\psi_1 \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}}(\partial\Omega_T)$, $\psi_2 \in \mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}(\partial\Omega_T)$ *and the corresponding compatibility conditions hold. The details are left to the interested reader.*

4. Proof of the Main Results

4.1. **Direct problems: solvability and estimates.** We begin this sections with consideration of the direct problems (1.1)-(1.6). The following theorem summarizes the results [19]- [21].

Theorem 4.1. *Let $T > 0$ be arbitrary fixed and $\beta \in (0,1)$. Then under the assumptions **h1-h6** the direct problem* (1.1)-(1.6) *with $\nu = \beta$ has a unique global classical solution $v \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\beta}(\bar{\Omega}_T)$ admitting the estimate*

$$\|v\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\beta}(\bar{\Omega}_T)} \leq C\Big[1 + \|g\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta}(\bar{\Omega}_T)} + \|v_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})}\Big].$$

Moreover, if in (1.1) $f \equiv 0$ *then*

$$\|v\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\beta}(\bar{\Omega}_T)} \leq C\Big[\|g\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta}(\bar{\Omega}_T)} + \|v_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})}\Big].$$

Here the generic positive bounded quantity C depends only on the parameters in the model (1.1)-(1.6).

Note that Theorem 4.1 is also valid if the assumptions **h2** and **h4** are substituted for their weaker versions:

h2*:

$$A_{ij}(x,t), A_i(x,t), B_i(x,t) \in \mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}\beta}(\bar{\Omega}_T),$$
$$A_0(x,t), B_0(x,t) \in \mathcal{C}^{\alpha,\frac{\alpha}{2}\beta}(\bar{\Omega}_T);$$

$$a_{ij}(x,t), b_{ij}(x,t), a_i(x,t), b_i(x,t), a_0(x,t), b_0(x,t) \in \mathcal{C}^{\alpha,\frac{\alpha}{2}\beta}(\bar{\Omega}_T),$$
$$c_i(x,t), d_i(x,t), c_0(x,t), d_0(x,t) \in \mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}\beta}(\bar{\Omega}_T).$$

h4*:

$$g(x,t) \in \mathcal{C}^{\alpha,\frac{\alpha}{2}\beta}(\bar{\Omega}_T).$$

Theorem 4.1 guarantees continuous dependence of the solution of (1.1)-(1.6) on g and v_0. Now we analyze the dependence of the solution on the order ν and start with some auxiliary estimates which play a key role in our further discussion.

Proposition 4.1. *Let $0 < \nu_1 < \nu_2 < 1$ and the function $U(t)$ have the fractional derivative of the order ν_2 satisfying regularity condition $\mathbf{D}_t^{\nu_2}U \in \mathcal{C}^{\gamma}[0,T]$ with some fixed $\gamma \in (0,1)$. Then for any fixed T and $t \in [0,T]$ the following hold*

(i):

$$|1 - \Gamma(\nu_2 - \nu_1 + 1)| \leq C(\nu_2 - \nu_1),$$

(ii):

$$|\Gamma(1 - \nu_2) - \Gamma(1 - \nu_1)| \leq C(\nu_2 - \nu_1),$$

(iii):

$$\Big|\int_0^t \Big[\frac{\tau^{-\nu_1}}{\Gamma(1-\nu_1)}-\frac{\tau^{-\nu_2}}{\Gamma(1-\nu_2)}\Big]d\tau\Big|\leq C(1+T)(\nu_2-\nu_1),$$

(iv):

$$|\mathbf{D}_t^{\nu_2}U-\mathbf{D}_t^{\nu_1}U|\leq C(\nu_2-\nu_1)[1+t+|\ln t|+t^{\nu_2-\nu_1+\gamma}](\|\mathbf{D}_t^{\nu_2}U\|_{\mathcal{C}[0,T]}+\langle\mathbf{D}_t^{\nu_2}U\rangle^{(\gamma)}_{t,[0,T]}),$$

(v):

$$\|\mathbf{D}_t^{\nu_2}U-\mathbf{D}_t^{\nu_1}U\|_{L^p(0,T)}\leq C(\nu_2-\nu_1)(\|\mathbf{D}_t^{\nu_2}U\|_{\mathcal{C}[0,T]}+\langle\mathbf{D}_t^{\nu_2}U\rangle^{(\gamma)}_{t,[0,T]})$$

with some generic positive bounded quantity C.

The proof of Proposition 4.1 is technical and left to Appendix. Theorem 4.1 and Proposition 4.1 allow us to estimate the stability of the solution of (1.1)-(1.6) with respect to the order ν.

Lemma 4.1. *Let the assumptions of Theorem 4.1 hold and $0<\beta_1<\beta_2<1$. If v_{β_1} and v_{β_2} solve (1.1)-(1.6) with $\nu=\beta_1$ and $\nu=\beta_2$, correspondingly, then*

i: *for each $\varepsilon\in(0,T)$*

$$\begin{aligned}&\|v_{\beta_1}-v_{\beta_2}\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\beta_1}(\bar{\Omega}\times[\varepsilon,T])}\\&\leq C(\beta_2-\beta_1)(1+|\ln\varepsilon|)[\|\mathbf{D}_t^{\beta_2}v_{\beta_2}\|_{\mathcal{C}(\bar{\Omega}_T)}+\langle\mathbf{D}_t^{\beta_2}v_{\beta_2}\rangle^{(\frac{\alpha\beta_2}{2})}_{t,\Omega_T}]\\&\leq C(\beta_2-\beta_1)(1+|\ln\varepsilon|)\Big[1+\|g\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta_2}(\bar{\Omega}_T)}+\|v_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})}\Big];\end{aligned}$$

ii: *in the one-dimensional case, i.e. $\Omega\subset\mathbb{R}$,*

$$\begin{aligned}&\|v_{\beta_1}-v_{\beta_2}\|_{\mathcal{C}(\bar{\Omega}_T)}+\|v_{\beta_1}-v_{\beta_2}\|_{\mathcal{C}([0,T],W^{1,2}(\Omega))}\\&\leq C(\beta_2-\beta_1)[\|\mathbf{D}_t^{\beta_2}v_{\beta_2}\|_{\mathcal{C}(\bar{\Omega}_T)}+\langle\mathbf{D}_t^{\beta_2}v_{\beta_2}\rangle^{(\frac{\alpha\beta_2}{2})}_{t,\Omega_T}]\\&\leq C(\beta_2-\beta_1)\Big[1+\|g\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta_2}(\bar{\Omega}_T)}+\|v_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})}\Big];\end{aligned}$$

iii: *in the multidimensional case, i.e. $\Omega\in\mathbb{R}^n$, $n\geq 2$, and for $p\geq 2$*

$$\begin{aligned}&\|v_{\beta_1}-v_{\beta_2}\|_{\mathcal{C}([0,T],L^p(\Omega))}+\|v_{\beta_1}-v_{\beta_2}\|_{\mathcal{C}([0,T],W^{1,2}(\Omega))}\\&\leq C(\beta_2-\beta_1)[\|v_{\beta_2}\|_{\mathcal{C}(\bar{\Omega}_T)}+\|\mathbf{D}_t^{\beta_2}v_{\beta_2}\|_{\mathcal{C}(\bar{\Omega}_T)}+\langle\mathbf{D}_t^{\beta_2}v_{\beta_2}\rangle^{(\frac{\alpha\beta_2}{2})}_{t,\Omega_T}]\\&\leq C(\beta_2-\beta_1)\Big[1+\|g\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta_2}(\bar{\Omega}_T)}+\|v_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})}\Big].\end{aligned}$$

Moreover, if in addition $f\equiv 0$ and $\mathcal{K},\mathcal{K}_1\in\mathcal{C}^1[0,T]$, then for $p>n+\frac{1}{\beta_1}$

$$\begin{aligned}&\|v_{\beta_1}-v_{\beta_2}\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta_1}(\bar{\Omega}_T)}+\|v_{\beta_1}-v_{\beta_2}\|_{L^p((0,T),W^{2,p}(\Omega))}\\&\leq C(\beta_2-\beta_1)\|\mathbf{D}_t^{\beta_2}v_{\beta_2}-\mathbf{D}_t^{\beta_1}v_{\beta_2}\|_{L^p(\Omega_T)}\\&\leq C(\beta_2-\beta_1)\Big[1+\|g\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta_2}(\bar{\Omega}_T)}+\|v_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})}\Big]\end{aligned}$$

.

