www.oeaw.ac.at

Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

A quasi-robust discretization error estimate for discontinuous Galerkin Isogeometric Analysis

S. Takacs

RICAM-Report 2019-02

www.ricam.oeaw.ac.at

A quasi-robust discretization error estimate for discontinuous Galerkin Isogeometric Analysis

Stefan Takacs[a]

[a] *Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences*

Abstract

Isogeometric Analysis is a spline-based discretization method to partial differential equations which shows the approximation power of a high-order method. The number of degrees of freedom, however, is as small as the number of degrees of freedom of a low-order method. This does not come for free as the original formulation of Isogeometric Analysis requires a global geometry function. Since this is too restrictive for many kinds of applications, the domain is usually decomposed into patches, where each patch is parameterized with its own geometry function. In simpler cases, the patches can be combined in a conforming way. However, for non-matching discretizations or for varying coefficients, a non-conforming discretization is desired. An symmetric interior penalty discontinuous Galerkin (SIPG) method for Isogeometric Analysis has been previously introduced. In the present paper, we give error estimates that only depend poly-logarithmically on the spline degree. This opens the door towards the construction and the analysis of fast linear solvers, particularly multigrid solvers for non-conforming multipatch Isogeometric Analysis.

Keywords: Isogeometric Analysis, multi-patch domains, symmetric interior penalty discontinuous Galerkin

1. Introduction

The original design goal of Isogeometric Analysis (IgA), [11], was to unite the world of computer aided design (CAD) and the world of finite element (FEM) simulation. In IgA, both the computational domain and the solution of the partial differential equation (PDE) are represented by spline functions, like tensor product B-splines or non-uniform rational B-splines (NURBS). This follows the design goal since such spline functions are also used in standard CAD systems to represent the geometric objects of interest.

Email address: `stefan.takacs@ricam.oeaw.ac.at` (Stefan Takacs)

The parameterization of the computational domain using just *one* tensor-product spline function, is possible only in simple cases. A necessary condition for this to be possible, is that the computational domain is topologically equivalent to the unit square or the unit cube. This might not be the case for more complicated computational domains. Such domains are typically decomposed into subdomains, in IgA called patches, where each of them is parameterized by its own geometry function. The standard approach is to set up a conforming discretization. For a standard Poisson problem, this means that the overall discretization needs to be continuous. For higher order problems, like the biharmonic problem, even more regularity is required; conforming discretizations in this case are rather hard to construct, cf. [12] and references therein.

Even for the Poisson problem, a conforming discretization requires the discretizations to agree on the interfaces. This excludes many cases of practical interest, like having different grid sizes or different spline degrees on the patches. Since such cases might be of interest, alternatives to conforming discretizations are of interest. One promising alternative are discontinuous Galerkin approaches, cf. [18, 2], particularly the symmetric interior penalty discontinuous Galerkin (SIPG) method [1]. The idea of applying this technique to couple patches in IgA, has been previously discussed in [13, 14].

Concerning the approximation error, in early IgA literature, only its dependence on the grid size has been studied, cf. [11, 3]. In recent publications [5, 24, 8, 19] also the dependence on the spline degree has been investigated. These error estimates are restricted to the single-patch case. In [22], the results from [24] on approximation errors for B-splines of maximum smoothness have been extended to the conforming multi-patch case.

For the case of discontinuous Galerkin discretizations, only error estimates in the grid size are known, cf. [14]. The goal of the present paper, is to present an error analysis in the grid size h, the spline degree p and patchwise constant diffusion coefficients α_k. We observe that under reasonable assumptions, the approximation error drops like h^{-1}, while being robust in the coefficients α_k. The dependence on the spline degree is only poly-logarithmically, cf. (15). This might be surprising as the penalization parameter has to grow like p^2 for the SIPG method to be well-posed.

The robust error estimates presented of this paper can be used to analyze multigrid solvers for discontinuous Galerkin multipatch discretizations, see [23] for solvers that show robust convergence behavior in numerical experiments.

The remainder of the paper is organized as follows. In Section 2, we introduce the model problem and give a detailed description of its discretization. A discussion of the existence of a unique solution and the discretization and the approximation error, is provided in Section 3. The proof of the approximation error estimate is given in Section 4. We provide numerical experiments that depict our estimates, in Section 5.

2. The model problem and its discretization

We consider the following *Poisson model problem.* Let $\Omega \subset \mathbb{R}^2$ be an open and simply connected Lipschitz domain. For any given source function $f \in L_2(\Omega)$, we are interested in the function $u \in H^{1,\circ}(\Omega) := H^1(\Omega) \cap L_2^\circ(\Omega)$ solving

$$(\alpha \nabla u, \nabla v)_{L_2(\Omega)} = (f, v)_{L_2(\Omega)} \qquad \text{for all } v \in H^{1,\circ}(\Omega), \tag{1}$$

where $\alpha > 0$ is piecewise constant. Here and in what follows, for any $r \in \mathbb{N} := \{1, 2, 3, \ldots\}$, $L_2(\Omega)$ and $H^r(\Omega)$ are the standard Lebesgue and Sobolev spaces with standard scalar products $(\cdot,\cdot)_{L_2(\Omega)}$, $(\cdot,\cdot)_{H^r(\Omega)} := (\nabla^r \cdot, \nabla^r \cdot)_{L_2(\Omega)}$, norms $\|\cdot\|_{L_2(\Omega)}$ and $\|\cdot\|_{H^r(\Omega)}$, and seminorms $|\cdot|_{H^r(\Omega)}$. The Lebesgue space of function with zero mean is given by $L_2^\circ(\Omega) := \{v \in L_2(\Omega) \,:\, (v, 1)_{L_2(\Omega)} = 0\}$.

The computational domain Ω is the union of K non-overlapping open patches Ω_k, i.e.,

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega_k} \quad \text{and} \quad \Omega_k \cap \Omega_l = \emptyset \text{ for any } k \neq l \tag{2}$$

holds, where $\overline{T}$ denotes the closure of T. We assume that the patches are constructed such that the coefficient function α is constant on each patch, i.e.,

$$\alpha = \alpha_k \quad \text{on} \quad \Omega_k, \quad \text{where} \quad \alpha_k \in \mathbb{R}^+.$$

Each patch Ω_k is represented by a bijective geometry function

$$G_k : \widehat{\Omega} := (0,1)^2 \to \Omega_k := G_k(\widehat{\Omega}) \subset \mathbb{R}^2,$$

which can be continuously extended to the closure of $\widehat{\Omega}$ such that $G_k(\overline{\widehat{\Omega}}) = \overline{\Omega_k}$. We use the notation

$$v_k := v|_{\Omega_k} \qquad \text{and} \qquad \widehat{v}_k := v_k \circ G_k$$

for any function v on Ω. If $v \in H^1(\Omega)$, we can use standard trace theorems to extend v_k to $\overline{\Omega_k}$ and to extend $\widehat{v}_k$ to $\overline{\widehat{\Omega}}$.

We assume that the mesh induced by the interfaces between the patches does not have any T-junctions, i.e., we assume as follows.

Assumption 1. *For any two patches Ω_k and Ω_l with $k \neq l$, the intersection $\partial\Omega_k \cap \partial\Omega_l$ is either (a) empty, (b) a common vertex, or (c) a common edge $I_{k,l} = I_{l,k}$ such that*

$$\widehat{I}_{k,l} := I_{k,l} \circ G_k \in \widehat{\mathcal{I}} := \{\{0\} \times (0,1), \{1\} \times (0,1), (0,1) \times \{0\}, (0,1) \times \{1\}\}. \tag{3}$$

Note that the pre-images $\widehat{I}_{k,l}$ and $\widehat{I}_{l,k}$ do not necessarily agree. We define

$$\begin{aligned} \mathcal{N}(k) &:= \{l \in \{1, \ldots, K\} \,:\, \Omega_k \text{ and } \Omega_l \text{ have common edge}\}, \\ \mathcal{N} &:= \{(k,l) \in \{1, \ldots, K\}^2 \,:\, k < l \text{ and } l \in \mathcal{N}(k)\}, \\ \mathcal{N}^* &:= \{(k,l) \in \{1, \ldots, K\}^2 \,:\, k > l \text{ and } l \in \mathcal{N}(k)\}, \end{aligned}$$

and the parameterization $\gamma_{k,l} : (0,1) \to \widehat{I}_{k,l}$ via

$$\gamma_{k,l}(t) := \begin{cases} (t,s) & \text{if} \quad \widehat{I}_{k,l} = (0,1) \times \{s\}, \quad s \in \{0,1\} \\ (s,t) & \text{if} \quad \widehat{I}_{k,l} = \{s\} \times (0,1), \quad s \in \{0,1\}. \end{cases} . \tag{4}$$

We assume that the geometry functions agree on the interface; this does not require any smoothness of the overall geometry function normal to the interface.

Assumption 2. *For all $(k,l) \in \mathcal{N}^*$ and $t \in (0,1)$, we have*

$$\gamma_{k,l}(t) = G_k^{-1} \circ G_l \circ \gamma_{l,k}(t) \quad \textit{or} \quad \gamma_{k,l}(t) = G_k^{-1} \circ G_l \circ \gamma_{l,k}(1-t).$$

Remark 1. We can reparameterize each patch such that this condition is satisfied. Assume to have two patches Ω_k and Ω_l, sharing the patch $I_{k,l} = G_k((0,1) \times \{0\}) = G_l((0,1) \times \{0\})$. Using

$$\widetilde{G}_k(x,y) := G_k(yx + (1-y)\rho(x), y), \quad \text{where} \quad (\rho(t), 0) := G_k^{-1} \circ G_l(t,0),$$

we obtain a reparameterization of G_k, which (a) matches the parameterization of Ω_l at the interface, (b) is unchanged on the other interfaces, and (c) keeps the patch Ω_k unchanged. By iteratively applying this approach to all patches, we obtain a discretization satisfying Assumption 2.

We assume that the geometry function is sufficiently smooth such that the following assumption holds.

Assumption 3. *There is a constant $C_G > 0$ such that the geometry functions G_k satisfy the estimate*

$$\sup_{x \in \widehat{\Omega}} \|\nabla^r G_k(x)\|_{\ell_2} \leq C_G \quad \textit{and} \quad \sup_{x \in \widehat{\Omega}} \|(\nabla^r G_k(x))^{-1}\|_{\ell_2} \leq C_G \quad \textit{for} \quad r \in \{1,2\}.$$

We assume full elliptic regularity.

Assumption 4. *The solution u of the model problem* (1) *is patch-wise H^2, i.e.,*

$$u_k \in H^2(\Omega_k)$$

holds for all $k = 1, \ldots, K$.

