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Abstract

In this article we consider the approximation of compact linear operators defined over tensor product Hilbert spaces. Necessary and sufficient conditions on the singular values of the problem under which we can or cannot achieve different notions of exponential tractability are given in [5]. In this paper, we use the new equivalency conditions shown in [1] to obtain these results in an alternative way. As opposed to the algebraic setting, quasi-polynomial tractability is not possible for non-trivial cases in the exponential setting.

1 Introduction and Preliminaries

Tractability of multivariate problems is the subject of a considerable number of articles and monographs in the field of Information-Based Complexity (IBC). For an introduction to IBC, we refer to the book [8]. For a recent overview of the state of the art in tractability studies we refer to the trilogy [2]–[4]. In this article we study tractability in the worst case setting for linear tensor product problems and for algorithms that use finitely many arbitrary continuous linear functionals.

The information complexity of a compact linear operator $S_d : \mathcal{H}_d \to \mathcal{G}_d$ is defined as the minimal number, $n(\varepsilon, S_d)$, of such linear functionals needed to find an ε-approximation. It is natural to ask how the information complexity of a given problem depends on both d and ε^{-1}.

In most of the literature on this subject, different notions of tractability are defined in terms of a relationship between $n(\varepsilon, S_d)$ and some powers of d and $\max(1, \varepsilon^{-1})$. This is called algebraic (ALG) tractability. For a complete overview of a wide range of results on algebraic tractability, see [2]–[4] On the other hand, a relatively recent stream of work defines different notions of tractability in terms of a relationship between $n(\varepsilon, S_d)$ and some powers of d and $1 + \log \max(1, \varepsilon^{-1})$. Now the complexity of the problem increases only logarithmically as the error tolerance vanishes. This situation is referred to as exponential (EXP) tractability, which is the subject of this article. Precise definitions of ALG and EXP tractabilities are given below.

General compact linear multivariate problems have been studied in the recent article [1]. Here, we deal with the case of tensor product problems for which the singular values of a d-variate problem are given as products of the singular values of univariate problems. Exponential tractability for tensor product problems has been studied in [5]. In this paper we re-prove the results of [5] by a different argument via the criteria presented in [1].

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Consider two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{G}_1$ and a compact linear solution operator, $S_1 : \mathcal{H}_1 \to \mathcal{G}_1$. Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}$, let

$$\mathcal{H}_d \;=\; \mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1 \quad \text{and} \quad \mathcal{G}_d \;=\; \mathcal{G}_1 \otimes \mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_1$$

be the d-fold tensor products of the spaces $\mathcal{H}_1$ and $\mathcal{G}_1$, respectively. Furthermore, let S_d be the linear tensor product operator,

$$S_d = S_1 \otimes S_1 \otimes \cdots \otimes S_1,$$

on $\mathcal{H}_d$. In this way, obtain a sequence of compact linear solution operators

$$\mathcal{S} \;=\; \{S_d : \mathcal{H}_d \to \mathcal{G}_d\}_{d \in \mathbb{N}}.$$

We now consider the problem of approximating $\{S_d(f)\}$ for f from the unit ball of $\mathcal{H}_d$ by means of algorithms $\{A_{d,n} : \mathcal{H}_d \to \mathcal{G}_d\}_{d \in \mathbb{N}, n \in \mathbb{N}_0}$. For $n = 0$, we set $A_{d,0} := 0$, and for $n \geq 1$, $A_{d,n}(f)$ depends on n continuous linear functionals $L_1(f), L_2(f), \ldots, L_n(f)$, so that

$$A_{d,n}(f) = \phi_n(L_1(f), L_2(f), \ldots, L_n(f)) \tag{1}$$

for some $\phi_n : \mathbb{C}^n \to \mathcal{G}_d$ or $\phi_n : \mathbb{R}^n \to \mathcal{G}_d$ and $L_j \in \mathcal{H}_d^*$. We allow an adaptive choice of $L_1, L_2, \ldots, L_n$ as well as n, i.e., $L_j = L_j(\cdot; L_1(f), L_2(f), \ldots, L_{j-1}(f))$ and n can be a function of the $L_j(f)$, see [8] and [2] for details. The error of a given algorithm $A_{d,n}$ is measured in the worst case setting, which means that we need to deal with

$$e(A_{d,n}) \;=\; \sup_{\substack{f \in \mathcal{H}_d \\ \|f\|_{\mathcal{H}_d} \leq 1}} \|S_d(f) - A_{d,n}(f)\|_{\mathcal{G}_d}.$$

However, to assess the difficulty of the approximation problem, we would not only like to study the worst case errors of particular algorithms, but consider a more general error measure. To this end, let

$$e(n, S_d) \;=\; \inf_{A_{d,n}} e(A_{d,n})$$

denote the nth minimal worst case error, where the infimum is extended over all admissible algorithms $A_{d,n}$ of the form (1). Then the information complexity $n(\varepsilon, S_d)$ is the minimal number n of continuous linear functionals needed to find an algorithm $A_{d,n}$ that approximates S_d with error at most ε. More precisely, we consider the absolute (ABS) and normalized (NOR) error criteria in which

$$\begin{aligned} n(\varepsilon, S_d) = n_{\text{ABS}}(\varepsilon, S_d) \;&= \min\{n : \; e(n, S_d) \leq \varepsilon\}, \\ n(\varepsilon, S_d) = n_{\text{NOR}}(\varepsilon, S_d) &= \min\{n : \; e(n, S_d) \leq \varepsilon \, \|S_d\|\}. \end{aligned}$$

It is known from [8] (see also [2]) that the information complexity is fully determined by the singular values of S_d, which are the same as the square roots of the eigenvalues of the compact self-adjoint and positive semi-definite linear operator $W_d = S_d^* S_d : \mathcal{H}_d \to \mathcal{H}_d$. We denote these eigenvalues by $\lambda_{d,1}, \lambda_{d,2}, \ldots$. Then it is known that the information complexity can be expressed in terms of the eigenvalues $\lambda_{d,j}$. Indeed,

$$n_{\text{ABS}}(\varepsilon, S_d) = \min\{n : \; \lambda_{d,n+1} \leq \varepsilon^2\}, \tag{2}$$

$$n_{\text{NOR}}(\varepsilon, S_d) = \min\{n : \; \lambda_{d,n+1} \leq \varepsilon^2 \lambda_{d,1}\}. \tag{3}$$

Clearly, $n_{\text{ABS}}(\varepsilon, S_d) = 0$ for $\varepsilon \geq \sqrt{\lambda_{d,1}} = \|S_d\|$, and $n_{\text{NOR}}(\varepsilon, S_d) = 0$ for $\varepsilon \geq 1$. Therefore for ABS we can restrict ourselves to $\varepsilon \in (0, \|S_d\|)$, whereas for NOR to $\varepsilon \in (0, 1)$. Since $\|S_d\|$ can be arbitrarily large, to deal simultaneously with ABS and NOR we consider $\varepsilon \in (0, \infty)$. It is known that $n_{\mathrm{ABS/NOR}}(\varepsilon, S_d)$ is finite for all $\varepsilon > 0$ iff S_d is compact, which justifies our assumption about the compactness of S_d.