Proof. We will carry out the detailed proof of Lemma 4.1 in the **DBC** case. The **III BC** case can be analyzed in a similar way.

Note that the difference

$$V(x,t)=v_{\beta_2}(x,t)-v_{\beta_1}(x,t)$$

solves the problem

$$\begin{cases} \mathbf{D}_t^{\beta_1} V - \mathcal{L}_1 V - \mathcal{K} \star \mathcal{L}_2 V = \bar{g}(x,t) + \bar{f}(x,t,V) & \text{in} \quad \Omega_T, \\ V(x,0) = 0 & \text{in} \quad \Omega, \\ V(x,t) = 0 & \text{on} \quad \partial\Omega_T, \end{cases} \tag{4.1}$$

where

$$\bar{f}(x,t,V) = f(x,t,v_{\beta_2}) - f(x,t,v_{\beta_2} - V) \text{ and } \bar{g}(x,t) = \mathbf{D}_t^{\beta_2} v_{\beta_2} - \mathbf{D}_t^{\beta_1} v_{\beta_2}.$$

Then the estimates of Theorem 4.1 and Proposition 4.1 immediately give the inequality **(i)**.

Next, we focus on getting bounds in the one-dimensional case and begin to evaluate V in $\mathcal{C}([0,T], W^{1,2}(\Omega))$. To this end, we recast here step-by-step the arguments from the proof of Lemma 5.1 [19].
For $t \in [0,T]$ we consider the functions

$$\Theta_{\beta_1}(t) = \frac{t^{\beta_1 - 1}}{\Gamma(\beta_1)},$$
$$W_1(t) = \|V(\cdot,t)\|^2_{L^2(\Omega)} + \Theta_{\beta_1} \star \|V_x(\cdot,t)\|^2_{L^2(\Omega)},$$
$$W_2(t) = \|V_x(\cdot,t)\|^2_{L^2(\Omega)} + \Theta_{\beta_1} \star \|V_{xx}(\cdot,t)\|^2_{L^2(\Omega)}.$$

Then, as in [19], for each $t \in [0,T]$ we have the bounds

$$W_1(t) \leq C[\Theta_{\beta_1} \star \|\bar{g}\|^2_{L^2(\Omega)} + [\Theta_{\beta_1} + |\mathcal{K}|] \star W_1],$$
$$W_2(t) \leq C[\Theta_{\beta_1} \star \|\bar{g}\|^2_{L^2(\Omega)} + [\Theta_{\beta_1} + |\mathcal{K}|] \star W_2].$$

After that, the easy verified embedding $(\Theta_{\beta_1} + |\mathcal{K}|) \in L^1(0,T)$ and Gronwall-type inequality (4.3) from [20] entail the estimate

$$|W_1| + |W_2| \leq C \sup_{[0,T]} (\Theta_{\beta_1} \star \|\bar{g}\|^2_{L^2(\Omega)}),$$

that can be rewritten in terms of the difference $(v_{\beta_2} - v_{\beta_1})$ as

$$\|v_{\beta_2} - v_{\beta_1}\|^2_{\mathcal{C}([0,T],W^{1,2}(\Omega))} \leq C \sup_{[0,T]} (\Theta_{\beta_1} \star \|\bar{g}\|^2_{L^2(\Omega)}). \tag{4.2}$$

Estimating the right-hand side by means of Proposition 4.1 and making use the estimates from Theorem 4.1, we enhance inequality (4.2) to

$$\begin{aligned} \|v_{\beta_2} - v_{\beta_1}\|_{\mathcal{C}([0,T],W^{1,2}(\Omega))} &\leq C(\beta_2 - \beta_1)[\|\mathbf{D}_t^{\beta_2} v_{\beta_2}\|_{\mathcal{C}(\bar{\Omega}_T)} + \langle \mathbf{D}_t^{\beta_2} v_{\beta_2} \rangle^{(\frac{\alpha\beta_2}{2})}_{t,\Omega_T}] \\ &\leq C(\beta_2 - \beta_1)\Big[1 + \|g\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta_2}(\bar{\Omega}_T)} + \|v_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})}\Big] \end{aligned} \tag{4.3}$$

with a generic positive bounded quantity C.

Finally, Sobolev embedding theorem (see, e.g. Section 5.4 [2]) and inequality (4.3) allow us to arrive at the estimate **(ii)** of this proposition. This completes our considerations in the one-dimensional case.

In the multidimensional case estimate (4.3) is obtained by the same arguments. Thus, in order to verify the statement **(iii)** of this proposition, we are left to estimate the function V in the space $\mathcal{C}([0,T], L^p(\Omega))$, $p \geq 2$.

Let $\bar{\mathcal{K}}$ be the conjugate kernel for $\mathcal{K}$ (the main properties of this kernel can be found in Proposition 4.4 of [20]) and let

$$\hat{g} = \bar{g} + \mathcal{K}(0)[\Theta_{1-\beta_1} - \Theta_{1-\beta_2}] \star v_{\beta_2} + \bar{\mathcal{K}}' \star [\Theta_{1-\beta_1} - \Theta_{1-\beta_2}] \star v_{\beta_2}.$$

Recasting the proof of Lemma 5.2 [20] in the case of problem (4.1) leads to the inequality

$$\int_{\Omega} |V|^p dx \leq Cp\Theta_{\beta_1} \star \|\hat{g}\|^p_{L^p(\Omega)} + C(p-1)\Theta_{\beta_1} \star \|V\|^p_{L^p(\Omega)}$$

with $p \geq 2$. Then, the straightforward calculations with aid of Proposition 4.1 provide the estimate

$$\int_{\Omega} |V|^p dx \leq C(\beta_2 - \beta_1)[\|\mathbf{D}_t^{\beta_2} v_{\beta_2}\|_{\mathcal{C}(\bar{\Omega}_T)} + \langle \mathbf{D}_t^{\beta_2} v_{\beta_2}\rangle^{(\frac{\alpha\beta_2}{2})}_{t,\bar{\Omega}_T} + \|v_{\beta_2}\|_{\mathcal{C}(\bar{\Omega}_T)}]^p$$
$$+ C(p-1)\Theta_{\beta_1} \star \|V\|^p_{L^p(\Omega)}.$$

Applying again Gronwal-type inequality and Theorem 4.1, we conclude

$$\begin{aligned}\|v_{\beta_2} - v_{\beta_1}\|_{\mathcal{C}([0,T],L^p(\Omega))} &= \|V\|_{\mathcal{C}([0,T],L^p(\Omega))}\\ &\leq C(\beta_2 - \beta_1)[\|\mathbf{D}_t^{\beta_2} v_{\beta_2}\|_{\mathcal{C}(\bar{\Omega}_T)} + \langle \mathbf{D}_t^{\beta_2} v_{\beta_2}\rangle^{(\frac{\alpha\beta_2}{2})}_{t,\bar{\Omega}_T} + \|v_{\beta_2}\|_{\mathcal{C}(\bar{\Omega}_T)}]\\ &\leq C(\beta_2 - \beta_1)\Big[1 + \|g\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta_2}(\bar{\Omega}_T)} + \|v_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})}\Big].\end{aligned}$$

