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Guaranteed and computable bounds of errors for semi-discrete approximations of the Biot problem

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Abstract

The paper is concerned with guaranteed and fully computable a posteriori error estimates for evolutionary problems associated with poroelastic media governed by the quasi-static linear Biot equations. It considers approximation errors, which arise in iterative methods used in implicit semi-discrete schemes. The derivation of the error bounds is based on the combination of estimates for contraction mappings for iterative schemes and a posteriori error estimates of the functional type for elliptic problems. For decoupling, fixed-stress split scheme is used. The estimates are applicable for any approximation from the admissible functional space and independent of the discretisation method. They are fully computable. Moreover, they do not contain mesh dependent constants, and provide reliable global estimates of the error measured in the energy norm.

Keywords: Biot problem, fixed-stress split scheme, functional error estimates, contraction mappings.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = \{2, 3\})$ be a bounded domain with Lipschitz continuous boundary Γ and $Q := \Omega \times (0,T)$ denote a space-time cylinder (where $T > 0$) with the lateral surface $\Sigma := \partial\Omega \times (0,T)$. The bottom and the top parts of Q are defined by $\Sigma_0 := \partial\Omega \times \{0\}$ and $\Sigma_T := \partial\Omega \times \{T\}$, respectively, so that $\partial Q = \Sigma \cup \Sigma_0 \cup \Sigma_T$.

The Biot model is a system describing flow and displacement in a porous medium (see, e.g., [5]). For modelling of the *solid displacement* $\boldsymbol{u}$ and the *fluid pressure* p, we consider the system that governs the coupling of elastic, isotropic porous medium saturated with slightly compressible viscous single-phase fluid

$$\begin{aligned} -\mathrm{div}\big(\lambda\,(\mathrm{div}\boldsymbol{u})\,\mathbb{I} + 2\,\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - \alpha\,p\,\mathbb{I}\big) &= \boldsymbol{f} \quad \text{in} \quad Q, \\ \partial_t\big(\beta\,p + \alpha\mathrm{div}\boldsymbol{u}\big) - \mathrm{div}\mathbb{K}\nabla p &= g \quad \text{in} \quad Q, \end{aligned} \tag{1.1}$$

where $\boldsymbol{f}$ and g are body force and volumetric fluid source, respectively. Here, the first equation follows from the balance of linear momentum for the *total Cauchy stress tensor* $\boldsymbol{\sigma}_{\rm por} := \boldsymbol{\sigma}(\boldsymbol{u}) - \alpha\,p\,\mathbb{I}$ that accounts both $\boldsymbol{u}$ and p with the dimensionless Biot-Willis coefficient $\alpha > 0$. Linear elastic tensor $\boldsymbol{\sigma}(\boldsymbol{u})$ is governed by the Hooke law

$$\boldsymbol{\sigma}(\boldsymbol{u}) := 2\,\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\,\mathrm{tr}\boldsymbol{\varepsilon}(\boldsymbol{u})\,\mathbb{I} = 2\,\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\,(\mathrm{div}\boldsymbol{u})\,\mathbb{I},$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\big(\nabla\,\boldsymbol{u} + (\nabla\,\boldsymbol{u})^{\rm T}\big)$ is the *strain tensor*, and $\lambda, \mu > 0$ are generally called the *Lamé constants*. The second equation is fluid mass conservation (continuity) equation in Q. In (1.1), β stands for the *storage coefficient* and $\mathbb{K}$ is the *permeability tensor*. We assume that $\mathbb{K}$ is symmetric, uniformly bounded, uniformly elliptic, anisotropic, and heterogeneous in space and constant in time. Let the smallest eigenvalue of $\mathbb{K}$ be $\lambda_{\mathbb{K}}$. The system is completed by using appropriate initial conditions, i.e., $p(x,0) = p_0$, $\boldsymbol{u}(x,0) = \boldsymbol{u}_0$ on Σ_0, and homogeneous Dirichlet boundary conditions $p_D = 0$ and $\boldsymbol{u}_D = \boldsymbol{0}$, even though all results are valid for more general assumptions.

In this work, we focus on an *iterative* scheme, named fixed stress split, to solve (1.1) (see, e.g., [15]). This approach is sequential, where at each step time step the flow problem is solved first, followed by solving the mechanics using previously recovered pressure (see, e.g., [6, 9] and the reference therein). The procedure is iterated until the desired accuracy is reached. A variation where one takes finer time steps for flow and coarse time step for mechanics, known as multi-rate schemes [2], can be considered.

The aim of this study is to provide a posteriori error estimates for the approximation of the system (1.1). In [10], the functional approach to the error control was used for the Barenblatt–Biot system. The current paper deals with more advanced quasi-static Biot model based on the elliptic-parabolic system of partial differential equations (PDEs). To our knowledge, it is the first study that derives guaranteed and computable bounds of errors for a coupled elasticity and flow model in a poroelastic medium. Such error controls are much needed in a coupled problem setting. An advantage of our approach is that it is based on only contraction of the iterative method and functional type estimates of individual equations. Both of these are available for other coupled models to which our results can be adapted. This underlines the significance of results in this paper. We refer to [1] for a different approach to getting a posteriori error estimates for Biot model.

2. Variational setting and discretisation

The Biot system of type (1.1) was analysed by a number of authors to establish existence, uniqueness, and regularity. The first theoretical results on the existence and uniqueness of a (weak) solution are presented in [17] for the case of $\beta = 0$. Further works in this direction can be found in [16]. Relevant to our work, [9] established contractive results in suitable norms for iterative coupling of (1.1). For an overview of the stability of existing iterative algorithms, we refer the reader to [6].