If all α_k are equal, we obtain $u \in H^2(\Omega)$ (and thus also Assumption 4) for domains Ω with a sufficiently smooth boundary, cf. [15], and for convex polygonal domains Ω, cf. [6, 7]. This case is of interest if the dG discretization is used to obtain a flexible combination of the patches. If not all values of α_k agree, in general $u \notin H^2(\Omega)$, but Assumption 4 might be satisfied under certain circumstances, cf. [16, 17] and others. The theory of this paper can be extended to cases where we only know $u_k \in H^{3/2+\epsilon}(\Omega_k)$ for some $\epsilon > 0$. For simplicity, we restrict ourselves to the case of full elliptic regularity (Assumption 4).

Having a representation of the domain, we introduce the isogeometric function space. Following [13, 14], we use a conforming isogeometric discretization for each patch and couple the contributions for the patches using a symmetric interior penalty discontinuous Galerkin (SIPG) method, cf. [1], as follows.

For the *univariate case*, the space of spline functions of degree $p \in \{2, 3, \ldots\}$ and size $h = 1/n$ with $n \in \mathbb{N}$ is given by

$$S_{p,h}(0,1) := \left\{ v \in H^p(0,1) : \; v|_{(ih,(i+1)h]} \in \mathbb{P}^p \text{ for all } j = 1, \ldots, n-1 \right\},$$

where $\mathbb{P}^p$ is the space of polynomials of degree p. On the *parameter domain* $\widehat{\Omega} := (0,1)^2$, we introduce tensor-product B-spline functions

$$S_{p,h}(\widehat{\Omega}) := S_{p,h}(0,1) \otimes S_{p,h}(0,1).$$

The *multi-patch function space* V_h is given by

$$V_h := \{u_h \in L_2^\circ(\Omega) \; : \; u_h \circ G_k \in S_{p_k,h_k}(\widehat{\Omega}) \text{ for } k = 1, \ldots, K, \}. \tag{5}$$

Note that the grid sizes h_k and the spline degrees p_k can be different for each of the patches. We define

$$p := \max_{k \in \{1,\ldots,K\}} p_k, \qquad p_{\min} := \min_{k \in \{1,\ldots,K\}} p_k, \qquad \text{and} \qquad h := \max_{k \in \{1,\ldots,K\}} h_k$$

to be the largest spline degree, the smallest spline degree and the largest grid size and assume there to be a constant $C_h > 0$ such that $h \leq C_h h_k$ for all $k \in \{1, \ldots, K\}$.

Following the assumption that u_h is a patchwise function, we define for each $r \in \mathbb{N}$ a broken Sobolev space

$$\mathcal{H}^r(\Omega) := \{v \in L_2(\Omega) \; : \; v_k \in H^r(\Omega)\},$$

with associated norms and scalar products

$$\|v\|_{\mathcal{H}^r(\Omega)} := (v,v)_{\mathcal{H}^r(\Omega)}^{1/2} \quad \text{and} \quad (u,v)_{\mathcal{H}^r(\Omega)} := \sum_{k=1}^{K} (u,v)_{H^r(\Omega_k)}$$

and weighted norms and scalar products

$$\|v\|_{\mathcal{H}^r_\alpha(\Omega)} := (v,v)_{\mathcal{H}^r_\alpha(\Omega)}^{1/2} \quad \text{and} \quad (u,v)_{\mathcal{H}^r_\alpha(\Omega)} := \sum_{k=1}^{K} \alpha_k (u,v)_{H^r(\Omega_k)}.$$

For each patch, we define on its boundary $\partial\Omega_k$ the outer normal vector $\mathbf{n}_k$. On each interface $I_{k,l}$, we define the jump operator $[\![\cdot]\!]$ by

$$[\![v]\!] := v_k - v_l \qquad \text{on } I_{k,l} = I_{l,k} \text{ where } (k,l) \in \mathcal{N}$$

and the average operator $\{\cdot\}$ by

$$\{v\} := \tfrac{1}{2}(v_k + v_l) \qquad \text{on } I_{k,l} = I_{l,k} \text{ where } (k,l) \in \mathcal{N}.$$

The discretization of the variational problem using the *symmetric interior penalty discontinuous Galerkin method* reads as follows. Find $u_h \in V_h$ such that

$$(u_h, v_h)_{A_h} = (f, v_h)_{L_2(\Omega)} \qquad \text{for all } v_h \in V_h, \tag{6}$$

where

$$\begin{aligned}
(u, v)_{A_h} &:= (u, v)_{\mathcal{H}^1_\alpha(\Omega)} - (u, v)_{B_h} - (v, u)_{B_h} + (u, v)_{C_h}, \\
(u, v)_{B_h} &:= \sum_{(k,l)\in\mathcal{N}} (\llbracket u \rrbracket, \{\alpha \nabla v\} \cdot \mathbf{n}_k)_{L_2(I_{k,l})}, \\
(u, v)_{C_h} &:= \frac{\sigma}{h} \sum_{(k,l)\in\mathcal{N}} \alpha_{k,l} (\llbracket u \rrbracket, \llbracket v \rrbracket)_{L_2(I_{k,l})}
\end{aligned}$$

for all $u, v \in \mathcal{H}^{2,\circ}(\Omega)$,

$$\alpha_{k,l} := \max\{\alpha_k, \alpha_l\}$$

and the penalty parameter $\sigma \geq \sigma_0 p^2 > 0$ is chosen sufficiently large.

Using a basis for the space V_h, we obtain a standard matrix-vector problem: Find $\underline{u}_h \in \mathbb{R}^N$ such that

$$A_h \underline{u}_h = \underline{f}_h. \tag{7}$$

Here and in what follows, $\underline{u}_h = [u_i]_{i=1}^N$ is the coefficient vector representing u_h with respect to the chosen basis, i.e., $u_h = \sum_{i=1}^N u_i \varphi_i$, and $\underline{f}_h = [(f, \varphi_i)_{L_2(\Omega)}]_{i=1}^N$ is the coefficient vector obtained by testing the right-hand-side functional with the basis functions.

As the dependence on the geometry function is not in the focus of this paper, unspecified constants might depend on C_G, C_I and C_h. Before we proceed, we introduce a convenient notation.

Definition 5. *Any generic constant $c > 0$ used within this paper is understood to be independent of the grid size h, the spline degree p and the number of patches K, but it might depend on the constants C_G, C_I and C_h.*

We use the notation $a \lesssim b$ if there is a generic constant c such that $a \leq cb$ and the notation $a \eqsim b$ if $a \lesssim b$ and $b \lesssim a$.

For symmetric positive definite matrices A and B, we write

$$A \leq B \qquad \text{if} \qquad \underline{v}_h^\top A \underline{v}_h \leq \underline{v}_h^\top B \underline{v}_h \quad \text{for all vectors } \underline{v}_h.$$

The notations $A \lesssim B$ and $A \eqsim B$ are defined analogously.

3. A discretization error estimate

In [13], it has been shown that the bilinear form $(\cdot, \cdot)_{A_h}$ is coercive and bounded in the dG-norm. For our further analysis, it is vital to know these conditions to be satisfied with constants that are independent of the spline degree p. Thus, we define the dG-norm via

$$\|u\|_{Q_h}^2 := (u,u)_{Q_h}, \qquad \text{where} \qquad (u,v)_{Q_h} := (u,v)_{\mathcal{H}^1_\alpha(\Omega)} + (u,v)_{C_h}$$

for all $u, v \in \mathcal{H}^{2,\circ}(\Omega)$. Note that we define the norm differently to [13], where the dG-norm was independent of p.

Before we proceed, we give some estimates on the geometry functions.

Lemma 6. *The geometry functions G_k satisfy*

$$\|v \circ G_k^{-1}\|_{H^r(\Omega_k)} \eqsim \|v\|_{H^r(\widehat{\Omega})} \quad \text{for all } v \in H^r(\widehat{\Omega}),\ r \in \{0,1,2\}, \quad \text{and}$$
$$\|v \circ G_k^{-1}\|_{L_2(I_{k,l})} \eqsim \|v\|_{L_2(\widehat{I}_{k,l})} \quad \text{for all } v \in H^1(\widehat{\Omega}).$$

For ease of notation, here and in what follows, we define $H^0 := L_2$.

Proof. The statements follow directly from the chain rule for differentiation, the substitution rule for integration and Assumption 3. □

Lemma 7. *The geometry functions G_k satisfy*

$$\|(\nabla v \circ G_k^{-1}) \cdot \mathbf{n}_k\|_{L_2(I_{k,l})} \lesssim \|\nabla v\|_{L_2(\widehat{I}_{k,l})} \quad \text{for all } v \in H^2(\widehat{\Omega}).$$

Proof. We have

$$\|(\nabla v \circ G_k^{-1}) \cdot \mathbf{n}_k\|_{L_2(I_{k,l})} \le \|\nabla v \circ G_k^{-1}\|_{L_2(I_{k,l})} \|\mathbf{n}_k\|_{L_\infty(I_{k,l})},$$

where certainly $\|\mathbf{n}_k\|_{L_\infty(I_{k,l})} = 1$ because the length of $\mathbf{n}_k$ is always 1. The estimate $\|\nabla v \circ G_k^{-1}\|_{L_2(I_{k,l})} \lesssim \|\nabla v\|_{L_2(\widehat{I}_{k,l})}$ follows directly from the chain rule for differentiation, the substitution rule for integration and Assumption 3. □

For σ sufficiently large, the symmetric bilinear form $(\cdot,\cdot)_{A_h}$ is coercive and bounded, i.e., a scalar product.

Theorem 8 (Coercivity and boundedness). *There is some $\sigma_0 > 0$ that only depends on C_G and C_I such that*

$$(u_h, u_h)_{A_h} \gtrsim \|u_h\|_{Q_h}^2 \quad \text{and} \quad (u_h, v_h)_{A_h} \lesssim \|u_h\|_{Q_h} \|v\|_{Q_h}$$

holds for all $u_h, v_h \in V_h$ and all $\sigma \ge p^2 \sigma_0$.

