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If A+A is small then AAA is superquadratic

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RICAM-Report 2018-37

IF $A+A$ IS SMALL THEN AAA IS SUPERQUADRATIC

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ABSTRACT. This note proves that there exists positive constants c_1 and c_2 such that for all finite $A \subset \mathbb{R}$ with $|A+A| \leq |A|^{1+c_1}$ we have $|AAA| \gg |A|^{2+c_2}$.

NOTATION

Throughout the paper, the standard notation $\ll, \gg$ is applied to positive quantities in the usual way. That is, $X \gg Y$ and $Y \ll X$ both mean that $X \geq cY$, for some absolute constant $c > 0$. The expression $X \approx Y$ means that both $X \gg Y$ and $X \ll Y$ hold. The notation $\lesssim$ and $\gtrsim$ is used to suppress both constant and logarithmic factors. To be precise, the expression $X \gtrsim Y$ or $Y \lesssim X$ means that $X \gg Y/(\log X)^c$, for some absolute constant $c > 0$. All logarithms have base 2.

For $A \subset \mathbb{R}$, the sumset of A is the set $A+A := \{a+b : a, b \in A\}$. The difference set $A-A$ and the product set AA are defined similarly. The k-fold sum set is $kA := \{a_1 + \cdots + a_k : a_1, \ldots, a_k \in A\}$.

1. INTRODUCTION

Let A be a finite set of real numbers such that $|A+A| \leq |A|^{1+c}$ where c is a small but positive real number. According to sum-product phenomena, we expect that the product set AA should be rather large. In fact, this "few sums implies many products" problem is well understood. For example, Elekes and Ruzsa [2] used the Szemerédi-Trotter Theorem to prove the bound

$$|A+A|^4|AA| \gg \frac{|A|^6}{\log |A|}, \tag{1.1}$$

which in particular gives

$$|A+A| \ll |A|^{1+c} \Rightarrow |AA| \gtrsim |A|^{2-4c}. \tag{1.2}$$

A better dependence on c can be obtained by using Solymosi's [8] sum-product estimate

$$|A+A|^2|AA| \gg \frac{|A|^4}{\log |A|}. \tag{1.3}$$

One may consider the same question with multifold product sets. To be precise we expect that there exists $c > 0$ such that the following weak form of the k-fold Erdős-Szemerédi conjecture holds:

$$|A+A| \leq |A|^{1+c} \Rightarrow |A^{(k)}| \gtrsim |A|^k, \tag{1.4}$$

where $A^{(k)}$ denotes the k-fold product set $\{a_1a_2\cdots a_k : a_1,\dots,a_k \in A\}$. However, it seems that this problem is not well understood at all, and we are not aware even of a bound of the form

$$|A+A| \leq |A|^{1+c_1} \Rightarrow |A^{(100)}| \gg |A|^{2+c_2}, \tag{1.5}$$

for some positive constants c_1 and c_2. The aim of this note is to prove such a superquadratic bound.

Theorem 1.1. *There exist positive constants c_1 and c_2 such that for all sufficiently large finite sets $A \subset \mathbb{R}$ with $|A+A| \leq |A|^{1+c_1}$, it follows that $|AAA| \geq |A|^{2+c_2}$.*

We will give two slightly different proofs of this result. The first proof uses a result of Shkredov and Zhelezov [7], and may be preferable for the reader familiar with [7]. The second is more self-contained and is presented in a quantitative form, although we do not pursue the best possible bounds given by our methods, instead preferring to simplify the proof at certain points. See the forthcoming Theorem 3.2 for a quantitative formulation of Theorem 1.1.

The two proofs are certainly similar, but we feet that they are different enough, and short enough, to warrant giving both in full. On common feature is that they both rely on a threshold-breaking bound for the additive energy of a set with small product set. The additive energy of A, denoted $E^+(A)$, is the number of solutions to the equation

$$a_1 + a_2 = a_3 + a_4, \qquad a_i \in A.$$

A simple application of the Cauchy-Schwarz inequality gives the much used bound

$$E^+(A) \geq \frac{|A|^4}{|A+A|}. \tag{1.6}$$

We will use the following result, which is [5, Theorem 3].