To discretise (1.1) w.r.t. to time, we represent the interval $[0,T]$ by a union of N sub-intervals $\mathcal{T}_N = \cup_{n=1}^N \overline{I^n}$, $I^n = (t^{n-1}, t^n)$. Let $\boldsymbol{u}^n(x) \in \boldsymbol{V}_0$ and $p^n(x) \in W_0$, where $\boldsymbol{V}_0 := \big\{ \boldsymbol{v} \in [H^1(\Omega)]^d \,|\, \boldsymbol{v}\big|_\Gamma = 0 \big\}$ and $W_0 := \big\{ w \in H^1(\Omega) \,|\, w\big|_\Gamma = 0 \big\}$, respectively, be a pair of solutions at the n-th moment in time. Then, the semi-discrete counterpart of (1.1) reads as: find the pair $(\boldsymbol{u}, p)^n \in \boldsymbol{V}_0 \times W_0$

$$2\,\mu\,(\boldsymbol{\varepsilon}(\boldsymbol{u}^n), \boldsymbol{\varepsilon}(\boldsymbol{v}))_\Omega + \lambda\,(\mathrm{div}\boldsymbol{u}^n, \mathrm{div}\boldsymbol{v})_\Omega + \alpha\,(\nabla p^n, \boldsymbol{v})_\Omega = (\boldsymbol{f}^n, \boldsymbol{v})_\Omega, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0, \tag{2.1}$$

$$(\mathbb{K}_{\tau^n} \nabla p^n, \nabla w)_\Omega + \beta\,(p^n, w)_\Omega + \alpha\,(\mathrm{div}\,\boldsymbol{u}^n, w)_\Omega = (\widetilde{g}^n, w)_\Omega, \quad \forall\, w_0 \in W_0, \tag{2.2}$$

where $\tau^n = t^n - t^{n-1}$ and $\widetilde{g}^n = \tau^n\, g^n + \beta\, p^{n-1} + \alpha\, \mathrm{div}\boldsymbol{u}^{n-1}$, $\mathbb{K}_{\tau^n} = \tau^n\,\mathbb{K}$, and $(\boldsymbol{u}, p)^{n-1} \in \boldsymbol{V}_0 \times W_0$ are given by the previous time-step. The initial values are set to be $(\boldsymbol{u}, p)^0 = (p_0, \boldsymbol{u}_0)$. Since from now on we deal only with the semi-discrete counterpart of Biot problem (2.1)–(2.2), we omit underscript Ω in the scalar product as well as the index n for the rest of the paper, i.e., we consider

$$2\,\mu\,(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda\,(\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{v}) + \alpha\,(\nabla p, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0, \tag{2.3}$$

$$(\mathbb{K}_\tau \nabla p, \nabla w) + \beta\,(p, w) + \alpha\,(\mathrm{div}\,\boldsymbol{u}, w) = (\widetilde{g}, w), \quad \forall\, w \in W_0. \tag{2.4}$$

We use fixed-stress split approach for solving (2.3)–(2.4) (see e.g., [9]). It decouples the Biot model into two classical sub-problems of linear elasticity and single-phase fluid flow. Let i denote the iteration number. First, we decouple the system applying the iterative procedure in order to obtain the pair $(\boldsymbol{u}, p)^i$. Next, each equation is discretised and we obtain the approximation $(\boldsymbol{u}, p)^i_h$ for $(\boldsymbol{u}, p)^i$.

The aim of this work is to derive a fully guaranteed a posteriori estimates of the error between the approximations $(\boldsymbol{u}^i_h, p^i_h) \in \boldsymbol{V}_0 \times W_0$ and the pair of the exact solutions $(\boldsymbol{u}, p)$ of the Biot system (2.3)–(2.4), i.e., $e_{\boldsymbol{u}} := \boldsymbol{u} - \boldsymbol{u}^i_h$ and $e_p := p - p^i_h$. Each error is measured in terms of the combined norm

$$\big|\!\big[(e_{\boldsymbol{u}}, e_p)\big]\!\big| := \|\!| e_{\boldsymbol{u}} |\!\|^2_{\boldsymbol{u}} + \|\!| e_p |\!\|^2_p, \qquad \|\!| e_{\boldsymbol{u}} |\!\|^2_{\boldsymbol{u}} := \|\, \boldsymbol{\varepsilon}(e_{\boldsymbol{u}}) \,\|^2_{2\mu} + \|\, \mathrm{div}(e_{\boldsymbol{u}}) \,\|^2_\lambda, \quad \|\!| e_p |\!\|^2_p := \|\, \nabla e_p \,\|^2_{\mathbb{K}_\tau} + \|\, e_p \,\|^2_\beta. \tag{2.5}$$

Here, the norm $\|v\|^2_\varrho$ is a standard ϱ-weighed L^2-norm for scalar- or vector-valued functions.

2.1. Main result

Theorem 1 presents the main result of this work, that is, an upper bound of the error (2.5).

Theorem 1. *For $(\boldsymbol{u}, p)^i_h$ approximating $(\boldsymbol{u}, p)$, the exact solutions of (2.3)–(2.4), we have the estimates*

$$\big|\!\big[(e_{\boldsymbol{u}}, e_p)\big]\!\big| \leq \overline{\mathrm{M}}_{\mathrm{it}} := \overline{\mathrm{M}}_{\boldsymbol{u}} + \overline{\mathrm{M}}_p, \quad \textit{where} \quad \overline{\mathrm{M}}_{\boldsymbol{u}} := 2\,\big(\overline{\mathrm{M}}^h_{\boldsymbol{u}} + \overline{\mathrm{M}}^i_{\boldsymbol{u}}\big), \quad \overline{\mathrm{M}}_p := 2\,\big(\overline{\mathrm{M}}^h_p + \overline{\mathrm{M}}^i_p\big). \tag{2.6}$$

Here, the approximation refers to the numerical computation of the fixed-stress split scheme given in Section 3. $\overline{\mathrm{M}}^h_p$ and $\overline{\mathrm{M}}^i_p$ are defined in Lemmas 2 and 5, whereas $\overline{\mathrm{M}}^h_{\boldsymbol{u}}$ and $\overline{\mathrm{M}}^i_{\boldsymbol{u}}$ follows from Lemmas 4 and 6, respectively.