Thus, to completes the proof, we are left to verify estimates in the linear case of (1.1). Recasting step-by-step the arguments from the proof of Proposition 5.4 [20] and applying the inequality above, we deduce

$$\begin{aligned}&\|V\|_{\mathcal{C}^{\alpha,\frac{\alpha\beta_1}{2}}(\bar{\Omega}_T)} + \|V\|_{L^p((0,T),W^{2,p}(\Omega))}\\ &\qquad\leq C[\|\mathbf{D}_t^{\beta_2} v_{\beta_2} - \mathbf{D}_t^{\beta_1} v_{\beta_2}\|_{L^p(\Omega_T)} + \|V\|_{\mathcal{C}([0,T],L^p(\Omega))}]\\ &\qquad\leq C(\beta_2 - \beta_1)\Big[1 + \|g\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\beta_2}(\bar{\Omega}_T)} + \|v_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})}\Big]\\ &\qquad\quad + C\|\mathbf{D}_t^{\beta_2} v_{\beta_2} - \mathbf{D}_t^{\beta_1} v_{\beta_2}\|_{L^p(\Omega_T)}\end{aligned}$$

with $p > \beta_1^{-1} + n$. The proof is finished by applying Proposition 4.1, statement **(iii)** of this Lemma and then Theorem 4.1 to estimate the last term in the right-hand side of the inequality above. □

As we see, Lemma 4.1 can provide the continuous dependence of the solution v on the order $\nu = \beta$ in the case of the Sobolev spaces, but this result does not hold in the case of $\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\beta}$. Most likely, in virtue of statement **(i)**, the stability could be expected in weighted fractional Hölder spaces. Analysis of this issue is left to a further work.

4.2. **Solvability of IP, explicit formula to ν.** The following result is a direct consequence of Theorem 4.1 and condition (1.7).

Proposition 4.2. *Let $\nu \in (0,1)$, and let the assumptions of Theorems 3.1 and 3.2 hold. If the pair (ν, v_ν) satisfies* (1.1)-(1.7)*, then*

$$\psi(t) = v_\nu(x_0,t) \in \mathcal{C}^{\nu}[0,t^\star],$$
$$\mathbf{D}_t^\nu \psi(t) = \mathbf{D}_t^\nu v_\nu(x_0,t) \in \mathcal{C}^{\frac{\alpha\nu}{2}}[0,t^\star],$$
$$\mathbf{D}_t^\nu \psi(0) = \mathcal{L}_1 v_0(x_0)|_{t=0} + f(x_0,0,v_0(x_0)) + g(x_0,0) \neq 0.$$

Now we begin to obtain the explicit formula for the order of the fractional derivative.

Lemma 4.2. *Let T be arbitrarily fixed and $\beta \in (0,1)$. Let a function $G(t)$ and its fractional derivative $\mathbf{D}_t^{\beta} G(t)$ be continuous on $[0,T]$. Then for each $t \in [0,T]$ the function $G(t)$ allows the following representation*

$$G(t) = G(0) + t^{\beta} \frac{\mathbf{D}_t^{\beta} G(0)}{\Gamma(1+\beta)} + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} [\mathbf{D}_\tau^{\beta} G(\tau) - \mathbf{D}_t^{\beta} G(0)] d\tau. \quad (4.4)$$

Assume in addition, that $\mathbf{D}_t^{\beta} G(0) \neq 0$ and

$$\omega(t) := \sup_{\tau \in [0,t]} |\mathbf{D}_\tau^{\beta} G(\tau) - \mathbf{D}_t^{\beta} G(0)| \to 0 \quad as \quad t \to 0,$$

then

$$\beta = \lim_{t \to 0} \frac{\ln |G(t) - G(0)|}{\ln t}. \quad (4.5)$$

Proof. The direct calculations based on the smoothness of the function G give (4.4). Further, equality (4.4) provides the relation

$$\ln |G(t) - G(0)| = \beta \ln t + \ln \left| \frac{\mathbf{D}_t^{\beta} G(0)}{\Gamma(1+\beta)} + \frac{t^{-\beta}}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} [\mathbf{D}_t^{\beta} G(\tau) - \mathbf{D}_t^{\beta} G(0)] d\tau \right|,$$

where

$$\left| \frac{t^{-\beta}}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} [\mathbf{D}_t^{\beta} G(\tau) - \mathbf{D}_t^{\beta} G(0)] d\tau \right| \leq \frac{\omega(t)}{\Gamma(1+\beta)} \underset{t \to 0}{\to} 0.$$

Thus, we achieve the representation

$$\frac{\ln |G(t) - G(0)|}{\ln t} = \beta + \frac{\ln \left| \frac{\mathbf{D}_t^{\beta} G(0)}{\Gamma(1+\beta)} + \frac{t^{-\beta}}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} [\mathbf{D}_t^{\beta} G(\tau) - \mathbf{D}_t^{\beta} G(0)] d\tau \right|}{\ln t} \underset{t \to 0}{\to} \beta,$$

that completes the proof of the lemma. □

Note that the right-hand side of (4.5) exists and is bounded under weaker conditions on the function $G(t)$.

Remark 4.1. *It is apparent that if the function $G \in \mathcal{C}^{\beta}[0,T]$, $\beta \in (0,1)$, then*

$$0 < \lim_{t \to 0} \frac{\ln |G(t) - G(0)|}{\ln t} \leq \beta.$$

Returning to the observation $\psi(t)$ and taking into account the condition **h4**, one can easily obtain

$$\sup_{\tau \in [0,t]} |\mathbf{D}_t^{\beta} \psi(\tau) - \mathbf{D}_t^{\beta} \psi(0)| \leq t^{\frac{\alpha\beta}{2}} \langle \mathbf{D}_t^{\beta} \psi \rangle_{t,[0,T]}^{(\frac{\alpha\beta}{2})} \underset{t \to 0}{\to} 0. \quad (4.6)$$

Therefore, relation (4.6) together with Lemma 4.2, Theorem 4.1 and Proposition 4.2 provide the existence of the solution (ν, v_ν) of (1.1)-(1.7), where

$$\nu = \beta = \lim_{t \to 0} \frac{\ln |\psi(t) - v_0(x_0)|}{\ln t}.$$

4.3. **Uniqueness in IP.** Here we discuss the uniqueness of the pair (ν, v_ν) satisfying problem (1.1)-(1.7).

Theorem 4.2. *Under the assumptions **h1-h6** and restriction* (3.5)*, the order* ν *can be identified uniquely by the observation data* $\psi(t)$ *using formula* (3.6)*, and there is the unique pair* (ν, v_ν) *satisfying problem* (1.1)-(1.7).

