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Order estimates of best orthogonal trigonometric approximations of classes of infinitely differentiable functions

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Abstract In this paper we establish exact order estimates for the best uniform orthogonal trigonometric approximations of the classes of 2π-periodic functions, whose (ψ,β)–derivatives belong to unit balls of spaces L_p, $1\leq p<\infty$, in the case, when the sequence $\psi(k)$ tends to zero faster, than any power function, but slower than geometric progression. Similar estimates are also established in the L_s-metric, $1<s\leq\infty$, for the classes of differentiable functions, which (ψ,β)–derivatives belong to unit ball of space L_1.

1 Introduction

Let C be a space of 2π–periodic continuous functions with the following norm: $\|f\|_C := \max\limits_{t\in[0,2\pi)}|f(t)|$; L_∞ be the space of 2π–periodic functions f, which are Lebesgue measurable and essentially bounded with the norm $\|f\|_\infty := \operatorname{ess}\sup\limits_t |f(t)|$ and L_p, $1\leq p<\infty$, be the space of 2π–periodic functions f summable to the power p on $[0,2\pi)$, with the norm $\|f\|_p := \Big(\int\limits_0^{2\pi}|f(t)|^p dt\Big)^{\frac{1}{p}}$.

Let $f:\mathbb{R}\to\mathbb{R}$ be the function from L_1, whose Fourier series is given by

$$\sum_{k=-\infty}^{\infty}\hat{f}(k)e^{ikx},$$

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where $\hat{f}(k)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}f(t)e^{-ikt}dt$ are the Fourier coefficients of the function f, $\psi(k)$ is an arbitrary fixed sequence of real numbers and β is a fixed real number. Then, if the series

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{\hat{f}(k)}{\psi(|k|)}e^{i(kx+\frac{\beta\pi}{2}\operatorname{sign}k)}$$

is the Fourier series of some function φ from L_1, then this function is called the (ψ,β)–derivative of the function f and is denoted by f^{ψ}_{β}. A set of functions f, whose (ψ,β)–derivatives exist, is denoted by L^{ψ}_{β} (see [16]).

Let

$$B^0_p:=\left\{\varphi\in L_p:\ ||\varphi||_p\leq 1,\ \varphi\perp 1\right\},\ \ 1\leq p\leq\infty.$$

If $f\in L^{\psi}_{\beta}$, and, at the same time $f^{\psi}_{\beta}\in B^0_p$, then we say that the function f belongs to the class $L^{\psi}_{\beta,p}$.

Denote $C^{\psi}_{\beta}=C\cap L^{\psi}_{\beta}$ and $C^{\psi}_{\beta,p}=C\cap L^{\psi}_{\beta,p}$.

By $\mathfrak{M}$ we denote the set of all convex (downward) continuous functions $\psi(t)$, $t\geq 1$, such that $\lim\limits_{t\to\infty}\psi(t)=0$. Assume that the sequence $\psi(k),\ k\in\mathbb{N}$, specifying the class $L^{\psi}_{\beta,p},\ 1\leq p\leq\infty$, is the restriction of the functions $\psi(t)$ from $\mathfrak{M}$ to the set of natural numbers.

Following Stepanets (see, e.g., [16]), by using the characteristic $\mu(\psi;t)$ of functions ψ from $\in\mathfrak{M}$ of the form

$$\mu(t)=\mu(\psi;t):=\frac{t}{\eta(t)-t},\tag{1}$$

where $\eta(t)=\eta(\psi;t):=\psi^{-1}\left(\psi(t)/2\right)$, ψ^{-1} is the function inverse to ψ, we select the following subsets of the set $\mathfrak{M}$:

$$\mathfrak{M}^+_{\infty}=\left\{\psi\in\mathfrak{M}:\quad \mu(\psi;t)\uparrow\infty\right\}.$$

$$\mathfrak{M}''_{\infty}=\left\{\psi\in\mathfrak{M}^+_{\infty}:\quad \exists K>0\ \ \eta(\psi;t)-t\geq K\quad t\geq 1\right\}.$$

The functions $\psi_{r,\alpha}(t)=\exp(-\alpha t^r)$ are typical representatives of the set $\mathfrak{M}^+_{\infty}$. Moreover, if $r\in(0,1]$, then $\psi_{r,\alpha}\in\mathfrak{M}''_{\infty}$. The classes $L^{\psi}_{\beta,p}$, generated by the functions $\psi=\psi_{r,\alpha}$ are denoted by $L^{\alpha,r}_{\beta,p}$.

If $\psi\in\mathfrak{M}^+_{\infty}$ then (see, e.g., [15, p. 97]) the function $\psi(t)$ vanishes faster than any power function, i.e.,

$$\lim_{t\to\infty}t^r\psi(t)=0\quad\forall r\in\mathbb{R}.$$

This implies that, under the condition $\psi\in\mathfrak{M}^+_{\infty}$, the Fourier series of any function f from $C^{\psi}_{\beta,p}$, $\beta\in\mathbb{R}$, can be differentiated infinitely many times and, as a result, we get uniformly convergent series. Hence, the classes $C^{\psi}_{\beta,p}$ with $\psi\in\mathfrak{M}^+_{\infty}$ consist of infinitely differentiable functions. On the other hand, as shown in [17, p. 1692], for any infinitely differentiable 2π–periodic function f, one can indicate a function from the set $\mathfrak{M}_\infty^+$ such that $f \in C^\psi_\beta$ for any $\beta \in \mathbb{R}$.