We now recall that the spaces $\mathcal{H}_d$ and $\mathcal{G}_d$ are tensor product spaces. It is known that the eigenvalues $\lambda_{d,j}$ of W_d are then given as products of the eigenvalues $\widetilde{\lambda}_j$ of the operator $W_1 = S_1^* S_1 : \mathcal{H}_1 \to \mathcal{H}_1$, i.e.,

$$\lambda_{d,j} = \prod_{\ell=1}^{d} \widetilde{\lambda}_{j_\ell}. \tag{4}$$

Without loss of generality, we assume that the $\widetilde{\lambda}_j$ are ordered, i.e., $\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \cdots$.

Although the $\widetilde{\lambda}_j$ are given by (4), the ordering of the $\widetilde{\lambda}_j$ does not easily imply the ordering of the $\lambda_{d,j}$ since the map $j \in \mathbb{N} \mapsto (j_1, \ldots, j_d) \in \mathbb{N}^d$ exists but does not have a simple explicit form. This makes the tractability analysis challenging.

We are ready to define exactly various notions of ALG and EXP tractabilities. To present them concisely, let

$$\mathsf{y} \in \{\mathrm{ABS}, \mathrm{NOR}\}, \quad \mathsf{z} = \begin{cases} \max(1, \varepsilon^{-1}), & \text{in the case of ALG,} \\ 1 + \log\, \max(1, \varepsilon^{-1}), & \text{in the case of EXP.} \end{cases}$$

These definitions are as follows.

The problem $\mathcal{S}$ is ...

- strongly polynomially tractable (SPT) if there are $C, q \geq 0$ such that
$$n_{\mathsf{y}}(\varepsilon, S_d) \leq C \mathsf{z}^q \quad \forall d \in \mathbb{N},\ \varepsilon \in (0, \infty),$$
- polynomially tractable (PT) if there are $C, p, q \geq 0$ such that
$$n_{\mathsf{y}}(\varepsilon, S_d) \leq C d^p \mathsf{z}^q \quad \forall d \in \mathbb{N},\ \varepsilon \in (0, \infty),$$
- quasi-polynomially tractable (QPT) if there are $C, p \geq 0$ such that
$$n_{\mathsf{y}}(\varepsilon, S_d) \leq C\, \exp\big(p\,(1 + \log\, d)(1 + \log\, \mathsf{z})\big) \quad \forall d \in \mathbb{N},\ \varepsilon \in (0, \infty),$$
- (s, t)-weakly tractable ((s, t)-WT) if
$$\lim_{d + \varepsilon^{-1} \to \infty} \frac{\log\, \max(1, n_{\mathsf{y}}(\varepsilon, S_d))}{d^t + \mathsf{z}^s} = 0,$$
- uniformly weakly tractable (UWT) if (s, t)-WT holds for all $s, t > 0$.

We use the prefix ALG- with the above tractability notions in the case $\mathsf{z} = \max(1, \varepsilon^{-1})$ and EXP- in the case $\mathsf{z} = 1 + \log\, \max(1, \varepsilon^{-1})$.

A recent article [1] provides necessary and sufficient conditions on the eigenvalues $\lambda_{d,j}$ of W_d for the various tractability notions above. For the special case of linear tensor product spaces considered here, it is natural to ask for conditions on the eigenvalues $\widetilde{\lambda}_j$ of W_1 such that we obtain the different kinds of exponential tractability. For results on algebraic tractability for tensor product spaces, see again [2]–[4] and the articles cited therein. The notion of (s, t)-WT was introduced in [7], and UWT was introduced in [6]. See also [9] and [1] for results on (s, t)-WT and UWT in the algebraic sense.

Finding necessary and sufficient conditions on the $\widetilde{\lambda}_j$ for the different kinds of exponential tractability turns out to be a technically difficult question. Necessary and sufficient conditions have been considered in the paper [5]. Here we re-prove the results in [5], using a completely different technique, namely using criteria that have been shown very recently in the paper [1]. In some cases, our new technique enables us to obtain the desired results using shorter and/or less technical arguments than those that were used in [5].

2 Results

In this section we show results on tractability conditions in terms of the eigenvalues of the operator W_1.

Note that, if all of the $\widetilde{\lambda}_j$ equal zero, then the operators S_d are all zero, and $n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) = 0$ for all $d \geq 1$. Furthermore, if only $\widetilde{\lambda}_1 > 0$ and $\widetilde{\lambda}_2 = \widetilde{\lambda}_3 = \cdots = 0$ (remember that the $\widetilde{\lambda}_j$ are ordered), it can be shown that $n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \leq 1$ for all $d \geq 1$. Hence, the problem is interesting only if at least two of the $\widetilde{\lambda}_j$ are positive, which we assume from now on.

Before we state our main result, we state two technical lemmas and a theorem proved elsewhere. The first lemma is well known.

Lemma 1 *For any $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \geq 0$ we have:*

- *For $s \geq 1$,*
$$\frac{1}{n^{s-1}}(a_1 + \cdots + a_n)^s \leq a_1^s + \cdots + a_n^s \leq (a_1 + \cdots + a_n)^s.$$
- *For $s \leq 1$*
$$(a_1 + \cdots + a_n)^s \leq a_1^s + \cdots + a_n^s \leq n^{1-s}(a_1 + \cdots + a_n)^s.$$

Lemma 2 *For all $k, n \in \mathbb{N}$ with $k < n$, it follows that*

$$\max\left\{\left(\frac{n}{k}\right)^k, \left(\frac{n}{n-k}\right)^{n-k}\right\} \leq \binom{n}{k} \leq \min\left\{\left(\frac{\mathrm{e}n}{k}\right)^k, \left(\frac{\mathrm{e}n}{n-k}\right)^{n-k}\right\}.$$