We note that $\left|\left[(e_{\boldsymbol{u}}, e_p)\right]\right|$ consists of two parts, i.e., $\| e_{\boldsymbol{u}} \|^2_{\boldsymbol{u}}$ and $\| e_p \|^2_p$. Since we are using an iterative approach, there are three pairs of relevant solution variables: the exact solution $(\boldsymbol{u}, p)$ of the semi-discrete scheme, the exact *iterative* solution of decoupled equation $(\boldsymbol{u}, p)^i$, and the numerical solution of the discrete scheme at i-th iteration $(\boldsymbol{u}, p)^i_h$. Majorant $\overline{\mathrm{M}}_{\mathrm{it}}$ follows from a triangle inequality applied to the error between the exact and numerical solution *via* the exact iterative solutions. Thus, the error of each unknown decomposes in two parts, which are controlled by (3.22) and (3.11) or (3.25) and (3.19) (depending on the variable).

The main ingredients used for Lemmas 2–5 are the contraction mappings estimates [3] and functional a posteriori error majorants for the elliptic problems (initially introduced in [11, 14]). To be precise, guaranteed bounds for the error in either pressure or displacement approximations (reconstructed on the i-th iteration) combine estimates used for the iterative schemes, $\overline{\mathrm{M}}^i_p$ and $\overline{\mathrm{M}}^i_{\boldsymbol{u}}$ derived in Subsection 3.3, and majorants $\overline{\mathrm{M}}^h_p$ and $\overline{\mathrm{M}}^h_{\boldsymbol{u}}$ in Subsection 3.2.

3. Iterative approach

The current section derives an upper bound of the error introduced by the iterative as well as the discretisation schemes. Subsection 3.1 recapitulates contractive properties of the chosen fixed-stress split approach (see, e.g., [9]). Whereas, Subsections 3.2–2.1 derive majorant of the error in the approximations $(\boldsymbol{u}, p)^i_h$ based on the functional types error bounds and estimates for contraction mappings.

3.1. Fixed-stress split iteration scheme

The decoupled system considered prior to the fixed-stress split approach follows from (2.1)–(2.2), i.e.,

$$(\mathbb{K}_\tau \nabla p^i, \nabla w) + \beta\,(p^i, w) + \alpha\,(\mathrm{div}\boldsymbol{u}^{i-1}, w) = (\widetilde{g}, w), \quad \forall\, w \in W_0, \tag{3.1}$$

$$2\,\mu\,(\boldsymbol{\varepsilon}(\boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda\,(\mathrm{div}\boldsymbol{u}^i, \mathrm{div}\boldsymbol{v}) + \alpha\,(\nabla p^i, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0, \tag{3.2}$$

where p^i is solved using $\boldsymbol{u}^{i-1}$, and and the mechanics $\boldsymbol{u}^i$ is reconstructed using p^i recovered earlier. To obtain the initial data, p^0 is set to the hydrostatic pressure, i.e., $\nabla p^0 = \beta_f g$, whereas $\boldsymbol{u}^0$ is reconstructed from (3.2) using p^0.

To stabilise (3.1)–(3.2), we fix the artificial *volumetric mean total stress* defined by the relation

$$\gamma\eta^i = \alpha \mathrm{div}\boldsymbol{u}^i - L\,p^i, \tag{3.3}$$

where γ and L are certain positive parameters. The optimal choice of the latter ones allows to obtain the contraction in the norm $\left|\left[(e_p, e_{\boldsymbol{u}})\right]\right|$ (cf.(2.5)). First, we consider the difference of i-th and (i-1)-th iterations in (3.1). Denote $\delta p^i = p^i - p^{i-1}$ and $\delta\boldsymbol{u}^i = \boldsymbol{u}^i - \boldsymbol{u}^{i-1}$, and add pressure stabilisation terms $L\delta p^i$ and $L\delta p^{i-1}$, $L > 0$ on both sides, respectively and using (3.3), we obtain

$$(\beta + L)\,(\delta p^i, w) + (\mathbb{K}_\tau \nabla \delta p^i, \nabla w) = -(\gamma\,\delta\eta^{i-1}, w). \tag{3.4}$$

Using analogous manipulation for (3.2), the following identity for the displacement increment holds:

$$2\,\mu\,(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda\,(\mathrm{div}\delta\boldsymbol{u}^i, \mathrm{div}\boldsymbol{v}) - \alpha\,(\delta p^i, \mathrm{div}\boldsymbol{v}) = 0. \tag{3.5}$$

Theorem 2 establishes a contraction-type inequality for the norm $\|\delta\eta^i\|^2$.

Theorem 2. [9] *With* $\gamma = \frac{\alpha}{\sqrt{\lambda}}$ *and* $L \geq \frac{\alpha^2}{2\,\lambda}$*, the fixed-stress split scheme* (3.4)–(3.5) *is a contraction given by*

$$\|\boldsymbol{\varepsilon}(\delta\boldsymbol{u}^i)\|^2_{2\mu} + q\,\|\nabla\delta p^i\|^2_{\mathbb{K}_\tau} + \|\delta\eta^i\|^2 \leq q^2 \|\delta\eta^{i-1}\|^2, \quad q = \tfrac{L}{\beta + L}.$$

Remark 1. *There are several suggestions to choose the parameter L. The physically motivated choice $L_{cl} = \frac{\alpha^2}{2\,\lambda}$ is considered in [6]. Whereas [9] suggested $L_{opt} = \frac{\alpha^2}{2\,(\lambda + 2\mu/d)}$ with the complete convergence analysis valid for homogenous Lamé parameters. Numerical investigations in [4] demonstrated, that the optimal value of L is not only dependent on mechanical material parameters but on the boundary conditions and material parameters associated with the fluid flow problem.*

Corollary 1. *Theorem 2 yields that* $\nabla \delta p^i = \nabla p^i - \nabla p^{i-1}$ *and* $\boldsymbol{\varepsilon}(\delta \boldsymbol{u}^i) = \boldsymbol{\varepsilon}(\boldsymbol{u}^i) - \boldsymbol{\varepsilon}(\boldsymbol{u}^{i-1})$ *are converging sequences, i.e.,*

$$\|\nabla \delta p^i\|^2_{\mathbb{K}_\tau} \leq q \|\delta \eta^{i-1}\|^2 \quad \textit{and} \quad \|\boldsymbol{\varepsilon}(\delta \boldsymbol{u}^i)\|^2_{2\mu} \leq q^2 \|\delta \eta^{i-1}\|^2, \quad \textit{respectively.}$$

The latter is used in the derivation of the error estimate for $|\!|\!| e_p |\!|\!|^2_p$, in particular, it yields the following Lemma 1.