For functions f from classes $L^\psi_{\beta,p}$ we consider: L_s–norms of deviations of the functions f from their partial Fourier sums of order $n-1$, i.e., the quantities

$$\|\rho_n(f;\cdot)\|_s = \|f(\cdot) - S_{n-1}(f;\cdot)\|_s, \;\; 1 \le s \le \infty, \tag{2}$$

where

$$S_{n-1}(f;x) = \sum_{k=-n+1}^{n-1} \hat{f}(k) e^{ikx};$$

and the best orthogonal trigonometric approximations of the functions f in metric of space L_s, i.e., the quantities of the form

$$e^\perp_m(f)_s = \inf_{\gamma_m} \|f(\cdot) - S_{\gamma_m}(f;\cdot)\|_s, \quad 1 \le s \le \infty, \tag{3}$$

where γ_m, $m \in \mathbb{N}$, is an arbitrary collection of m integer numbers, and

$$S_{\gamma_m}(f;x) = \sum_{k\in\gamma_m} \hat{f}(k) e^{ikx}.$$

We set

$$\mathscr{E}_n(L^\psi_{\beta,p})_s = \sup_{f\in L^\psi_{\beta,p}} \|\rho_n(f;\cdot)\|_s, \;\; 1 \le p,s \le \infty, \tag{4}$$

$$e^\perp_n(L^\psi_{\beta,p})_s = \sup_{f\in L^\psi_{\beta,p}} e^\perp_n(f)_s, \; 1 \le p,s \le \infty. \tag{5}$$

It is clear, that if $f \in C$, then

$$\|\rho_n(f;x)\|_\infty = \|\rho_n(f;x)\|_C, \quad e^\perp_m(f)_C = e^\perp_m(f)_\infty.$$

That is why

$$\mathscr{E}_n(C^\psi_{\beta,p})_C = \mathscr{E}_n(C^\psi_{\beta,p})_\infty, \quad e^\perp_m(C^\psi_{\beta,p})_C = e^\perp_m(C^\psi_{\beta,p})_\infty.$$

The following inequalities follow from given above definitions (4) and (5)

$$e^\perp_{2n}(L^\psi_{\beta,p})_s \le e^\perp_{2n-1}(L^\psi_{\beta,p})_s \le \mathscr{E}_n(L^\psi_{\beta,p})_s, \; 1 \le p,s \le \infty. \tag{6}$$

In the case when $\psi(k) = k^{-r}$, $r > 0$, the classes $L^\psi_{\beta,p}$, $1 \le p \le \infty$, $\beta \in \mathbb{R}$, are well–known Weyl–Nagy classes $W^r_{\beta,p}$. For these classes, the order estimates of quantities $e^\perp_n(L^\psi_{\beta,p})_s$ are known for $1 < p,s < \infty$ (see [4], [5]), for $1 \le p < \infty$, $s = \infty$, $r > \frac{1}{p}$ and also for $p = 1$, $1 < s < \infty$, $r > \frac{1}{s'}$, $\frac{1}{s} + \frac{1}{s'} = 1$ (see [5], [6]).

In the case, when $\psi(k)$ tends to zero not faster than some power function, order estimates for quantities (5) were established in [1], [10], [12]–[14]. In the case, when

$\psi(k)$ tends to zero not slower than geometric progression, exact order estimates for $e_n^{\perp}(L^{\psi}_{\beta,p})_s$ were found in [11] for all $1 \le p,s \le \infty$.

Our aim is to establish the exact-order estimates of $e_n^{\perp}(L^{\psi}_{\beta,p})_{\infty}$, $1 \le p < \infty$, and $e_n^{\perp}(L^{\psi}_{\beta,1})_s$, $1 < s < \infty$, in the case, when ψ decreases faster than any power function, but slower than geometric progression ($\psi \in \mathfrak{M}''_{\infty}$).

2 Best orthogonal trigonometric approximations of the classes $L^{\psi}_{\beta,p}$, $1 < p < \infty$, in the metric of space L_{∞}

We write $a_n \asymp b_n$ to mean that there exist positive constants C_1 and C_2 independent of n such that $C_1 a_n \le b_n \le C_2 a_n$ for all n.

Theorem 1. *Let $1 < p < \infty$, $\psi \in \mathfrak{M}''_{\infty}$ and the function $\frac{\psi(t)}{|\psi'(t)|} \uparrow \infty$ as $t \to \infty$. Then, for all $\beta \in \mathbb{R}$ the following order estimates hold*

$$e_{2n-1}^{\perp}(L^{\psi}_{\beta,p})_{\infty} \asymp e_{2n}^{\perp}(L^{\psi}_{\beta,p})_{\infty} \asymp \psi(n)(\eta(n)-n)^{\frac{1}{p}}. \tag{7}$$

In Theorem 1 and further we will assume $\psi'(t) := \psi'(t+0)$.

Proof. According to Theorem 1 from [8] under conditions $\psi \in \mathfrak{M}^{+}_{\infty}$, $\beta \in \mathbb{R}$, $1 \le p < \infty$, for $n \in \mathbb{N}$, such that $\eta(n) - n \ge a > 2$, $\mu(n) \ge b > 2$ the following estimate is true

$$\mathscr{E}_n(L^{\psi}_{\beta,p})_{\infty} \le K_{a,b}\ (2p)^{1-\frac{1}{p}}\psi(n)(\eta(n)-n)^{\frac{1}{p}}, \tag{8}$$

where

$$K_{a,b} = \frac{1}{\pi}\max\left\{\frac{2b}{b-2} + \frac{1}{a},\ 2\pi\right\}.$$

It should be noticed, that condition $\frac{\psi(t)}{|\psi'(t)|} \to \infty$ as $t \to \infty$ implies that always exists $n_0 \in \mathbb{N}$, such that for all $n > n_0$, $n \in \mathbb{N}$ the following inequalities take place: $\eta(\psi,n) - n \ge a > 2$ and $\mu(\psi,n) \ge b > 2$. This fact follows from Remark 3.13.1 [16], which says, that for every $\psi \in \mathfrak{M}^{+}_{\infty}$

$$K_1(\eta(t)-t) \le \frac{\psi(t)}{|\psi'(t)|} \le K_2(\eta(t)-t), K_1,\ K_2 > 0, \tag{9}$$

and from the definition of quantity $\mu(\psi,t)$.