Proof. It is easy to see that

$$\left(\frac{n}{k}\right)^k = \frac{n}{k} \cdots \frac{n}{k} \leq \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \binom{n}{k} = \binom{n}{n-k}$$
$$\leq \frac{n^k}{k!} = \left(\frac{n}{k}\right)^k \frac{k^k}{k!} \leq \left(\frac{\mathrm{e}n}{k}\right)^k,$$

and the estimates of the lemma follow. □ □

Theorem 1 *[1, Theorem 3] $\mathcal{S}$ is* EXP-(s,t)-WT-ABS/NOR *iff*

$$\sup_{d \in \mathbb{N}} \sigma_{\mathrm{EWT}}(d, s, t, c) < \infty \quad \forall c > 0, \tag{5}$$

where

$$\sigma_{\mathrm{EWT}}(d, s, t, c) := \exp(-cd^t) \sum_{j=1}^{\infty} \exp\left(-c\left[1 + \log\left(2\max\left(1, \frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right)\right]^s\right),$$

where

$$\mathrm{CRI}_d = \begin{cases} 1 & \textit{for } \mathrm{ABS}, \\ \lambda_{d,1} & \textit{for } \mathrm{NOR}. \end{cases}$$

We now state and prove the main result of this article.

Theorem 2 *Let*

$$\widetilde{\lambda}_1 \ge \widetilde{\lambda}_2 > 0.$$

Consider the conditions

$$\lim_{n\to\infty} \frac{\log \widetilde{\lambda}_n^{-1}}{(\log n)^{1/\min(s,t)}} = \infty, \tag{6}$$

$$\lim_{n\to\infty} \frac{\log \widetilde{\lambda}_n^{-1}}{(\log n)^{1/s}} = \infty, \tag{7}$$

$$\lim_{n\to\infty} \frac{\log \widetilde{\lambda}_n^{-1}}{(\log n)^{1/\eta}} = \infty, \tag{8}$$

where η is given below.

EXP-(s,t)-WT-ABS *holds iff one of the following conditions is true:*

- ***(A.1):*** $t > 1$, $s > 1$, $\widetilde{\lambda}_1 > 1$, *and* (6) *holds or*
- ***(A.2):*** $t > 1$, $s \ge 1$, $\widetilde{\lambda}_1 \le 1$, *and* (7) *holds or*
- ***(A.3):*** $t > 1$, $s < 1$, *and* (8) *holds with* $\eta = s(t-1)/(t-s)$ *or*
- ***(A.4):*** $t \le 1$, $s > 1$, $\widetilde{\lambda}_1 \le 1$, $\widetilde{\lambda}_2 < 1$, *and* (7) *holds.*

EXP-(s,t)-WT-NOR *holds iff one of the following conditions is true:*

- ***(N.1):*** $t > 1$, $s \ge 1$, *and* (7) *holds or*
- ***(N.2):*** $t > 1$, $s < 1$, *and* (8) *holds with* $\eta = s(t-1)/(t-s)$ *or*
- ***(N.3):*** $t \le 1$, $s > 1$, $\widetilde{\lambda}_1 > \widetilde{\lambda}_2$, *and* (7) *holds.*

Furthermore, EXP-UWT, EXP-QPT, EXP-PT, *and* EXP-SPT *do not hold under any conditions on $\widetilde{\lambda}_j$, i.e., even for $\widetilde{\lambda}_3 = \widetilde{\lambda}_4 = \cdots = 0$.*

Proof. We know from Theorem 1 that EXP-(s,t)-WT holds iff

$$\sup_{d\in\mathbb{N}} \sigma_{\,\mathrm{EWT}}(d,s,t,c) < \infty \quad \forall c > 0, \tag{9}$$

where

$$\begin{aligned} &\sigma_{\,\mathrm{EWT}}(d,s,t,c) \\ &:= \sum_{j=1}^{\infty} \exp\left(-c\left\{d^t + [\log(2\mathrm{e}) + \log\left(\max\left(1, \mathrm{CRI}_d/\lambda_{d,j}\right)\right)]^s\right\}\right) \qquad (10) \\ &= \sum_{j_1=1}^{\infty} \cdots \sum_{j_d=1}^{\infty} \exp\left(-c\left\{d^t + \left[\log\left(2\mathrm{e}\max\left(1, \prod_{\ell=1}^{d} \mathrm{CRI}/\widetilde{\lambda}_{j_\ell}\right)\right)\right]^s\right\}\right), \qquad (11) \end{aligned}$$

where

$$\mathrm{CRI}_d = \begin{cases} 1 & \text{for ABS,} \\ \lambda_{d,1} & \text{for NOR,} \end{cases} \qquad \text{and} \qquad \mathrm{CRI} = \begin{cases} 1 & \text{for ABS,} \\ \widetilde{\lambda}_1 & \text{for NOR.} \end{cases}$$

We first show the necessity of the conditions on the eigenvalues $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ for ABS and NOR, and then the necessity of the conditions (6), (7), or (8), depending on the different cases. Then, we show the sufficient conditions for all the cases (A.1)–(A.4) and (N.1)–(N.3).

NECESSARY CONDITIONS:

Case I: $t \le 1$ & $\widetilde{\lambda}_1 > 1 \implies$ **NO EXP-(s,t)-WT-ABS** Choose the smallest non-negative r such that $\widetilde{\lambda}_2 \ge \widetilde{\lambda}_1^{-r}$, and for every $d > r+2$, let $k = \lfloor d/(r+2) \rfloor$. Then it follows that

$$k \le \frac{d}{r+2}, \qquad d - k - kr \ge d\left[1 - \frac{r+1}{r+2}\right] = \frac{d}{r+2}.$$

Focusing on just these eigenvalues of the form

$$\lambda_{d,j} = \widetilde{\lambda}_1^{d-k}\widetilde{\lambda}_2^{k} = \widetilde{\lambda}_1^{d-k-kr}\widetilde{\lambda}_1^{kr}\widetilde{\lambda}_2^{k} \ge \widetilde{\lambda}_1^{d-k-kr}\widetilde{\lambda}_1^{kr}\widetilde{\lambda}_1^{-kr} = \widetilde{\lambda}_1^{d-k-kr} \ge \widetilde{\lambda}_1^{d/(r+2)} > 1,$$

$\sigma_{\,\mathrm{EWT}}(d,s,t,c)$ has the following lower bound via (10) and Lemma 2.

$$\begin{aligned}
\sigma_{\,\mathrm{EWT}}(d,s,t,c) &\ge \binom{d}{k}\exp\left(-c\left\{d^t + \left[\log(2\mathrm{e}) + \log\left(\max\left(1,\lambda_{d,j}^{-1}\right)\right)\right]^s\right\}\right)\\
&\ge \left(\frac{d}{k}\right)^k \exp\left(-c\left\{d^t + [\log(2\mathrm{e})]^s\right\}\right)\\
&\ge (r+2)^{d/(r+2)-1}\exp\left(-c\left\{d^t + [\log(2\mathrm{e})]^s\right\}\right)\\
&= \frac{\left[(r+2)^{1/(r+2)}\right]^d}{r+2}\exp\left(-c\left\{d^t + [\log(2\mathrm{e})]^s\right\}\right).
\end{aligned}$$

Since $(r+2)^{1/(r+2)} > 1$ and $t \le 1$, then $\sigma_{\,\mathrm{EWT}}(d,s,t,c) \to \infty$ for small c as $d \to \infty$, regardless of the value of s. Hence, we do not have EXP-(s,t)-WT-ABS.