Proof. Assume contrary that there are two quantities ν_1 and ν_2 defined with (3.6) and corresponding to the same observation data (1.7) and the same right-hand sides and coefficients in model (1.1)-(1.6). We denote by v_{ν_1} and v_{ν_2} the corresponding solutions of direct problem (1.1)-(1.6). To be specific, we assume that $0 < \nu_1 < \nu_2 < 1$. Proposition 4.2 and condition (3.5) provide

$$\begin{aligned} \psi(t) \in \mathcal{C}^{\nu_2}[0, t^*], \quad \mathbf{D}_t^{\nu_i}\psi(t) \in \mathcal{C}^{\alpha\nu_i/2}[0, t^*], \\ \psi(0) = v_0(x_0) \quad \text{and} \quad \mathbf{D}_t^{\nu_i}\psi(0) \neq 0. \end{aligned} \tag{4.7}$$

Then Lemma 4.2 allows the representations

$$\psi(t) = v_0(x_0) + \frac{t^{\nu_i}}{\Gamma(1+\nu_i)}\mathbf{D}_t^{\nu_i}\psi(0) + \frac{1}{\Gamma(\nu_i)}\int_0^t \frac{[\mathbf{D}_\tau^{\nu_i}\psi(\tau) - \mathbf{D}_\tau^{\nu_i}\psi(0)]}{(t-\tau)^{1-\nu_i}}d\tau, \quad i = 1, 2,$$

telling us that for $t \in [0, t^\star]$ the equality holds

$$\begin{aligned} &\frac{\mathbf{D}_t^{\nu_1}\psi(0)}{\Gamma(1+\nu_1)} + \frac{t^{-\nu_1}}{\Gamma(\nu_1)}\int_0^t \frac{[\mathbf{D}_\tau^{\nu_1}\psi(\tau) - \mathbf{D}_\tau^{\nu_1}\psi(0)]}{(t-\tau)^{1-\nu_1}}d\tau \\ &= \frac{t^{\nu_2-\nu_1}}{\Gamma(1+\nu_2)}\mathbf{D}_t^{\nu_2}\psi(0) + \frac{t^{-\nu_1}}{\Gamma(\nu_2)}\int_0^t \frac{[\mathbf{D}_\tau^{\nu_2}\psi(\tau) - \mathbf{D}_\tau^{\nu_2}\psi(0)]}{(t-\tau)^{1-\nu_2}}d\tau. \end{aligned} \tag{4.8}$$

It is apparent that for sufficiently small t we have

$$\frac{t^{-\nu_1}}{\Gamma(\nu_1)}\int_0^t \frac{[\mathbf{D}_\tau^{\nu_1}\psi(\tau) - \mathbf{D}_\tau^{\nu_1}\psi(0)]}{(t-\tau)^{1-\nu_1}}d\tau = O(t^{\alpha\nu_1/2}),$$

$$\frac{t^{-\nu_1}}{\Gamma(\nu_2)}\int_0^t \frac{[\mathbf{D}_\tau^{\nu_2}\psi(\tau) - \mathbf{D}_\tau^{\nu_2}\psi(0)]}{(t-\tau)^{1-\nu_2}}d\tau = O(t^{\nu_2-\nu_1+\alpha\nu_2/2}).$$

Then, passing to the limit in (4.8) as $t \to 0$, we obtain the equality

$$\frac{\mathbf{D}_t^{\nu_1}\psi(0)}{\Gamma(1+\nu_1)} = 0,$$

that contradicts conclusion (4.7) above.

This contradiction is resolved as soon as we admit $\nu_1 = \nu_2$. After that, Theorem 4.1 ensures the equality $v_{\nu_1} = v_{\nu_2}$, that completes the proof. □

4.4. **Error estimate for noisy data.** Here we analyze formula (3.6) in the case, where $\psi(t)$ is replaced by noisy data $\psi_\delta(t)$.

We assume that

$$|\psi(t) - \psi_\delta(t)| \leq C_1\delta\Phi(t), \quad 0 \leq t \leq t^\star, \tag{4.9}$$

where $\Phi(t)$ is some nonnegative function, δ denotes the noise level and C_1 is some constant.

In the paper, we focus on the analysis of two types of the noise. The first-type noise (**FTN**) model corresponds to the case when

$$\Phi(0) = 0 \quad \text{and} \quad \Phi(t)t^{-\gamma} < C^{\star} \quad \text{for all} \quad \gamma \in (0,1), \tag{4.10}$$

while the second type noise (**STN**) model is characterized by

$$\Phi(0) = 0 \quad \text{and} \quad \Phi(t)t^{-\gamma} < |\ln t| \quad \text{for some fixed} \quad \gamma \in [\nu,1). \tag{4.11}$$

Remark 4.2. *It is easy to see that the function*

$$\Phi(t) = \begin{cases} e^{-1/t}|\ln t| & \text{in } \textbf{\textit{FTN}} \text{ case,} \\ t^{\gamma}|\ln t| & \text{in } \textbf{\textit{STN}} \text{ case,} \end{cases}$$

satisfies conditions (4.10) *and* (4.11)*.*

In the sequel we will deal with

$$\nu_\delta = \lim_{t\to 0} \frac{\ln|\psi_\delta(t) - v_0(x_0)|}{\ln t}.$$

Proposition 4.3. *Let the assumptions of Theorems 3.1 and 3.2 hold. If inequalities* (4.9)-(4.11) *are satisfied with* C_1 *and* $\delta \in (0,1)$ *such that*

$$\frac{C_1}{||\mathcal{L}_1 v_0(x_0)|_{t=0} + f(x_0,0,v_0(x_0)) + g(x_0,0)| - C_1\delta|} < 1,$$

then

$$|\nu - \nu_\delta| = 0 \quad \text{in } \textbf{\textit{FTN}} \text{ case,}$$

and

$$|\nu - \nu_\delta| \le \frac{C_1\delta}{||\mathcal{L}_1 v_0(x_0)|_{t=0} + f(x_0,0,v_0(x_0)) + g(x_0,0)| - C_1\delta|} \quad \text{in } \textbf{\textit{STN}} \text{ case.}$$

Proof. In view of condition (3.5), we can assume for simplicity that

$$C_0 := \mathcal{L}_1 v_0(x_0)|_{t=0} + f(x_0,0,v_0(x_0)) + g(x_0,0) > 0.$$

Consider the quantity

$$\Delta := \left| \frac{\psi(t) - \psi_\delta(t)}{\psi_\delta(t) - v_0(x_0)} \right|,$$

which due to (4.9) allows the bound

$$\Delta \le \frac{C_1\delta\Phi(t)}{||\psi(t) - v_0(x_0)| - |\psi_\delta(t) - \psi(t)||} \le \frac{C_1\delta\Phi(t)}{||\psi(t) - v_0(x_0)| - C_1\delta\Phi(t)|}. \tag{4.12}$$

Let us denote

$$C_2 = \langle \mathbf{D}_t^{\nu}\psi\rangle_{t,[0,t^\star]}^{(\alpha\nu/2)}$$

and use representation (4.4) with $G(t) = \psi(t)$. Then

$$\begin{aligned} |\psi(t) - v_0(x_0)| &\ge \left| \frac{t^{\nu}\mathbf{D}_t^{\nu}\psi(0)}{\Gamma(1+\nu)} - \frac{C_2 t^{\nu+\alpha\nu/2}}{\Gamma(1+\nu)} \right| \\ &= \left| \frac{t^{\nu}C_0}{\Gamma(1+\nu)} - \frac{C_2 t^{\nu+\alpha\nu/2}}{\Gamma(1+\nu)} \right|. \end{aligned}$$

This estimate together with (4.9) and (4.12) give

$$\Delta \le \frac{C_1\delta t^{-\nu}\Phi(t)}{||C_0 - C_2 t^{\alpha\nu/2}| - C_1\delta t^{-\nu}\Phi(t)|}. \tag{4.13}$$

The conditions on C_1 and δ ensure that

$$\lim_{t\to 0}\frac{C_1\delta}{||C_0 - C_2 t^{\alpha\nu/2}| - C_1\delta|} = \frac{C_1\delta}{|C_0 - C_1\delta|}.$$

We are now in the position to evaluate $|\nu - \nu_\delta|$. In the case of **STN**, keeping in mind estimate (4.13), we conclude

$$\begin{aligned}
|\nu - \nu_\delta| &= \left|\lim_{t\to 0}\frac{\ln|1+\Delta|}{\ln t}\right| \\
&\leq \lim_{t\to 0}\frac{C_1\delta\Phi(t)t^{-\nu}}{|\ln t|||C_0 - C_2 t^{\alpha\nu/2}| - C_1\delta|} \\
&\leq \frac{C_1\delta}{|C_0 - C_1\delta|}.
\end{aligned}$$

In the **FTN** case, we have for some positive quantity $\gamma \in (\nu, 1)$

$$\begin{aligned}
|\nu - \nu_\delta| &\leq \lim_{t\to 0}\frac{C_1\delta\Phi(t)t^{-\nu}}{|\ln t|||C_0 - C_2 t^{\alpha\nu/2}| - C_1\delta\Phi(t)t^{-\nu}|} \\
&\leq \lim_{t\to 0}\frac{C_1\delta C^\star t^{\gamma-\nu}}{|\ln t|||C_0 - C_2 t^{\alpha\nu/2}| - C_1\delta C^\star t^{\gamma-\nu}|} \\
&= 0.
\end{aligned}$$

□

5. Regularized formulation

In this section, we discuss a regularization of the reconstruction formula (3.6). Recall that this formula has been proven assuming enough smoothness of the observation data, while in practice only noisy discrete measurements are usually available. Therefore, the obtained formula should be regularized to deal with such data. Here we propose a way of combining formula (3.6) with a regularization procedure.

