Adaptive Space-Time Isogeometric Analysis for Parabolic Evolution Problems

U. Langer, S. Matculevich, S. Repin

RICAM-Report 2018-23
Adaptive Space-Time Isogeometric Analysis for Parabolic Evolution Problems
Ulrich Langer, Svetlana Matculevich and Sergey Repin

Abstract. The paper is concerned with locally stabilized space-time IgA approximations to initial boundary value problems of the parabolic type. Originally, similar schemes (but weighted with a global mesh parameter) were presented and studied by U. Langer, M. Neu-
müller, and S. Moore (2016). The current work devises a localised version of this scheme. The localization of the stabilizations enables local mesh refinement that is one of the main ingredi-
ents of adaptive algorithms. We establish coercivity, boundedness, and consistency of the cor-
responding bilinear form. Using these fundamental properties together with the corresponding approximation error estimates for B-splines, we show that the space-time IgA solutions gener-
atated by the new scheme satisfy asymptotically optimal a priori discretization error estimates. The adaptive mesh refinement algorithm proposed in the paper is based on a posteriori error estimates of the functional type that has been rigorously studied in earlier works by S. Repin (2002) and U. Langer, S. Matculevich, and S. Repin (2017). Numerical results presented in the paper confirm the improved convergence of global approximation errors. Moreover, these results also confirm local efficiency of the error indicators produced by the error majorants.

Keywords. parabolic initial-boundary value problems, locally stabilized space-time isoge-
ometric analysis, a priori and a posteriori estimates of approximation errors.


1 Introduction

Time-dependent problems governed by parabolic partial differential equations (PDEs) are typical models in many scientific and engineering applications, e.g., heat conduction and diffusion, changing in time processes in social and life sciences, etc. This fact triggers their active investigation in modelling, mathematical analysis, and numerical solution. This paper is focused on the numerical treatment of parabolic problems by means of Isogeometric Analysis (IgA) [30] combined with a full space-time approach that treats time as yet another variable; see [20] and [63] for time-parallel and space-
time methods. Due to the fast development of parallel computers, this approach to quantitative analysis of evolutionary problems has became quite natural. Moreover, this way of treating evolutionary systems is not affected by the curse of sequentiality typical for time-marching schemes. Various versions of the space–time method can be efficiently used in combination with parallelisation methods; see, e.g., [20, 21, 44, 28].

The research is supported by the Austrian Science Fund (FWF) through the NFN S117-03 project.
This paper uses the idea similar to that applied in [44] for the derivation of the globally stabilized space-time scheme. It is based on testing the corresponding integral identity with the help of ‘time-upwind’ test functions, which are motivated by the space-time streamline diffusion method studied in [27, 32, 33]. In contrast to the scheme presented in [44], this work is focused on element-wise analysis that leads to a locally stabilized space-time IgA scheme.

One of the attractive features of the IgA method is high accuracy and flexibility of approximations obtained due to the high smoothness of the respective basis functions. This fact allows a user to combine space-time schemes with IgA technologies, and construct fully-adaptive schemes aiming to tackle problems generated by industrial applications; several earlier studied examples can be found in [65, 66].

Construction of effective adaptive refinement techniques is highly important for the design of fast and efficient numerical methods for solving PDEs. Adaptivity relies strongly on the reliable and locally quantitatively efficient a posteriori error estimation. We refer to [1, 3, 55, 46] for the overview of different error estimators. An efficient error indicator supposes to identify the areas, where discretization errors are excessively high, in order to refine the mesh and minimise local errors. A smart combination of solvers and error indicators could potentially provide a fully automated refinement algorithm taking into account special features of the problem, and generating a discretisation that produces approximate solutions with the desired accuracy.

Due to a tensor-product setting of IgA splines, mesh refinement has global effects, including a large percentage of superfluous control points. Challenges, arising along with these disadvantages, have triggered the development of local refinement techniques for IgA, such as truncated B-splines (T-splines) (introduced in [60, 61] and analysed in [5, 6, 58, 59]), hierarchical (HB-splines) [18, 36] and truncated hierarchical B-splines (THB-splines) [68, 23], patchwork splines (PB-splines) [16], locally refined splines (LR-splines) [13, 9], polynomial splines over hierarchical T-meshes (PHB-splines) [49, 69], etc. In the case of elliptic boundary value problems, local refinement IgA techniques were combined with some a posteriori error estimation approaches in several publications (a posteriori error estimates using the hierarchical bases in [14, 68], residual-based a posteriori error estimators and their modifications in [31, 69, 10, 37], and goal-oriented error estimators in [67, 11, 38, 39]).

In this paper, we deduce fully guaranteed error estimates in terms of several global norms equivalent to the norm of the functional space containing the corresponding generalised solution. These estimates do not use mesh-dependent constants (which must be recalculated in the process of mesh adaptation), and include only global constants characterising the geometry. Henceforth, we shortly call them error majorants. A posteriori error estimates of this type were originally introduced in [52, 53] and later applied to various problems; see [55, 46] and reference therein. These estimates are valid for any approximation from the admissible functional space. They do not use special properties of approximations (e.g., Galerkin orthogonality) or/and additional requirements for the exact solution (e.g., extra regularity beyond the minimal energy...
class that guarantees the existence of the unique generalised solution) and are valid for any approximation from the admissible functional space. Moreover, the majorant also generates efficient indicators of local (element-wise) error distribution over the domain.

We present a new localised space-time IgA scheme, where the adaptivity is driven by the functional type a posteriori error estimates. By exploiting the universality and efficiency of these error estimates as well as taking an advantage of smoothness of the IgA approximations, we aim at constructing fast fully adaptive space-time methods that could tackle complicated problems inspired by industrial applications. These two techniques were already combined in application to elliptic problems in [34] and [47] using tensor-based splines and THB-splines [23, 24, 22], respectively. Both papers confirmed that the majorants provide not only reliable and efficient upper bounds of the total energy error but a quantitatively sharp indicator of local element-wise errors.

For the time-dependent problems, the simplest form of such error bounds was derived for the heat equation in [54] and tested for the generalised diffusion equation in [19]. Majorants for approximations to the evolutionary convection-diffusion problem having jumps in time were considered in [56]. In [48], authors study the majorant’s robustness to a drastic change in values of the reaction parameter in evolutionary reaction-diffusion problems and provide the comparison of upper bound to newly introduced minorant of the error. Another extensive discussion on the numerical properties of the above-mentioned error estimates w.r.t. both time-marching and space-time methods can be found in [29].

Paper [43], that proceeds the current study, presents new functional-type a posteriori error estimates in a context of globally weighed space-time IgA schemes introduced in [44]. It illustrates the reliability and efficiency of functional a posterior error estimates for IgA solutions w.r.t several examples exhibiting different features and reports on the computing cost for these bounds. Moreover, the numerical examples discussed in [43] demonstrate the efficiency of the space-time THB-spline-based adaptive procedure. Therefore, the importance of locally stabilized space-time IgA schemes as well as the investigation of their numerical properties are rather inevitable in the context of the construction of fully adaptive schemes for initial-boundary value problems (I-BVPs).