Using inequalities (6) and (8), we obtain

$$e_{2n}^{\perp}(L^{\psi}_{\beta,p})_{\infty} \le e_{2n-1}^{\perp}(L^{\psi}_{\beta,p})_{\infty} \le K_{a,b}\ (2p)^{1-\frac{1}{p}}\psi(n)(\eta(n)-n)^{\frac{1}{p}}. \tag{10}$$

Let us find the lower estimate for the quantity $e^{\perp}_{2n}(L^{\psi}_{\beta,p})_{\infty}$. With this purpose we construct the function

$$f^*_{p,n}(t)=f^*_{p,n}(\psi;t):=\frac{\lambda_p}{\psi(n)(\eta(n)-n)^{\frac{1}{p'}}}\Big(\frac{1}{2}\psi(1)\psi(2n)+$$

$$+\sum_{k=1}^{n-1}\psi(k)\psi(2n-k)\cos kt+\sum_{k=n}^{2n}\psi^2(k)\cos kt\Big),\ \ \frac{1}{p}+\frac{1}{p'}=1. \tag{11}$$

Let us show that $f^*_{p,n}\in L^{\psi}_{\beta,p}$. The definition of (ψ,β)–derivative yields

$$(f^*_{p,n}(t))^{\psi}_{\beta}=\frac{\lambda_p}{\psi(n)(\eta(n)-n)^{\frac{1}{p'}}}\Big(\sum_{k=1}^{n-1}\psi(2n-k)\cos\Big(kt+\frac{\beta\pi}{2}\Big)$$

$$+\sum_{k=n}^{2n}\psi(k)\cos\Big(kt+\frac{\beta\pi}{2}\Big)\Big). \tag{12}$$

Obviously

$$\big|(f^*_{p,n}(t))^{\psi}_{\beta}\big|\le\frac{\lambda_p}{\psi(n)(\eta(n)-n)^{\frac{1}{p'}}}\Big(\sum_{k=1}^{n-1}\psi(2n-k)+\sum_{k=n}^{2n}\psi(k)\Big)<$$

$$<\frac{2\lambda_p}{\psi(n)(\eta(n)-n)^{\frac{1}{p'}}}\sum_{k=n}^{2n}\psi(k)\le\frac{2\lambda_p}{\psi(n)(\eta(n)-n)^{\frac{1}{p'}}}\Big(\psi(n)+\int\limits_{n}^{\infty}\psi(u)du\Big). \tag{13}$$

To estimate the integral from the right part of formula (13), we use the following statement [7, p. 500].

Proposition 1. *If $\psi\in\mathfrak{M}^+_{\infty}$, then for arbitrary $m\in\mathbb{N}$, such that $\mu(\psi,m)>2$ the following condition holds*

$$\int\limits_{m}^{\infty}\psi(u)du\le\frac{2}{1-\frac{2}{\mu(m)}}\psi(m)(\eta(m)-m). \tag{14}$$

Formulas (13) and (14) imply that

$$\big|(f^*_{p,n}(t))^{\psi}_{\beta}\big|\le\frac{2\lambda_p}{\psi(n)(\eta(n)-n)^{\frac{1}{p'}}}\Big(\psi(n)+\frac{2b}{b-2}\psi(n)(\eta(n)-n)\Big)<$$

$$<\frac{5\lambda_p b}{b-2}(\eta(n)-n)^{\frac{1}{p}}. \tag{15}$$

We denote

$$D_{k,\beta}(t):=\frac{1}{2}\cos\frac{\beta\pi}{2}+\sum_{j=1}^{k}\cos\Big(jt+\frac{\beta\pi}{2}\Big). \tag{16}$$

Applying Abel transform, we have

$$\sum_{k=1}^{n-1}\psi(2n-k)\cos\left(kt+\frac{\beta\pi}{2}\right)=\sum_{k=1}^{n-2}(\psi(2n-k)-\psi(2n-k+1))D_{k,\beta}(t)$$
$$+\psi(n+1)D_{n-1,\beta}(t)-\psi(2n-1)\frac{1}{2}\cos\frac{\beta\pi}{2} \quad (17)$$

and

$$\sum_{k=n}^{2n}\psi(k)\cos\left(kt+\frac{\beta\pi}{2}\right)=\sum_{k=n}^{2n-1}(\psi(k)-\psi(k+1))D_{k,\beta}(t)$$
$$+\psi(2n)D_{2n,\beta}(t)-\psi(n)D_{n-1,\beta}(t). \quad (18)$$

Since

$$\sum_{k=0}^{N-1}\sin(\gamma+kt)=\sin\left(\gamma+\frac{N-1}{2}t\right)\sin\frac{Nt}{2}\frac{1}{\sin\frac{t}{2}} \quad (19)$$