Theorem 1.2. *Let $A \subset \mathbb{R}$ and $|AA| \leq M|A|$. Then*

$$E^+(A) \ll M^{8/5}|A|^{49/20}\log^{1/5}|A|\,.$$

The proof of Theorem 1.2 uses the Szemerédi-Trotter Theorem, as well as higher energy tools, which have been used to push past several thresholds for sum-product type problems in recent years. Note that a weaker bound of the form $E^+(A) \ll_M |A|^{5/2}$ can be obtained by a more straightforward application of the Szemerédi-Trotter Theorem, and is implicitly contained in the work of Elekes [1]. However, for both proofs of the main theorem of this paper, it is crucial that the exponent $49/20$ is strictly less than this threshold of $5/2$.

1.1. **Other tools.** Some other well-known results that are used in our proofs are collected here. The following two forms of the Plünnecke-Ruzsa inequality are applied. See Petridis [6] for short proofs.

Lemma 1.3. *Let X be a finite set in an additive abelian group. Then*

$$|kX - lX| \leq \frac{|X+X|^{k+l}}{|X|^{k+l-1}}.$$

Lemma 1.4. *Let X and Y be finite subsets in an additive abelian group. Then*

$$|kX| \leq \frac{|X+Y|^k}{|Y|^{k-1}}.$$

The ratio set of A is the set $A/A = \{a/b : a, b \in A\}$. We need an estimate for the ratio set when the sum set is small. Solymosi's estimate (1.3) also holds when AA is replaced with A/A. The following variant, which removes the logarithmic factor, was observed by Li and Shen [4]: for any finite set $A \subset \mathbb{R}$,

$$|A+A|^2|A/A| \gg |A|^4. \tag{1.7}$$

Given finite sets $A, B \in \mathbb{R}$, let $T(A,A,B)$ denote the number of solutions to the equation

$$\frac{a_1 - b}{a_2 - b} = \frac{a_1' - b'}{a_2' - b'}, \quad a_1, a_2, a_1', a_2' \in A, b, b' \in B.$$

The notation $T(A)$ is used as shorthand for $T(A,A,A)$. This is essentially the number of collinear triples in the point set $A \times A$, an observation which Jones [3] used to prove that

$$T(A) \ll |A|^4 \log |A|. \tag{1.8}$$

2. The first proof

Given finite sets B, C and $x \in \mathbb{R}$, the notation $r_{B+C}(x)$ is used for the number of representations of x as an element of $B+C$. That is,

$$r_{B+C}(x) := |\{(b,c) \in B \times C : b + c = x\}|.$$

In our first proof of Theorem 1.1, we will need the following lemma, which is essentially contained in [7] (see Remark 1 therein). We omit the proof, since all of the necessary details can be found in [7], but remark that a crucial ingredient is a threshold-breaking bound for the additive energy of a similar form to Theorem 1.2.

For sets $B, C, X \subset \mathbb{R}$, define

$$\sigma_X(B,C) := \sum_{x \in X} r_{B+C}(x),$$

and let $\sigma_X(B) = \sigma_X(B,C)$.

Lemma 2.1. *There exist positive constants c and c' such that the following holds. For any finite $X \subset \mathbb{R}$ such that $|XX| \leq |X|^{1+c}$, and for any finite $B \subset \mathbb{R}$,*

$$\sigma_X(B) \lesssim |B|^{\frac{17}{10}}|X|^{\frac{3}{20}-c'}.$$

We will first prove a slightly weaker version of Theorem 1.1, using the 4-fold rather than 3-fold product set.

Theorem 2.2. *There exist positive constants c_1 and c_2 such that for all sufficiently large finite sets $A \subset \mathbb{R}$ with $|A+A| \leq |A|^{1+c_1}$, it follows that $|AAAA| \geq |A|^{2+c_2}$.*

Proof. Let c and c' be the positive constants given by Lemma 2.1. The constant c_1 must be sufficiently small compared to c and c': taking $c_1 := \frac{1}{4}\min\{c, c'\}$ would comfortably suffice. We can take $c_2 = c$.