Case II: $t \le 1$ & $\widetilde{\lambda}_1 \ge \widetilde{\lambda}_2 \ge 1 \implies$ **NO EXP-(s,t)-WT-ABS** We have 2^d eigenvalues no smaller than 1. Therefore

$$\sigma_{\,\mathrm{EWT}}(d,s,t,c) \ge 2^d \exp\left(-c(d^t + [\log(2e)]^s)\right) \to \infty$$

for small c independently of s. Hence, we do not have EXP-(s,t)-WT-ABS.

Case III: $t \le 1$ & $s \le 1 \implies$ **NO EXP-(s,t)-WT-ABS** We have 2^d eigenvalues no smaller than $\widetilde{\lambda}_2^d$. We then have

$$\begin{aligned}
\sigma_{\,\mathrm{EWT}}(d,s,t,c) &\ge 2^d \exp\left(-c\left(d^t + \left[\log\left(2\mathrm{e}\max(1,\widetilde{\lambda}_2^{-d})\right)\right]^s\right)\right)\\
&= \exp\left(d\log 2 - cd^t - c\left[\log(2\mathrm{e}) + d\log\,\max(1,\widetilde{\lambda}_2^{-1})\right]^s\right).
\end{aligned}$$

Since $s,t \le 1$ and since c can be arbitrarily small, we see that this latter term is not bounded for $d \to \infty$. Hence, we do not have EXP-(s,t)-WT-ABS.

From the analysis of all these cases, we see that EXP-(s,t)-WT-ABS may only hold when $t > 1$, or when $t \le 1 < s$, $\widetilde{\lambda}_1 \le 1$, and $\widetilde{\lambda}_2 < 1$. This completes the proof of the necessary conditions on $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ for ABS.

We turn to the necessary conditions on $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ for NOR. This corresponds to considering the ratios $\lambda_{d,1}/\lambda_{d,j}$ which are at least 1. We know that EXP-(s,t)-WT-NOR holds iff $\sup_{d\in\mathbb{N}} \sigma_{\,\mathrm{EWT}}(d,s,t,c) < \infty$ for all $c > 0$, where

$$\begin{aligned}
\sigma_{\,\mathrm{EWT}}(d,s,t,c) &= \sum_{j=1}^{\infty}\exp\left(-c\left\{d^t + \left[\log(2\mathrm{e}) + \log\left(\frac{\lambda_{d,1}}{\lambda_{d,j}}\right)\right)\right]^s\right)\\
&= \sum_{j_1=1}^{\infty}\cdots\sum_{j_d=1}^{\infty}\exp\left(-c\left\{d^t + \left[\log\left(2\mathrm{e}\prod_{\ell=1}^{d}\frac{\widetilde{\lambda}_1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s\right\}\right).
\end{aligned}$$

Hence, it is the same as ABS if we assume that $\widetilde{\lambda}_1 = 1$. Using the previous results on necessary conditions for ABS with $\widetilde{\lambda}_1 = 1$ we obtain the results for the parameters s, t, $\widetilde{\lambda}_1$, and $\widetilde{\lambda}_2$ for NOR.

Next, we show the necessity of the conditions (6), (7), or (8), depending on the different cases.

Necessity of (6)**:** The necessity of (6) for the corresponding subcases follows from Items L1 and L2 of Lemma 1 in [5]. We remark that these are only based on general definitions, and do not require the technical results used in the proof of the main theorem of [5].

Necessity of (7)**:** Assume first that EXP-(s,t)-WT-ABS/NOR holds and that the parameters $t, s, \widetilde{\lambda}_1$, and $\widetilde{\lambda}_2$ are as in Case (A.2), (A.4), (N.1), or (N.3), respectively. We prove that (7) holds. Take $d = 1$. Then we know that

$$\lim_{\varepsilon\to 0} \frac{\log\ \max(1, n_{\text{ABS/NOR}}(\varepsilon, S_1))}{(\log \varepsilon^{-1})^s} = 0.$$

This means that for any (small) positive β there is a positive ε_β such that

$$\log \max\left(1, n_{\text{ABS/NOR}}(\varepsilon, S_1)\right) \le \beta(\log\ \varepsilon^{-1})^s \quad \text{for all } \varepsilon \le \varepsilon_\beta,$$

and equivalently

$$\varepsilon^2 \le \exp\left(-2/\beta^{1/s} \left(\log \max(1, n_{\text{ABS/NOR}}(\varepsilon, S_1))\right)^{1/s}\right) \quad \text{for all } \varepsilon \le \varepsilon_\beta.$$

Let $n = n_{\text{ABS/NOR}}(\varepsilon, S_1)$. Since $\widetilde{\lambda}_{n+1} \le \varepsilon^2 \text{CRI}$, with CRI $= 1$ for ABS and CRI $= \widetilde{\lambda}_1$ for NOR, we obtain for $n \ge \max(1, n_{\text{ABS/NOR}}(\varepsilon_\beta, S_1))$,

$$\log \widetilde{\lambda}_{n+1}^{-1} \ge \frac{2}{\beta^{1/s}}\ (\log\ n)^{1/s} + \log\ \text{CRI}^{-1}.$$

Since $2/\beta^{1/s}$ can be arbitrarily large, this yields (6).

Necessity of (8)**:** The necessity of (8) for the corresponding subcases follows from Items L1 and L2 of Lemma 1 in [5]. We remark that these are only based on general definitions, and do not require the technical results used in the proof of the main theorem of [5].

SUFFICIENT CONDITIONS:

For technical reasons, we begin the proof by showing Case (A.2).

(A.2): $t > 1, s \ge 1,$ $\widetilde{\lambda}_1 \le 1$ & (7) $\implies$ **EXP-**(s,t)**-WT-ABS** Due to the assumption that $\widetilde{\lambda}_1 \le 1$, it is clear that $\widetilde{\lambda}_j^{-1} \ge 1$ for all j.