(see, e.g., [2, p.43]), for $N=k+1$, $\gamma=(\beta-1)\frac{\pi}{2}$, the following inequality holds

$$|D_{k,\beta}(t)|=\left|\frac{\cos\left(\frac{kt}{2}+\frac{\beta\pi}{2}\right)\sin\frac{k+1}{2}t}{\sin\frac{t}{2}}-\frac{1}{2}\cos\frac{\beta\pi}{2}\right|$$
$$=\left|\frac{\sin\left((k+\frac{1}{2})t+\frac{\beta\pi}{2}\right)-\cos\frac{t}{2}\sin\frac{\beta\pi}{2}}{2\sin\frac{t}{2}}\right|\leq\frac{\pi}{t},\quad 0<|t|\leq\pi. \quad (20)$$

According to (12), (17), (18) and (20), we obtain

$$\left|(f^*_{p,n}(t))^{\psi}_{\beta}\right|\leq\frac{\lambda_p}{\psi(n)(\eta(n)-n)^{\frac{1}{p'}}}\frac{\pi}{|t|}\left(\sum_{k=1}^{n-2}|\psi(2n-k)-\psi(2n-k-1)|+\psi(n+1)\right.$$
$$\left.+\psi(2n-1)+\sum_{k=n}^{2n-1}|\psi(k)-\psi(k+1)|+\psi(2n)+\psi(n)\right)$$
$$=\frac{\lambda_p}{\psi(n)(\eta(n)-n)^{\frac{1}{p'}}}\frac{2\pi}{|t|}(\psi(n+1)+\psi(n))\leq\frac{4\pi\lambda_p}{(\eta(n)-n)^{\frac{1}{p'}}}\frac{1}{|t|}. \quad (21)$$

So, (15) and (21) imply

$$\left\|(f^*_{p,n}(t))^{\psi}_{\beta}\right\|_p$$
$$\leq\lambda_p\max\left\{\frac{5b}{b-2},\,4\pi\right\}\left(\int\limits_{|t|\leq\frac{1}{\eta(n)-n}}(\eta(n)-n)dt+\frac{1}{(\eta(n)-n)^{\frac{p}{p'}}}\int\limits_{\frac{1}{\eta(n)-n}\leq|t|\leq\pi}\frac{dt}{|t|^p}\right)^{\frac{1}{p}}$$
$$\leq 2\lambda_p\max\left\{\frac{5b}{b-2},\,4\pi\right\}\left(1+\frac{1}{p-1}\right)^{\frac{1}{p}}=2\lambda_p\max\left\{\frac{5b}{b-2},\,4\pi\right\}(p')^{\frac{1}{p}}.$$

Hence, for

$$\lambda_p = \frac{1}{2(p')^{\frac{1}{p}} \max\left\{\frac{5b}{b-2},\, 4\pi\right\}}$$

the embedding $f^*_{p,n} \in L^{\psi}_{\beta,p}$ is true.

Let us consider the quantity

$$I_1 := \inf_{\gamma_{2n}} \left| \int_{-\pi}^{\pi} (f^*_{p,n}(t) - S_{\gamma_{2n}}(f^*_{p,n};t)) V_{2n}(t)dt \right|, \tag{22}$$

where V_{2n} are de la Vallée-Poisson kernels of the form

$$V_m(t) := \frac{1}{2} + \sum_{k=1}^{m} \cos kt + 2 \sum_{k=m+1}^{2m-1} \left(1 - \frac{k}{2m}\right) \cos kt,\ m \in \mathbb{N}. \tag{23}$$

Proposition A1.1 from [3] implies

$$I_1 \leq \inf_{\gamma_{2n}} \|f^*_p(t) - S_{\gamma_{2n}}(f^*_{p,n};t)\|_\infty \|V_{2n}\|_1 = e^{\perp}_{2n}(f^*_{p,n})_\infty \|V_{2n}\|_1. \tag{24}$$

Since (see, e.g., [18, p.247])

$$\|V_m\|_1 \leq 3\pi,\ \ m \in \mathbb{N}, \tag{25}$$

from (24) and (25) we can write down the estimate

$$e^{\perp}_{2n}(f^*_{p,n})_\infty \geq \frac{1}{3\pi} I_1. \tag{26}$$

Notice, that

$$f^*_{p,n}(t) - S_{\gamma_{2n}}(f^*_{p,n};t) = \frac{\lambda_p}{2\psi(n)(\eta(n)-n)^{\frac{1}{p'}}} \left(\sum_{\substack{|k|\leq n-1,\\ k\notin\gamma_{2n}}} \psi(|k|)\psi(2n-|k|)e^{ikt} + \sum_{\substack{n\leq|k|\leq 2n,\\ k\notin\gamma_{2n}}} \psi^2(|k|)e^{ikt} \right), \tag{27}$$

where $\psi(0) := \psi(1)$

Whereas

$$\int_{-\pi}^{\pi} e^{ikt} e^{imt} dt = \begin{cases} 0, & k+m \neq 0, \\ 2\pi, & k+m = 0, \end{cases} \quad k,m \in \mathbb{Z}, \tag{28}$$

and taking into account (23), we obtain

$$\int_{-\pi}^{\pi} (f^*_{p,n}(t) - S_{\gamma_{2n}}(f^*_{p,n};t)) V_{2n}(t)dt \tag{29}$$

$$= \frac{\lambda_p}{4\psi(n)(\eta(n)-n)^{\frac{1}{p'}}} \int\limits_{-\pi}^{\pi} \Big(\sum_{\substack{0\le k\le n-1,\\ k\notin\gamma_{2n}}} \psi(k)\psi(2n-k)e^{ikt} + \sum_{\substack{-n+1\le k\le -1,\\ k\notin\gamma_{2n}}} \psi(|k|)\psi(2n-|k|)e^{ikt}$$