It is worth mentioning that the availability of only discrete observations is the more problematic issue than the presence of noise in continuous data $\psi_\delta(t)$, because in view of Proposition 4.3, such continuous noisy data in principle allow us to estimate the order ν rather accurately.

5.1. **Algorithm of reconstruction.** Assume that we are able to observe the solution $v(x,t)$ of (1.1)-(1.6) at some location $x = x_0 \in \bar{\Omega}$ and at time moments t_k, $k = 1, 2, ..., N$, $0 < t_1 < t_2 < ... < t_N \leq t^\star$, but these observations are blurred by an additive noise so, that what we actually observe is

$$\psi_{\delta,k} = v(x_0, t_k) + \delta_k, \quad k = 1, 2, ..., N.$$

Moreover, initial condition (1.2) allows us to know the value $\psi_0 = v(x_0, 0) = v_0(x_0)$.

To be able to use such discrete noisy information in formula (3.6), a reconstruction algorithm should at first approximately recover the function $\psi(t) = v(x_0, t)$ from the values $\psi_{\delta,k}$, $k = 0, 1, ..., N$, where with a bit abuse of symbols we define

$$\psi_{\delta,0} = \psi_0 = v_0(x_0).$$

At this point, it is important to note that in view of Proposition 4.2 and Lemma 4.2 the following asymptotic holds true for $t \leq t^\star$:

$$\psi(t) = v_0(x_0) + O(t^\nu).$$

This tells us that our target function should be square integrable on $(0, t_N)$, $t_N \le t^\star$, with unbounded weight $w(t) = t^{-\gamma}$, $\gamma \in (0,1)$. Therefore, it is natural to approximate the function $\psi(t)$ by elements of the space $L^2_{t^{-\gamma}}(0, t_N)$.

Then according to the methodology of the Tikhonov regularization, the above mentioned approximate recovery of $\psi(t)$ from noisy values $\{\psi_{\delta,k}\}_{k=0}^N$ can be performed by minimizing a penalized least squares functional

$$\sum_{k=0}^{N} \left(\psi(t_k) - \psi_{\delta,k}\right)^2 + \lambda \|\psi\|^2_{L^2_{t^{-\gamma}}(0,t_N)} \to \min, \tag{5.1}$$

where λ is a regularization parameter.

Since the Jacobi polynomials $\mathcal{P}_m^{(0,-\gamma)}(t/t_N)$, $t \in (0, t_N)$, constitute an orthogonal system in $L^2_{t^{-\gamma}}(0, t_N)$, it is natural to look for the minimizer of (5.1) in the form of a linear combination of $\mathcal{P}_m^{(0,-\gamma)}(t/t_N)$. Moreover, having a series of initial guesses $\nu_1, \nu_2, ..., \nu_{\mathcal{J}}$ for the value of ν, one may incorporate the functions t^{ν_j}, $j = 1, 2, ..., \mathcal{J}$, into the basis in which minimization problem (5.1) should be solved.

Then an approximate minimizer of (5.1) can be written as

$$\psi_{\delta,\lambda}(t) = \sum_{j=1}^{\mathcal{J}} c_j t^{\nu_j} + \sum_{j=\mathcal{J}+1}^{P} c_j \mathcal{P}^{(0,-\gamma)}_{j-\mathcal{J}-1}(t/t_N), \tag{5.2}$$

where the coefficients c_j can be found from the corresponding system of linear algebraic equations written in the matrix form as follows:

$$(A^T A + \lambda \mathcal{H})\vec{c} = A^T \vec{\psi}_\delta,$$

where
$\vec{c} = (c_1, c_2, ..., c_P)^T$, $\vec{\psi}_\delta = (\psi_{\delta,0}, \psi_{\delta,1}, ..., \psi_{\delta,N})^T$, $A = \{A_{i,j}\}_{i=0,j=1}^{N\ \ P}$, $A_{ij} = e_j(t_i)$, $\mathcal{H} = \{\mathcal{H}\}_{l,m=1}^P$, $\mathcal{H}_{l,m} = \int\limits_0^{t_N} t^{-\gamma} e_l(t) e_m(t) dt$, and

$$e_l(t) = \begin{cases} t^{\nu_l}, & l = 1, 2, ..., \mathcal{J}, \\ \mathcal{P}^{(0,-\gamma)}_{l-\mathcal{J}-1}(t/t_N), & l = 1 + \mathcal{J}, ..., P. \end{cases}$$

Note that even after the approximate recovery of $\psi(t)$ in the form of $\psi_{\delta,\lambda}(t)$, the problem of calculating limit in reconstruction formula (3.6) is an ill-posed one and needs to be regularized.

As an approximate value of that limit one can take the quantity

$$\nu_\delta(\lambda, \tilde{t}) = \frac{\ln |\psi_{\delta,\lambda}(\tilde{t}) - \psi_0|}{\ln \tilde{t}}$$

computed at some point $t = \tilde{t}$ that is sufficiently close to zero and playing the role of a regularized parameter.

Thus, the regularized approximate value $\nu_\delta(\lambda, \tilde{t})$ of the memory order ν suggested by the proposed algorithm depends on two regularization parameters λ and $\tilde{t}$ that need to be properly chosen. Since in practice the amplitudes δ_k of the noise perturbations are usually unknown, one should rely on the so-called noise level-free regularization parameter choice rules.

The quasi-optimality criterion [32] is one of the simplest and the oldest, but still quite efficient strategy among such rules. Its version for the choice of multiple regularization parameters, such as λ and $\tilde{t}$, has been discussed in [11, 23].

To implement the quasi-optimality criterion in the present context one should consider two geometric sequences of regularization parameters values:

$$\lambda = \lambda_i = \lambda_1 q_1^{i-1}, \quad i = 1, 2, ..., M_1;$$

$$\tilde{t} = \tilde{t}_j = \tilde{t}_1 q_2^{j-1}, \quad j = 1, 2, ..., M_2;$$

$$0 < q_1, \ q_2 < 1.$$

Then the values $\nu_\delta(\lambda_i, \tilde{t}_j)$ should be computed as described above.
Next, for each $\tilde{t}_j$ one needs to find $\lambda_{i_j} \in \{\lambda_i\}_{i=1}^{M_1}$ such that

$$\left|\nu_\delta(\lambda_{i_j}, \tilde{t}_j) - \nu_\delta(\lambda_{i_j-1}, \tilde{t}_j)\right| = \min\{|\nu_\delta(\lambda_i, \tilde{t}_j) - \nu_\delta(\lambda_{i-1}, \tilde{t}_j)|, \quad i = 2, 3, ..., M_1\}.$$

Finally, $\tilde{t}_{j_0}$ is selected from $\{\tilde{t}_j\}_{j=1}^{M_2}$ such that

$$\left|\nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0}) - \nu_\delta(\lambda_{i_{j_0-1}}, \tilde{t}_{j_0-1})\right| = \min\{|\nu_\delta(\lambda_{i_j}, \tilde{t}_j) - \nu_\delta(\lambda_{i_{j-1}}, \tilde{t}_{j-1})|, \ j = 2, 3, ..., M_2\}.$$

The value $\nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0})$ is the output of the proposed algorithm. In the next subsection we illustrate the performance of the algorithm by a series of numerical tests.