We then have

$$
\begin{aligned}
\sigma_{\mathrm{EWT}}(d,s,t,c) &= \exp(-c\,d^t)\sum_{j_1=1}^{\infty}\cdots\sum_{j_d=1}^{\infty}\exp\left(-c\left[\log(2\mathrm{e})+\log\left(\prod_{\ell=1}^{d}\frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s\right)\\
&\le \exp(-c\,d^t)\sum_{j_1=1}^{\infty}\cdots\sum_{j_d=1}^{\infty}\exp\left(-c\left[\log\left(\prod_{\ell=1}^{d}\frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s\right)\\
&= \exp(-c\,d^t)\sum_{j_1=1}^{\infty}\cdots\sum_{j_d=1}^{\infty}\exp\left(-c\left[\sum_{\ell=1}^{d}\log\left(\frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s\right)\\
&\le \exp(-c\,d^t)\sum_{j_1=1}^{\infty}\cdots\sum_{j_d=1}^{\infty}\exp\left(-c\sum_{\ell=1}^{d}\left[\log\left(\frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s\right),
\end{aligned}
$$

where we used $s \ge 1$ and Lemma 1 in the last step. This yields

$$
\sigma_{\mathrm{EWT}}(d,s,t,c) \le \exp(-c\,d^t)\left(\sum_{j=1}^{\infty}\exp\left(-c\left[\log\left(1/\widetilde{\lambda}_j\right)\right]^s\right)\right)^d .
$$

Since Condition (7) holds, we know that

$$
\log\left(1/\widetilde{\lambda}_j\right) = h_j(\log(j+1))^{1/s},
$$

where $(h_j)_{j\ge 1}$ is a sequence with $\lim_{j\to\infty} h_j = \infty$, i.e.,

$$
\begin{aligned}
\sigma_{\mathrm{EWT}}(d,s,t,c) &\le \exp(-c\,d^t)\left(\sum_{j=1}^{\infty}\exp\left(-c\left[h_j(\log(j+1))^{1/s}\right]^s\right)\right)^d\\
&= \exp(-c\,d^t)\left(\sum_{j=1}^{\infty}\exp\left(-c\,h_j^s\log(j+1)\right)\right)^d\\
&= \exp(-c\,d^t)\left(\sum_{j=1}^{\infty}\exp\left(\log\left(1/(j+1)^{c\,h_j^s}\right)\right)\right)^d\\
&= \exp(-c\,d^t)\left(\sum_{j=1}^{\infty}\left(\frac{1}{j+1}\right)^{c\,h_j^s}\right)^d\\
&= \exp(-c\,d^t)\,A_c^d,
\end{aligned}
$$

where $A_c = \sum_{j=1}^{\infty}\left(\frac{1}{j+1}\right)^{c\,h_j^s}$ is well defined and independent of d since $c\,h_j^s$ is greater than one for sufficiently large j, and the series is convergent. Hence,

$$
\sigma_{\mathrm{EWT}}(d,s,t,c) \le \exp(-c\,d^t)\,\exp\left(d\log A_c\right).
$$

As $t > 1$, we obtain EXP-(s,t)-WT-NOR.

We now show Case (A.1), and the other cases in the same order as they are stated in the theorem.

(A.1): $t > 1, s > 1$, $\widetilde{\lambda}_1 > 1$ & (6) $\implies$ EXP-(s,t)-WT-ABS

Subcase (A.1.1): $s \le t$ We define $\beta_j := \widetilde{\lambda}_j/\widetilde{\lambda}_1$ for $j \ge 1$, and consider the information complexity with respect to β_j instead of $\widetilde{\lambda}_j$. We denote the information complexity with respect to the sequence $\boldsymbol{\beta} = (\beta_j)_{j\ge 1}$ by $n^{(\boldsymbol{\beta})}_{\rm ABS}$, and that with respect to the sequence $\boldsymbol{\lambda} = (\widetilde{\lambda}_j)_{j\ge1}$ by $n^{(\boldsymbol{\lambda})}_{\rm ABS}$. Then it is straightforward to see that

$$n^{(\boldsymbol{\lambda})}_{\rm ABS}(\varepsilon, S_d) = n^{(\boldsymbol{\beta})}_{\rm ABS}(\varepsilon/\widetilde{\lambda}_1^{d/2}, S_d)$$

Since $s \le t$, we have, by Lemma 1,

$$\begin{aligned} d^t + \left(d \log(\widetilde{\lambda}_1^{1/2}) + \log(1/\varepsilon)\right)^s &\le d^t + 2^{s-1}\, d^s\, (\log(1/\varepsilon))^s + 2^{s-1}\, (\log(1/\varepsilon))^s \\ &\le C_s(d^t + (\log(1/\varepsilon))^s) \end{aligned}$$

for some positive constant C_s depending on s, but not on d or ε. This implies

$$\frac{\log\, n^{(\boldsymbol{\lambda})}_{\rm ABS}(\varepsilon, S_d)}{d^t + (1+\log\, \max(1,\varepsilon^{-1}))^s} \le \frac{\log\, n^{(\boldsymbol{\beta})}_{\rm ABS}(\varepsilon/\widetilde{\lambda}_1^{d/2}, S_d)}{\widetilde{C}_s\left(d^t + \left(1+\log\, \max(1,\varepsilon^{-1}\, \widetilde{\lambda}_1^{d/2})\right)^s\right)}$$

for some positive constant $\widetilde{C}_s$. This means that EXP-(s,t)-WT-ABS holds with respect to $\boldsymbol{\lambda}$ if it holds with respect to $\boldsymbol{\beta}$. However, as $\beta_j \le 1$ for all $j \ge 1$, and since in this subcase $\min(s,t) = s$, the result follows from case (A.2) above.

Subcase (A.1.2): $s > t$ Assume that (6) holds with $\min(s,t) = t$. We need to show that

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\log\, n^{(\boldsymbol{\lambda})}_{\rm ABS}(\varepsilon, S_d)}{d^t + (1+\log\, \max(1,\varepsilon^{-1}))^s} = 0.$$

However, note that

$$\frac{\log\, n^{(\boldsymbol{\lambda})}_{\rm ABS}(\varepsilon, S_d)}{d^t + (1+\log\, \max(1,\varepsilon^{-1}))^s} \le \frac{\log\, n^{(\boldsymbol{\lambda})}_{\rm ABS}(\varepsilon, S_d)}{d^t + (1+\log\, \max(1,\varepsilon^{-1}))^t},$$

and that

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\log\, n^{(\boldsymbol{\lambda})}_{\rm ABS}(\varepsilon, S_d)}{d^t + (1+\log\, \max(1,\varepsilon^{-1}))^t} = 0$$

by Case (A.1.1). This shows the result.