This work is organized as follows: Section 2 defines the model evolutionary problem and recapitulates notation and functional spaces used throughout the paper. Section 3 presents a concise overview of the IgA framework and respective notions and definitions. Furthermore, it presents the globally stabilized space-time IgA scheme from [44] and discusses its main properties. Section 4 introduces the new locally stabilized version of the space-time IgA scheme, and provides the proofs of coercivity, boundedness, and consistency of the bilinear form corresponding to the IgA scheme. We also establish a priori error estimates for the considered class of approximations. The last section is dedicated to a posteriori estimates and practical aspects of the efficient combination of locally stabilized scheme and functional error majorants as well as their application to a series of numerical examples possessing different features.
2 Space-time variational formulation

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ be a bounded domain with Lipschitz continuous boundary $\partial \Omega$ and $(0,T)$, $0 < T < +\infty$ be a given time interval. By $Q := \Omega \times (0,T)$ and $\overline{Q} := Q \cup \partial Q$ we denote the space-time cylinder and its closure, respectively. The lateral surface of $Q$ is defined as $\partial Q := \Sigma \cup \Sigma_0 \cup \Sigma_T$, where $\Sigma = \partial \Omega \times (0,T)$, $\Sigma_0 = \Omega \times \{0\}$ and $\Sigma_T = \Omega \times \{T\}$.

We discuss an approach to adaptive space-time IgA approximations of evolutionary problems using the classical model of the linear parabolic initial-boundary value problem: find $u : \overline{Q} \to \mathbb{R}$ satisfying the equations

$$\partial_t u - \Delta_x u = f \quad \text{in} \ Q, \quad u = 0 \quad \text{on} \ \Sigma, \quad u = u_0 \quad \text{on} \ \Sigma_0,$$

where $\partial_t$ denotes the time derivative, $\Delta_x$ is the spatial Laplace operator, $f$ is a source function, and $u_0(x)$ is a given initial state.

Let us now introduce the functions spaces that we need in the following. The norm and scalar product in the Lebesgue space $L_2(Q)$ of square-integrable functions in the space-time cylinder $Q$ are denoted by $\|v\|_Q := \|v\|_{L_2(Q)}$ and $(v,w)_Q := \int_Q v(x,t)w(x,t)dxdt$, $v, w \in L_2(Q)$, respectively, with the corresponding changes for spaces of vector-valued fields. By $H^s(Q)$, $s \geq 1$, we denote standard Sobolev spaces supplied with the norm $\|v\|_{H^s(Q)} := (\int_Q \sum_{|\alpha| \leq s} \partial^\alpha v dx dt)^{1/2}$ for $s \in \mathbb{N} \cup 0$, where $\alpha := \{\alpha_1, \ldots, \alpha_d\}$ is a multi-index, and $\partial^\alpha v := \partial^{\alpha_1} v / \partial t^{\alpha_1} \cdots \partial^{\alpha_d}$. Then, $\|v\|_{H^s(Q)} := (\int_Q \sum_{|\alpha| = s} \partial^\alpha v dx dt)^{1/2}$ denotes the $H^s$-seminorm. Next, we introduce the following spaces

$$V_0^{1,0} := H_0^{1,0}(Q) := \{ u \in L_2(Q) : \nabla_x u \in [L_2(Q)]^d, u = 0 \text{ on } \Sigma \},$$

$$V_0^{1} := H_0^{1}(Q) := \{ u \in V_0^{1,0} : \partial_t u \in L_2(Q), u = 0 \text{ on } \Sigma_T \},$$

$$V_0^{1} := H_0^{1}(Q) := \{ u \in V_0^{1,0} : \partial_t u \in L_2(Q), u = 0 \text{ on } \Sigma_0 \},$$

and

$$V_0^{\Delta_x,1} := H_0^{\Delta_x,1}(Q) := \{ u \in V_0^{1,0} : \Delta_x u \in L_2(Q), \partial_t u \in L_2(Q) \},$$

where the latter is equipped with the norm $\|w\|_{V_0^{\Delta_x,1}}^2 := \|\Delta_x w\|_Q^2 + \|\partial_t w\|_Q^2$. Finally,

$$V_0^{s} := H^s(Q) \cap V_0^{1},$$

and

$$H^{\text{div}_x,0}(Q) := \{ y \in [L_2(Q)]^d : \text{div}_x y \in L_2(Q) \}$$

equipped with a scalar product $(v,w)_{\text{div}_x,0} := (v,w)_Q + (\text{div}_x v, \text{div}_x w)_Q$. Since $\Omega$ is bounded, we have the Friedrichs inequality $\|w\|_{\Omega} \leq C_F \|\nabla_w w\|_{\Omega}$ for all $w \in H_0^{1}(\Omega)$, which also implies $\|w\|_Q \leq C_F \|\nabla_w w\|_{\Omega}$ for all $w \in V_0^{1,0}(Q)$. 

It is proven in [41] that the standard space-time variational formulation of the initial-boundary value problem (2.1), find \( u \in V_{0}^{1,0} \)

\[
a(u, w) = \ell(w), \quad \forall w \in V_{0,0}^{1}, \tag{2.2}
\]

with the bilinear form

\[
a(u, w) := (\nabla_{x} u, \nabla_{x} w)_{Q} - (u, \partial_{t} w)_{Q},
\]

and the linear form

\[
\ell(w) := (f, w)_{Q} + (u_{0}, w)_{\Sigma_{0}},
\]

has a unique solution provided that \( f \in L_{2,1}(Q) := \{ v \in L_{1}(Q) \mid \int_{0}^{T} \| v(t) \|_{\Omega} \, dt < \infty \} \) and \( u_{0} \in L_{2}(\Omega) \). Here and later on, \( (u_{0}, w)_{\Sigma_{0}} := \int_{\Sigma_{0}} u_{0}(x) \, w(x,0) \, dx = \int_{\Omega} u_{0}(x) \, w(x,0) \, dx \).

Moreover, if \( f \in L_{2}(Q) \) and \( u_{0} \in H_{0}^{1}(\Omega) \), then problem (2.2) is uniquely solvable in \( V_{0}^{\Delta x,1} \), and the solution \( u \) continuously depends on \( t \) in the norm of the space \( H_{0}^{1}(\Omega) \) (see, e.g., [40] and [41, Theorem 2.1]). Furthermore, according to [41, Remark 2.2], \( \| u_{x}(\cdot, t) \|_{\Omega}^{2} \) is an absolutely continuous function of \( t \in [0, T] \) for any \( u \in V_{0}^{\Delta x,1} \).

Throughout the paper, we assume that \( f \in L_{2}(Q) \) and \( u_{0} \in H_{0}^{1}(\Sigma_{0}) \), i.e., we know that the solution \( u \) of the space-time variational problem (2.2) belongs to \( V_{0}^{\Delta x,1} \). In this case, without a loss of generality, we can assume homogeneous initial conditions \( u_{0} = 0 \); cf. also [28].

### 3 IgA framework

For the convenience of the reader, we recall the general concept of the IgA technology, the definition of B-splines, NURBS, and THB-splines, and their use in the geometrical representation of the space-time cylinder \( Q \), as well as the construction of the IgA trial and discretization spaces, which are used to approximate solutions satisfying the variational formulation of (2.2).