Lemma 1 (Contractive mapping estimates). *Let the assumptions of Theorem 2 and Corollary 1 hold. Then,*

$$\|\nabla(p - p^i)\|^2_{\mathbb{K}_\tau} \leq \tfrac{q}{(1-q)^2} \|\eta^i - \eta^{i-1}\|^2, \tag{3.6}$$

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}^i)\|_{2\mu} \leq \tfrac{q^2}{(1-q)^2} \|\eta^i - \eta^{i-1}\|^2. \tag{3.7}$$

Proof: Consider

$$\begin{aligned}\|\nabla(p^{i+m} - p^i)\|_{\mathbb{K}_\tau} &\leq \|\delta \nabla p^{i+m}\|_{\mathbb{K}_\tau} + \ldots + \|\delta \nabla p^{i+1}\|_{\mathbb{K}_\tau} \\ &\leq q\,(\|\delta \eta^{i+m-1}\| + \ldots + \|\delta \eta^i\|) \leq \sqrt{q}\,(q^m + \ldots + 1)\,\|\eta^i - \eta^{i-1}\|.\end{aligned}$$

Note that $(q^m + q^{m-1} + \ldots + 1) \to \frac{1}{1-q}$ as $m \to \infty$, which yields (3.6). Inequality (3.7) is proved analogously. □

3.2. Error majorants for discretisation errors

First, we focus on the error incorporated in approximations $(\boldsymbol{u}, p)^i_h$ to the exact solutions $(\boldsymbol{u}, p)^i$ of (3.4)–(3.5). We aim to derive computable and reliable estimates of the errors

$$e^i_{p,h} := p^i - p^i_h \quad \text{and} \quad e^i_{\boldsymbol{u},h} := \boldsymbol{u}^i - \boldsymbol{u}^i_h \tag{3.8}$$

measured in the terms of $\|e^i_{p,h}\|^2_p$ and $|\!|\!| e^i_{\boldsymbol{u},h} |\!|\!|^2_{\boldsymbol{u}}$ (cf. (2.5)). For this purpose, we rewrite (3.1) and (3.2) as

$$(\mathbb{K}_\tau \nabla p^i, \nabla w) + (\beta + L)(p^i, w) = (\widetilde{g} - \gamma\, \eta^{i-1}, w), \quad \forall\, w \in W_0, \tag{3.9}$$

$$2\mu\, \big(\boldsymbol{\varepsilon}(\boldsymbol{u}^i), \boldsymbol{\varepsilon}(\boldsymbol{v})\big) + \lambda(\mathrm{div}\boldsymbol{u}^i, \mathrm{div}\boldsymbol{v}) = (\boldsymbol{f}^i - \alpha \nabla p^i, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0 \tag{3.10}$$

with BCs $p^i = 0$ and $\boldsymbol{u}^i = \boldsymbol{0}$ on Γ. First, Lemma 2 derives an upper bound of $e^i_{p,h}$ measured in the energy norm.

Lemma 2. *For any* $p^i_h \in W_0$ *approximating* $p^i \in W_0$ *in* (3.9)*, and any auxiliary vector-valued function* $\boldsymbol{y}^i \in H_{\mathrm{div}}(\Omega) := \big\{\, \boldsymbol{y}^i \in [L(\Omega)]^d \mid \mathrm{div}\boldsymbol{y}^i \in L^2(\Omega) \big\}$*, we have the following estimate*

$$|\!|\!| e^i_{p,h} |\!|\!|^2_p \leq \overline{\mathrm{M}}^h_p(p^i_h, \boldsymbol{y}^i) := (1 + \zeta)\, \|\mathbf{r}_{\mathrm{d}}(p^i_h, \boldsymbol{z}^i)\|^2_{\mathbb{K}_\tau^{-1}} + (1 + \tfrac{1}{\zeta})\, \tfrac{C^2_\Omega}{\beta + L} \|\mathbf{r}_{\mathrm{eq}}(p^i_h, \boldsymbol{z}^i)\|^2_\Omega. \tag{3.11}$$

Here, $|\!|\!| e^i_{p,h} |\!|\!|^2_p$ *is defined in* (2.5)*,* $\mathbf{r}_{\mathrm{d},\mathbb{K}_\tau} := \boldsymbol{y}^i - \mathbb{K}_\tau \nabla p^i_h$*, and* $\mathbf{r}_{\mathrm{eq}} := \widetilde{g} - \gamma\, \eta^{i-1} - (\beta + L)\, p^i_h + \mathrm{div}\, \boldsymbol{y}^i$*, where* $\widetilde{g}$ *is defined in* (2.2)*, parameter* $\zeta \geq 0$*,* $C_\Omega = \frac{C_{\mathrm{F}}}{\sqrt{\lambda_{\mathbb{K}_\tau}}}$ *depends on the constant in the Friedrichs inequality* $\|w\|_\Omega \leq C_{\mathrm{F}} |\, w\,|_\Omega$*,* $\forall w \in W_0$*, and* $\lambda_{\mathbb{K}_\tau}$*, the minimum eigenvalue of the permeability tensor* $\mathbb{K}_\tau$*.*

Proof: The majorant $\overline{\mathrm{M}}^h_p(p^i_h, \boldsymbol{y}^i)$ follows from [13, Section 2]. □

Remark 2. *To obtain the sharpest guaranteed bounds, functional* $\overline{\mathrm{M}}_P$ *must be optimised w.r.t.* $\boldsymbol{y}^i$ *and* ζ *iteratively. This generates in an auxiliary variational problem w.r.t.* $\boldsymbol{y}^i$*. Alternatively, one can consider the mixed formulation of* (3.4)*-*(3.5) *and reconstruct* $(p^i_h, \boldsymbol{y}^i)$ *simultaneously using one of the well-developed mixed methods. This way, both variables required for the reconstruction of* $\overline{\mathrm{M}}_p$ *are available, and no additional post-processing (computational overhead) is required.*

Lemma 2 yields the estimate of the $e^i_{p,h}$ (cf. (3.8)) measured in terms of L^2-norm.