Proof. Note that $(u_h, v_h)_{A_h} = (u_h, v_h)_{Q_h} - (u_h, v_h)_{B_h} - (v_h, u_h)_{B_h}$. Using Lemma 7, [22, Lemma 4.4], [20, Corollary 3.94] and Lemma 6, we obtain

$$\|\nabla v_h \cdot \mathbf{n}_k\|_{L_2(I_{k,l})}^2 \lesssim \|\nabla (v_h \circ G_k) \cdot \mathbf{n}_k\|_{L_2(\widehat{I}_{k,l})}^2 \lesssim \|v_h \circ G_k\|_{H^2(\widehat{\Omega})} \|v_h \circ G_k\|_{H^1(\widehat{\Omega})}$$
$$\lesssim \frac{p^2}{h} \|v_h \circ G_k\|_{H^1(\widehat{\Omega})}^2 \lesssim \frac{p^2}{h} \|v_h\|_{H^1(\Omega_k)}^2 \tag{8}$$

for all $v_h \in V_h$, $k = 1, \dots, K$ and $l \in \mathcal{N}(k)$. As $V_h \subset H_0^1(\Omega)$, the Poincaré inequality (see, e.g., [20, Theorem A.25]) yields also

$$\|\nabla v_h \cdot \mathbf{n}_k\|_{L_2(I_{k,l})}^2 \lesssim \frac{p^2}{h} |v_h|_{H^1(\Omega_k)}^2.$$

The Cauchy-Schwarz inequality, the triangle inequality, (8), $\alpha_{k,l}^{-1}\alpha_k \leq 1$, and $|\mathcal{N}(k)| \leq 4$ yield

$$\begin{aligned}
&|(u_h, v_h)_{B_h}| \\
&\leq \left(\sum_{(k,l)\in\mathcal{N}} \alpha_{k,l} \|[\![u_h]\!]\|_{L_2(I_{k,l})}^2 \right)^{1/2} \left(\sum_{(k,l)\in\mathcal{N}} \alpha_{k,l}^{-1} \|\{\alpha \nabla v_h\} \cdot \mathbf{n}_k\|_{L_2(I_{k,l})}^2 \right)^{1/2} \\
&\lesssim \left(\sum_{(k,l)\in\mathcal{N}} \alpha_{k,l} \|[\![u_h]\!]\|_{L_2(I_{k,l})}^2 \right)^{1/2} \left(\sum_{k=1}^{K} \sum_{j\in\mathcal{N}(k)} \alpha_{k,l}^{-1} \alpha_k^2 \|\nabla v_h \cdot \mathbf{n}_k\|_{L_2(I_{k,l})}^2 \right)^{1/2} \\
&\lesssim \left(\frac{p^2}{h} \right)^{1/2} \left(\sum_{(k,l)\in\mathcal{N}} \alpha_{k,l} \|[\![u_h]\!]\|_{L_2(I_{k,l})}^2 \right)^{1/2} \left(\sum_{k=1}^{K} \alpha_k |v_h|_{H^1(\Omega_k)}^2 \right)^{1/2} \\
&\leq p\,\sigma^{-1/2} \|u_h\|_{Q_h} \|v_h\|_{Q_h}
\end{aligned} \tag{9}$$

for all $u_h, v_h \in V_h$. Let $c_0 \approx 1$ be the hidden constant, i.e., such that

$$|(u_h, v_h)_{B_h}| \leq c_0\, p\, \sigma^{-1/2} \|u_h\|_{Q_h} \|v_h\|_{Q_h}. \tag{10}$$

For $\sigma \geq 16\, c_0\, p^2$, we obtain

$$(u_h, u_h)_{A_h} = \|u_h\|_{Q_h}^2 - 2(u_h, u_h)_{B_h} \geq \frac{1}{2} \|u_h\|_{Q_h}^2,$$

i.e., coercivity. Using (9) and Cauchy-Schwarz inequality, we obtain further

$$(u_h, v_h)_{A_h} = (u_h, v_h)_{Q_h} - (u_h, v_h)_{B_h} - (v_h, u_h)_{B_h} \leq \frac{3}{2} \|u_h\|_{Q_h} \|v_h\|_{Q_h},$$

i.e., boundedness. □

As we have boundedness and coercivity (Theorem 8), the Lax Milgram theorem (see, e.g., [20, Theorem 1.24]) yields states existence and uniqueness of a solution, i.e., the following statement.

Theorem 9 (Existence and uniqueness). *If σ is chosen as in Theorem 8, the problem* (6) *has exactly one solution $u_h \in V_h$.*

The following theorem shows that the solution of the original problem also satisfies the discretized bilinear form.

Theorem 10 (Consistency). *The solution $u \in H^{1,\circ}(\Omega) \cap \mathcal{H}^2(\Omega)$ of the original problem* (1) *satisfies*

$$(u, v_h)_{A_h} = (f, v_h)_{L_2(\Omega)} \quad \textit{for all } v_h \in V_h.$$

For a proof, see, e.g., [18, Proposition 2.9]; the proof requires elliptic regularity (cf. Assumption 4).

If boundedness of the bilinear form $(\cdot,\cdot)_{A_h}$ was also satisfied for $u \in \mathcal{H}^{2,\circ}(\Omega)$, Ceá's Lemma (see, e.g., [20, Theorem 2.19.iii]) would allow to bound the discretization error. However, the bilinear form is not bounded in the norm $\|\cdot\|_{Q_h}$, but only in the stronger norm $\|\cdot\|_{Q_h^+}$, given by

$$\|u\|_{Q_h^+}^2 := \|u\|_{Q_h}^2 + \frac{h^2}{\sigma^2}|u|_{\mathcal{H}^2_\alpha(\Omega)}^2. \tag{11}$$

Theorem 11. *There is some $\sigma_0 > 0$ that only depends on C_G and C_I such that*

$$(u, v_h)_{A_h} \lesssim \|u\|_{Q_h^+}\|v_h\|_{Q_h}$$

holds for all $u \in \mathcal{H}^{2,\circ}(\Omega)$, all $v_h \in V_h$ all $\sigma \geq p^2\sigma_0$.

Proof. Let $u \in \mathcal{H}^{2,\circ}(\Omega)$ and $v_h \in V_h$ be arbitrarily but fixed and assume $\sigma \geq 16c_0p^2$, where c_0 is as in (10). Note that the arguments from (9) also hold if the first parameter of the bilinear form $(\cdot,\cdot)_{B_h}$ is not in V_h. So, we obtain

$$|(u, v_h)_{B_h}| \leq \frac{1}{4}\|u\|_{Q_h}\|v_h\|_{Q_h}.$$

Using Lemma 7, [22, Lemma 4.4], Lemma 6 and the Poincaré inequality, we obtain

$$\begin{aligned}
&\|\nabla v \cdot \mathbf{n}_k\|_{L_2(I_{k,l})}^2 \lesssim \|\nabla(v \circ G_k)\cdot \mathbf{n}_k\|_{L_2(\widehat{I}_{k,l})}^2 \lesssim \|v \circ G_k\|_{H^2(\widehat{\Omega})}\|v \circ G_k\|_{H^1(\widehat{\Omega})} \\
&\lesssim \|v\|_{H^2(\Omega_k)}\|v\|_{H^1(\Omega_k)} \leq \frac{1}{\beta}\|v\|_{H^2(\Omega_k)}^2 + \beta\|v\|_{H^1(\Omega_k)}^2 \leq \frac{1}{\beta}|v|_{H^2(\Omega_k)}^2 + \beta|v|_{H^1(\Omega_k)}^2
\end{aligned} \tag{12}$$

for all $v \in H^2(\Omega_k)$, all $k = 1,\ldots,K$, all $l \in \mathcal{N}(k)$ and all $\beta > 1$. Using this estimate, $\alpha_{k,l}^{-1}\alpha_k \leq 1$, and $|\mathcal{N}(k)| \leq 4$, we obtain for $\beta := h^{-2}\sigma$

$$\begin{aligned}
&|(v_h, u)_{B_h}| \\
&\quad\leq \left(\frac{\sigma}{h}\sum_{(k,l)\in\mathcal{N}} \alpha_{k,l}\|[\![v_h]\!]\|_{L_2(I_{k,l})}^2\right)^{1/2} \left(\frac{h}{\sigma}\sum_{k=1}^K\sum_{l\in\mathcal{N}(k)} \frac{\alpha_k^2}{\alpha_{k,l}}\|\nabla u\cdot \mathbf{n}_k\|_{L_2(I_{k,l})}^2\right)^{1/2} \\
&\quad\leq \|v_h\|_{Q_h}\left(\sum_{k=1}^K \alpha_k|u|_{H^1(\Omega_k)}^2 + \frac{h^2}{\sigma^2}\sum_{k=1}^K \alpha_k|u|_{H^2(\Omega_k)}^2\right)^{1/2} \\
&\quad\lesssim \|v_h\|_{Q_h}\|u\|_{Q_h^+}.
\end{aligned}$$

Using these estimates, we obtain

$$(u, v_h)_{A_h} = (u, v_h)_{Q_h} - (u, v_h)_{B_h} - (v_h, u)_{B_h} \lesssim \|u\|_{Q_h^+}\|v_h\|_{Q_h},$$

which finishes the proof. $\square$

Using consistency (Theorem 10), coercivity and boundedness (Theorems 8 and 11), we can bound the discretization error using a the approximation error.

Theorem 12 (Discretization error estimate). *Provided the assumptions of Theorems 8 and 10, the estimate*

$$\|u - u_h\|_{Q_h} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_{Q_h^+}$$

holds, where u is the solution of the original problem (1) *and u_h is the solution of the discrete problem* (6).

Proof. For any $v_h \in V_h$, the triangle inequality yields

$$\|u - u_h\|_{Q_h} \leq \|u - v_h\|_{Q_h} + \|v_h - u_h\|_{Q_h}. \tag{13}$$

Theorem 10 and Galerkin orthogonality yield $(u - u_h, w_h)_{A_h} = 0$ for all $w_h \in V_h$. So, we obtain using Theorems 8 and 11 that

$$\|v_h - u_h\|_{Q_h}^2 \lesssim (v_h - u_h, v_h - u_h)_{A_h} = (v_h - u, v_h - u_h)_{A_h} \lesssim \|v_h - u\|_{Q_h^+} \|v_h - u_h\|_{Q_h},$$

which shows $\|v_h - u_h\|_{Q_h} \lesssim \|u - v_h\|_{Q_h^+}$. Together with (13), this shows $\|u - u_h\|_{Q_h} \lesssim \|u - v_h\|_{Q_h^+}$. Since this holds for all $v_h \in V_h$, this finishes the proof. □

It is rather straight forward to give approximation error estimates that bound the approximation error as follows:

$$\inf_{v_h \in V_h} \|u - v_h\|_{Q_h^+} \lesssim \sigma^{1/2}\, h |u|_{\mathcal{H}^2(\Omega)}$$

for all $u \in \mathcal{H}^{2,\circ}(\Omega)$. If σ is chosen in an optimal way, this yields a result of the form

$$\inf_{v_h \in V_h} \|u - v_h\|_{Q_h^+} \lesssim \sigma_0^{1/2}\, p^2\, h |u|_{\mathcal{H}^2(\Omega)},$$

i.e., a quadratic increase in the spline degree p. Using a refined analysis, we obtain as follows.