$$+ \sum_{\substack{n\le k\le 2n,\\ k\notin\gamma_{2n}}} \psi^2(k)e^{ikt} + \sum_{\substack{-2n\le k\le -n,\\ k\notin\gamma_{2n}}} \psi^2(|k|)e^{ikt} \Big) \times$$

$$\times \Big(\sum_{0\le k\le 2n} e^{ikt} + \sum_{-2n\le k\le -1} e^{ikt} + 2\sum_{2n+1\le |k|\le 4n-1} \Big(1-\frac{|k|}{4n}\Big)e^{ikt}\Big)dt \qquad (30)$$

$$= \frac{\lambda_p \pi}{2\psi(n)(\eta(n)-n)^{\frac{1}{p'}}} \Big(\sum_{\substack{|k|\le n-1,\\ k\notin\gamma_{2n}}} \psi(|k|)\psi(2n-|k|) + \sum_{\substack{n\le |k|\le 2n,\\ k\notin\gamma_{2n}}} \psi^2(|k|)\Big). \qquad (31)$$

The function $\phi_n(t) := \psi(t)\psi(2n-t)$ decreases for $t\in[1,n]$. Indeed

$$\phi_n'(t) = |\psi'(t)||\psi'(2n-t)|\Big(\frac{\psi(t)}{|\psi'(t)|} - \frac{\psi(2n-t)}{|\psi'(2n-t)|}\Big) \le 0,$$

because $\frac{\psi(t)}{|\psi'(t)|}\uparrow\infty$ for large n.

Thus, the monotonicity of function $\phi_n(t)$ and (31) imply

$$I_1 = \frac{\pi\lambda_p}{2\psi(n)(\eta(n)-n)^{\frac{1}{p'}}} \Big(\psi^2(n) + \sum_{n+1\le |k|\le 2n} \psi^2(|k|)\Big)$$

$$> \frac{\pi\lambda_p}{2\psi(n)(\eta(n)-n)^{\frac{1}{p'}}} \sum_{k=n}^{2n} \psi^2(k) \ge \frac{\pi\lambda_p}{2\psi(n)(\eta(n)-n)^{\frac{1}{p'}}} \int\limits_n^{\eta(n)} \psi^2(t)dt$$

$$> \frac{\pi\lambda_p}{2\psi(n)(\eta(n)-n)^{\frac{1}{p'}}} \psi^2(\eta(n))(\eta(n)-n) = \frac{\pi\lambda_p}{8}\psi(n)(\eta(n)-n)^{\frac{1}{p}}. \qquad (32)$$

By considering (26) and (32) we can write

$$e_{2n}^{\perp}(L^{\psi}_{\beta,p})_{\infty} \ge e_{2n}^{\perp}(f^{*}_{p,n})_{\infty} \ge \frac{1}{3\pi} I_1 \ge \frac{\lambda_p}{24}\psi(n)(\eta(n)-n)^{\frac{1}{p}}. \qquad (33)$$

Theorem 1 is proved.

In fact in the proof of Theorem 1 we obtained estimates with constants in explicit form.

Proposition 2. *Let $\psi\in\mathfrak{M}^{+}_{\infty}$, $\beta\in\mathbb{R}$, $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$, and the function $\frac{\psi(t)}{|\psi'(t)|}$ increases monotonically. Then for $n\in\mathbb{N}$, such that $\mu(\psi,n)\ge b>2$ and $\eta(\psi,n)-n\ge a>2$, the following estimates hold*

$$K_{b,p}\psi(n)(\eta(n)-n)^{\frac{1}{p}} \le e_{2n}^{\perp}(L^{\psi}_{\beta,p})_{\infty} \le e_{2n-1}^{\perp}(L^{\psi}_{\beta,p})_{\infty} \le K_{a,b,p}\psi(n)(\eta(n)-n)^{\frac{1}{p}}, \qquad (34)$$

where

$$K_{a,b,p} = \frac{1}{\pi} \max\left\{ \frac{2b}{b-2} + \frac{1}{a},\ 2\pi \right\} (2p)^{\frac{1}{p'}}. \tag{35}$$

$$K_{b,p} = \frac{1}{48 \max\left\{ \frac{5b}{b-2},\ 4\pi \right\} (p')^{\frac{1}{p}}}. \tag{36}$$

3 Best orthogonal trigonometric approximations of the classes $L^{\psi}_{\beta,1}$ in the metric of space L_∞

Theorem 2. *Let $\psi \in \mathfrak{M}^{+}_{\infty}$. Then for all $\beta \in \mathbb{R}$ order estimates are true*

$$e^{\perp}_{2n-1}(L^{\psi}_{\beta,1})_\infty \asymp e^{\perp}_{2n}(L^{\psi}_{\beta,1})_\infty \asymp \psi(n)(\eta(n)-n). \tag{37}$$

Proof. According to formula (48) from [18] under conditions $\psi \in \mathfrak{M}$, $\sum\limits_{k=1}^{\infty} \psi(k) < \infty$, $\beta \in \mathbb{R}$, for all $n \in \mathbb{N}$ the following estimate holds

$$\mathscr{E}_n(L^{\psi}_{\beta,1})_\infty \le \frac{1}{\pi} \sum_{k=n}^{\infty} \psi(k). \tag{38}$$