(A.3): $t > 1$, $s < 1$, & (8) with $\eta = s(t-1)/(t-s)$ $\implies$ EXP-(s,t)-WT-ABS

Subcase (A.3.1): $\widetilde{\lambda}_1 \le 1$ We study the expression

$$\sigma_{\rm EWT}(d,s,t,c) \le \exp(-c\, d^t) \sum_{j_1=1}^{\infty} \cdots \sum_{j_d=1}^{\infty} \exp\left(-c\left[\sum_{\ell=1}^{d} \log\left(1/\widetilde{\lambda}_{j_\ell}\right)\right]^s\right).$$

Note that the definition of η together with $s < 1$ implies that $\eta < s < 1$, and $1/\eta > 1/s > 1$. Note furthermore that Condition (8) implies

$$\log\left(1/\widetilde{\lambda}_j\right) = h_j(\log(j+1))^{1/\eta} = h_j\, (\log(j+1))^{1/\eta - 1/s}\, (\log(j+1))^{1/s}, \qquad (12)$$

where $(h_j)_{j\ge 1}$ is a sequence with $\lim_{j\to\infty} h_j = \infty$.

Using (12) and the second item in Lemma 1, we obtain

$$\begin{aligned}\left[\sum_{\ell=1}^{d} \log\left(1/\widetilde{\lambda}_{j_\ell}\right)\right]^s &= \left[\sum_{\ell=1}^{d} h_{j_\ell}\,(\log(j_\ell+1))^{1/\eta-1/s}\,(\log(j_\ell+1))^{1/s}\right]^s \\ &\ge d^{s-1}\sum_{\ell=1}^{d}(\log(j_\ell+1))^{(s-\eta)/\eta}\,h_{j_\ell}^s\,\log(j_\ell+1).\end{aligned}$$

Consequently,

$$\begin{aligned}\sigma_{\,\mathrm{EWT}}(d,s,t,c) &\le \exp(-c\,d^t) \\ &\quad\times\sum_{j_1=1}^{\infty}\cdots\sum_{j_d=1}^{\infty}\exp\left(-c\,d^{s-1}\sum_{\ell=1}^{d}(\log(j_\ell+1))^{(s-\eta)/\eta}\,h_{j_\ell}^s\,\log(j_\ell+1)\right) \\ &= \exp(-c\,d^t)\left(\sum_{j=1}^{\infty}\exp\left(-c\,d^{s-1}\,(\log(j+1))^{(s-\eta)/\eta}\,h_j^s\,\log(j+1)\right)\right)^d.\end{aligned}$$

Note that

$$d^{s-1}\,(\log(j+1))^{(s-\eta)/\eta} \ge (c/2)^{\eta/(s-\eta)}$$

if and only if

$$j \ge \left\lceil \exp\left((c/2)\,d^{(1-s)\eta/(s-\eta)}\right) - 1\right\rceil =: J_0 = J_0(c,d,s,\eta).$$

This implies

$$\sigma_{\,\mathrm{EWT}}(d,s,t,c) \le \exp(-c\,d^t)\left(J_0 + \sum_{j=J_0}^{\infty}\exp\left(-c^{s/(s-\eta)}2^{s/(\eta-s)}\,h_j^s\,\log(j+1)\right)\right)^d.$$

In the same way as in case (A.2), we conclude that there exists a positive constant A_c such that

$$\begin{aligned}\sigma_{\,\mathrm{EWT}}(d,s,t,c) &\le \exp(-c\,d^t)\,(J_0 + A_c)^d \\ &= \exp(-c\,d^t)\,\left(\exp\left((c/2)\,d^{(1-s)\eta/(s-\eta)}\right) + A_c\right)^d.\end{aligned}$$

Since A_c is independent of d, for sufficiently large d,

$$\begin{aligned}\sigma_{\,\mathrm{EWT}}(d,s,t,c) &\le \exp(-c\,d^t)\,\left(2\exp\left((c/2)\,d^{(1-s)\eta/(s-\eta)}\right)\right)^d \\ &= \exp(-c\,d^t)\,2^d\,\exp\left((c/2)\,d^{(1-s)\eta/(s-\eta)+1}\right).\end{aligned}$$

It is easily checked that $(1-s)\eta/(s-\eta)+1 = t$, so we obtain

$$\sigma_{\,\mathrm{EWT}}(d,s,t,c) \le \exp(-c\,d^t)\,2^d\exp((c/2)\,d^t) = \exp(-(c/2)\,d^t)\,\exp(d\log 2).$$

As $t > 1$, we obtain EXP-(s,t)-WT-ABS.

Subcase (A.3.2): $\widetilde{\lambda}_1 > 1$ We again define $\beta_j := \widetilde{\lambda}_j/\widetilde{\lambda}_1$ for $j \geq 1$, and consider the information complexity with respect to β_j instead of $\widetilde{\lambda}_j$. Then, as in Case (A.1.1),

$$n^{(\boldsymbol{\lambda})}_{\mathrm{ABS}}(\varepsilon, S_d) = n^{(\boldsymbol{\beta})}_{\mathrm{ABS}}(\varepsilon/\widetilde{\lambda}_1^{d/2}, S_d)$$

Since $t > 1$ and $s < 1$, we have

$$d^t + \left(d \log(\widetilde{\lambda}_1^{1/2}) + \log(1/\varepsilon)\right)^s \approx d^t + (\log(1/\varepsilon))^s,$$

and this implies that EXP-(s,t)-WT-ABS holds with respect to $\boldsymbol{\beta}$ if and only if it holds with respect to $\boldsymbol{\lambda}$. However, as $\beta_j \leq 1$ for all $j \geq 1$, the result now follows from Case (A.3.1), similar to Case (A.1.1).

(A.4): $t \leq 1$**,** $s > 1$**,** $\widetilde{\lambda}_1 \leq 1$**,** $\widetilde{\lambda}_2 < 1$ **& (7)** $\Longrightarrow$ **EXP-**(s,t)**-WT-ABS** Due to the assumption that $\widetilde{\lambda}_1 \leq 1$, we again have $\widetilde{\lambda}_j^{-1} \geq 1$ for all j, such that

$$\sigma_{\mathrm{EWT}}(d,s,t,c) \quad \leq \quad \exp(-c\, d^t) \sum_{j_1=1}^{\infty} \cdots \sum_{j_d=1}^{\infty} \exp\left(-c\left[\log\left(\prod_{\ell=1}^{d} \frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s\right).$$

Note that the approach taken in Case (A.2) does not work now since $t \leq 1$.