Theorem 13 (Approximation error estimate). *Provided $h \leq 1$, and $\sigma \gtrsim p^2$, the estimate*

$$\inf_{v_h \in V_h} \|u - v_h\|_{Q_h^+} \lesssim (\ln \sigma)^2\, \sigma^{1/(2p_{\min}-1)}\, h |u|_{\mathcal{H}^2_\alpha(\Omega)} \tag{14}$$

holds for all $u \in H^{1,\circ}(\Omega) \cap \mathcal{H}^2(\Omega)$.

The proof is given at the end of the next section.

If we consider the case $p \eqsim p_{\min}$ and if we do not consider over-penalization, i.e., we assume $\sigma \eqsim p^2$, we obtain using Theorem 12 the estimate

$$\|u - u_h\|_{Q_h} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_{Q_h^+} \lesssim (\ln p)^2\, h |u|_{\mathcal{H}^2_\alpha(\Omega)}, \tag{15}$$

where u is the solution of the original problem and u_h is the solution of the problem discretized with the proposed SIPG approach.

4. Proof of the approximation error estimate

Before we can give the proof, we give some auxiliary results. This section is organized as follows. In Section 4.1, we give patch-wise projectors and estimates for them. We introduce a mollifying operator and give estimates for that operator in Section 4.2. Finally, in Section 4.3, we give the proof for the approximation error estimate.

4.1. Patch-wise projectors

As first step, we recall the projection operators from [22, Sections 3.1 and 3.2]. Let $\Pi_{p,h}$ be the $H^1_D(0,1)$-orthogonal projection into $S_{p,h}(0,1)$, where

$$(u,v)_{H^1_D(0,1)} = (u',v')_{L_2(0,1)} + u(0)v(0).$$

[22, Lemma 3.1] states that $(\Pi_{p,h}u)(0) = u(0)$ and $(\Pi_{p,h}u)(1) = u(1)$. Using $0 = (u-\Pi_{p,h}u, x^2)_{H^1_D(0,1)} = 2((u-\Pi_{p,h}u)', x)_{L_2(0,1)} = -2(u-\Pi_{p,h}u, 1)_{L_2(0,1)} + u(1) - (\Pi_{p,h}u)(1)$, we obtain for $p \geq 2$ and $u \in H^1(0,1)$ that

$$(u - \Pi_{p,h}u, 1)_{L_2(0,1)} = 0. \tag{16}$$

The next step is to consider the multivariate case, more precisely the parameter domain $\widehat{\Omega} = (0,1)^2$. Let $\Pi^x_{p,h} : H^2(\widehat{\Omega}) \to H^2(\widehat{\Omega})$ and $\Pi^y_{p,h} : H^2(\widehat{\Omega}) \to H^2(\widehat{\Omega})$ be given by

$$(\Pi^x_{p,h}u)(x,y) = (\Pi_{p,h}u(\cdot,y))(x) \quad \text{and} \quad (\Pi^y_{p,h}u)(x,y) = (\Pi_{p,h}u(x,\cdot))(y)$$

and let $\widehat{\Pi}_k : H^2(\widehat{\Omega}) \to S_{p,h}(\widehat{\Omega})$ be such that

$$\widehat{\Pi}_k = \Pi^x_{p_k,h_k}\Pi^y_{p_k,h_k}. \tag{17}$$

For the physical domain, define $\Pi : H^{1,\circ}(\Omega) \cap \mathcal{H}^2(\Omega) \to V_h$ to be such that

$$(\Pi v)|_{\Omega_k} = (\widehat{\Pi}_k(v \circ G_k)) \circ G_k^{-1} \quad \text{for all } v \in \mathcal{H}^2(\Omega) \text{ and } k = 1,\dots,K.$$

Observe that we obtain using (16) that

$$(u - \widehat{\Pi}_k u, 1)_{L_2(\widehat{\Omega})} = 0, \qquad \widehat{\Pi}_k c = c \qquad \text{and} \qquad \Pi c = c. \tag{18}$$

for all $c \in \mathbb{R}$.

The projectors $\widehat{\Pi}_k$ satisfy robust error estimates and are almost stable in H^2.

Lemma 14. *$|(I - \widehat{\Pi}_k)u|_{H^1(\widehat{\Omega})} \lesssim h_k|u|_{H^2(\widehat{\Omega})}$ holds for all $u \in H^2(\widehat{\Omega})$.*

Proof. This result follows directly from [22, Theorem 3.3].

Lemma 15. *$|(I - \widehat{\Pi}_k)u|_{H^2(\widehat{\Omega})} \lesssim p_k^2|u|_{H^2(\widehat{\Omega})}$ holds for all $u \in H^2(\widehat{\Omega})$.*

Proof. The proof is analogous to the proof of [9, Theorem 4]. Let $\widehat{R}_k$ be the $H^{2,\circ}(\widehat{\Omega})$-orthogonal projection into $S_{p_k,h_k}(\widehat{\Omega})$, where the scalar product $(\cdot,\cdot)_{H^{2,\circ}(\widehat{\Omega})}$ is given by

$$(u,v)_{H^{2,\circ}(\widehat{\Omega})} := (u,v)_{H^2(\widehat{\Omega})} + (u,1)_{L_2(\widehat{\Omega})}(v,1)_{L_2(\widehat{\Omega})} + (\nabla u,1)_{L_2(\widehat{\Omega})}(\nabla v,1)_{L_2(\widehat{\Omega})}. \tag{19}$$

Using [24, Theorem 7.1] and an Aubin-Nitsche trick duality argument, which is completely analogous to that in the proof of [21, Theorem 7], we obtain

$$|(I-\widehat{R}_k)u|_{H^1(\widehat{\Omega})} \lesssim h_k|u|_{H^2(\widehat{\Omega})}; \tag{20}$$

here we use that we have H^3-regularity on each convex polygonal domain, cf. [4], like on the parameter domain $\widehat{\Omega}$. The triangle inequality yields

$$\begin{aligned}|(I-\widehat{\Pi}_k)u|_{H^2(\widehat{\Omega})} &\le |(I-\widehat{R}_k)u|_{H^2(\widehat{\Omega})} + |(\widehat{R}_k-\widehat{\Pi}_k)u|_{H^2(\widehat{\Omega})}\\ &\le |u|_{H^2(\widehat{\Omega})} + |(\widehat{R}_k-\widehat{\Pi}_k)u|_{H^2(\widehat{\Omega})}.\end{aligned}$$

Note that $(R_k-\Pi_k)u \in S_{p_k,h_k}(\widehat{\Omega})$, so a standard inverse estimate [20, Corollary 3.94] yields

$$\begin{aligned}|(I-\widehat{\Pi}_k)u|_{H^2(\widehat{\Omega})} &\lesssim |u|_{H^2(\widehat{\Omega})} + h_k^{-1}p_k^2|(\widehat{R}_k-\widehat{\Pi}_k)u|_{H^1(\widehat{\Omega})}\\ &\le |u|_{H^2(\widehat{\Omega})} + h_k^{-1}p_k^2|(I-\widehat{\Pi}_k)u|_{H^1(\widehat{\Omega})} + h_k^{-1}p_k^2|(I-\widehat{R}_k)u|_{H^1(\widehat{\Omega})}.\end{aligned}$$

Using [22, Theorem 3.3] and (20), we obtain further

$$|(I-\widehat{\Pi}_k)u|_{H^2(\widehat{\Omega})} \lesssim |u|_{H^2(\widehat{\Omega})} + p_k^2|u|_{H^2(\widehat{\Omega})} + p_k^2|u|_{H^2(\widehat{\Omega})},$$

which shows the desired result. □

A corresponding result is also true for the univariate case.

Lemma 16. $|(I-\Pi_{p,h})u|_{H^2(0,1)} \lesssim p^2|u|_{H^2(0,1)}$ *holds for all* $u \in H^2(0,1)$.

Proof. The proof is analogous to the proof Lemma 15. □

On the interfaces, we have the following approximation error estimate.

Lemma 17. $\|(I-\widehat{\Pi}_k)u\|^2_{L_2(\widehat{I}_{k,l})} \lesssim 2^r h^{2r}|u|^2_{H^r(\widehat{I}_{k,l})}$ *holds for all* $u \in H^2(\widehat{\Omega}) \cap H^r(\widehat{I}_{k,l})$ *with* $r \in \{1,2,3,\ldots,p_{\min}\}$ *and all* $(k,l) \in \mathcal{N}\cup\mathcal{N}^*$.

Proof. Without loss of generality, we assume $\widehat{I}_{k,l} = (0,1)\times\{0\}$. [22, Theorem 3.4] states that $((I-\widehat{\Pi}_k)u)(\cdot,0) = (I-\Pi_{p,h})(u(\cdot,0))$. So, we have

$$\|(I-\widehat{\Pi}_k)u\|^2_{L_2(\widehat{I}_{k,l})} = \|(I-\Pi_{p,h})u(\cdot,0)\|^2_{L_2(0,1)}.$$

Using [22, Eq. (3.4)], [10, Lemma 8] and that $\Pi_{p,h}$ minimizes the H^1-seminorm, we further obtain

$$\|(I-\widehat{\Pi}_k)u\|^2_{L_2(\widehat{I}_{k,l})} \lesssim h^2 \inf_{v_h\in S_{p,h}(0,1)} |u(\cdot,0)-v_h|^2_{H^1(0,1)}.$$

[24, Theorem 7.3] yields the desired result. □

4.2. A mollifying operator

A second step of the proof is the introduction of a particular mollification operator for the interfaces.