Using Proposition 1, we have

$$e^{\perp}_{2n}(L^{\psi}_{\beta,1})_\infty \le e^{\perp}_{2n-1}(L^{\psi}_{\beta,1})_\infty \le \mathscr{E}_n(L^{\psi}_{\beta,1})_\infty \le \frac{1}{\pi} \sum_{k=n}^{\infty} \psi(k)$$
$$\le \frac{1}{\pi}\left(\psi(n) + \int_n^{\infty} \psi(u)du \right) \le \frac{\psi(n)}{\pi}\left(1 + \frac{b}{b-2}(\eta(n)-n) \right). \tag{39}$$

Let us find the lower estimate for the quantity $e^{\perp}_{2n}(L^{\psi}_{\beta,1})_\infty$.
We consider the quantity

$$I_2 := \inf_{\gamma_{2n}} \left| \int_{-\pi}^{\pi} (f^*_{2n}(t) - S_{\gamma_{2n}}(f^*_{2n};t)) V_{2n}(t) dt \right|, \tag{40}$$

where V_m are de la Vallée-Poisson kernels of the form (23), and

$$f^*_m(t) = f^*_m(\psi;t) := \frac{1}{5\pi m}\Big(\frac{1}{2}\psi(1) + \sum_{k=1}^{m} k\psi(k)\cos kt + \sum_{k=m+1}^{2m} (2m+1-k)\psi(k)\cos kt \Big). \tag{41}$$

In [18, p. 263–265] it was shown that $\|(f^*_m)^{\psi}_{\beta}\|_1 \le 1$, i.e., f^*_m belongs to the class $L^{\psi}_{\beta,1}$ for all $m \in \mathbb{N}$.

Using Proposition A1.1 from [3] and inequality (25), we have

$$I_2 \leq \inf_{\gamma_{2n}} \|f^*_{2n}(t) - S_{\gamma_{2n}}(f^*_{2n};t)\|_\infty \|V_{2n}\|_1 \leq 3\pi\, e^\perp_{2n}(f^*_{2n})_\infty. \tag{42}$$

Assuming again $\psi(0) := \psi(1)$, from (23) and (41), we derive

$$I_2 = \frac{1}{20\pi n} \inf_{\gamma_{2n}} \Bigg| \int_{-\pi}^{\pi} \Big(\sum_{\substack{|k|\leq 2n,\\ k\notin\gamma_{2n}}} |k|\psi(|k|)e^{ikt} + \sum_{\substack{2n+1\leq|k|\leq 4n,\\ k\notin\gamma_{2n}}} (4n+1-|k|)\psi(|k|)e^{ikt}\Big) \times$$

$$\times \Big(\sum_{|k|\leq 2n} e^{ikt} + 2 \sum_{2n+1\leq|k|\leq 4n-1} \Big(1-\frac{|k|}{4n}\Big)e^{ikt}\Big)dt \Bigg|$$

$$= \frac{1}{10n} \inf_{\gamma_{2n}} \Bigg(\sum_{\substack{|k|\leq 2n,\\ k\notin\gamma_{2n}}} |k|\psi(|k|) + \sum_{\substack{2n+1\leq|k|\leq 4n,\\ k\notin\gamma_{2n}}} \Big(1-\frac{|k|}{4n}\Big)(4n+1-|k|)\psi(|k|)\Bigg)$$

$$> \frac{1}{10n} \inf_{\gamma_{2n}} \sum_{\substack{|k|\leq 2n,\\ k\notin\gamma_{2n}}} |k|\psi(|k|) = \frac{1}{10n}\Bigg(n\psi(n) + 2\sum_{k=n+1}^{2n} k\psi(k)\Bigg)$$

$$> \frac{1}{10}\sum_{k=n}^{2n}\psi(k) > \frac{1}{10}\int_{n}^{\eta(n)} \psi(t)dt > \frac{1}{20}\psi(n)(\eta(n)-n), \tag{43}$$

where we have used, that function $t\psi(t)$ decreases monotonically from some number t_0. Indeed,

$$(t\psi(t))' = |\psi'(t)|t\Big(\frac{\psi(t)}{|\psi'(t)|} - 1\Big),$$

and relations (1) and (9) yield

$$\frac{\psi(t)}{|\psi'(t)|} \asymp \frac{\eta(t)-t}{t} = \frac{1}{\mu(t)} \to 0, \text{ as } t\to\infty \text{ for } \psi \in \mathfrak{M}^+_\infty.$$

Formulas (42) and (43) imply, that for $n > t_0$

$$e^\perp_{2n}(L^\psi_{\beta,1})_\infty \geq e^\perp_{2n}(f^*_{2n})_\infty \geq \frac{1}{3\pi}I_2 > \frac{1}{60\pi}\psi(n)(\eta(n)-n).$$

Theorem 2 is proved.

Corollary 1. *Let $r\in(0,1)$, $\alpha>0$, $1\leq p<\infty$ and $\beta\in\mathbb{R}$. Then for all $n\in\mathbb{N}$ the following estimates are true*

$$e^\perp_n(L^{\alpha,r}_{\beta,p})_\infty \asymp \exp(-\alpha n^r)n^{\frac{1-r}{p}}. \tag{44}$$

4 Best orthogonal trigonometric approximations of the classes $L^{\psi}_{\beta,1}$ in the metric of spaces L_s, $1<s<\infty$

Theorem 3. *Let $1<s<\infty$, $\psi\in\mathfrak{M}''_{\infty}$ and function $\frac{\psi(t)}{|\psi'(t)|}\uparrow\infty$ as $t\to\infty$. Then for all $\beta\in\mathbb{R}$ order estimates hold*

$$e^{\perp}_{2n-1}(L^{\psi}_{\beta,1})_s \asymp e^{\perp}_{2n}(L^{\psi}_{\beta,1})_s \asymp \psi(n)(\eta(n)-n)^{\frac{1}{s'}},\ \ \frac{1}{s}+\frac{1}{s'}=1. \tag{45}$$