Suppose that $|A+A| \leq |A|^{1+c_1}$. Recall from (1.3) that $|AA| \gtrsim |A|^{2-2c_1}$. It would be sufficient to show that $|(AA)(AA)| \geq |AA|^{1+c}$, as we would then have

$$|AAAA| \geq |AA|^{1+c} \gtrsim |A|^{(1+c)(2-2c_1)} \geq |A|^{2+c_2}.$$

Suppose for a contradiction that this is not true and we have $|(AA)(AA)| \leq |AA|^{1+c}$. Note that, for any $a \in A$ we have $A \subset a^{-1}AA$. Define $X := a^{-1}AA$. By our assumption, $|XX| \leq |X|^{1+c}$. Write $S := A+A$. Then, notice that

$$\sigma_X(-A, S) = \sum_{x \in X} r_{S-A}(x) \geq \sum_{x \in A} r_{S-A}(x) \geq |A|^2,$$

since $r_{S-A}(x) \geq |A|$ for all $x \in A$ (because of the solutions $x = (x+y) - y$). On the other hand, by Lemma 2.1,

$$|A|^2 \leq \sigma_X(-A, S) \leq \sigma_X(S \cup -A) \lesssim |S|^{17/10}|X|^{3/20-c'}.$$

With our earlier choice of c_1, this implies that $|X| \geq |A|^{2+\epsilon}$ for some constant $\epsilon > 0$, which is a contradiction.

□

It is then a straightforward task to use the Plünnecke-Ruzsa Theorem to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let c_1 be a sufficiently small positive constant (no larger than the c_1 in the statement of Theorem 2.2). So, $|AAAA| \geq |A|^{2+c_2}$, where c_2 is the constant from the statement of Theorem 2.2. Applying Lemma 1.4 with $X = A$ and $Y = AA$ gives

$$|AAAA| \leq \frac{|(AA)A|^4}{|AA|^3}$$

Also, $|AA| \gtrsim |A|^{2-2c_1}$ by (1.3). Putting all of this together gives

$$|AAA| \gtrsim |A|^{2+\frac{1}{4}(c_2-6c_1)}.$$

Choosing c_1 sufficiently small gives $|AAA| \gtrsim |A|^{2+\frac{c_2}{8}}$. This completes the proof.

$\square$

3. The second proof

Similarly to the previous section, we will first prove a superquadratic bound involving more variables.

Lemma 3.1. *Let $A \subset \mathbb{R}$ be finite and write $|A+A| = K|A|$. Then*

$$\left|\frac{AAAA}{AAAA}\right| \gtrsim \frac{|A|^{\frac{100}{49}}}{K^{\frac{40}{7}}}.$$

Proof. Let $S := A + A$. Consider the set

$$\mathcal{A} := \left\{(b,b',c,c') \in S \times S \times A \times A \ : \ b-c, b-c', b'-c, b'-c' \in A\right\}.$$

By the Cauchy-Schwarz inequality,

$$|\mathcal{A}| = \sum_{c,c' \in A}\left(\sum_{b \in S} A(b-c)A(b-c')\right)^2 \geq |A|^{-2}\left(\sum_{c,c' \in A}\sum_{b \in S} A(b-c)A(b-c')\right)^2$$
$$= E^+(A)^2|A|^{-2} \overset{(1.6)}{\geq} |A|^6|S|^{-2} := X.$$

Write $|\mathcal{A}(b,b'c)| := \sum_{c'} \mathcal{A}(b,b',c,c')$. In these terms

$$\begin{aligned} X \leq |\mathcal{A}| &= \sum_{(b,b',c) \in S \times S \times A} |\mathcal{A}(b,b'c)| \\ &= \sum_{(b,b',c) \in S \times S \times A : |\mathcal{A}(b,b',c)| \leq \frac{X}{2|A||S|^2}} |\mathcal{A}(b,b',c)| + \sum_{(b,b',c) \in S \times S \times A : |\mathcal{A}(b,b',c)| > \frac{X}{2|A||S|^2}} |\mathcal{A}(b,b'c)| \\ &\leq \frac{X}{2} + \sum_{(b,b',c) \in S \times S \times A : |\mathcal{A}(b,b',c)| > \frac{X}{2|A||S|^2}} |\mathcal{A}(b,b'c)|. \end{aligned}$$

This implies that

$$\frac{X}{2} \leq \sum_{b,b',c \ : \ |\mathcal{A}(b,b',c)| > X/(2|A||S|^2)} |\mathcal{A}(b,b',c)| \leq |A| \sum_{b,b',c \ : \ |\mathcal{A}(b,b',c)| > X/(2|A||S|^2)} 1. \tag{3.1}$$

Define

$$n(x) := \left|\left\{(b,b',c) \in S \times S \times A : x = \frac{b-c}{b'-c}, |\mathcal{A}(b,b',c)| > \frac{X}{2|A||S|^2}\right\}\right|.$$