Let \( p \geq 2 \) be the polynomial degree, and let \( n \) denote the number of basis functions used to construct a B-spline curve. The knot-vector in one dimension is a non-decreasing set of coordinates in the parameter domain, written as \( \Xi = \{ \xi_{1}, \ldots, \xi_{n+p+1} \} \), \( \xi_{i} \in \mathbb{R} \), where \( \xi_{1} = 0 \) and \( \xi_{n+p+1} = 1 \). The knots can be repeated, and the multiplicity of the \( i \)-th knot is indicated by \( m_{i} \). Throughout the paper, we consider only open knot vectors, i.e., the multiplicity \( m_{1} \) and \( m_{n+p+1} \) of the first and the last knots, respectively, is equal to \( p + 1 \). In the case of the one-dimensional parametric domain \( \hat{Q} = (0,1) \), there is an underlying mesh of elements \( \hat{K} \in \hat{K}_{h} \) such that each of them is constructed by the distinct neighbouring knots. The global size of \( \hat{K}_{h} \) is denoted by \( \hat{h} := \max_{\hat{K} \in \hat{K}_{h}} \{ \hat{h}_{\hat{K}} \} \), where \( \hat{h}_{\hat{K}} := \text{diam}(\hat{K}) \). For the time being, we assume locally quasi-uniform meshes, i.e., the ratio of two neighbouring elements \( \hat{K} \) and \( \hat{K}' \) satisfies the inequality \( c_{1} \leq \hat{h}_{\hat{K}}/\hat{h}_{\hat{K}'} \leq c_{2} \), where \( c_{1}, c_{2} \) are positive constants.
The univariate B-spline basis functions $\hat{B}_{i,p} : \hat{Q} \to \mathbb{R}$ are defined by means of the Cox-de Boor formula $\hat{B}_{i,p}(\xi) := \frac{\xi - \xi_{i-1}}{\xi_{i+p} - \xi_{i-1}} \hat{B}_{i-1,p}(\xi) + \frac{\xi_{i+p} - \xi}{\xi_{i+p} - \xi_{i-1}} \hat{B}_{i+1,p-1}(\xi)$, with $\hat{B}_{i,0}(\xi) := \{1 \text{ if } \xi_i \leq \xi \leq \xi_{i+1}, \text{ and } 0 \text{ otherwise}\}$, where a division by zero is defined to be zero. One of the most crucial properties of these basis functions is their $C^{p-1}$ continuity. For that, we define the knot-vector dependent on the space-time direction. Furthermore, we introduce $\mathcal{I} = \{i = (i_1, \ldots, i_{d+1}) : i_\alpha = 1, \ldots, n_\alpha; \alpha = 1, \ldots, d + 1\}$, the set used to number basis number functions, and multi-indices standing for the order of polynomials $\mathbb{P} := (p_1, \ldots, p_{d+1})$. The tensor-product of the univariate B-spline basis functions generates a multivariate splines defined as $\hat{B}_{i,p}(\xi) := \prod_{\alpha=1}^{d+1} \hat{B}_{i_\alpha,p_\alpha}(\xi_\alpha)$, where $\xi = (\xi_1, \ldots, \xi_{d+1}) \in \hat{Q}$. The univariate and multivariate NURBS basis functions are defined in the parametric domain by means of the $p$-dimensional knot-set $\mathbb{K}$. For the given $\mathbb{P} := (p_1, \ldots, p_{d+1})$ and for any $i \in \mathcal{I}$, NURBS are defined as follows: $\hat{R}_{i,p} : \hat{Q} \to \mathbb{R}$, $\hat{R}_{i,p}(\xi) := \frac{w_i \hat{B}_{i,p}(\xi)}{W(\xi)}$, with a weighting function $W : \hat{Q} \to \mathbb{R}$, $W(\xi) := \sum_{i \in \mathcal{I}} w_i B_{i,p}(\xi)$, where $w_i > 0$ are real numbers and $\sum_{i \in \mathcal{I}} w_i = 1$.

In the association with the knot-vectors $\Xi_\alpha$, $\alpha = 1, \ldots, d + 1$, we define a mesh $\hat{K}_h$ partitioning $\hat{Q}$ into $d + 1$-dimensional open knot spans (elements)

$$\hat{K}_h = \hat{K}_h(\Xi_1, \ldots, \Xi_{d+1}) := \left\{ \hat{Q} = \otimes_{\alpha=1}^{d+1} (\xi_\alpha_{i_\alpha}, \xi_\alpha_{i_\alpha+1}) : \hat{Q} \neq \emptyset, \; p^\alpha + 1 \leq i^\alpha \leq n^\alpha - 1 \right\}.$$

A non-empty element $\hat{K} = \otimes_{\alpha=1}^{d+1} (\xi_\alpha_{i_\alpha}, \xi_\alpha_{i_\alpha+1}) \in \hat{K}_h$ is characterized by its diameter $\hat{h}_\hat{K}$. To $\hat{K}$, we associate $\hat{K} \in \hat{Q}$ defined as

$$\hat{K} = \otimes_{\alpha=1}^{d+1} (\xi_\alpha_{i_\alpha-p_\alpha}, \xi_\alpha_{i_\alpha+p_\alpha+1}) \in \hat{K}_h.$$

The set $\hat{K}$ represents the support extension of $\hat{K}$ and is constructed by the union of the supports of basis functions intersecting with $\hat{K}$.

The physical space-time domain $Q \subset \mathbb{R}^{d+1}$ is defined from the parametric domain $\hat{Q} = (0,1)^{d+1}$ by the geometrical mapping:

$$\Phi : \hat{Q} \to Q := \Phi(\hat{Q}) \subset \mathbb{R}^{d+1}, \quad \Phi(\xi) := \sum_{i \in \mathcal{I}} \hat{R}_{i,p}(\xi) \mathbf{P}_i.$$
where \( \{ P_i \}_{i \in I} \in \mathbb{R}^{d+1} \) are the control points. For the simplicity, we assume below the same polynomial degree for all directions, i.e., \( p_\alpha = p \), for all \( \alpha = 1, \ldots, d + 1 \).

For each \( \hat{K} \in \hat{K}_h \) and \( \hat{K} \), we obtain an element and a support extension on the physical domain

\[
K = \Phi(\hat{K}) := \{ \Phi(\xi) : \xi \in \hat{K} \} \quad \text{and} \quad \hat{K} := \Phi(\hat{K}), \tag{3.1}
\]

respectively. The physical mesh \( \mathcal{K}_h \) is defined on the space-time cylinder \( Q \) as follows

\[
\mathcal{K}_h := \{ K = \Phi(\hat{K}) : \hat{K} \in \hat{K}_h \}.
\]

The global mesh size is denoted by

\[
h := \max_{K \in \mathcal{K}_h} \{ h_K \}, \quad h_K := \| \nabla_x \Phi \|_{L^\infty(K)} \hat{h}_{\hat{K}}. \tag{3.2}
\]

Moreover, we assume that the physical mesh is also quasi-uniform, i.e.,

\[
h_K \leq h \leq C_u h_K. \tag{3.3}
\]

The set of facets corresponding to the discretisation \( \mathcal{K}_h \) is denoted by \( \mathcal{E}_h \) and can be split into the inner facets

\[
\mathcal{E}_h^I = \{ E \in \mathcal{E}_h^K : \exists K, K' \in \mathcal{K}_h : E = \partial K \cap \partial K' \wedge E \not\subset \partial Q \},
\]

and the facets intersecting with the boundary, namely,

\[
\mathcal{E}_h^{\partial Q} = \{ E \in \mathcal{E}_h^K : \exists K, K' \in \mathcal{K}_h : E = \partial K \cap \partial K' \wedge E \cap \partial Q \neq \emptyset \}.
\]