For $(k,l) \in \mathcal{N}$, let $\Upsilon_{k,l}$ be given by $\Upsilon_{k,l} v := v \circ \gamma_{k,l}$. For $(k,l) \in \mathcal{N}^*$, we define $\Upsilon_{k,l} v := v \circ G_k^{-1} \circ G_l \circ \gamma_{l,k}$, i.e., we have $(\Upsilon_{k,l} v)(t) = v(\gamma_{l,k}(t))$ or $(\Upsilon_{k,l} v)(t) = v(\gamma_{l,k}(1-t))$, cf. Assumption 2. For all cases, $\Upsilon_{k,l}$ is a bijective function $H^s(0,1) \to H^s(\widehat{I}_{k,l})$ and

$$|u|_{H^s(0,1)} \eqsim |\Upsilon_{k,l} u|_{H^s(\widehat{I}_{k,l})} \tag{21}$$

holds for all s. For $v \in H^s(\widehat{\Omega})$, we define the abbreviated notation $\Upsilon_{k,l}^{-1} v := \Upsilon_{k,l}^{-1}(v|_{\widehat{I}_{k,l}})$ and observe

$$\Upsilon_{k,l}^{-1} u \in H^{3/2}(\widehat{I}_{k,l}) \quad \text{for all} \quad u \in H^2(\widehat{\Omega}).$$

For $(k,l) \in \mathcal{N} \cup \mathcal{N}^*$, we define *extension operators* $\Xi_{k,l} : H^s(\widehat{I}_{k,l}) \to H^s(\widehat{\Omega})$ by

$$(\Xi_{k,l} w)(x,y) := \begin{cases} \phi(x) w(0,y) & \text{if } \widehat{I}_{k,l} = \{0\} \times (0,1) \\ \phi(1-x) w(1,y) & \text{if } \widehat{I}_{k,l} = \{1\} \times (0,1) \\ \phi(y) w(x,0) & \text{if } \widehat{I}_{k,l} = (0,1) \times \{0\} \\ \phi(1-y) w(x,1) & \text{if } \widehat{I}_{k,l} = (0,1) \times \{1\} \end{cases},$$

where

$$\phi(x) := \max\{0, 1 - \eta^{-1} x\} \quad \text{and} \quad \eta \in (0,1). \tag{22}$$

Now, define for each patch Ω_k a mollifying operator $\widehat{\mathcal{M}}_k$ by

$$\widehat{\mathcal{M}}_k := I - \sum_{l \in \mathcal{N}(k)} \Xi_{k,l} \Upsilon_{k,l} (I - \Pi_{r,\eta}) \Upsilon_{k,l}^{-1}. \tag{23}$$

The combination of the patch local operators yields a global operator $\mathcal{M}$:

$$(\mathcal{M} u)|_{\Omega_k} := (\widehat{\mathcal{M}}_k (u \circ G_k)) \circ G_k^{-1}. \tag{24}$$

Observe that $\mathcal{M}$ preserves constants, i.e.,

$$\mathcal{M} c = c \quad \text{for all} \quad c \in \mathbb{R}. \tag{25}$$

Lemma 18. *For all $(k,l) \in \mathcal{N} \cup \mathcal{N}^*$ and all $u \in H_0^1(\widehat{I}_{k,l}) := \{u \in H^1(\widehat{I}_{k,l}) : u = 0 \text{ on } \partial \widehat{I}_{k,l}\}$, we have*

$$\Xi_{k,l} u = 0 \quad \text{on} \quad \partial\widehat{\Omega} \setminus \widehat{I}_{k,l} \quad \text{and} \quad \Xi_{k,l} u = u \quad \text{on} \quad \widehat{I}_{k,l}.$$

Proof. Assume without loss of generality that $\widehat{I}_{k,l} = \{0\} \times (0,1)$. For this case, we have

$$(\Xi_{k,l}u)(x,y) = \phi(x)u(0,y).$$

As $u \in H_0^1(\widehat{I}_{k,l})$, we obtain $u(0,0) = u(0,1) = 0$. This shows the first statement for the two boundary segments adjacent to $\widehat{I}_{k,l}$, i.e., $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$. Since $\eta < 1$ yields $\phi(1) = 0$, we also have the first statement for the boundary segment $\{1\} \times (0,1)$. This finishes the proof for the first statement. The proof for the second statement follows directly from $\phi(0) = 1$. □

Lemma 19. *$\Upsilon_{k,l}^{-1}\widehat{\mathcal{M}}_k = \Pi_{r,\eta}\Upsilon_{k,l}^{-1}$ holds for all $(k,l) \in \mathcal{N} \cup \mathcal{N}^*$.*

Proof. (23) implies $\Upsilon_{k,l}^{-1}\widehat{\mathcal{M}}_k = \Upsilon_{k,l}^{-1} - \sum_{j\in\mathcal{N}(k)} \Upsilon_{k,l}^{-1}\Xi_{k,j}\Upsilon_{k,j}(I-\Pi_{r,\eta})\Upsilon_{k,j}^{-1}$. Observe that the projector $\Pi_{r,\eta}$ is interpolatory on the boundary ([22, Lemma 3.1]). So, $(I-\Pi_{r,\eta})$ maps into $H_0^1(0,1)$ and $\Upsilon_{k,j}(I-\Pi_{r,\eta})$ maps into $H_0^1(\widehat{I}_{k,j})$. Therefore, Lemma 18 yields $\Upsilon_{k,l}^{-1}\widehat{M}_k = \Upsilon_{k,l}^{-1} - \Upsilon_{k,l}^{-1}\Upsilon_{k,l}(I-\Pi_{r,\eta})\Upsilon_{k,l}^{-1}$, which immediately implies the desired result. □

Before we proceed, we give a certain trace like estimate.

Lemma 20. *The estimate*

$$\Psi(u) := \inf_{v\in H^1(\widehat{I}_{k,l})} \|u-v\|^2_{L_2(\widehat{I}_{k,l})} + \theta^2|v|^2_{H^1(\widehat{I}_{k,l})} \lesssim \theta|u|^2_{H^1(\widehat{\Omega})}$$

holds for all $u \in H^1(\widehat{\Omega})$ and $(k,l) \in \mathcal{N} \cup \mathcal{N}^*$ and all $\theta > 0$.

Proof. A trace theorem [22, Lemma 4.4] yields

$$\Psi(u) \lesssim \inf_{v\in H^2(\widehat{\Omega})} \|u-v\|_{L_2(\widehat{\Omega})}|u-v|_{H^1(\widehat{\Omega})} + \theta^2|v|_{H^1(\widehat{\Omega})}|v|_{H^2(\widehat{\Omega})}. \tag{26}$$

Case 1. Assume $\theta < 1$. In this case, we choose v to be the H^1-orthogonal projection of u into $S_{3,\lceil\theta^{-1}\rceil^{-1}}(\widehat{\Omega})$. Since the spline degree of that space is fixed, we obtain using a standard inverse inequality ([20, Corollary 3.94]) and a standard approximation error estimate (like from [24]) that

$$\Psi(u) \lesssim (\lceil\theta^{-1}\rceil^{-1} + \theta^2\lceil\theta^{-1}\rceil)|v|^2_{H^1(\widehat{\Omega})} \lesssim \theta|v|^2_{H^1(\widehat{\Omega})}.$$

Case 2. Assume $\theta \geq 1$. In this case, we choose $v := (u,1)_{L_2(\Omega)}$ and obtain from (26) directly

$$\Psi(u) \lesssim \|u-v\|_{H^1(\widehat{\Omega})}|u|_{H^1(\widehat{\Omega})}.$$

In this case, the Poincaré inequality finishes the proof. □

As a next step, we show that the mollifier constructs functions that are very smooth on the interfaces.

Lemma 21. *The estimate $|\widehat{\mathcal{M}}_k\widehat{u}_k|^2_{H^r(\widehat{I}_{k,l})} \lesssim (2\sqrt{3}r^2\eta^{-1})^{2r-3}r^2|\widehat{u}_k|^2_{H^2(\widehat{\Omega})}$ holds for all $\widehat{u}_k \in H^r(\widehat{\Omega})$ and all $(k,l) \in \mathcal{N} \cup \mathcal{N}^*$.*

Proof. We have using (21) and Lemma 19

$$|\widehat{\mathcal{M}}_k \widehat{u}_k|^2_{H^r(\widehat{I}_{k,l})} \eqsim |\Upsilon_{k,l}^{-1} \widehat{\mathcal{M}}_k \widehat{u}_k|^2_{H^r(0,1)} = |\Pi_{r,\eta} \Upsilon_{k,l}^{-1} \widehat{u}_k|^2_{H^r(0,1)}.$$

Now, a standard inverse estimate ([20, Corollary 3.94]) yields

$$|\widehat{\mathcal{M}}_k \widehat{u}_k|^2_{H^r(\widehat{I}_{k,l})} \lesssim \psi^{2(r-s)} |\Pi_{r,\eta} \Upsilon_{k,l}^{-1} \widehat{u}_k|^2_{H^s(0,1)} \quad \text{for} \quad s \in \{1,2\},$$

where $\psi := 2\sqrt{3} r^2 \eta^{-1}$. Lemma 16 and the H^1-stability of $\Pi_{r,\eta}$ yield

$$|\Pi_{r,\eta} w|^2_{H^2(0,1)} \lesssim r^4 |w|^2_{H^2(0,1)} \quad \text{and} \quad |\Pi_{r,\eta} w|^2_{H^1(0,1)} \le |w|^2_{H^1(0,1)},$$

so we obtain

$$|\widehat{\mathcal{M}}_k \widehat{u}_k|^2_{H^r(\widehat{I}_{k,l})} \lesssim \psi^{2r-2} \inf_{v \in H^2(0,1)} \big(|\Upsilon_{k,l}^{-1} \widehat{u}_k - v|^2_{H^1(0,1)} + \psi^{-2} r^4 |v|^2_{H^2(0,1)} \big).$$

Using (21), we obtain

$$|\widehat{\mathcal{M}}_k \widehat{u}_k|^2_{H^r(\widehat{I}_{k,l})} \lesssim \psi^{2r-2} \inf_{v \in H^2(\widehat{I}_{k,l})} \big(|\widehat{u}_k - v|^2_{H^1(\widehat{I}_{k,l})} + \psi^{-2} r^4 |v|^2_{H^2(\widehat{I}_{k,l})} \big).$$

By applying Lemma 20 to the derivative of $\widehat{u}_k$, we obtain the desired result. □

Lemma 22. *$\|[\![(I-\mathcal{M})u]\!]\|_{L_2(I_{k,l})} = 0$ holds for all $u \in H^{1,\circ}(\Omega) \cap \mathcal{H}^2(\Omega)$ and $(k,l) \in \mathcal{N}$.*

Proof. Let $u \in H^{1,\circ}(\Omega) \cap \mathcal{H}^2(\Omega)$ be arbitrary but fixed.