5.2. **Numerical experiments.** Three different numerical examples corresponding to the final moment $T = 0.1$ and the time to measurements $t^\star = 0.012$ are treated below by the proposed algorithm.

At first, we consider problem (1.1)-(1.6) in the one-dimensional case $\Omega := (0, L)$:

$$\begin{cases} \mathbf{D}_t^\nu v - a(x,t)v_{xx} + \tilde{a}(x,t)v_x - \int\limits_0^t \mathcal{K}(t-s)b(x,s)v_{xx}(x,s)ds \\ = f(x,t,v) + g(x,t) \qquad \text{in} \quad (0,L)\times(0,T), \\ v(x,0) = v_0(x), \qquad x \in [0,L], \\ v_x(0,t) = v_x(L,t) = 0, \qquad t \in [0,T]. \end{cases} \tag{5.3}$$

In general, it is problematic to find the solution of (5.3) in the analytical form. Therefore, to generate the synthetic test data, we have to solve problem (5.3) numerically using the computational scheme described briefly below.

Introducing the space-time mesh with nodes

$$x_k = kh, \quad \tau_j = j\tau, \quad k = 0, 1, \ldots, \tilde{N}, \quad j = 0, 1, \ldots, \tilde{M}, \quad h = L/\tilde{N}, \quad \tau = T/\tilde{M},$$

and approximating the differential equation from (5.3) at each level τ_{j+1}, we derive the following finite-difference scheme:

$$\tau^{-\nu}\sum_{m=0}^{j+1}(v_k^{j+1-m}-v_0(x_k))\rho_m-\frac{a_k^{j+1}}{h^2}(v_{k-1}^{j+1}-2v_k^{j+1}+v_{k+1}^{j+1})+\frac{\tilde{a}_k^{j+1}}{2h}(v_{k+1}^{j+1}-v_{k-1}^{j+1})$$

$$=\sum_{m=0}^{j}\left(b_k^m\frac{v_{k-1}^m-2v_k^m+v_{k+1}^m}{h^2}+b_k^{m+1}\frac{v_{k-1}^{m+1}-2v_k^{m+1}+v_{k+1}^{m+1}}{h^2}\right)\frac{\tilde{\mathcal{K}}_{m,j}}{2} \tag{5.4}$$

$$+f(x_k,\tau_j,v_k^j)+g(x_k,\tau_{j+1}),\quad k=1,\dots,\tilde{N}-1,\quad j=0,1,\dots,\tilde{M}-1,$$

where we denote the finite-difference approximation of the function v at the point (x_k,τ_j) by v_k^j and put

$$a_k^{j+1}=a(x_k,\tau_{j+1}),\quad \tilde{a}_k^{j+1}=\tilde{a}(x_k,\tau_{j+1}),\quad b_k^j=b(x_k,\tau_j),$$
$$\rho_m=(-1)^m\binom{\nu}{m},\quad \tilde{\mathcal{K}}_{m,j}=\int_{\tau_m}^{\tau_{m+1}}\mathcal{K}(\tau_{j+1}-s)ds.$$

Here we use the second-order finite-difference formulas to approximate derivatives v_x and v_{xx}; the Grünwald-Letnikov formula [7,18,33] to approximate the derivative $\mathbf{D}_t^\nu v$; the trapezoid-rule to approximate the integrals in the sum

$$\sum_{m=0}^{j}\int_{\tau_m}^{\tau_{m+1}}\mathcal{K}(\tau_{j+1}-s)b(x,s)v_{xx}(x,s)ds.$$

It is worth noting that an improvement in the accuracy of calculations can be achieved by Richardson extrapolation and finite element method with mass lumping (see [7], [27]). We also use two fictitious mesh points outside the spatial domain to approximate the Neumann boundary conditions with the second order of accuracy (see, e.g. [14]).

In all our tests the noisy measurements are simulated according to (4.9), i.e.,

$$\psi_{\delta,k}=v(x_0,t_k)+C_1\delta\Phi(t_k),\quad k=1,2,...,21,$$

where $C_1=0.3$, $\delta=0.1$, and we examine the case $\Phi(t)=t|\ln t|$ corresponding to (4.10), as well as the case $\Phi(t)=t^\nu|\ln t|$ mentioned in Remark 4.2. The solution $v(x,t)$ and the space location x_0 are changing from example to example. Moreover, we consider four different distributions of the observation time moment t_k. Namely,

$$\begin{aligned}
&\mathbf{C1:}\quad t_k=(99+k)\tau,\quad k=1,2,...,21,\\
&\mathbf{C2:}\quad t_1=50\tau,\quad t_2=51\tau,\quad t_k=(99+k)\tau,\quad k=3,4,...,21,\\
&\mathbf{C3:}\quad t_k=(9+k)\tau,\quad k=1,2,...,21,\\
&\mathbf{C4:}\quad t_1=5\tau,\quad t_2=6\tau,\quad t_k=(9+k)\tau,\quad k=3,4,...,21,
\end{aligned} \tag{5.5}$$

were $\tau=10^{-4}$ that corresponds to $\tilde{M}=10^3$ in scheme (5.4).

The sequences of the regularization parameters of our reconstruction algorithm are chosen as follows:

$$\lambda_i=2^{1-i},\quad i=1,2,\dots,60,\quad \tilde{t}_j=2^{1-j}t_N,\quad j=1,2,\dots,10.$$

The approximate minimizer $\psi_{\delta,\lambda}(t)$ has the form (5.2) with $\mathcal{J}=3$, $\nu_1=0.1$, $\nu_2=0.4$, $\nu_3=0.7$, $\gamma=0.99$, $P=9$ and $N=21$, i.e., $t_N=t_{21}$.

The absolute errors $|\nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0}) - \nu|$ for the analyzed examples are shown in Tables 1-6. The final step of the reconstruction algorithm is illustrated by Figure 1 displacing the sequence $\nu_\delta(\lambda_{i_j}, \tilde{t}_j)$ and the values $\nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0})$ chosen by the quasi-optimality criterion, which in the considered example (see Example 2 below) is the best or almost the best approximation of the real ν among $\nu_\delta(\lambda_{i_j}, \tilde{t}_j)$.

Example 1. *Consider problem* (5.3) *with* $L = 1$ *and*

$$a(x,t) = \cos\pi x/2 + t, \quad \tilde{a}(x,t) = x + t, \quad b(x,t) = t^{1/3} + \sin\pi x,$$

$$\mathcal{K}(t) = t^{-1/3}, \quad v_0(x) = \cos\pi x, \quad f(x,t,v) = xt\sin(v^2),$$

$$g(x,t) = 1 + \pi^2\Big(\cos\frac{\pi x}{2} + t + \frac{3t^{2/3}\sin(\pi x)}{2} + \frac{t\pi}{3\sin\pi/3}\Big)\cos\pi x$$

$$-(x+t)\pi\sin\pi x - xt\sin\Big(\cos\pi x + \frac{t^\nu}{\Gamma(1+\nu)}\Big)^2.$$

In this example, we test our reconstruction algorithm for different values of the memory order $\nu = 0.05$, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9.