The latter one, in particular, contains inside the sets

\[
\mathcal{E}_h^\Sigma = \{ E \in \mathcal{E}_h^K : \exists K, K' \in \mathcal{K}_h : E = \partial K \cap \partial K' \wedge E \cap \Sigma \neq \emptyset \} \quad \text{and} \quad \mathcal{E}_h^{\Sigma_T} = \{ E \in \mathcal{E}_h^K : \exists K, K' \in \mathcal{K}_h : E = \partial K \cap \partial K' \wedge E \cap \Sigma_T \neq \emptyset \}.
\]

Let \( \mathcal{E}_h^K \) denote the set of facets of the local element \( K \in \mathcal{K}_h \), i.e.,

\[
\mathcal{E}_h^K := \{ E \in \mathcal{E}_h : E \cap \partial K \neq \emptyset, K \in \mathcal{K}_h \}.
\]

The discretisation spaces on \( Q \) are constructed by a push-forward of the basis functions defined on the parametric domain

\[
V_h := \text{span} \{ \phi_{h,i} := \hat{S}_h^p \circ \Phi^{-1}, (\ell, i) \in I \}_{i \in I}, \tag{3.4}
\]

where \( \hat{S}_h^p \) is the space of splines (e.g., B-splines, NURBS, THB-splines) of a degree \( p \), and \( \Phi \) is assumed to be invertible in \( Q \), with smooth inverse on each element \( K \in \mathcal{K}_h \) (see [4, 8] and references therein). Moreover, we introduce the subspace \( V_{0h} := V_h \cap V_{0,0}^1 \) for the functions satisfying homogeneous initial and boundary conditions.

Let us recall two fundamental inequalities, i.e., scaled trace and inverse inequalities, that are important for the derivation of a priori discretization error estimates for the space-time IgA scheme presented in the further sections.
Lemma 3.1. [17, Theorem 3.2] Let $K \in \mathcal{K}_h$. Then the scaled trace inequality

$$\|v\|_{\partial K} \leq C_{tr} h_K^{-1/2}(\|v\|_K + h_K \|\nabla v\|_K)$$  \hspace{1cm} (3.5)

holds for all $v \in H^1(K)$, where $\nabla = (\nabla_x, \partial_t)$, $h_K$ is a local mesh size (cf. (3.2)), and $C_{tr}$ is a positive constant independent of $K \in \mathcal{K}_h$.

Lemma 3.2. [4, Theorem 4.1] Let $K \in \mathcal{K}_h$. Then the inverse inequalities

$$\|\nabla_x v_h\|_K \leq C_{int,1} h_K^{-1} \|v_h\|_K, \quad \text{and}$$

$$\|v_h\|_{\partial K} \leq C_{int,0} h_K^{-1/2} \|v_h\|_K$$  \hspace{1cm} (3.6)

hold for all $v_h \in V_h$, where $C_{int,0}$ and $C_{int,1}$ are positive constants independent of $K \in \mathcal{K}_h$, and $h_K := \text{diam}_{K \in \mathcal{K}_h}$ is a local mesh size.

For the completeness, we recall fundamental results on the approximation properties of spaces generated by NURBS using [4, Section 3]. It states the existences of a projection operator that provides an asymptotically optimal approximation result.

Lemma 3.3. [4, Theorem 3.1] Let $\ell, s \in \mathbb{N}$ be $0 \leq \ell \leq s \leq p+1$, $u \in V_{0,0}^s$, and $K$ and $h$ are elements defined in (3.1). Then there exists a projection operator $\Pi_h : V_{0,0}^s \rightarrow V_{0h}$ and a constant $C_s > 0$ such that

$$|v - \Pi_h v|^2_{H^\ell(K)} \leq C_{l,s}^2 h_K^{2(s-\ell)} \sum_{i=0}^{s} c_K^{2i} |v|^2_{H^{i}(K)} \quad \forall v \in L_2(Q) \cap H^\ell(K),$$  \hspace{1cm} (3.8)

where $C_{l,s}$ is a dimensionless shape constant dependent on $s, \ell,$ and $p$, the shape regularity of $K$, described by $\Phi$ and its gradient, $h_K$ is a local mesh size (cf. (3.2)), and $c_K := \|\nabla_x \Phi\|_{L_\infty(\Phi^{-1}(\hat{K}))}$.

Unlike the classical finite element spaces of degree $p$, Lemma 3.3 provides the bound, where the $\ell^\text{th}$-order seminorm of the error $u - \Pi_h u$ is controlled by the full $s^\text{th}$-order norm of $u$. In particular, the following formulations of (3.8) will be used:

$$\|u - \Pi_h u\|_{L_2(K)}^2 \leq C_{0,s}^2 h_K^{2s} \sum_{i=0}^{s} c_K^{2i} |u|^2_{H^i(K)},$$  \hspace{1cm} (3.9)

$$|u - \Pi_h u|^2_{H^1(K)} \leq C_{1,s}^2 h_K^{2(s-1)} c_K^{-2} \sum_{i=0}^{s} c_K^{2i} |u|^2_{H^i(K)},$$  \hspace{1cm} (3.10)

$$|u - \Pi_h u|^2_{H^2(K)} \leq C_{2,s}^2 h_K^{2(s-2)} c_K^{-4} \sum_{i=0}^{s} c_K^{2i} |u|^2_{H^i(K)},$$  \hspace{1cm} (3.11)

for any $v \in L_2(Q) \cap H^\ell(K)$. 


Globally stabilized space-time IgA scheme for parabolic equations have been presented and analysed in [44], where the authors proved its efficiency for fixed and moving spatial computational domains. In particular, it was shown that the corresponding discrete bilinear form is elliptic w.r.t. a discrete energy norm, bounded, consistent, and that generated IgA approximations satisfy a priori discretisation error estimate. In order to derive a \textit{globally stabilized discrete IgA space-time scheme}, the authors considered time-upwind test function $v_h + \delta_h \partial_t v_h$, $\delta_h = \theta h$, $v_h \in V_{0h}$, such that $\theta > 0$ is an auxiliary constant and $h$ is the global mesh-size (cf. (3.2)). This implies the discrete stabilized space-time IgA scheme: find $u_h \in V_{0h}$ satisfying

$$a_h(u_h, v_h) = l_h(v_h), \quad \forall v_h \in V_{0h}$$

where

$$a_h(u_h, v_h) := (\partial_t u_h, v_h)_Q + (\nabla_x u_h, \nabla_x v_h)_Q + \delta_h \left( (\partial_t u_h, \partial_t v_h)_Q + (\nabla_x u_h, \partial_t (\nabla_x v_h))_Q \right)$$

$$l_h(v_h) := (f, v_h + \delta_h \partial_t v_h)_Q.$$  

Combining coercivity and boundedness properties of $a_h(\cdot, \cdot)$ with the consistency of the scheme and approximation results for IgA spaces, we obtain the corresponding a priori error estimate w.r.t. the norm

$$|||v_h|||^2_h := \|\nabla_x v_h\|_Q^2 + \delta_h \|\partial_t v_h\|_Q^2 + \|v_h\|_{\Sigma_T}^2 + \delta_h \|\nabla_x v_h\|_{\Sigma_T}^2,$$

which is presented in Theorem 3.4 below.