We obtain using the definition of $\Upsilon_{k,l}$ and $\Upsilon_{l,k}$ and Lemma 6 that

$$\begin{aligned} \|[\![w]\!]\|_{L_2(I_{k,l})} &= \|w_k - w_l\|_{L_2(I_{k,l})} \eqsim \|\widehat{w}_k - \widehat{w}_l \circ G_l^{-1} \circ G_k\|_{L_2(\widehat{I}_{k,l})} \\ &= \|\Upsilon_{k,l}^{-1}(\widehat{w}_k - \widehat{w}_l \circ G_l^{-1} \circ G_k)\|_{L_2(0,1)} \\ &= \|\Upsilon_{k,l}^{-1} \widehat{w}_k - \Upsilon_{l,k}^{-1} \widehat{w}_l\|_{L_2(0,1)} \end{aligned} \tag{27}$$

holds, where $\widehat{w}_k := w_k \circ G_k$ and $\widehat{w}_l := w_l \circ G_l$. Since $u \in H^1(\Omega)$, a standard trace theorem yields

$$\Upsilon_{k,l}^{-1} \widehat{u}_k - \Upsilon_{l,k}^{-1} \widehat{u}_l = 0. \tag{28}$$

Thus, (27) implies $\|[\![u]\!]\|_{L_2(I_{k,l})} = 0$. By plugging $\mathcal{M}u$ into (27), we obtain using Lemma 19

$$\|[\![\mathcal{M}u]\!]\|_{L_2(I_{k,l})} \eqsim \|\Pi_{r,\eta}(\Upsilon_{k,l}^{-1} \widehat{u}_k - \Upsilon_{l,k}^{-1} \widehat{u}_l)\|_{L_2(0,1)}.$$

Using (28) and $\Pi_{r,\eta} 0 = 0$, we obtain $\|[\![\mathcal{M}u]\!]\|_{L_2(I_{k,l})} = 0$ and consequently also $\|[\![(I-\mathcal{M})u]\!]\|_{L_2(I_{k,l})} = 0$. □

Lemma 23. *The estimate $\|\widehat{\Pi}_k (I - \widehat{\mathcal{M}}_k) u\|^2_{H^{1,\circ}(\widehat{\Omega})} \lesssim (1+\eta^2 h^{-2}) h^2 |u|^2_{H^2(\widehat{\Omega})}$ holds for all $u \in H^2(\widehat{\Omega})$ and $k = 1, \ldots, K$.*

Proof. Using the definition of $\widehat{\mathcal{M}}_k$ and of the $H^{1,\circ}$-norm, we obtain

$$\begin{aligned}\|\widehat{\Pi}_k(I-\widehat{\mathcal{M}}_k)u\|_{H^{1,\circ}(\widehat{\Omega})} &\le \sum_{l\in\mathcal{N}(k)} \|\widehat{\Pi}_k\Xi_{k,l}\Upsilon_{k,l}(I-\Pi_{r,\eta})\Upsilon_{k,l}^{-1}u\|_{H^{1,\circ}(\widehat{\Omega})}\\ &\lesssim \sum_{l\in\mathcal{N}(k)} (\Psi_{x,l}+\Psi_{y,l}+\Psi_{\circ,l}),\end{aligned}$$

where $\Psi_{\square,l} := \|\frac{\partial}{\partial\square}\widehat{\Pi}_k\Xi_{k,l}\Upsilon_{k,l}(I-\Pi_{r,\eta})\Upsilon_{k,l}^{-1}u\|_{L_2(\widehat{\Omega})}$ for $\square\in\{x,y\}$ and $\Psi_{\circ,l} := (\widehat{\Pi}_k\Xi_{k,l}\Upsilon_{k,l}(I-\Pi_{r,\eta})\Upsilon_{k,l}^{-1}u,1)_{L_2(\widehat{\Omega})}$. We estimate the terms $\Psi_{x,l}$, $\Psi_{y,l}$ and $\Psi_{\circ,l}$ separately. Let without loss of generality $\widehat{I}_{k,l}=\{0\}\times(0,1)$.

Step 1. Using (17) and the H^1-stability of the $H^{1,D}$-orthogonal projection, and $w:=\Upsilon_{k,l}(I-\Pi_{r,\eta})\Upsilon_{k,l}^{-1}u$, we obtain

$$\begin{aligned}\Psi_{x,l}^2 &= \|\tfrac{\partial}{\partial x}\Pi_{p,h}^x\Pi_{p,h}^y\Xi_{k,l}w\|_{L_2(\widehat{\Omega})}^2 \le \|\tfrac{\partial}{\partial x}\Pi_{p,h}^y\Xi_{k,l}w\|_{L_2(\widehat{\Omega})}^2\\ &= \int_0^1\int_0^1 \big(\phi'(x)\ (\Pi_{p,h}w(0,\cdot))(y)\big)^2\,\mathrm{d}x\,\mathrm{d}y\\ &= |\phi|_{H^1(0,1)}^2\|\Pi_{p,h}(w(0,\cdot))\|_{L_2(0,1)}^2 \eqsim \hat{\Psi}_{x,l}^2 := \eta^{-1}\|\Pi_{p,h}(w(0,\cdot))\|_{L_2(0,1)}^2,\end{aligned}$$

where we use $|\phi|_{H^1(0,1)}^2 \eqsim \eta^{-1}$. The triangle inequality yields

$$\Psi_{x,l}^2 \le \hat{\Psi}_{x,l}^2 \lesssim \eta^{-1}\|w(0,\cdot)\|_{L_2(0,1)}^2 + \eta^{-1}\|(I-\Pi_{p,h})(w(0,\cdot))\|_{L_2(0,1)}^2.$$

The approximation error estimate [22, Theorem 3.2] yields

$$\Psi_{x,l}^2 \le \hat{\Psi}_{x,l}^2 \lesssim \eta^{-1}\|w(0,\cdot)\|_{L_2(0,1)}^2 + \eta^{-1}h^2|w(0,\cdot)|_{H^1(0,1)}^2.$$

The definition of w and (21) yield

$$\Psi_{x,l}^2 \le \hat{\Psi}_{x,l}^2 \lesssim \eta^{-1}\|(I-\Pi_{r,\eta})\Upsilon_{k,l}^{-1}u\|_{L_2(0,1)}^2 + \eta^{-1}h^2|(I-\Pi_{r,\eta})\Upsilon_{k,l}^{-1}u|_{H^1(0,1)}^2.$$

Now, the approximation error estimates [22, Theorems 3.1 and 3.2] yield

$$\Psi_{x,l}^2 \le \hat{\Psi}_{x,l}^2 \lesssim (\eta+\eta^{-1}h^2)\Big(\inf_{v\in H^2(0,1)} |\Upsilon_{k,l}^{-1}u - v|_{H^1(0,1)}^2 + \eta^2|v|_{H^2(0,1)}^2\Big).$$

The equation (21) yields further

$$\Psi_{x,l}^2 \le \hat{\Psi}_{x,l}^2 \lesssim (\eta+\eta^{-1}h^2)\Big(\inf_{v\in H^2(\widehat{I}_{k,l})} |u-v|_{H^1(\widehat{I}_{k,l})}^2 + \eta^2|v|_{H^2(\widehat{I}_{k,l})}^2\Big).$$

Now, Lemma 20 applied to the derivative of u yields

$$\Psi_{x,l}^2 \le \hat{\Psi}_{x,l}^2 \lesssim (\eta+\eta^{-1}h^2)\eta|u|_{H^2(\widehat{\Omega})}^2 = (1+\eta^2h^{-2})h^2|u|_{H^2(\widehat{\Omega})}^2.$$

Step 2. Using (17) and the H^1-stability of the $H^{1,D}$-orthogonal projection and $w:=\Upsilon_{k,l}(I-\Pi_{r,\eta})\Upsilon_{k,l}^{-1}u$, we obtain

$$\begin{aligned}\Psi_{y,l}^2 &= \|\tfrac{\partial}{\partial y}\Pi_{p,h}^x\Pi_{p,h}^y\Xi_{k,l}w\|_{L_2(\widehat{\Omega})}^2 \le \|\tfrac{\partial}{\partial y}\Pi_{p,h}^x\Xi_{k,l}w\|_{L_2(\widehat{\Omega})}^2\\ &= \int_0^1\int_0^1 \big((\Pi_{p,h}\phi)(x)\ \tfrac{\partial}{\partial y}w(0,y)\big)^2\,\mathrm{d}x\,\mathrm{d}y \eqsim \|\Pi_{p,h}\phi\|_{L_2(0,1)}^2|w|_{H^1(\widehat{I}_{k,l})}^2,\end{aligned}$$

Using $\|\Pi_{p,h}\phi\|^2_{L_2(0,1)} \lesssim \|\phi\|^2_{L_2(0,1)}+\|(I-\Pi_{p,h})\phi\|^2_{L_2(0,1)} \lesssim \|\phi\|^2_{L_2(0,1)}+h^2|\phi|^2_{H^1(0,1)} \eqsim \eta + h^2\eta^{-1} = (\eta^2 h^{-2}+1)h^2\eta^{-1}$ and the definition of w, we obtain

$$\Psi^2_{y,l} \lesssim (\eta^2 h^{-2}+1)h^2\eta^{-1}|\Upsilon_{k,l}(I-\Pi_{r,\eta})\Upsilon^{-1}_{k,l}u|^2_{H^1(\widehat{I}_{k,l})}.$$

Using (21), we obtain further

$$\Psi^2_{y,l} \lesssim (\eta^2 h^{-2}+1)h^2\eta^{-1}|(I-\Pi_{r,\eta})\Upsilon^{-1}_{k,l}u|^2_{H^1(0,1)}.$$

Using the H^1-stability of $\Pi_{r,\eta}$ and the approximation error estimate [22, Theorem 3.1], we obtain

$$\Psi^2_{y,l} \lesssim (\eta^2 h^{-2}+1)h^2\big(\inf_{v\in H^2(0,1)} \eta^{-1}|\Upsilon^{-1}_{k,l}u-v|^2_{H^1(0,1)} + \eta|v|^2_{H^2(0,1)}\big).$$

Using (21) and Lemma 20 applied to the derivative of $\Upsilon^{-1}_{k,l}u$, we obtain

$$\begin{aligned}\Psi^2_{y,l} &\lesssim (\eta^2 h^{-2}+1)h^2\big(\inf_{v\in H^2(\widehat{I}_{k,l})} \eta^{-1}|u-v|^2_{H^1(\widehat{I}_{k,l})} + \eta|v|^2_{H^2(\widehat{I}_{k,l})}\big)\\ &\lesssim (\eta^2 h^{-2}+1)h^2|u|^2_{H^2(\widehat{\Omega})}.\end{aligned}$$

Step 3. Using $w := \Upsilon_{k,l}(I-\Pi_{r,\eta})\Upsilon^{-1}_{k,l}u$ and (16), we obtain

$$\begin{aligned}\Psi^2_{\circ,l} &= (\Pi^x_{p,h}\Pi^y_{p,h}\Xi_{k,l}w,1)^2_{L_2(\widehat{\Omega})} = \int_0^1\int_0^1 (\Pi_{p,h}\phi)(x)(\Pi_{p,h}w(0,\cdot))(y)\mathrm{d}x\mathrm{d}y\\ &= \int_0^1 (\Pi_{p,h}\phi)(x)\mathrm{d}x \int_0^1 (\Pi_{p,h}w(0,\cdot))(y)\mathrm{d}y\\ &= \int_0^1 \phi(x)\mathrm{d}x \int_0^1 (\Pi_{p,h}w(0,\cdot))(y)\mathrm{d}y \le \frac{\eta}{2}\,\|\Pi_{p,h}w(0,\cdot)\|^2_{L_2(0,1)} = \frac{\eta^2}{2}\hat{\Psi}^2_{x,l}.\end{aligned}$$

Using the estimate for $\hat{\Psi}^2_{x,l}$, and using $\eta \le 1$, we further obtain

$$\Psi^2_{\circ,l} \lesssim (1+\eta^2 h^{-2})h^2|u|^2_{H^2(\widehat{\Omega})}.$$

Concluding step. As we have bounded $\Psi_{x,l}$, $\Psi_{y,l}$ and $\Psi_{\circ,l}$, we also obtain the corresponding bound for $\|\widehat{\Pi}_k(I-\widehat{\mathcal{M}}_k)u\|^2_{H^{1,\circ}(\widehat{\Omega})}$. □

4.3. The approximation error estimate

The following three lemmas give approximation error estimates (14) for the choice $u_h := \Pi\mathcal{M}u$ separately for the individual parts of $\|\cdot\|_{Q_h^+}$.