In the following we write $[d]$ for the index set $\{1,2,\ldots,d\}$ and, for $\mathfrak{u} \subseteq [d]$, $\overline{\mathfrak{u}} = [d] \setminus \mathfrak{u}$. Due to (7), we can find $m \in \mathbb{N}$ such that

$$\widetilde{\lambda}_j^{-1} \geq \widetilde{\lambda}_2^{-1} \exp\left((\log j)^{1/s}\,(2/c)^{1/s}\right) \qquad \forall j \geq m+1. \tag{13}$$

Now we study

$$\sum_{j_1=1}^{\infty} \cdots \sum_{j_d=1}^{\infty} \exp\left(-c\left[\log\left(\prod_{\ell=1}^{d} \frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s\right) =$$
$$= \sum_{\substack{\mathfrak{u}\subseteq[d] \\ \mathfrak{u}=\{v_1,\ldots,v_{|\mathfrak{u}|}\} \\ \overline{\mathfrak{u}}=\{w_1,\ldots,w_{d-|\mathfrak{u}|}\}}} \sum_{j_{v_1}=1}^{m} \cdots \sum_{j_{v_{|\mathfrak{u}|}}=1}^{m} \sum_{j_{w_1}=m+1}^{\infty} \cdots \sum_{j_{w_{d-|\mathfrak{u}|}}=m+1}^{\infty} \exp\left(-c\left[\log\left(\prod_{\ell=1}^{d} \frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s\right). \tag{14}$$

Since $s > 1$, we have, for fixed $\mathfrak{u} \subseteq [d]$,

$$\left[\log\left(\prod_{\ell=1}^{d} \frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s \geq \left[\log\left(\prod_{\ell\in\mathfrak{u}} \frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s + \left[\log\left(\prod_{\ell\in\overline{\mathfrak{u}}} \frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s,$$

so the expression in (14) is bounded by

$$\sum_{\substack{\mathfrak{u}\subseteq[d] \\ \mathfrak{u}=\{v_1,\ldots,v_{|\mathfrak{u}|}\} \\ \overline{\mathfrak{u}}=\{w_1,\ldots,w_{d-|\mathfrak{u}|}\}}} \sum_{j_{v_1}=1}^{m} \cdots \sum_{j_{v_{|\mathfrak{u}|}}=1}^{m} \exp\left(-c\left[\log\left(\prod_{\ell\in\mathfrak{u}} \widetilde{\lambda}_{j_\ell}^{-1}\right)\right]^s\right)$$
$$\times \sum_{j_{w_1}=m+1}^{\infty} \cdots \sum_{j_{w_{d-|\mathfrak{u}|}}=m+1}^{\infty} \exp\left(-c\left[\log\left(\prod_{\ell\in\overline{\mathfrak{u}}} \widetilde{\lambda}_{j_\ell}^{-1}\right)\right]^s\right).$$

We first study

$$A_{\mathfrak{u}} := \sum_{j_{v_1}=1}^{m} \cdots \sum_{j_{v_{|\mathfrak{u}|}}=1}^{m} \exp\left(-c\left[\log\left(\prod_{\ell\in\mathfrak{u}} \frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^s\right).$$

There are a total of m^d terms of the form $\prod_{\ell\in\mathfrak{u}} \widetilde{\lambda}_{j_\ell}^{-1}$ for $j_{v_1},\dots,j_{v_{|\mathfrak{u}|}} \in \{1,\dots,m\}$ in $A_{\mathfrak{u}}$. For $k=0,\dots,|\mathfrak{u}|$ there are $(m-1)^k\binom{|\mathfrak{u}|}{k}$ of these terms containing the factor of $\widetilde{\lambda}_1^{-1}$ exactly $|\mathfrak{u}|-k$ times. Such terms are bounded below by $\widetilde{\lambda}_2^{-k}$, so we obtain

$$A_{\mathfrak{u}} \le \sum_{k=0}^{|\mathfrak{u}|} \binom{|\mathfrak{u}|}{k} (m-1)^k \exp\left(-c\left[\log(\widetilde{\lambda}_2^{-k})\right]^s\right).$$

We bound $\binom{|\mathfrak{u}|}{k}$ by $(\mathrm{e}\,|\mathfrak{u}|/k)^k$ due to Lemma 2. Hence, we have

$$A_{\mathfrak{u}} \le 1 + |\mathfrak{u}| \max_{k=1,\dots,|\mathfrak{u}|} \exp(f(k)) \le 1 + \exp\big(\log(|\mathfrak{u}|) + f(k_{\max})\big),$$

where

$$\begin{aligned} f(k) &= k + k\log(|\mathfrak{u}|/k) + k\log(m-1) - ck^s\left[\log(\widetilde{\lambda}_2^{-1})\right]^s,\\ f'(k) &= \log(|\mathfrak{u}|/k) + \log(m-1) - csk^{s-1}\left[\log(\widetilde{\lambda}_2^{-1})\right],\\ f(k_{\max}) &= \max_{k\in[1,|\mathfrak{u}|]} f(k) \ge \max_{k=1,\dots,|\mathfrak{u}|} f(k). \end{aligned}$$

For $|\mathfrak{u}|$ large enough, we have

$$\begin{aligned} f'(1) &= \log(|\mathfrak{u}|) + \log(m-1) - cs(\log(\widetilde{\lambda}_2^{-1}))^s > 0,\\ f'(d) &= \log(m-1) - cs\,|\mathfrak{u}|^{s-1}(\log(\widetilde{\lambda}_2^{-1}))^s < 0, \end{aligned}$$

hence, the maximum occurs in the interior. By setting the $f'(k)=0$, we obtain

$$\begin{aligned} 0 &= \log(|\mathfrak{u}|/k_{\max}) + \log(m-1) - csk_{\max}^{s-1}(\log(\widetilde{\lambda}_2^{-1}))^s,\\ f(k_{\max}) &= k_{\max} + k_{\max}\log(|\mathfrak{u}|/k_{\max}) + k_{\max}\log(m-1) - ck_{\max}^s\left(\log(\widetilde{\lambda}_2^{-1})\right)^s\\ &= k_{\max} + c\,(s-1)\,k_{\max}^s\left(\log(\widetilde{\lambda}_2^{-1})\right)^s. \end{aligned}$$