**Theorem 3.4.** [44, 42] Let $u \in V^s_0 := V^s(Q) \cap V^{1,0}_0$, $s \in \mathbb{N}$, $s \geq 2$, be the exact solution to (2.2), and let $u_h \in V_{0h}$ be the solution to (3.12) with some fixed parameter $\theta > 0$. Then, the following a priori discretization error estimate

$$\|u - u_h\|_h \leq C h^{r-1} \|u\|_{H^r(Q)}$$

holds with $r = \min\{s, p + 1\}$ and some generic constant $C > 0$ independent of $h$.

### 4 Locally stabilized IgA schemes

In the current section, we assume that $p \geq 2$ and $m \leq p - 1$, which yields that $V_{0h} \subset C^1(Q)$ providing the inclusion $V_{0h} \subset V^{\Delta x,1}_{0,0} := V^{\Delta x,1}_0 \cap V^{0,1}_{0,0}$. We know that the solution $u$ of (2.2) belongs to $V^{\Delta x,1}_{0,0}$ provided that $f \in L^2(Q)$. In this case, for all $K \in \mathcal{K}_h$, we can write the PDE $\partial_t u - \Delta_x u = f$ in $K$ and can multiply it with the localized test functions

$$v_h + \delta_K \partial_t v_h, \quad \delta_K = \theta_K h_K, \quad \theta_K > 0, \quad h_K := \text{diam}(K),$$
such that
\[
(\partial_t u - \Delta u, v_h + \delta_K \partial_t v_h)_K = (f, v_h + \delta_K \partial_t v_h)_K, \quad \forall v_h \in V_0h.
\]

By summing up all the elements in \( \mathcal{K}_h \), we obtain the relation
\[
(\partial_t u - \Delta u, v_h)_Q + \sum_{K \in \mathcal{K}_h} \delta_K \left( (\partial_t u - \Delta u, \partial_t v_h)_K \right) = (f, v_h)_Q + \sum_{K \in \mathcal{K}_h} \delta_K (f, \partial_t v_h)_K.
\]

The integration by parts w.r.t. to the space variable yields
\[
\ell_{\text{loc}, h}(v_h) := (f, v_h)_Q + \sum_{K \in \mathcal{K}_h} \delta_K \left( (\partial_t u, v_h)_K + (\nabla_x u, \nabla_x \partial_t v_h)_K - \langle n^K_x \cdot \nabla_x u, \partial_t v_h \rangle_{\partial K} \right) =: a_{\text{loc}, h}(u, v_h),
\]

where \( n^K_x \) is an external normal vector to \( \partial K \). Here, the last term is nothing else but a duality product \( \langle \cdot, \cdot \rangle_{\partial K} = \langle \cdot, \cdot \rangle_{H^{-1/2}(\partial K) \times H^{1/2}(\partial K)} : H^{-1/2}(\partial K) \times H^{1/2}(\partial K) \to \mathbb{R} \), and \( H^{-1/2} \) is dual space to \( H^{1/2} \). Thus, we arrive at the finite dimensional problem: find \( u_h \in V_0h \) satisfying the identity
\[
a_{\text{loc}, h}(u_h, v_h) = \ell_{\text{loc}, h}(v_h), \quad \forall u_h, v_h \in V_0h, \tag{4.1}
\]

where the bilinear form \( a_{\text{loc}, h}(u_h, v_h) \) can be written as follows
\[
a_{\text{loc}, h}(u_h, v_h) := (\partial_t u_h, v_h)_Q + (\nabla_x u_h, \nabla_x v_h)_Q
\]
\[
+ \sum_{K \in \mathcal{K}_h} \delta_K \left( (\partial_t u_h, \partial_t v_h)_K + (\nabla_x u_h, \nabla_x \partial_t v_h)_K - \langle n^K_x \cdot \nabla_x u_h, \partial_t v_h \rangle_{\partial K} \right)
\]
\[
- \sum_{K \in \mathcal{K}_h} \delta_K \sum_{E \in \mathcal{E}_h^I \subset \mathcal{E}_h^{\partial Q}} (n^E_x \cdot \nabla_x u_h, \partial_t v_h)_E.
\]

Due to the assumptions \( v_h|_{\Sigma} = 0 \) and \( n^E_x|_{\Sigma_T \cup \Sigma_T} = 0 \), contributions of the terms \( (n^E_x \cdot \nabla_x u_h, \partial_t v_h)_E \in \mathcal{E}_h^{\partial Q} \) vanishes.

### 4.1 Coercivity

**Lemma 4.1.** Let the parameters \( \theta_K \) be sufficiently small, i.e., \( \theta_K \in \left( 0, \frac{h_K}{d C_{\text{int}, 1}} \right) \), where \( C_{\text{int}, 1} \) is the interpolation constant in (3.6) associated with \( K \in \mathcal{K}_h \). Then, the bilinear form \( a_{\text{loc}, h}(\cdot, \cdot) : V_0h \times V_0h \to \mathbb{R} \) is \( V_0h \)-coercive w.r.t. to the norm
\[
\|v_h\|_{\text{loc}, h}^2 := \|\nabla_x v_h\|_Q^2 + \frac{1}{2} \|v_h\|_{\Sigma_T}^2 + \sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t v_h\|_K^2, \tag{4.2}
\]

i.e., there exists a constant \( \mu_{\text{loc}, c} > 0 \) such that
\[
a_{\text{loc}, h}(v_h, v_h) \geq \mu_{\text{loc}, c} \|v_h\|_{\text{loc}, h}^2, \quad \forall v_h \in V_0h. \tag{4.3}
\]
Proof. Integration by parts of $a_{\text{loc},h}(v_h, v_h)$ yields

$$a_{\text{loc},h}(v_h, v_h) := (\partial_t v_h, v_h)_Q + (\nabla_x v_h, \nabla_x v_h)_Q$$

$$+ \sum_{K \in \mathcal{K}_h} \delta_K \left\{ (\partial_t v_h, \partial_t v_h)_K + (\nabla_x v_h, \nabla_x \partial_t v_h)_K - \sum_{E \in \mathcal{E}^l_h} (\mathbf{n}_x \cdot \nabla_x u_h, \partial_t v_h)_E \right\}$$

$$= \frac{1}{2} \|v_h\|_{L^2_{\Sigma_T}}^2 + \|\nabla_x v_h\|_Q^2 + \sum_{K \in \mathcal{K}_h} \delta_K \left\{ \|\partial_t v_h\|_K^2 - (\Delta_x v_h, \partial_t v_h)_K \right\}. \quad (4.4)$$