Lemma 24. $|(I-\Pi\mathcal{M})u|^2_{\mathcal{H}^1_\alpha(\Omega)} \le (1+\eta^2 h^{-2})h^2|u|^2_{\mathcal{H}^2_\alpha(\Omega)}$ *holds for all* $u \in H^{1,\circ}(\Omega)\cap\mathcal{H}^2(\Omega)$.

Proof. First note that the Poincaré inequality yields

$$\|\cdot\|^2_{H^1(\widehat\Omega)} \eqsim \|\cdot\|^2_{H^{1,\circ}(\widehat\Omega)} := |\cdot|^2_{H^1(\widehat\Omega)} + (\cdot,1)^2_{L_2(\widehat\Omega)}. \tag{29}$$

Let $u \in H^{1,\circ}(\Omega)\cap\mathcal{H}^2(\Omega)$ be arbitrary but fixed and let $\widehat u_k := u\circ G_k$. Using Lemma 6, the triangle inequality, (29) and (18), we obtain

$$|(I-\Pi\mathcal{M})u|^2_{\mathcal{H}^1_\alpha(\Omega)} \lesssim \sum_{k=1}^K \alpha_k |(I-\widehat\Pi_k)\widehat u_k|^2_{H^1(\widehat\Omega)} + \sum_{k=1}^K \alpha_k \|\widehat\Pi_k(I-\widehat{\mathcal{M}}_k)\widehat u_k\|^2_{H^{1,\circ}(\widehat\Omega)}.$$

We further obtain using Lemma 14 and Lemma 23,

$$|(I-\Pi\mathcal{M})u|^2_{\mathcal{H}^1(\Omega)} \lesssim h^2\sum_{k=1}^K \alpha_k|\widehat u_k|^2_{H^2(\widehat\Omega)} + \sum_{k=1}^K (1+\eta^2h^{-2})h^2\alpha_k\|\widehat u_k\|^2_{H^2(\widehat\Omega)}.$$

Lemma 6, (18), (25) and the Poincaré inequality finish the proof. □

Lemma 25. *$|(I-\Pi\mathcal{M})u|^2_{\mathcal{H}^2_\alpha(\Omega)} \le (1+\eta^2h^{-2})p^4|u|^2_{\mathcal{H}^2_\alpha(\Omega)}$ holds for all $u\in H^{1,\circ}(\Omega)\cap\mathcal{H}^2(\Omega)$.*

Proof. Using Lemma 6, the triangle inequality, (29), (18) and a standard inverse inequality ([20, Corollary 3.94]), we obtain

$$\begin{aligned}
|(I-\Pi\mathcal{M})u|^2_{\mathcal{H}^2_\alpha(\Omega)} &\lesssim \sum_{k=1}^K \alpha_k\|(I-\widehat\Pi_k\widehat{\mathcal{M}}_k)\widehat u_k\|^2_{H^2(\widehat\Omega)}\\
&\lesssim \sum_{k=1}^K \alpha_k\|(I-\widehat\Pi_k)\widehat u_k\|^2_{H^2(\widehat\Omega)} + \sum_{k=1}^K \alpha_k\|\widehat\Pi_k(I-\widehat{\mathcal{M}}_k)\widehat u_k\|^2_{H^2(\widehat\Omega)}\\
&\lesssim \sum_{k=1}^K \alpha_k|(I-\widehat\Pi_k)\widehat u_k|^2_{H^2(\widehat\Omega)} + p^4h^{-2}\sum_{k=1}^K \alpha_k\|\widehat\Pi_k(I-\widehat{\mathcal{M}}_k)\widehat u_k\|^2_{H^{1,\circ}(\widehat\Omega)}.
\end{aligned}$$

By again applying Lemma 15 and Lemma 23, we obtain

$$\begin{aligned}
|(I-\Pi\mathcal{M})u|^2_{\mathcal{H}^2_\alpha(\Omega)} &\lesssim p^2\sum_{k=1}^K \alpha_k|\widehat u_k|^2_{H^2(\widehat\Omega)} + p^4(1+\eta^2h^{-2})\sum_{k=1}^K \alpha_k\|\widehat u_k\|^2_{H^2(\widehat\Omega)}\\
&\lesssim p^4(1+\eta^2h^{-2})\sum_{k=1}^K \alpha_k\|\widehat u_k\|^2_{H^2(\widehat\Omega)}.
\end{aligned}$$

Lemma 6, (18), (25) and the Poincaré inequality finish the proof. □

Lemma 26. *$\sum_{(k,l)\in\mathcal{I}}\alpha_{k,l}\|[\![(I-\Pi\mathcal{M})u]\!]\|^2_{L_2(I_{k,l})} \le 2^rh^{2r}(2\sqrt{3}r^2\eta^{-1})^{2r-3}r^2|u|^2_{\mathcal{H}^2_\alpha(\Omega)}$ holds for all $u\in H^{1,\circ}(\Omega)\cap\mathcal{H}^2(\Omega)$.*

Proof. Let $u \in H^{1,\circ}(\Omega) \cap \mathcal{H}^2(\Omega)$ be arbitrary but fixed and let $\widehat{u}_k := u \circ G_k$. Observe that the triangle inequality, Lemma 22 and Lemma 6 yield

$$\begin{aligned}\sum_{(k,l)\in\mathcal{N}} \alpha_{k,l} \|[\![(I-\Pi\mathcal{M})u]\!]\|^2_{L_2(I_{k,l})} &\lesssim \sum_{(k,l)\in\mathcal{N}} \alpha_{k,l} \|[\![(I-\Pi)\mathcal{M}u]\!]\|^2_{L_2(I_{k,l})} \\ \lesssim \sum_{(k,l)\in\mathcal{N}\cup\mathcal{N}^*} \alpha_{k,l} \|((I-\Pi)\mathcal{M}u)|_{\Omega_k}\|^2_{L_2(I_{k,l})} \\ \lesssim \sum_{(k,l)\in\mathcal{N}\cup\mathcal{N}^*} \alpha_{k,l} \|(I-\widehat{\Pi}_k)\widehat{\mathcal{M}}_k \widehat{u}_k\|^2_{L_2(\widehat{I}_{k,l})}.\end{aligned}$$

Lemma 17 yields

$$\sum_{(k,l)\in\mathcal{I}} \alpha_{k,l} \|[\![(I-\Pi\mathcal{M})u]\!]\|^2_{L_2(I_{k,l})} \lesssim 2^r h^{2r} \sum_{(k,l)\in\mathcal{N}\cup\mathcal{N}^*} \alpha_{k,l} |\widehat{\mathcal{M}}_k \widehat{u}_k|^2_{H^r(\widehat{I}_{k,l})}.$$

Since Assumption 2 yields $|\widehat{\mathcal{M}}_k \widehat{u}_k|^2_{H^r(\widehat{I}_{k,l})} \eqsim |\widehat{\mathcal{M}}_l \widehat{u}_l|^2_{H^r(\widehat{I}_{l,k})}$, we obtain

$$\alpha_{k,l}|\widehat{\mathcal{M}}_k \widehat{u}_k|^2_{H^r(\widehat{I}_{k,l})} + \alpha_{k,l}|\widehat{\mathcal{M}}_l \widehat{u}_l|^2_{H^r(\widehat{I}_{l,k})} \lesssim \alpha_k|\widehat{\mathcal{M}}_k \widehat{u}_k|^2_{H^r(\widehat{I}_{k,l})} + \alpha_l|\widehat{\mathcal{M}}_l \widehat{u}_l|^2_{H^r(\widehat{I}_{l,k})}$$

and therefore also

$$\sum_{(k,l)\in\mathcal{I}} \alpha_{k,l} \|[\![(I-\Pi\mathcal{M})u]\!]\|^2_{L_2(I_{k,l})} \lesssim 2^r h^{2r} \sum_{(k,l)\in\mathcal{N}\cup\mathcal{N}^*} \alpha_k |\widehat{\mathcal{M}}_k \widehat{u}_k|^2_{H^r(\widehat{I}_{k,l})}.$$

Now, Lemma 21 yields

$$\sum_{(k,l)\in\mathcal{I}} \alpha_{k,l} \|[\![(I-\Pi\mathcal{M})u]\!]\|^2_{L_2(I_{k,l})} \lesssim 2^r h^{2r} (2\sqrt{3}r^2\eta^{-1})^{2r-3} r^2 \sum_{k=1}^{K} \alpha_k |\widehat{u}_k|^2_{H^2(\widehat{\Omega})}.$$

Lemma 6 finishes the proof. □

Finally, we can show Theorem 13.

Proof (of Theorem 13). Let $u \in \mathcal{H}^{2,\circ}(\Omega) \cap H^1(\Omega)$ be arbitrary but fixed and define $u_h := \Pi\mathcal{M}u$.

First, we show that

$$\|u-u_h\|^2_{Q_h^+} \lesssim \underbrace{\left(1+\eta_0^2 h^{-2} + \sigma(8\sqrt{6}r^2 h\eta_0^{-1})^{2r-3}r^2\right)}_{\Psi :=} h^2 |u|^2_{\mathcal{H}^2_\alpha(\Omega)} \tag{30}$$

holds for any $r \in \{2,\ldots,p_{\min}\}$ and all $\eta_0 > 0$.

Case 1. Assume $\eta_0 \leq 1$. In this, case we define $\eta := \lceil \eta_0^{-1} \rceil^{-1}$ and observe

$$\frac{1}{2}\eta_0 \leq \eta \leq \eta_0.$$

(11) and Lemmas 24, 25, and 26 and $\sigma \gtrsim p^2$ yield

$$\|u-u_h\|^2_{Q_h^+} \lesssim \left(1+\eta^2 h^{-2} + \frac{\sigma}{h^3} 2^r h^{2r} (2\sqrt{3} r^2 \eta^{-1})^{2r-3} r^2\right) h^2 |u|^2_{\mathcal{H}^2_\alpha(\Omega)}$$

and thus (30).