It is easy to verify that the function $v(x,t) = \cos\pi x + \frac{t^\nu}{\Gamma(1+\nu)}$ solves direct problem (5.3) with the parameters specified above. The solution is observed at the space location $x_0 = 0.70$. Then, simple calculations show that for this example the error bound provided by Proposition 4.3 in **STN** case has the value

$$\frac{C_1\delta}{||\mathcal{L}_1 v_0(x_0)|_{t=0} + f(x_0, 0, v_0(x_0)) + g(x_0, 0)| - C_1\delta|} = 0.031. \tag{5.6}$$

Although in Example 1 the analytic form of the solution $v(x,t)$ is known, we generate synthetic noisy data by using numerical scheme (5.4). Of course, this increases the noise level and makes the test even harder. Nevertheless, as it can be seen from Tables 1 and 2, the accuracy of our reconstruction algorithm has the order predicted by Proposition 4.3 and bound (5.6). This is remarkable, because Proposition 4.3 presupposes the availability of continuous data, while our algorithm operates only with discrete data and does not use any information about the noise level.

$\lvert\nu - \nu_\delta(\lambda_{i_{j_0}}, \tilde{t}_{j_0})\rvert$				ν
C1	**C2**	**C3**	**C4**	
0.0046	0.0047	0.0031	0.0031	0.05
0.0117	0.0005	0.0044	0.0044	0.1
0.0169	0.0091	0.0096	0.0092	0.2
0.0226	0.0162	0.0119	0.0110	0.3
0.0254	0.0213	0.0134	0.0106	0.4
0.0260	0.0220	0.0144	0.0128	0.5
0.0260	0.0217	0.0148	0.0122	0.6
0.0252	0.0210	0.0140	0.0115	0.7
0.0244	0.0237	0.0151	0.0110	0.8
0.0256	0.0242	0.0187	0.0155	0.9

Table 1: The absolute error in Example 1 for the noise model with $\Phi(t) = t|\ln t|$

$\lvert\nu-\nu_\delta(\lambda_{i_{j_0}},\tilde{t}_{j_0})\rvert$				ν
C1	**C2**	**C3**	**C4**	
0.0325	0.0316	0.0297	0.0190	0.05
0.0341	0.0337	0.0307	0.0301	0.1
0.0417	0.0327	0.0337	0.0333	0.2
0.0467	0.0311	0.0359	0.0347	0.3
0.0500	0.0453	0.0359	0.0325	0.4
0.0519	0.0453	0.0394	0.0364	0.5
0.0477	0.0439	0.0383	0.0371	0.6
0.0453	0.0415	0.0380	0.0355	0.7
0.0411	0.0400	0.0346	0.0299	0.8
0.0367	0.0345	0.0317	0.0301	0.9

Table 2: The absolute error in Example 1 for the noise model with $\Phi(t)=t^\nu\lvert\ln t\rvert$

Example 2. *Consider problem* (5.3) *with* $L=1$ *and* $\nu=0.3,\ 0.9,$ *and*

$$a(x,t)=1,\quad \tilde{a}(x,t)=0,$$

$$\mathcal{K}=0,\quad b(x,t)=1,$$

$$f(x,t,v)=0\quad g(x,t)=100t(-2x^3+3x^2)\quad \textit{and}\quad v_0(x)=-\frac{2}{3}x^3+x^2+1.$$

In this example, an analytic form of the solution is unknown and we use the numerical scheme (5.4) to generate measurements $v(x_0,t)$ at the point $x_0=0.20$, for which

$$\frac{C_1\delta}{\lvert\lvert\mathcal{L}_1v_0(x_0)|_{t=0}+f(x_0,0,v_0(x_0))+g(x_0,0)\rvert-C_1\delta\rvert}=0.026$$

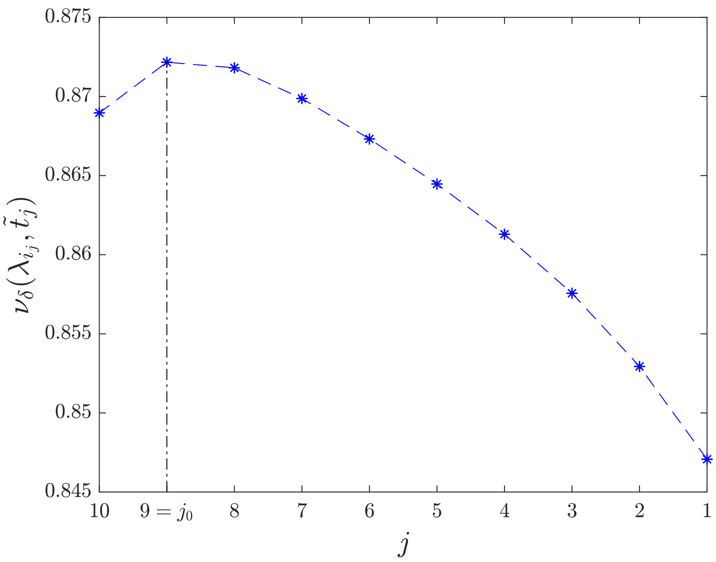
and, hence, condition (3.5) holds.

For this example the corresponding absolute errors are listed in Tables 3, 4. Moreover, in Figure 1 we demonstrate the work of our algorithm in the final step for the order $\nu=0.9$ and **C4** distribution. These plots show the dependence of the value $\nu_\delta(\lambda_{i_j},\tilde{t}_j)$ (where the quantity λ_{i_j} is chosen by the quasi-optimality criterion for each $\tilde{t}_j$) on the time step j, $j=1,2,...10$.

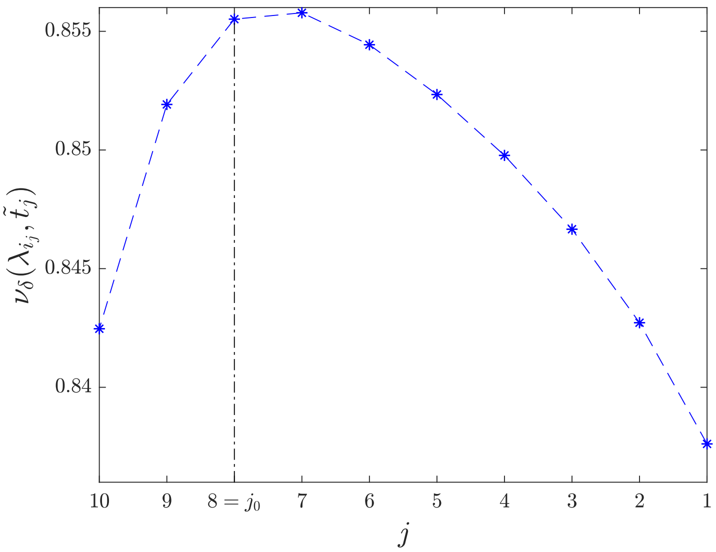
As one can see from Figure 1, in the considered example the quasi-optimality criterion provides the best (for noise model with $\Phi=t\lvert\ln t\rvert$) or almost the best (for noise model with $\Phi=t^\nu\lvert\ln t\rvert$) choice among the constructed approximations.

$\lvert\nu-\nu_\delta(\lambda_{i_{j_0}},\tilde{t}_{j_0})\rvert$				ν
C1	**C2**	**C3**	**C4**	
0.0557	0.0448	0.0429	0.0390	0.3
0.0659	0.0615	0.0524	0.0445	0.9

Table 3: The absolute error in Example 2 for the noise model with $\Phi(t)=t^\nu\lvert\ln t\rvert$



(a) noise model with $\Phi(t) = t|\ln t|$



(b) noise model with $\Phi(t) = t^\nu|\ln t|$

Figure 1: The final step in the regularized reconstruction algorithm for Example 2, $\nu = 0.9$, with (a) noise model with $\Phi(t) = t|\ln t|$, (b) noise model with $\Phi(t) = t^\nu|\ln t|$.