Here, and later on, we assume that $E \in \mathcal{E}^K_h$ and therefore omit repeating it. In order to prove coercivity, we need to estimate the last term in (4.4). By using (3.6) and Young inequality, we arrive at

$$\sum_{K \in \mathcal{K}_h} \delta_K (\Delta_x v_h, \partial_t v_h)_K \leq \left( \sum_{K \in \mathcal{K}_h} \delta_K \|\Delta_x v_h\|_K^2 \right)^{1/2} \left( \sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t v_h\|_K^2 \right)^{1/2}$$

$$\leq \left( \sum_{K \in \mathcal{K}_h} \delta_K \left( \sum_{l=1}^d \|\partial_x^2 v_h\|_K^2 \right)^{1/2} \left( \sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t v_h\|_K^2 \right)^{1/2}$$

$$\leq \left( \sum_{K \in \mathcal{K}_h} \delta_K \left( \sum_{l=1}^d \theta_K h_K \left( C_{int,1}^2 \|\nabla_x v_h\|_Q^2 \right) \right)^{1/2} \left( \sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t v_h\|_K^2 \right)^{1/2}$$

$$\leq \frac{d}{2} \max_{K \in \mathcal{K}_h} \left( \frac{\theta_K}{h_K} C_{int,1}^2 \right) \left( \|\nabla_x v_h\|_Q^2 + \sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t v_h\|_K^2 \right).$$

Therefore, $a_{\text{loc},h}(v_h, v_h)$ can be bounded from below as follows:

$$a_{\text{loc},h}(v_h, v_h) \geq \frac{1}{2} \|v_h\|_{L^2_{\Sigma_T}}^2 + \left( 1 - \frac{d}{2} \max_{K \in \mathcal{K}_h} \frac{\theta_K}{h_K} C_{int,1}^2 \right) \left\{ \sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t v_h\|_K^2 + \|\nabla_x v_h\|_Q^2 \right\}$$

$$\geq \frac{1}{2} \|v_h\|_{L^2_{\Sigma_T}}^2 + \left( 1 - \frac{d}{2} \max_{K \in \mathcal{K}_h} \frac{\theta_K}{h_K} C_{int,1}^2 \right) \left\{ \sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t v_h\|_K^2 + \|\nabla_x v_h\|_Q^2 \right\}$$

$$\geq \frac{1}{2} \|v_h\|_{L^2_{\Sigma_T}}^2,$$

provided that $\theta_K \in \left( 0, \frac{h_K}{d C_{int,1}} \right)$ for $K \in \mathcal{K}_h$. \qed

Remark 4.2. Computation of the constants $C_{int,1}$ in the inverse inequalities corresponds to the question of accurate estimation of maximal eigenvalues for generalised eigenvalue problems for considered differential equations. In [35], the authors applied symbolic computation methods to this problem defined on the square elements and were able to improve the previously known upper bounds in [57].
The second term is estimated by means of the Hölder inequality, i.e., whereas the third one is treated as follows:

Assume that Lemma 4.3.

Equipped with the norm localised bilinear form To prove a priori error bounds, we need to show the uniform boundedness of the linear system generated by the bilinear form is positive definite.

4.2 Boundedness

To prove a priori error bounds, we need to show the uniform boundedness of the localised bilinear form $a_{loc,h}(\cdot, \cdot)$ on $V_{0h, *} \times V_{0h}$, where $V_{0h, *}: = V_{0,0}^1 \cap V_{0,0}^{\Delta_x,1} + V_{0h}$ is equipped with the norm

$$\|v\|_{loc,h, *}^2 := \|v\|_{loc,h}^2 + \sum_{K \in \mathcal{K}_h} (\delta_K^{-1} \|v\|_K^2 + \delta_K \|\Delta_x v\|_K^2).$$

Lemma 4.3. Assume that $\theta_K \in (0, \frac{h_K}{d \epsilon_{int}}]$, $K \in \mathcal{K}_h$. Then, the bilinear form $a_{loc,h}(\cdot, \cdot)$ is uniformly bounded on $V_{0h, *} \times V_{0h}$, i.e., there exists a positive constant $\mu_{loc,b}$ that does not depend on $h_K$ such that

$$|a_{loc,h}(u, v_h)| \leq \mu_{loc,b} \|u\|_{loc,h, *} \|v_h\|_{loc,h}, \quad \forall u \in V_{0h, *}, \quad \forall v_h \in V_{0h}. \quad (4.5)$$

Proof. We estimate $a_{loc,h}(u, v_h)$ term by term. For the first one, we apply integration by parts w.r.t. time and the Cauchy inequality:

$$(\partial_t u, v_h)_Q = (u, v_h)_{\Sigma_T} - (u, \partial_t v_h)_Q \leq \|u\|_{\Sigma_T} \|v_h\|_{\Sigma_T} + \sum_{K \in \mathcal{K}_h} \delta_K^{-1/2} \|u\|_K \delta_K^{1/2} \|\partial_t v_h\|_K$$

$$\leq \left[\|u\|_{\Sigma_T}^2 + \sum_{K \in \mathcal{K}_h} \delta_K^{-1} \|u\|_K^2\right]^{1/2} \left[\|v_h\|_{\Sigma_T}^2 + \sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t v_h\|_K^2\right]^{1/2}. $$

The second term is estimated by means of the Hölder inequality, i.e.,

$$(\nabla_x u, \nabla_x v_h)_Q \leq \|\nabla_x u\| \|\nabla_x v_h\|,$$

whereas the third one is treated as follows:

$$\sum_{K \in \mathcal{K}_h} \delta_K (\partial_t u, \partial_t v_h)_K \leq \left[\sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t u\|_K^2\right]^{1/2} \left[\sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t v_h\|_K^2\right]^{1/2}. $$

If we consider result of (4.4), the last term can be estimated as

$$- \sum_{K \in \mathcal{K}_h} \delta_K (\Delta_x u, \partial_t v_h)_K \leq \left[\sum_{K \in \mathcal{K}_h} \delta_K \|\Delta_x u\|_K^2\right]^{1/2} \left[\sum_{K \in \mathcal{K}_h} \delta_K \|\partial_t v_h\|_K^2\right]^{1/2}. $$
By combining the obtained results, the bilinear form can be bounded as

$$|a_{loc,h}(u, v_h)| \leq \left[ \|u\|_{H^s(\Omega)}^2 + \|\nabla_x u\|_{Q}^2 + \sum_{K \in K_h} \left\{ \delta_K^{-1} \|u\|_{K}^2 + \delta_K (\|\partial_t u\|_{K}^2 + \|\Delta_x u\|_{K}^2) \right\} \right]^{1/2}$$

$$\times \left[ \|v_h\|_{H^s(\Omega)}^2 + \|\nabla_x v_h\|_{Q}^2 + 3 \sum_{K \in K_h} \delta_K \|\partial_t v_h\|_{K}^2 \right]^{1/2}$$

$$\leq \mu_{loc,b} \|u\|_{loc,h} \|v_h\|_{loc,h},$$

where $\mu_{loc,b} = 3$. \hfill \Box

### 4.3 Approximation properties

The estimate (3.8) implies a priori estimates of the interpolation error $u - \Pi_h u$, measured in terms of the $L_2$-norm and the discrete norms $\| \cdot \|_{loc,h}$ and $\| \cdot \|_{loc,h,*}$, which we later need in order to obtain an a priori estimate for $u - u_h$.