Case 2. Assume $\eta_0 > 1$. Define

$$W := \{u \in H^1(\Omega) \ : \ u \circ G_k \in S_{1,1}(\widehat{\Omega}) \text{ for all } k = 1, \dots, K\},$$

i.e., the set of all globally continuous functions which are locally just linear. Observe that $W \subseteq V_h$. Using u and w being continuous, we obtain

$$\|u-w\|^2_{Q_h^+} = |u-w|^2_{\mathcal{H}^1_\alpha(\Omega)} + \frac{h^2}{\sigma^2} |u-w|^2_{\mathcal{H}^2_\alpha(\Omega)}.$$

For the choice

$$w \in H^1(\Omega) \quad \text{with} \quad w|_{\Omega_k} := w_k = \widehat{w}_k \circ G_k^{-1},$$

where

$$\widehat{w}_k(x,y) := \sum_{i=0}^{1} \sum_{j=0}^{1} \widehat{\phi}_i(x) \phi_j(y) \widehat{u}_k(i,j) \quad \text{where} \quad \widehat{\phi}_0(t) := 1-t \quad \text{and} \quad \widehat{\phi}_1(t) := t,$$

we further obtain using standard approximation error estimates and Lemma 6

$$\inf_{v_h \in V_h} \|u-v_h\|^2_{Q_h^+} \le \|u-w+c\|^2_{Q_h^+} = \|u-w\|^2_{Q_h^+} \lesssim (1+h^2\sigma^{-2}) |u|^2_{\mathcal{H}^2_\alpha(\Omega)},$$

where $c := (w,1)_{L_2(\Omega)}/(1,1)_{L_2(\Omega)}$. Using $\sigma^{-2} \le p^{-4} \le \frac{1}{16}$ and $h \le 1$ and $\eta_0 > 1$, we have

$$\inf_{v_h \in V_h} \|u-v_h\|^2_{Q_h^+} \lesssim \left(\eta_0^2 + \frac{h^2}{16}\right) |u|^2_{\mathcal{H}^2_\alpha(\Omega)} \le (1+\eta_0^2 h^{-2}) h^2 |u|^2_{\mathcal{H}^2_\alpha(\Omega)},$$

which shows (30) also for the second case.

Finally, we show that Ψ is such that the desired bound (14) follows. We again consider two cases.

Case 1. Assume $\ln(\sigma^{1/2}) \le p_{\min}$. In this case, we choose

$$r := \max\{2, \lceil \ln(\sigma^{1/2}) \rceil\} \quad \text{and} \quad \eta_0 := 8\sqrt{6}\mathbf{e} r^2 h,$$

where $\mathbf{e} \sim 2.718$ is Euler's number, and obtain

$$\Psi \lesssim r^4 + \sigma \mathbf{e}^{-2r} \eqsim r^4 \lesssim \big(\ln\sigma\big)^4 \lesssim \big(\ln\sigma\big)^4 \sigma^{2/(2p_{\min}-1)},$$

which finishes the proof for Case 1.

Case 2. Assume $\ln(\sigma^{1/2}) \ge p_{\min}$. In this case, we choose

$$r := p_{\min} \quad \text{and} \quad \eta_0 := 8\sqrt{6}\sigma^{1/(2r-1)} r^2 h$$

and obtain immediately

$$\Psi = 1 + (8\sqrt{6})^2 \sigma^{2/(2r-1)} r^4 + \sigma^{2/(2r-1)} r^2 \lesssim \sigma^{2/(2p_{\min}-1)} (\ln\sigma)^4,$$

which finishes the proof for Case 2. □

5. Numerical Experiments

We depict the results of this paper with numerical results. We choose the Yeti footprint, cf. Figure 1, as computational domain Ω. The domain is decomposed into 21 patches as depicted in Figure 1.

Figure 1: The Yeti footprint

We solve the Poisson equation

$$-\Delta u = 2\,\pi^2 \sin(x\,\pi)\ \sin(y\,\pi) \text{ on } \Omega \quad \text{and} \quad u = g \text{ on } \partial\Omega,$$

where

$$g(x,y) = \sin(x\,\pi)\ \sin(y\,\pi)$$

is the exact solution. For various values of the spline degree p, we introduce a coarse discretization space V_{h_0} for $\ell = 0$. Then, we refine that space uniformly for $\ell = 1, 2, \ldots$. In Table 1, depict the discretization errors $e_{\ell,p}$ and the corresponding rates $r_{\ell,p}$, given by

$$e_{\ell,p} := \|u_{h_\ell,p} - g\|_{Q_h} \qquad \text{and} \qquad r_{\ell,p} := \frac{\|u_{h_{\ell-1},p} - g\|_{Q_h}}{\|u_{h_\ell,p} - g\|_{Q_h}}.$$

	$p=2$		$p=4$		$p=6$		$p=8$		$p=10$	
ℓ	$e_{\ell,p}$	$r_{\ell,p}$	$e_{\ell,p}$	$r_{\ell,p}$	$e_{\ell,p}$	$r_{\ell,p}$	$e_{\ell,p}$	$r_{\ell,p}$	$e_{\ell,p}$	$r_{\ell,p}$
2	0.03272		0.01515		0.01504		0.01397		0.01309	
3	0.00741	4.4	0.00431	3.5	0.00493	3.1	0.00516	2.7	0.00520	2.5
4	0.00178	4.2	0.00144	3.0	0.00168	2.9	0.00179	2.9	0.00185	2.8
5	0.00044	4.1	0.00050	2.9	0.00059	2.9	0.00062	2.9	0.00065	2.9
6	0.00011	4.0	0.00018	2.8	0.00021	2.8	0.00022	3.0	0.00023	2.8

Table 1: Discretization errors

The numerical experiments show that the error decreases like $h_\ell \approx 2^{-\ell}$ or even better and that the error only grows slightly with the spline degree. This coincides with the discretization error analysis since the effect of the logarithmic dependence on the spline degree cannot be observed for any reasonable choice of the spline degree. The observation that the error decreases faster than h_ℓ is a consequence of the fact that the solution of the original problem is smoother than just $H^2(\Omega)$.

Acknowledgments. The author was partially supported by the Austrian Science Fund (FWF): grant S117, and by the bilateral project DNTS-Austria 01/3/2017 (WTZ BG 03/2017), funded by Bulgarian National Science Fund and OeAD (Austria).

References

[1] D. Arnold, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal. **19** (1982), no. 4, 742 – 760.

[2] D. Arnold, F. Brezzi, B. Cockburn, and L. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal. **39** (2002), no. 5, 1749 – 1779.

[3] Y. Bazilevs, L. Beirão da Veiga, J. A. Cottrell, T. J. R. Hughes, and G. Sangalli, *Isogeometric analysis: approximation, stability and error estimates for h-refined meshes*, Math. Models Methods Appl. Sci **16** (2006), no. 07, 1031 – 1090.

[4] H. Blum, R. Rannacher, and R. Leis, *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Meth. Appl. Sci. **2** (1980), no. 4, 556 – 581.

[5] L. Beirão da Veiga, A. Buffa, J. Rivas, and G. Sangalli, *Some estimates for h-p-k-refinement in isogeometric analysis*, Numer. Math. **118** (2011), no. 2, 271 – 305.

[6] M. Dauge, *Elliptic boundary value problems on corner domains. Smoothness and asymptotics of solutions*, Lecture Notes in Mathematics, 1341. Berlin etc.: Springer-Verlag, 1988.

[7] ———, *Neumann and mixed problems on curvilinear polyhedra*, Integr. Equat. Operat. Theor. **15** (1992), 227 – 261.

[8] M. Floater and E. Sande, *Optimal spline spaces of higher degree for L_2 n-widths*, J. Approx. Theor. **216** (2017), 1 – 15.

[9] C. Hofreither and S. Takacs, *Robust multigrid for isogeometric analysis based on stable splittings of spline spaces*, SIAM J. Numer. Anal. **4** (2017), no. 55, 2004 – 2024.

[10] C. Hofreither, S. Takacs, and W. Zulehner, *A robust multigrid method for isogeometric analysis in two dimensions using boundary correction*, Comput. Methods Appl. Mech. Eng. **316** (2017), 22 – 42.

[11] T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs, *Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement*, Comput. Methods Appl. Mech. Eng. **194** (2005), no. 39-41, 4135 – 4195.

[12] M. Kapl, G. Sangalli, and T. Takacs, *Isogeometric analysis with c1 functions on unstructured quadrilateral meshes*, (2018), arXiv: 1812.09088.

[13] U. Langer, A. Mantzaflaris, S. Moore, and I. Toulopoulos, *Multipatch discontinuous Galerkin Isogeometric Analysis*, Isogeometric Analysis and Applications 2014 (B. Jüttler and B. Simeon, eds.), Springer International Publishing, 2015, pp. 1 – 32.

[14] U. Langer and I. Toulopoulos, *Analysis of multipatch discontinuous galerkin iga approximations to elliptic boundary value problems*, Computing and Visualization in Science **17** (2015), no. 5, 217 – 233.

[15] J. Necas, *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris, 1967.

[16] M. Petzoldt, *Regularity and error estimators for elliptic problems with discontinuous coefficients*, Ph.D. thesis, Freie Universität Berlin, Weierstraß–Institut für Angewandte Analysis und Stochastik, 2001.

[17] ______, *A posteriori error estimators for elliptic equations with discontinuous coefficients*, Advances in Computational Mathematics **16** (2002), no. 1, 47 – 75.

[18] B. Rivière, *Discontinuous Galerkin methods for solving elliptic and parabolic equations*, Society for Industrial and Applied Mathematics, 2008.

[19] E. Sande, C. Manni, and H. Speleers, *Sharp error estimates for spline approximation: explicit constants, n-widths, and eigenfunction convergence*, (2018), arXiv: 1810.13418.

[20] C. Schwab, *p- and hp-finite element methods: Theory and applications in solid and fluid mechanics*, Numerical Mathematics and Scientific Computation, Clarendon Press, Oxford, 1998.

[21] J. Sogn and S. Takacs, *Robust multigrid solvers for the biharmonic problem in isogeometric analysis*, Computers and Mathematics with Applications **77** (2018), no. 1, 105 – 124.

[22] S. Takacs, *Robust approximation error estimates and multigrid solvers for isogeometric multi-patch discretizations*, Math. Models Methods Appl. Sci **28** (2018), no. 10, 1899 – 1928.

[23] ______, *Fast multigrid solvers for conforming and non-conforming multi-patch isogeometric analysis*, (2019), arXiv: 1902.01818.

[24] S. Takacs and T. Takacs, *Approximation error estimates and inverse inequalities for B-splines of maximum smoothness*, Math. Models Methods Appl. Sci **26** (2016), no. 07, 1411 – 1445.