Corollary 2. *Assume that Lemma 2 holds. Then, for any* $p^i_h \in W_0$ *approximating* $p^i \in W_0$ *in* (3.9)*, the estimate*

$$\|e^i_{p,h}\|^2 \leq \overline{\mathrm{M}}^h_{p,L^2}(p^i_h, \boldsymbol{y}^i) \equiv \overline{\mathrm{M}}^h_{p,L^2}(i) := \tfrac{C^2_{\mathrm{F}}}{\lambda_{\mathbb{K}_\tau} + C^2_{\mathrm{F}} \beta}\, \overline{\mathrm{M}}^h_p(p^i_h, \boldsymbol{y}^i) \tag{3.12}$$

holds, where $\overline{\mathrm{M}}^h_p(p^i_h, \boldsymbol{y}^i)$ *is defined in* (3.11)*.*

Proof: By means of the Friedrichs inequality and property of permeability tensor $\mathbb{K}_\tau$, we obtain

$$|\!|\!| e^i_{p,h} |\!|\!|^2_p \geq \lambda_{\mathbb{K}_\tau} C_{\rm F}^{-2} \| e^i_p \|^2 + \beta \, \| e^i_{p,h} \|^2_\beta \geq \big(\lambda_{\mathbb{K}_\tau} C_{\rm F}^{-2} + \beta \big) \| \, e^i_{p,h} \, \|^2.$$

□

Next, we consider estimates for the error $e^i_{\boldsymbol{u},h} := \boldsymbol{u}^i - \boldsymbol{u}^i_h$, (cf. (3.8)). Since p^i_h is, in fact, used instead of p^i, the original problem (3.10) is replaced by

$$2\mu \left(\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right) + \lambda ({\rm div} \tilde{\boldsymbol{u}}^i, {\rm div} \boldsymbol{v}) = (\boldsymbol{f}^i - \alpha \nabla p^i_h, \boldsymbol{v}), \quad \forall \, \boldsymbol{v} \in \boldsymbol{V_0}, \tag{3.13}$$

with a perturbed right-hand side. Therefore, $\boldsymbol{u}^i_h$ is an approximation of $\tilde{\boldsymbol{u}}^i$ instead of $\boldsymbol{u}^i$. In other words, $e^i_{\boldsymbol{u},h}$ is composed of the error $\boldsymbol{u}^i - \tilde{\boldsymbol{u}}^i$ generated due to replacing (3.10) by (3.13) and the error $\tilde{\boldsymbol{u}}^i - \boldsymbol{u}^i_h$ arising because (3.13) is solved approximately. By means of the triangle inequality, $e^i_{\boldsymbol{u},h}$ is estimated by these errors as follows:

$$\| \, \boldsymbol{\varepsilon}(e^i_{\boldsymbol{u},h}) \, \|^2_{2\mu} + \| \, {\rm div}(e^i_{\boldsymbol{u},h}) \, \|^2_\lambda =: |\!|\!| \, e^i_{\boldsymbol{u},h} \, |\!|\!|^2_{\boldsymbol{u}} \leq 2 \, |\!|\!| \, \boldsymbol{u}^i - \tilde{\boldsymbol{u}}^i |\!|\!|^2_{\boldsymbol{u}} + 2 \, |\!|\!| \, \tilde{\boldsymbol{u}}^i - \boldsymbol{u}^i_h \, |\!|\!|^2_{\boldsymbol{u}}. \tag{3.14}$$

Here, $|\!|\!| \, \tilde{\boldsymbol{u}}^i - \boldsymbol{u}^i_h \, |\!|\!|^2_{\boldsymbol{u}}$ can be estimated by the functional majorant for a class of the elasticity problems (see Lemma 3), whereas the bound for $|\!|\!| \, \boldsymbol{u}^i - \tilde{\boldsymbol{u}}^i |\!|\!|^2_{\boldsymbol{u}}$ follows from the difference of (3.10) and (3.13) (see Lemma 4).

Lemma 3. *For any* $\boldsymbol{u}^i_h \in \boldsymbol{V_0}$ *approximating* $\tilde{\boldsymbol{u}}^i \in \boldsymbol{V_0}$ *in* (3.13)*, and any auxiliary tensor-valued function* $\boldsymbol{\tau}^i \in [\mathcal{T}_{\rm Div}(\Omega)]^{d \times d} := \left\{ \boldsymbol{\tau}^i \in [L^2(\Omega)]^{d \times d} \; \middle| \; {\rm Div} \boldsymbol{\tau}^i \in [L^2(\Omega)]^d \right\}$*, we have the estimate*

$$|\!|\!| \tilde{\boldsymbol{u}}^i - \boldsymbol{u}^i_h |\!|\!|^2_{\boldsymbol{u}} \leq \overline{\rm M}^h_{\tilde{u}}((\boldsymbol{u}, p)^i_h, \boldsymbol{\tau}^i) := (1 + \chi) \, \| \mathbf{r}_{\rm d, \mathbb{L}}(\boldsymbol{u}^i_h, \boldsymbol{\tau}^i) \|^2_{\mathbb{L}^{-1}} + (1 + \tfrac{1}{\chi}) \, C^2_{\rm K} \| \mathbf{r}_{\rm eq}(\boldsymbol{u}^i_h, \boldsymbol{\tau}^i) \|^2 \tag{3.15}$$

where $\mathbf{r}_{\rm d, \mathbb{L}}(\boldsymbol{u}^i_h, \boldsymbol{\tau}^i) := \boldsymbol{\tau}^i - \mathbb{L} \boldsymbol{\varepsilon}(\boldsymbol{u}^i_h)$ *with* $\mathbb{L} \, \boldsymbol{\varepsilon}(\boldsymbol{u}) := 2 \, \mu \, \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda \, {\rm div}(\boldsymbol{u})$ *and* $\mathbf{r}_{\rm eq}(\boldsymbol{\tau}^i) := \boldsymbol{f}^i - \alpha \nabla p^i_h + {\rm Div} \boldsymbol{\tau}^i$*,* $\chi > 0$*,* $C_{\rm K}$ *is a constant in the Korn first inequality* $\| \boldsymbol{w} \|_{[H^1(\Omega)]^d} \leq C_{\rm K} \| \boldsymbol{\varepsilon}(\boldsymbol{w}) \|_{[L^2(\Omega)]^{d \times d}}$*,* $\forall \boldsymbol{w} \in \boldsymbol{V_0}$*, and* α*,* μ*,* λ *are parameters of the Biot model.*