**Lemma 4.4.** Let $l, s \in \mathbb{N}$ be $1 \leq l \leq s \leq p + 1$, and $u \in V_{0, \Omega}^s$. Then, there exists a projection operator $\Pi_h : V_{0, \Omega}^s \to V_{0h}$ (see Lemma 3.3) and positive constants $C_1, C_2$, and $C_3$, such that the following a priori error estimates hold

$$\|u - \Pi_h u\|_{loc,h}^2 \leq C_1 \left( \sum_{K \in K_h} h_K^{2(s-1)} \sum_{i=0}^s c_K^{2i} |u|_{H^i(K)}^2 \right), \quad (4.6)$$

$$\|u - \Pi_h u\|_{loc,h,*}^2 \leq C_2 \left( \sum_{K \in K_h} h_K^{2(s-1)} \sum_{i=0}^s c_K^{2i} |u|_{H^i(K)}^2 \right), \quad (4.7)$$

for all $u \in L_2(\Omega) \cap H^s(K)$.

**Proof.** To prove (4.6) and (4.7), we need to provide estimates for each term in the norm $\|u - \Pi_h u\|_{loc,h}$. In order to bound the first term, we use (3.10), i.e.,

$$\|\nabla_x (u - \Pi_h u)\|_{Q}^2 \leq \sum_{K \in K_h} |u - \Pi_h u|_{H^1(K)}^2$$

$$\leq C_{1,s} \max_{K \in K_h} c_K^{-2} \sum_{K \in K_h} h_K^{2(s-1)} \sum_{i=0}^s c_K^{2i} |u|_{H^s(K)}^2. \quad (4.8)$$

For the next one, we use (3.3), (3.10), and by similar approach derive:

$$\sum_{K \in K_h} \delta_K \|\partial_t (u - \Pi_h u)\|_{K}^2 \leq \max_{K \in K_h} \delta_K \sum_{K \in K_h} |u - \Pi_h u|_{H^1(K)}^2$$

$$\leq C_{1,s} \max_{K \in K_h} \left\{ \delta_K c_K^{-2} \right\} \sum_{K \in K_h} h_K^{2(s-1)} \sum_{i=0}^s c_K^{2i} |u|_{H^s(K)}^2. \quad (4.9)$$
Let $\mathcal{K}_h^{\Sigma_T} := \{ K ∈ \mathcal{K}_h | \partial K ∩ \Sigma_T ≠ ∅ \}$. By applying (3.5), (3.3), (3.9), and (3.10), the estimate of the part of the norm on $\Sigma_T$ reads as

\[ \| u - \Pi_h u \|^2_{\Sigma_T} = \sum_{E ∈ \mathcal{E}_h^{\Sigma_T}} \| u - \Pi_h u \|^2_E \]

\[ \leq \sum_{K ∈ \mathcal{K}_h^{\Sigma_T}} C_{tr}^2 \left( h_K^{-1} \| u - \Pi_h u \|^2_K + h_K \| u - \Pi_h u \|^2_{H^1(K)} \right) \]

\[ \leq \max_{K ∈ \mathcal{K}_h^{\Sigma_T}} \{ C_{tr}^2 \} C_u \left( h^{-1} \sum_{K ∈ \mathcal{K}_h^{\Sigma_T}} \| u - \Pi_h u \|^2_K + h \sum_{K ∈ \mathcal{K}_h^{\Sigma_T}} \| u - \Pi_h u \|^2_{H^1(K)} \right) \]

\[ \leq \max_{K ∈ \mathcal{K}_h^{\Sigma_T}} \{ C_{tr}^2 \} C_u \left( \sum_{K ∈ \mathcal{K}_h^{\Sigma_T}} \sum_{i=0}^{s} h_K^{2s-1} \sum_{i=0}^{s} c_K^2 |u|_{H^i(K)} \right. \]

\[ \left. + C_{1,s}^2 \sum_{K ∈ \mathcal{K}_h^{\Sigma_T}} h_K^{2s-1} \sum_{i=0}^{s} c_K^2 |u|_{H^i(K)} \right) \]

\[ \leq C_u \max_{K ∈ \mathcal{K}_h^{\Sigma_T}} \{ C_{tr}^2 (C_{0,s}^2 + C_{1,s}^2 c_K^{-2}) \} \sum_{K ∈ \mathcal{K}_h^{\Sigma_T}} h_K^{2s-1} \sum_{i=0}^{s} c_K^2 |u|_{H^i(K)} \]

\[ \leq C_{\Sigma_T} \sum_{K ∈ \mathcal{K}_h^{\Sigma_T}} h_K^{2s-1} \sum_{i=0}^{s} c_K^2 |u|_{H^i(K)}, \quad (4.10) \]

where

\[ C_{\Sigma_T} = C_u \max_{K ∈ \mathcal{K}_h^{\Sigma_T}} \left\{ C_{tr}^2 (C_{0,s}^2 + C_{1,s}^2 c_K^{-2}) \right\}. \quad (4.11) \]

Combining (4.8)–(4.10), we obtain the bound

\[ \| u - \Pi_h u \|^2_{\text{loc},h} \leq C_1 \sum_{K ∈ \mathcal{K}_h} h_K^{2(s-1)} \sum_{i=0}^{s} c_K^2 |u|_{H^i(K)}, \quad (4.12) \]

where $C_1 = \max_{K ∈ \mathcal{K}_h} \left\{ C_{1,s}^2 (1 + δ_K^{-1} c_K^{-2} + C_{\Sigma_T} \right\}$ with constant $C_{\Sigma_T}$ defined in (4.11).

In order to prove (4.7), we need to estimate $\sum_{K ∈ \mathcal{K}_h} \delta_K^{-1} \| u \|^2_K$ and $δ_K \|Δx v\|^2_K$ included into $\| u \|^2_{\text{loc},h,*}$. First, using (3.9), we obtain

\[ \sum_{K ∈ \mathcal{K}_h} \delta_K^{-1} \| u - \Pi_h u \|^2_K \leq C_{0,s}^2 C_u \max_{K ∈ \mathcal{K}_h} \frac{h_K}{δ_K} \sum_{K ∈ \mathcal{K}_h} h_K^{2(s-1)} \sum_{i=0}^{s} c_K^2 |u|_{H^i(K)}. \]
By accounting $\theta_K \leq \frac{h_K}{d C_{int}^2}$ and (3.11), the second term is bounded as follows

$$\sum_{K \in K_h} \delta_K \|\Delta x (u - \Pi_h u)\|^2_K \leq \sum_{K \in K_h} \frac{h_K}{d C_{int}^2} h_K d \|u - \Pi_h u\|^2_{H^2(K)}$$

$$\leq C^2_{2,s} \sum_{K \in K_h} C_{int}^{-2} h_K^2 c_K^{-4} h_K^{2(s-2)} \sum_{i=0}^s C^2_K \|u\|^2_{H^s(K)}$$

$$\leq C^2_{2,s} \max_{K \in K_h} \left\{ C_{int}^{-2} c_K^{-4} \right\} \sum_{K \in K_h} h_K^{2(s-1)} \sum_{i=0}^s C^2_K \|u\|^2_{H^s(K)}.$$

Thus, we obtain

$$\|u - \Pi_h u\|^2_{loc,h,+} \leq C_2 \max_{K \in K_h} \frac{h_K}{\theta_K} \sum_{K \in K_h} h_K^{2(s-1)} \sum_{i=0}^s C^2_K \|u\|^2_{H^i(K)},$$

where

$$C_2 = \max_{K \in K_h} \left\{ C_{\Sigma_T} + C^2_{2,s} C_{int}^{-2} c_K^{-4} + C^2_{0,s} C_a \frac{h_K}{\theta_K} \right\},$$

(4.13)

where $C_{\Sigma_T}$ is defined in (4.11).