Proof. According to Theorem 2 from [8] under conditions $\psi\in\mathfrak{M}^{+}_{\infty}$, $\beta\in\mathbb{R}$, $1<s\le\infty$ for $n\in\mathbb{N}$, such that $\eta(n)-n\ge a>2$, $\mu(n)\ge b>2$ the following estimate holds

$$\mathscr{E}_n(L^{\psi}_{\beta,1})_s \le K_{a,b}\ (2s')^{\frac{1}{s}}\psi(n)(\eta(n)-n)^{\frac{1}{s'}}. \tag{46}$$

Since, $\frac{\psi(t)}{|\psi'(t)|}\uparrow\infty$, then as it was noticed in the proof of Theorem 1, exists number n_0, such that for all $n>n_0$ inequalities $\eta(n)-n\ge a>2$, $\mu(n)\ge b>2$ hold.

Using inequalities (6) and (46), we get

$$e^{\perp}_{2n}(L^{\psi}_{\beta,1})_s \le e^{\perp}_{2n-1}(L^{\psi}_{\beta,1})_s \le K_{a,b,s'}\ \left(2s'\right)^{\frac{1}{s}}\ \psi(n)(\eta(n)-n)^{\frac{1}{s'}}. \tag{47}$$

Let us find the lower estimate of the quantity $e^{\perp}_{2n}(L^{\psi}_{\beta,1})_s$.

We consider the quantity

$$I_3 := \inf_{\gamma_{2n}}\left|\int\limits_{-\pi}^{\pi}(f^{**}_{2n}(t)-S_{\gamma_{2n}}(f^{**}_{2n};t))f^{*}_{s',n}(t)dt\right|, \tag{48}$$

where

$$f^{**}_m(t)=\frac{1}{3\pi}V_m(t),$$

and $f^{*}_{s',n}$ is defined by formula (11).

On the basis of Proposition A1.1 from [3] we derive

$$I_3 \le \inf_{\gamma_{2n}}\|f^{**}_{2n}(t)-S_{\gamma_{2n}}(f^{**}_{2n};t)\|_s\|f^{*}_{s'}\|_{s'} \le e^{\perp}_{2n}(f^{**}_{2n})_s. \tag{49}$$

On other hand, using formulas (28), we write

$$I_3=\frac{\lambda_{s'}}{12\pi\psi(n)(\eta(n)-n)^{\frac{1}{s}}}\inf_{\gamma_{2n}}\left|\int\limits_{-\pi}^{\pi}\Bigg(\sum_{\substack{|k|\le 2n,\\ k\notin\gamma_{2n}}}e^{ikt}+2\sum_{\substack{2n+1\le|k|\le 4n-1,\\ k\notin\gamma_{2n}}}\Big(1-\frac{|k|}{4n}\Big)e^{ikt}\Bigg)\times\right.$$
$$\left.\times\Bigg(\sum_{|k|\le n-1}\psi(|k|)\psi(2n-|k|)e^{ikt}+\sum_{n\le|k|\le 2n}\psi^2(|k|)e^{ikt}\Bigg)dt\right|$$

$$= \frac{\lambda_{s'}}{6\psi(n)(\eta(n)-n)^{\frac{1}{s}}} \inf_{\gamma_{2n}} \Big(\sum_{\substack{|k|\le n-1,\\ k\notin\gamma_{2n}}} \psi(|k|)\psi(2n-|k|) + \sum_{\substack{n\le|k|\le 2n,\\ k\notin\gamma_{2n}}} \psi^2(|k|) \Big)$$

$$= \frac{\lambda_{s'}}{6\psi(n)(\eta(n)-n)^{\frac{1}{s}}} \Big(\psi^2(n) + 2\sum_{k=n+1}^{2n} \psi^2(k) \Big) > \frac{\lambda}{6\pi\psi(n)(\eta(n)-n)^{\frac{1}{s}}} \sum_{k=n}^{2n} \psi^2(k)$$

$$> \frac{\lambda_{s'}}{6\psi(n)(\eta(n)-n)^{\frac{1}{s}}} \int_{n}^{\eta(n)} \psi^2(t)dt > \frac{\lambda_{s'}}{24}\psi(n)(\eta(n)-n)^{\frac{1}{s'}}. \tag{50}$$

Hence, formulas (49) and (50) imply

$$e^{\perp}_{2n}(L^{\psi}_{\beta,1})_s \ge e^{\perp}_{2n}(f^{**}_{s'})_s \ge I_3 \ge \frac{\lambda_{s'}}{24}\psi(n)(\eta(n)-n)^{\frac{1}{s'}}. \tag{51}$$

Theorem 3 is proved.

In fact in the proof of Theorem 3 we obtained estimates with constants in explicit form.

Proposition 3. *Let $\psi \in \mathfrak{M}^{+}_{\infty}$, $\beta \in \mathbb{R}$, $1 \le p < \infty$ and function $\frac{\psi(t)}{|\psi'(t)|}$ increases monotonically. Then for all $n \in \mathbb{N}$, such that $\mu(\psi,n) \ge b > 2$ and $\eta(\psi,n) - n \ge a > 2$, the following estimates are true*

$$K_{b,s'}\psi(n)(\eta(n)-n)^{\frac{1}{s'}} \le e^{\perp}_{2n}(L^{\psi}_{\beta,1})_s \le e^{\perp}_{2n-1}(L^{\psi}_{\beta,1})_s$$
$$\le K_{a,b,s'}\psi(n)(\eta(n)-n)^{\frac{1}{s'}},$$

where $K_{a,b,s'}$ and $K_{b,s'}$ are defined by formulas (35) and (36) respectively.