Proof: For the simplicity of exposition, let us assume the following representation of the elasticity tensor

$$\mathbb{L} \, \boldsymbol{\varepsilon}(\boldsymbol{u}) := 2 \, \mu \, \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda \, {\rm div}(\boldsymbol{u}). \tag{3.16}$$

in (3.13). Then, the derivation of an a posteriori error estimate for the problem

$$\left(\mathbb{L} \, \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}^i), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right) = (\boldsymbol{f}^i - \alpha \nabla p^i_h, \boldsymbol{v}), \quad \forall \, \boldsymbol{v} \in \boldsymbol{V_0}, \tag{3.17}$$

follows the approach discussed in [12, Section 5.2]. In particular, for any $\chi > 0$, the problem (3.17) yields an estimate

$$\| \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}^i - \boldsymbol{u}^i_h) \|_{\mathbb{L}} \leq (1 + \chi) \, \| \mathbf{r}_{\rm d, \mathbb{L}}(\boldsymbol{u}^i_h, \boldsymbol{\tau}^i) \|^2_{\mathbb{L}^{-1}} + (1 + \tfrac{1}{\chi}) \, C^2_{\rm K} \, \| \mathbf{r}_{\rm eq}(p^i_h, \boldsymbol{\tau}^i) \|^2 \tag{3.18}$$

where $\mathbf{r}_{\rm d, \mathbb{L}}(\boldsymbol{u}^i_h, \boldsymbol{\tau}^i) := \boldsymbol{\tau}^i - \mathbb{L} \boldsymbol{\varepsilon}(\boldsymbol{u}^i_h)$. We note that $\mathbb{L}^{-1} \boldsymbol{\tau}^i := \frac{1}{2\mu} \big(\tau^i - \frac{\lambda}{3\lambda + 2\mu} {\rm tr}(\boldsymbol{\tau}^i) \mathbb{I} \big)$, such that the first term on the right-hand side of (3.18) yields

$$\| \mathbf{r}_{\rm d, \mathbb{L}}(p^i_h, \boldsymbol{\tau}^i) \|^2_{\mathbb{L}^{-1}} = 2 \, \mu \, | \boldsymbol{\varepsilon}(\boldsymbol{u}^i_h) |^2 + \lambda \, | {\rm div} \boldsymbol{u}^i_h |^2 + \tfrac{1}{2\mu} \left(| \boldsymbol{\tau}^i |^2 - \tfrac{\lambda}{3\lambda + 2\mu} | {\rm tr}(\boldsymbol{\tau}^i) |^2 \right) - 2 \, \boldsymbol{\varepsilon}(\boldsymbol{u}^i_h) : \tau =: {\rm r_d}(\boldsymbol{u}^i_h, \boldsymbol{\tau}^i).$$

□

Remark 3. *The structure of* (3.15) *is typical for this class of a posteriori estimates. It contains two terms, where* $\| \mathbf{r}_{\rm d, \mathbb{L}} \|^2_{\mathbb{L}^{-1}}$ *usually contains the major part of the error and* $\| \mathbf{r}_{\rm eq} \|^2$ *is the reliability term, which guarantees the reliability. If the reconstruction of numerical fluxes is done sufficiently accurately, i.e., close to exact stress* $\mathbb{L} \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}^i)$*, then the first term dominates and can be used as an efficient indicator of local errors (see, e.g., [8] and references therein).*

Lemma 4 derives an estimate of $|\!|\!| e^i_{\boldsymbol{u},h} |\!|\!|^2_{\boldsymbol{u}}$ (cf. (3.8)). Below, $(\boldsymbol{\tau}, \boldsymbol{z})^i \in H_{\rm div}(\Omega) \times [\mathcal{T}_{\rm Div}(\Omega)]^{d \times d}$ denote the pair of auxiliary functions corresponding to $(\boldsymbol{u}, p)^i_h \in \boldsymbol{V_0} \times W_0$.

Lemma 4. *For any $p_h^i \in W_0$ approximating $p^i \in W_0$ in (3.9), any $\boldsymbol{u}_h^i \in \boldsymbol{V}$ approximating $\boldsymbol{u}^i \in \boldsymbol{V}$ in (3.10) and satisfying (3.13), and any auxiliary functions $\boldsymbol{z}^i \in H_{\rm div}(\Omega)$ and $\boldsymbol{\tau}^i \in [\mathcal{T}_{\rm Div}(\Omega)]^{d\times d}$, the estimate*

$$\| e_{\boldsymbol{u}}^i \|_{\boldsymbol{u}}^2 \le \overline{\rm M}_u^h((\boldsymbol{u},p)_h^i,(\boldsymbol{\tau},\boldsymbol{z})^i) \equiv \overline{\rm M}_u^h(i) := \tfrac{2\,\lambda\,\eta^2\,\alpha^2}{2\,\eta\,\lambda-1}\,\overline{\rm M}_{p,L^2}^h(p_h^i,\boldsymbol{z}^i) + 2\,\overline{\rm M}_{\tilde{u}}^h(\boldsymbol{u}_h^i,\boldsymbol{\tau}^i) \tag{3.19}$$

holds, where $\eta \ge \frac{1}{2\,\lambda}$. Here, $\overline{\rm M}_{p,L^2}^h$ and $\overline{\rm M}_{\tilde{\boldsymbol{u}}}$ are defined in (3.12) and (3.15), respectively, and α and λ are characteristics of the Biot model.