$\lvert\nu-\nu_\delta(\lambda_{i_{j_0}},\tilde{t}_{j_0})\rvert$				ν
C1	**C2**	**C3**	**C4**	
0.0908	0.0880	0.0221	0.0165	0.3
0.0568	0.0545	0.0424	0.0278	0.9

Table 4: The absolute error in Example 2 for the noise model with $\Phi(t)=t\lvert\ln t\rvert$

Example 3. *In this example we consider problem* (1.1)-(1.6) *in the two-dimensional domain* $\Omega=(0,1)\times(0,1)$*:*

$$\begin{cases} \mathbf{D}_t^\nu v - v_{xx} - v_{yy} - \frac{t^{-\nu}}{\Gamma(1-\nu)}\star[v_{xx}+v_{yy}] = [\cos\pi x+\cos\pi y][\Gamma(1+\nu)+\pi^2(1+t^\nu) \\ +\pi^2 t(1+\Gamma(1+\nu)) + \frac{(1+\pi^2)t^{1-\nu}}{\Gamma(2-\nu)} + \frac{\pi^2 t^{2-\nu}}{\Gamma(3-\nu)}] \qquad \text{in} \quad \Omega\times(0,T), \\ v(x,y,0)=\cos\pi x+\cos\pi y, \qquad (x,y)\in[0,1]\times[0,1], \\ v_x(0,y,t)=v_x(1,y,t)=0, \qquad t\in[0,T],\ y\in[0,1], \\ v_y(x,0,t)=v_y(x,1,t)=0, \qquad t\in[0,T],\ x\in[0,1]. \end{cases}$$

In this case, the exact solution is represented as $v(x,y,t)=(\cos\pi x+\cos\pi y)(1+t+t^\nu)$, and this solution will be observed in the point $(x_0,y_0)=(0.65,0.65)$. In this point we have

$$\frac{C_1\delta}{\lvert\lvert\mathcal{L}_1 v_0(x_0)\rvert_{t=0}+f(x_0,0,v_0(x_0))+g(x_0,0)\rvert - C_1\delta\rvert} = \begin{cases} 0.0373 & \text{for} \quad \nu=0.2, \\ 0.0387 & \text{for} \quad \nu=0.5, \\ 0.0355 & \text{for} \quad \nu=0.9, \end{cases}$$

which mean that condition (3.5) is satisfied.

Tables 5 and 6 list the outputs for this example in the case of the noises generated with noise models $\Phi(t)=t^\nu\lvert\ln t\rvert$ and $\Phi(t)=t\lvert\ln t\rvert$, correspondingly.

$\lvert\nu-\nu_\delta(\lambda_{i_{j_0}},\tilde{t}_{j_0})\rvert$				ν
C1	**C2**	**C3**	**C4**	
0.0507	0.0513	0.0510	0.0496	0.2
0.0376	0.0456	0.0474	0.0485	0.5
0.0509	0.0391	0.0143	0.0009	0.9

Table 5: The absolute error in Example 3 for the noise model with $\Phi(t)=t^\nu\lvert\ln t\rvert$

$\lvert\nu-\nu_\delta(\lambda_{i_{j_0}},\tilde{t}_{j_0})\rvert$				ν
C1	**C2**	**C3**	**C4**	
0.0163	0.0162	0.0104	0.0102	0.2
0.0073	0.0089	0.0107	0.0096	0.5
0.0596	0.0490	0.0258	0.0095	0.9

Table 6: The absolute error in Example 3 for the noise model with $\Phi(t)=t\lvert\ln t\rvert$

6. Discussion and conclusion

In this paper, we propose an approach to reconstruct the semilinear subdiffusion order. To this end, analyzing boundary value problems for the nonautonomous semilinear subdiffusion equations with memory terms in the fractional Hölder spaces, we obtain an explicit reconstruction formula for the order ν in terms of the smooth observation data for small time. Then, based on the Tikhonov regularization scheme and the quasi-optimality criterion, we construct the computational algorithm to find the order ν from noisy discrete measurements.

The computational results demonstrate that the proposed method effectively determines the unknown memory order ν. Moreover, from the methodological view point, the proposed approach can provide an efficient numerical technique for estimating the limit of the ratio of two unbounded noisy functions.

7. Appendix: Proof of Proposition 4.1

First, we note that the statements **(i)**-**(iii)** have been obtained in the proof of Proposition 1 [22]. Moreover, the statement **(v)** follows immediately from statement **(iv)**.

Hence, we are left to verify the inequality **(iv)**. To this end, we first rewrite the difference $(\mathbf{D}_t^{\nu_2}U - \mathbf{D}_t^{\nu_1}U)$ as follows:

$$\begin{aligned}\mathbf{D}_t^{\nu_2}U(t) - \mathbf{D}_t^{\nu_1}U(t) &= \mathbf{D}_t^{\nu_2}U(t) - \Theta_{\nu_2-\nu_1}\star\mathbf{D}_t^{\nu_2}U(t) \\ &= \Big[1 - \frac{t^{\nu_2-\nu_1}}{\Gamma(\nu_2-\nu_1+1)}\Big]\mathbf{D}_t^{\nu_2}U(t) \\ &+ \int_0^t \frac{(t-\tau)^{\nu_2-\nu_1-1}}{\Gamma(\nu_2-\nu_1)}[\mathbf{D}_t^{\nu_2}U(t) - \mathbf{D}_\tau^{\nu_2}U(\tau)]d\tau \\ &\equiv \sum_{j=1}^{3}\mathcal{R}_j,\end{aligned} \tag{7.1}$$

where we put

$$\begin{aligned}\mathcal{R}_1 &= \frac{\Gamma(\nu_2-\nu_1+1)-1}{\Gamma(\nu_2-\nu_1+1)}\mathbf{D}_t^{\nu_2}U(t), \\ \mathcal{R}_2 &= \frac{1-t^{\nu_2-\nu_1}}{\Gamma(\nu_2-\nu_1+1)}\mathbf{D}_t^{\nu_2}U(t), \\ \mathcal{R}_3 &= \int_0^t \frac{(t-\tau)^{\nu_2-\nu_1-1}}{\Gamma(\nu_2-\nu_1)}[\mathbf{D}_t^{\nu_2}U(t) - \mathbf{D}_\tau^{\nu_2}U(\tau)]d\tau.\end{aligned}$$

Now it is enough to estimate each term $\mathcal{R}_j$ separately.

- By inequalities **(i)** in Proposition 4.1,

$$\mathcal{R}_1 \leq C(\nu_2-\nu_1)\|\mathbf{D}_t^{\nu_2}U\|_{\mathcal{C}[0,T]},$$

where C is the positive constant.

- Concerning $\mathcal{R}_2$, the simple straightforward calculations give

$$|1-t^{\nu_2-\nu_1}| \leq C(\nu_2-\nu_1)\begin{cases} t & \text{if} \quad t>1, \\ |\ln t| & \text{if} \quad t\leq 1,\end{cases}$$

and from this inequality, we easily draw the estimate

$$\mathcal{R}_2 \leq C(\nu_2 - \nu_1)(t + |\ln t|)\|\mathbf{D}_t^{\nu_2} U\|_{\mathcal{C}[0,T]}.$$

• For $\mathcal{R}_3$, we have

$$\mathcal{R}_3 \leq C\Big|\int_0^t \frac{(t-\tau)^{\nu_2-\nu_1-1+\gamma}}{\Gamma(\nu_2-\nu_1)} d\tau\Big| \langle \mathbf{D}_t^{\nu_2} U\rangle^{(\gamma)}_{t,[0,T]}$$

$$\leq C(\nu_2-\nu_1)\frac{t^{\nu_2-\nu_1+\gamma}}{(\nu_2-\nu_1+\gamma)\Gamma(1+\nu_2-\nu_1)}\langle \mathbf{D}_t^{\nu_2} U\rangle^{(\gamma)}_{t,[0,T]}.$$

Summarizing the above estimates, we obtain the statement **(iv)**.

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