Corollary 2. *Let $r \in (0,1)$, $\alpha > 0$, $1 < s < \infty$ and $\beta \in \mathbb{R}$. Then for all $n \in \mathbb{N}$ the following estimates are true*

$$e^{\perp}_{n}(L^{\alpha,r}_{\beta,1})_s \asymp \exp(-\alpha n^r) n^{\frac{1-r}{s'}}, \ \ \frac{1}{s} + \frac{1}{s'} = 1. \tag{52}$$

Note, that functions
1) $e^{-\alpha t^r} t^{\gamma}$, $\alpha > 0$, $r \in (0,1]$, $\gamma \le 0$;
2) $e^{-\alpha t^r} \ln^{\gamma}(t+K)$, $\alpha > 0$, $r \in (0,1]$, $\gamma \le 0$, $K > e-1$,
etc., can be regarded as examples of functions ψ, which satisfy the conditions of Theorem 1 and Theorem 3.

Remark 1. It should be noticed, that from Theorem 1–3 it follows that the orders of quantities $e^{\perp}_{n}(L^{\psi}_{\beta,p})_s$ for $1 \le p < \infty$, $s = \infty$ and $p = 1$, $1 < s < \infty$, coincide with orders of the best approximations $E_n(L^{\psi}_{\beta,p})_s$ (see [9]).

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References

1. A. S. Fedorenko, *On the best m-term trigonometric and orthogonal trigonometric approximations of functions from the classes* $L^{\psi}_{\beta,p}$, Ukr. Math. J., **51**:12 (1999), 1945–1949.
2. I. S. Gradshtein, I. M. Ryzhik, *Tables of Integrals, Sums, Series, and Products* [in Russian], Fizmatgiz, Moscow (1963).
3. N. P. Korneichuk, *Exact Constants in Approximation Theory*, **38**, Cambridge Univ. Press, Cambridge, New York (1990).
4. A. S. Romanyuk, *Approximation of classes of periodic functions of many variables*, Mat. Zametki, **71**:1 (2002), 109121 .
5. A. S. Romanyuk, *Best trigonometric approximations of the classes of periodic functions of many variables in a uniform metric*, Mat. Zametki, **81**:2 (2007), 247261 .
6. A. S. Romanyuk, *Approximate Characteristics of Classes of Periodic Functions of Many Variables* [in Russian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv (2012).
7. A. S. Serdyuk, *Approximation by interpolation trigonometric polynomials on classes of periodic analytic functions*, Ukr. Mat. Zh., **64**:5 (2012), 698712; English translation: Ukr. Math. J., **64**:5, (2012), 797815.
8. A. S. Serdyuk, T. A. Stepaniuk, *Order estimates for the best approximation and approximation by Fourier sums of classes of infinitely differentiable functions*, Zb. Pr. Inst. Mat. NAN Ukr. **10**:1 (2013), 255-282. [in Ukrainian]
9. A. S. Serdyuk, T. A. Stepanyuk, *Estimates for the best approximations of the classes of innately differentiable functions in uniform and integral metrics*, Ukr. Mat. Zh., 66:9 (2014), 1244–1256.
10. A. S. Serdyuk, T. A. Stepaniuk, *Order estimates for the best orthogonal trigonometric approximations of the classes of convolutions of periodic functions of low smoothness*, Ukr. Math. J., **67**:7 (2015), 1-24.
11. A. S. Serdyuk, T. A. Stepaniuk, *Estimates of the best m-term trigonometric approximations of classes of analytic functions*, Dopov. Nats. Akad. Nauk Ukr., Mat. Pryr. Tekh. Nauky, No. 2 (2015), 32–37. [in Ukrainian]
12. A. S. Serdyuk, T. A. Stepanyuk, *Estimates for approximations by Fourier sums, best approximations and best orthogonal trigonometric approximations of the classes of* (ψ,β)*–differentiable functions*, Bull. Soc. Sci. Lettres Lodz. Ser. Rech. Deform., vol. 66, No 2 (2016), 35–43.
13. V. V. Shkapa, *Estimates of the best M-term and orthogonal trigonometric approximations of functions from the classes* $L^{\psi}_{\beta,p}$ *in a uniform metric*, Differential Equations and Related Problems [in Ukrainian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv, **11**:2 (2014), 305–317.

14. V. V. Shkapa, *Best orthogonal trigonometric approximations of functions from the classes* $L^{\psi}_{\beta,1}$, Approximation Theory of Functions and Related Problems [in Ukrainian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv, **11**:3 (2014), 315–329.
15. A. I. Stepanets, *Classification and Approximation of Periodic Functions* [in Russian], Naukova Dumka, Kiev (1987).
16. A. I. Stepanets, *Methods of Approximation Theory*, VSP: Leiden, Boston (2005).
17. A. I. Stepanets, A. S. Serdyuk, A. L. Shidlich, *Classification of infinitely differentiable periodic functions*, Ukr. Mat. Zh. 60, No. 12, (2008)1686–1708.
18. T. A. Stepaniuk, *Estimates of the best approximations and approximations of Fourier sums of classes of convolutions of periodic functions of not high smoothness in integral metrics*, [in Ukrainian] Zb. Pr. Inst. Mat. NAN Ukr. **11**:3 (2014), 241–269.