Proof: To estimate the first term on the right-hand side of (3.14), we consider the difference of (3.10) and (3.13):

$$2\mu\,(\boldsymbol{\varepsilon}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i),\boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda({\rm div}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i),{\rm div}\boldsymbol{v}) = -\alpha(p^i-p_h^i,{\rm div}\boldsymbol{v}).$$

By choosing $\boldsymbol{v} = \boldsymbol{u}^i - \tilde{\boldsymbol{u}}^i$, we obtain the identity $\|\boldsymbol{\varepsilon}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\|_{2\mu}^2 + \|\,{\rm div}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\,\|_\lambda^2 = -\alpha(p^i-p_h^i,{\rm div}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i))$. The latter one can be estimated by the Cauchy inequality as follows:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\|_{2\mu}^2 + \|\,{\rm div}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\,\|_\lambda^2 \le \alpha\|e_p^i\|\|{\rm div}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\|, \quad \forall\alpha>0.$$

By using the Young inequality with $\eta \ge \frac{1}{2\lambda}$ and (3.12), we arrive at

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\|_{2\mu}^2 + (\lambda - \tfrac{1}{2\,\eta})\,\|\,{\rm div}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\,\|^2 \le \tfrac{\eta\,\alpha^2}{2}\,\overline{\rm M}_{p,L^2}^h(p_h^i,\boldsymbol{z}^i).$$

By means of $\|\boldsymbol{\varepsilon}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\|_{2\mu}^2 + \|\,{\rm div}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\,\|_\lambda^2 \le \frac{2\,\eta\,\lambda}{2\,\eta\,\lambda-1}\Big(\|\boldsymbol{\varepsilon}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\|_{2\mu}^2 + (\lambda-\frac{1}{2\,\eta})\,\|\,{\rm div}(\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i)\,\|^2\Big)$, we obtain

$$\|\boldsymbol{u}^i-\tilde{\boldsymbol{u}}^i\|_{\boldsymbol{u}}^2 \le \tfrac{\lambda\,\eta^2\,\alpha^2}{2\,\eta\,\lambda-1}\,\overline{\rm M}_{p,L^2}^h(p_h^i,\boldsymbol{z}^i). \tag{3.20}$$

Combining (3.15) and (3.20), we arrive at (3.19). □

In addition to (3.19), Corollary 3 provides the majorant for the error measured in terms of $\|{\rm div}\cdot\|^2$-norm.

Corollary 3. *Assume that Lemma 4 holds. Then, for any $p_h^i \in W_0$ approximating $p^i \in W_0$ in (3.9), any $\boldsymbol{u}_h^i \in \boldsymbol{V}$ approximating $\boldsymbol{u}^i$ in (3.10) and satisfying (3.13), we have*

$$\|\,{\rm div}(e_{\boldsymbol{u},h}^i)\,\|^2 \le \overline{\rm M}_{u,{\rm div}}^h((\boldsymbol{u},p)_h^i,(\boldsymbol{\tau},\boldsymbol{z})^i) \equiv \overline{\rm M}_{u,{\rm div}}^h(i) := \tfrac{1}{2\mu\,d+\lambda}\left(\tfrac{\lambda\,\eta^2\,\alpha^2}{(2\,\eta\,\lambda-1)}\,\overline{\rm M}_{p,L^2}^h(p_h^i,\boldsymbol{y}^i) + 2\,\overline{\rm M}_{\tilde{u}}(\boldsymbol{u}_h^i,\boldsymbol{\tau}^i)\right),$$

where $\overline{\rm M}_{p,L^2}^h(p_h^i,\boldsymbol{y}^i)$ and $\overline{\rm M}_{\tilde{u}}((\boldsymbol{u},p)_h^i,\boldsymbol{\tau}^i)$ are defined in (3.12) and (3.15), respectively, and μ is a parameter of the Biot model.

Proof: By using inequality $\|\,e_{\boldsymbol{u}}\,\|_{\boldsymbol{u}}^2 \le (2\,\mu + d\,\lambda)\,\|\,\boldsymbol{\varepsilon}(e_{\boldsymbol{u}})\,\|^2$ and substituting it in (3.19), we arrive at

$$(2\mu\,d+\lambda)\|\,{\rm div}(e_{\boldsymbol{u},h}^i)\,\|^2 \le \tfrac{\eta^2\,\alpha^2\,\lambda}{2\,\eta\,\lambda-1}\,\overline{\rm M}_{p,L^2}^h(p_h^i,\boldsymbol{y}^i) + 2\,\overline{\rm M}_{\tilde{u}}((\boldsymbol{u},p)_h^i,\boldsymbol{\tau}^i).$$

□

3.3. Error majorants for the iterative errors

In this section, we derive guaranteed bounds of the errors

$$e_p := p - p^i \quad \text{and} \quad e_{\boldsymbol{u}} := \boldsymbol{u} - \boldsymbol{u}^i \tag{3.21}$$

occurring in the pressure and the displacement, respectively, as the results of application of contractive scheme (2.3)–(2.4) to the system (3.1)–(3.2). We begin with Lemma 5 that presents a bound of e_p on the i-th iteration.

Lemma 5. *The difference between $p^i \in W_0$ and $p \in W_0$ in (2.4) is subject to the estimate*

$$\begin{aligned} \|\,e_p\,\|_p^2 \le & \overline{\rm M}_p^i\big((\boldsymbol{u},p)_h^{i-1},(\boldsymbol{\tau},\boldsymbol{z})^{i-1},(\boldsymbol{u},p)_h^i,(\boldsymbol{\tau},\boldsymbol{z})^i\big) := \tfrac{3\,q}{(1-q)^2}\left(\tfrac{C_{\rm F}^2\,\beta}{\lambda_{{\rm K}_\tau}}+1\right)\overline{\rm m}_{i,i\text{-}1}, \qquad \textit{where} \\ & \overline{\rm m}_{i,i\text{-}1} := \|\eta_h^i - \eta_h^{i-1}\|^2 + \tfrac{\lambda}{2}\left(\overline{\rm M}_{u,{\rm div}}^h(i) + \overline{\rm M}_{u,{\rm div}}^h(i\text{-}1)\right) + \tfrac{L}{4}\left(\overline{\rm M}_{p,L^2}^h(i) + \overline{\rm M}_{p,L^2}^h(i\text{-}1)\right), \end{aligned} \tag{3.22}$$