The nonlinear equation defining $k_{\max}$ above implies that

$$k_{\max} = \mathcal{O}\left((\log(|\mathfrak{u}|))^{1/(s-1)}\right), \quad \text{and} \quad f(k_{\max}) = \mathcal{O}\left((\log(|\mathfrak{u}|))^{s/(s-1)}\right).$$

Consequently,

$$A_{\mathfrak{u}} \le \exp\left(\log(|\mathfrak{u}|) + \mathcal{O}\left((\log(|\mathfrak{u}|))^{s/(s-1)}\right)\right).$$

Due to the choice of m in (13), we obtain

$$\begin{aligned} \left[\log\left(\prod_{\ell\in\overline{\mathfrak{u}}} \widetilde{\lambda}_{j_\ell}^{-1}\right)\right]^s &\ge \left[\log\left(\prod_{\ell\in\overline{\mathfrak{u}}} \widetilde{\lambda}_2^{-1}\right) + \log\left(\prod_{\ell\in\overline{\mathfrak{u}}} \exp\left((\log j_\ell)^{1/s}(2/c)^{1/s}\right)\right)\right]^s\\ &= \left[|\overline{\mathfrak{u}}|\log\left(\widetilde{\lambda}_2^{-1}\right) + \sum_{\ell\in\overline{\mathfrak{u}}}(\log j_\ell)^{1/s}(2/c)^{1/s}\right]^s\\ &\ge |\overline{\mathfrak{u}}|^s\left[\log\left(\widetilde{\lambda}_2^{-1}\right)\right]^s + \sum_{\ell\in\overline{\mathfrak{u}}}\frac{1}{c}\log\left(j_\ell^2\right). \end{aligned}$$

Consequently,

$$
\begin{aligned}
B_{\mathfrak{u}} &:= \sum_{j_{w_1}=m+1}^{\infty} \cdots \sum_{j_{w_{d-|\mathfrak{u}|}}=m+1}^{\infty} \exp\left(-c\left[\log\left(\prod_{\ell\in\overline{\mathfrak{u}}} \widetilde{\lambda}_{j_\ell}^{-1}\right)\right]^{s}\right) \\
&\le \exp\left(-c\,|\overline{\mathfrak{u}}|^{s}\left[\log\left(\widetilde{\lambda}_2^{-1}\right)\right]^{s}\right) \sum_{j_{w_1}=m+1}^{\infty} \cdots \sum_{j_{w_{d-|\mathfrak{u}|}}=m+1}^{\infty} \exp\left(-c\sum_{\ell\in\overline{\mathfrak{u}}} \frac{1}{c}\log\left(j_\ell^2\right)\right) \\
&= \exp\left(-c\,|\overline{\mathfrak{u}}|^{s}\left[\log\left(\widetilde{\lambda}_2^{-1}\right)\right]^{s}\right) \sum_{j_{w_1}=m+1}^{\infty} \frac{1}{j_{w_1}^{2}} \cdots \sum_{j_{w_{d-|\mathfrak{u}|}}=m+1}^{\infty} \frac{1}{j_{w_{d-|\mathfrak{u}|}}^{2}} \\
&\le \exp\left(-c\,|\overline{\mathfrak{u}}|^{s}\left[\log\left(\widetilde{\lambda}_2^{-1}\right)\right]^{s}\right) (\zeta(2))^{|\overline{\mathfrak{u}}|}
\end{aligned}
$$

In total, we obtain

$$
\begin{aligned}
&\sum_{j_1=1}^{\infty} \cdots \sum_{j_d=1}^{\infty} \exp\left(-c\left[\log\left(\prod_{\ell=1}^{d} \frac{1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^{s}\right) \le \\
&\le \sum_{\mathfrak{u}\subseteq[d]} \exp\left(-c\,|\overline{\mathfrak{u}}|^{s}\left[\log\left(\widetilde{\lambda}_2^{-1}\right)\right]^{s} + |\overline{\mathfrak{u}}|\log(\zeta(2)) + \log(|\mathfrak{u}|) + \mathcal{O}\left((\log(|\mathfrak{u}|))^{s/(s-1)}\right)\right) \\
&\le \exp\left(\log(d) + \mathcal{O}\left((\log(d))^{s/(s-1)}\right)\right) \sum_{\overline{\mathfrak{u}}\subseteq[d]} \exp\left(-c\,|\overline{\mathfrak{u}}|^{s}\left[\log\left(\widetilde{\lambda}_2^{-1}\right)\right]^{s} + |\overline{\mathfrak{u}}|\log(\zeta(2))\right),
\end{aligned}
$$

where we used $\overline{\mathfrak{u}} = [d]\setminus\mathfrak{u}$ in the last step.

Similarly as in the analysis of $A_{\mathfrak{u}}$, we see that the sum in the latter expression is bounded by

$$
\exp\left(\log(d) + \mathcal{O}\left((\log(d))^{s/(s-1)}\right)\right).
$$

This term grows slower with d than $\exp(-cd^t)$, so we obtain EXP-(s,t)-WT-ABS, as desired.

(N.1): $t>1, s\ge 1$ & (7) $\implies$ EXP-(s,t)-WT-NOR Note that in this case we have

$$
\sigma_{\mathrm{EWT}}(d,s,t,c) = \exp(-c\,d^t) \sum_{j_1=1}^{\infty} \cdots \sum_{j_d=1}^{\infty} \exp\left(-c\left[\log(2e) + \log\left(\prod_{\ell=1}^{d} \frac{\widetilde{\lambda}_1}{\widetilde{\lambda}_{j_\ell}}\right)\right]^{s}\right),
$$

since $\widetilde{\lambda}_1/\widetilde{\lambda}_j \ge 1$ for all j. The rest of the argument is analogous to that in Case (A.2).

(N.2): $t>1$, $s<1$, & (8) with $\eta = s(t-1)/(t-s)$ $\implies$ EXP-(s,t)-WT-NOR This case can be treated in a similar way as Case (A.3), Subcase (A.3.1).

(N.3): $t\le 1$, $s>1$, $\widetilde{\lambda}_1 > \widetilde{\lambda}_2$, & (7) $\implies$ EXP-(s,t)-WT-NOR This case can be treated in a similar way as Case (A.4).

Regarding all other tractability notions, we know from above that we do not have EXP-(s,t)-WT when $t\le 1$ and $s\le 1$. Since EXP-(s,t)-WT is a weaker tractability notion than all other tractability notions considered here, we cannot have any other stronger kind of tractability.

This completes the proof of Theorem 2.

□

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