Inequality (3.1) gives $\sum_x n(x) \geq \frac{X}{2|A|}$. Therefore, by the Cauchy–Schwarz inequality

$$X^2/(2|A|)^2 \leq \sum_x n^2(x) \cdot \left| \left\{ \frac{b-c}{b'-c} \ : \ |\mathcal{A}(c,b',b)| \geq X/(2|A||S|^2) \right\} \right|$$
$$\leq T(S,S,A) \cdot \left| \left\{ \frac{b-c}{b'-c} \ : \ |\mathcal{A}(c,b',b)| \geq X/(2|A||S|^2) \right\} \right|$$
$$\ll |S|^4 \log|A| \cdot \left| \left\{ \frac{b-c}{b'-c} \ : \ |\mathcal{A}(c,b',b)| \geq X/(2|A||S|^2) \right\} \right| .$$

Here we have used the trivial bound $T(S,S,A) \leq T(S \cup A)$ and applied (1.8). Define R to be the set on the right-hand side of the last inequality, so

$$|R| \gg \frac{X^2}{|A|^2|S|^4 \log|A|} = \frac{|A|^2}{K^8 \log|A|}. \tag{3.2}$$

Now, consider the identity

$$1 - \frac{b-c}{b'-c} = \frac{b'-c'}{b'-c} - \frac{b-c'}{b'-c}. \tag{3.3}$$

From (3.2) it follows that equation (3.3) gives at least

$$|R| \cdot X/(2|A||S|^2) \gg \frac{|A|^3}{K^{12}\log|A|}$$

solutions to the equation

$$1 - \alpha_1 = \alpha_2 - \alpha_3\,, \qquad \alpha_1, \alpha_2, \alpha_3 \in A/A\,. \tag{3.4}$$

Indeed, for any $r \in R$ we fix a representation $r = \frac{b-c}{b'-c}$. There are at least $X/(2|A||S|^2)$ elements c' such that $(b,b',c,c') \in \mathcal{A}$. As c' varies, so do the elements $\frac{b'-c'}{b'-c}$ and $\frac{b-c'}{b'-c}$, and so the solutions to (3.4) obtained in this way are all distinct.

Further multiplying equation (3.4) by any $a \in A/A$, we obtain that

$$E^+(AA/AA) \gg \frac{|A|^3}{K^{12}\log|A|}|A/A| \overset{(1.7)}{\gg} \frac{|A|^5}{K^{14}\log|A|}.$$

On the other hand, Theorem 1.2 implies that

$$E^+(AA/AA) \lesssim \left| \frac{AAAA}{AAAA} \right|^{\frac{49}{20}} .$$

Combining the last two inequalities gives

$$\left| \frac{AAAA}{AAAA} \right| \gtrsim \frac{|A|^{\frac{100}{49}}}{K^{\frac{40}{7}}},$$

as required

□

We are now ready to prove the following quantitative form of Theorem 1.1.

Theorem 3.2. *Let $A \subset \mathbb{R}$ be finite and write $|A+A| = K|A|$. Then*

$$|AAA| \gtrsim \frac{|A|^{2+\frac{1}{392}}}{K^{\frac{125}{56}}}.$$

Proof. Applying Lemma 1.3 in the multiplicative setting with $X = AA$ and $k = l = 2$ gives

$$|AAAA|^4 \geq |AA|^3 \left|\frac{AAAA}{AAAA}\right|. \tag{3.5}$$

Applying Lemma 1.4 with $X = AA$ and $Y = A$ gives

$$|AAAA| \leq \frac{|AAA|^4}{|AA|^3}. \tag{3.6}$$

Combining these two inequalities with Lemma 3.1, we have

$$|AAA|^{16} \geq |AA|^{12}|AAAA|^4 \geq |AA|^{15}\left|\frac{AAAA}{AAAA}\right| \gtrsim |AA|^{15} \cdot \frac{|A|^{\frac{100}{49}}}{K^{\frac{40}{7}}} \overset{(1.3)}{\gtrsim} \frac{|A|^{30}}{K^{30}} \cdot \frac{|A|^{\frac{100}{49}}}{K^{\frac{40}{7}}},$$

and a rearrangement completes the proof. □

Acknowledgements

Oliver Roche-Newton was partially supported by the Austrian Science Fund FWF Project P 30405-N32 as well as by the Austrian Science Fund (FWF), Project F5507-N26, which is a part of the Special Research Program Quasi-Monte Carlo Methods: Theory and Applications. We are grateful to George Shakan, Audie Warren and Dmitry Zhelezov for various helpful conversations and advice.

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