where $\overline{\mathrm{M}}^h_{u,\mathrm{div}}(i)$ *and* $\overline{\mathrm{M}}^h_{p,L^2}(i)$ *are defined in Corollaries 2 and 3 for any* $\boldsymbol{z}^i \in H_{\Gamma^p_N}(\Omega, \mathrm{div})$ *and* $\boldsymbol{\tau}^i \in [\mathcal{T}_{\mathrm{Div}}(\Omega)]^{d\times d}$*, respectively,* $q = \frac{L}{\beta+L}$*, and* $\eta^i_h = \frac{\alpha}{\gamma}\mathrm{div}\boldsymbol{u}^i_h - \frac{L}{\gamma}\, p^i_h$*,* $L = \frac{\alpha^2}{2\,\lambda}$*,* $\forall p^i_h \in W_0, \forall \boldsymbol{u}^i_h \in \boldsymbol{V_0}$*. Here,* α*,* β*,* λ*,* μ_f*,* $C^{\mathrm{F}}_{\Gamma^p_D}$*, and* $\lambda_{\mathbb{K}_\tau}$ *are parameters of the Biot model.*

Proof: We note that for the error e_p (cf. (3.21)) the following estimate

$$\|\!| e^i_p |\!\|^2_p = \| e^i_p \|^2_\beta + \| \nabla e^i_p \|^2_{\mathbb{K}_\tau} \;\leq \Big(\tfrac{C^2_{\mathrm{F}}\, \beta}{\lambda_{\mathbb{K}_\tau}} + 1 \Big) \| \nabla e^i_p \|^2_{\mathbb{K}_\tau}$$

holds. The estimate of $\| \nabla e^i \|^2_{\mathbb{K}_\tau}$ follows from (3.6). To bound $\|\eta^i - \eta^{i-1}\|^2$ on the right-hand side of (3.6), we add and extract η^{i-1}_h and η^i_h as well as apply triangular inequality. As the result, it yields

$$\|\!| e^i_p |\!\|^2_p \leq \tfrac{3\,q}{(1-q)^2} \Big(\tfrac{C^2_{\mathrm{F}}\, \beta}{\lambda_{\mathbb{K}_\tau}} + 1 \Big) \big(\|\eta^i_h - \eta^{i-1}_h\|^2 + \|\eta^i - \eta^i_h\|^2 + \|\eta^{i-1} - \eta^{i-1}_h\|^2 \big). \tag{3.23}$$

Here, the first term $\|\eta^i_h - \eta^{i-1}_h\|^2$ is fully computable, and by means of (3.3) and Corollaries 2 and 3, we derive the estimate for the second and third terms, i.e.,

$$\|\eta^i - \eta^i_h\|^2 \leq \tfrac{1}{2\gamma^2} \big(\alpha^2 \|\mathrm{div}(e^i_{\boldsymbol{u}})\|^2 + L^2 \|e^i_p\|^2 \big) \leq \tfrac{1}{2\gamma^2} \Big(\alpha^2\, \overline{\mathrm{M}}^h_{u,\mathrm{div}}((\boldsymbol{u}, p)^i_h, \boldsymbol{\tau}^i, \boldsymbol{z}^i) + L^2\, \overline{\mathrm{M}}^h_{p,L^2}(p^i_h, \boldsymbol{z}^i) \Big). \tag{3.24}$$

Therefore, by summarising (3.23) and (3.24), as well as substituting $\lambda = \frac{\alpha^2}{2\,L}$ and $\gamma^2 = 2\,L$, we arrive at (3.22). □

To derive an upper bound for $e_{\boldsymbol{u}}$ (cf. (3.21)) we exploit the analogous idea used in Lemma 5 (see Lemma 6).

Lemma 6. *The difference between* $\boldsymbol{u}^i \in \boldsymbol{V_0}$ *and* $\boldsymbol{u} \in \boldsymbol{V_0}$ *in* (2.3) *is subject to the estimate*

$$\|\!| e_{\boldsymbol{u}} |\!\|^2_{\boldsymbol{u}} \leq \overline{\mathrm{M}}^i_u \big((\boldsymbol{u}, p)^{i-1}_h, (\boldsymbol{\tau}, \boldsymbol{z})^{i-1}, (\boldsymbol{u}, p)^i_h, (\boldsymbol{\tau}, \boldsymbol{z})^i \big) := (1 + \tfrac{d\,\lambda}{2\mu}) \tfrac{3\,q^2}{(1-q)^2}\, \overline{\mathrm{m}}_{i,i\text{-}1} \tag{3.25}$$

where $\overline{\mathrm{M}}^h_{u,\mathrm{div}}$ *and* $\overline{\mathrm{M}}^h_{p,L^2}$ *are defined in Corollaries 2 and 3 for any* $\boldsymbol{z}^i \in H_{\mathrm{div}}(\Omega)$ *and* $\boldsymbol{\tau}^i \in [\mathcal{T}_{\mathrm{Div}}(\Omega)]^{d\times d}$*, respectively, and* $\overline{\mathrm{m}}_{\mathrm{i,i\text{-}1}}$ *is defined in* (3.22)*,* $q = \frac{L}{\beta+L}$*, and* $\eta^i_h = \frac{\alpha}{\gamma}\mathrm{div}\boldsymbol{u}^i_h - \frac{L}{\gamma}\, p^i_h$*,* $L = \frac{\alpha^2}{2\,\lambda}$*,* $\forall p^i_h \in W_0, \boldsymbol{u}^i_h \in \boldsymbol{V_0}$*. Here,* λ*,* μ*,* α *are parameters of the Biot model.*

Putting together the above Lemmas, we obtain our main result as given in Theorem 1.

4. Conclusion

We presented a technique to the error control for the Biot problem in the poroelastic medium. We analysed semi–discrete counterpart of the variational Biot system decoupled by a fixed stress iterative scheme. The main result is a majorant $\overline{\mathrm{M}}_{\mathrm{it}}$ that provides a guaranteed bound of the error in the corresponding approximations. The derivation combines estimates for contraction mappings and functional a posteriori error majorants for the elliptic problems. The detailed report with an overview of the related works and derivation of the model as well as the more thorough proofs can be found in [7].

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