4.4 Consistency

Lemma 4.5. If the solution $u \in V_0^{1,0}$ of (2.2) also belongs to $V_0^{\Delta x,1}$, then it satisfies the consistency identity

$$a_{loc,h}(u, v_h) = \ell_{loc,h}(v_h), \quad v_h \in V_{0h}.$$  (4.14)

Proof. Since $u \in V_0^{\Delta x,1}$, by integration by parts in (2.2) w.r.t. to $x$ and $t$ as well as density arguments, we obtain $u_t - \Delta x u = f$ in $L_2(Q)$ and $u|_{\Sigma} = 0$. The consistency identity $a_{loc,h}(u, v_h) = \ell_{loc,h}(v_h), \quad v_h \in V_{0h}$ is derived along with the discrete space-time formulation (4.1).

4.5 A priori estimates of approximation errors

Lemma 4.6. Let $u \in V_0^{\Delta x,1}$ be an exact solution of (2.2), and $u_h \in V_{0h}$ (with $p \geq 2$) be an approximate solution generated by (4.1). Then, the best approximation estimate

$$\|u - u_h\|_{loc,h} \leq (1 + \frac{\mu_{loc,b}}{\mu_{loc,c}}) \inf_{v_h \in V_{0h}} \|u - v_h\|_{loc,h,+}$$

(4.15)

holds. Here, $\mu_{loc,c}$ and $\mu_{loc,b}$ are positive constants from Lemmas 4.1 and 4.3, respectively, that do not depend on $h_K$.
Proof. The Galerkin orthogonality
\[ a_{\text{loc},h}(u - u_h, v_h) = 0. \] (4.16)
follows from (4.14). Applying the triangle inequality, we estimate the discretisation error \( u - u_h \) as follows:
\[ \| u - u_h \|_{\text{loc},h} \leq \| u - \Pi_h u \|_{\text{loc},h} + \| \Pi_h u - u_h \|_{\text{loc},h}. \] (4.17)
The first term on the RHS of (4.17) can easily be estimated by means of Lemma 4.4. For the estimation of \( \| \Pi_h u - u_h \|_{\text{loc},h} \), we first use the \( V_0h \)-ellipticity of \( a_{\text{loc},h}(\cdot, \cdot) \) w.r.t. the norm \( \| \cdot \|_{\text{loc},h} \) (see Lemma 4.1), i.e.,
\[ \mu_{\text{loc},c} \| \Pi_h u - u_h \|_{\text{loc},h} \leq a_{\text{loc},h}(\Pi_h u - u_h, \Pi_h u - u_h). \]
Next, by means of the Galerkin orthogonality (4.16), we obtain
\[ \mu_{\text{loc},c} \| \Pi_h u - u_h \|_{\text{loc},h} \leq a_{\text{loc},h}(\Pi_h u - u_h, \Pi_h u - u_h) \]
\[ = a_{\text{loc},h}(\Pi_h u - u, \Pi_h u - u_h). \]
Finally, we apply Lemma 4.1 and obtain the estimate
\[ \mu_{\text{loc},c} \| \Pi_h u - u_h \|_{\text{loc},h} \leq \mu_{\text{loc},b} \| \Pi_h u - u \|_{\text{loc},h} \| \Pi_h u - u \|_{\text{loc},h}, \]
which automatically yields
\[ \| \Pi_h u - u_h \|_{\text{loc},h} \leq \frac{\mu_{\text{loc},b}}{\mu_{\text{loc},c}} \| \Pi_h u - u \|_{\text{loc},h}. \] (4.18)
Combining \( \| \Pi_h u - u \|_{\text{loc},h} \leq \| \Pi_h u - u \|_{\text{loc},h}, (4.18), \) and (4.17), we arrive at
\[ \| u - u_h \|_{\text{loc},h} \leq (1 + \frac{\mu_{\text{loc},b}}{\mu_{\text{loc},c}}) \| u - \Pi_h u \|_{\text{loc},h}. \]
\( \square \)

Theorem 4.7. Let \( p \geq 2, u \in V_0^s, s \geq 2 \), be an exact solution of (2.2), and \( u_h \in V_0h \) be an approximate solution of (4.1) with \( \theta_K \in \left(0, \frac{h_K}{d c^2 \text{int}_1}\right] \), \( K \in \mathcal{K}_h \). Then, the discretisation error estimate
\[ \| u - u_h \|_{\text{loc},h}^2 \leq C \sum_{K \in \mathcal{K}_h} h_K^{2(s-1)} \sum_{i=0}^{s} c_K^2 |u|_{H^i(K)}^2 \]
hold, where \( C = (1 + \frac{\mu_{\text{loc},b}}{\mu_{\text{loc},c}}) C_2 \) is a constant independent of \( h \), \( r = \min\{s, p + 1\} \), and \( p \) denotes the polynomial degree of the THB-splines, \( \mu_{\text{loc},b} \) and \( \mu_{\text{loc},c} \) are constant in boundedness (4.3) and coercivity (4.5) inequalities, respectively.

Proof. Application of estimate (4.7) yields
\[ \| u - u_h \|_{\text{loc},h} \leq (1 + \frac{\mu_{\text{loc},b}}{\mu_{\text{loc},c}}) C_2 \sum_{K \in \mathcal{K}_h} h_K^{2(s-1)} \sum_{i=0}^{s} c_K^2 |u|_{H^i(K)}^2, \]
where \( C_2 \) is defined in (4.13).
\( \square \)
5 A posteriori error estimates and numerical experiments

In this section, we discuss the implementation of the numerical scheme discussed above and the estimates used to control the quality of approximations. Numerical experiments present the error order of convergence (e.o.c.) in terms of the error norm (4.2). Also, we discuss computational properties of the majorants $\bar{M}_1$ and $\bar{M}_\parallel$ that follow from [54] and of the error identity $\bar{\eta}$ [2]. Moreover, we compare time expenditures that are required for getting approximations of the solution with the time spent for computing efficient error bounds.

Let $u_h$ denote an approximation of $u$. We assume that $u_h \in V_{0h} := V_h \cap V_{0,0}^{\Delta x,1}$ (cf. (3.4)), and define

$$u_h(x,t) = u_h(x_1, \ldots, x_{d+1}) = \sum_{i \in I} u_{h,i} \phi_{h,i}(x_1, \ldots, x_{d+1}),$$

where $u_{h,i} := [u_{h,i}]_{i \in I} \in \mathbb{R}^{|I|}$ contains free parameters to be defined (it is the vector of degrees of freedom (d.o.f.) or, in the IgA community, vector of control points). This vector is generated by the linear system

$$K_h u_h = f_h, \quad K_h := [a_{loc,h}(\phi_{h,i}, \phi_{h,j})]_{i,j \in I}, \quad f_h := [l_{loc,h}(\phi_{h,i})]_{i \in I}. \tag{5.1}$$

The system (5.1) is solved by means of the sparse direct LU factorisations. This choice of the solution method is motivated by our intention to provide a fair comparison of time expenditures used for solving the system generating $u_h$ and $y_h$ (for the majorant $\bar{M}_1$) as well as $w_h$ (for $\bar{M}_\parallel$). Due to properties of IgA approximations the condition $u_h \in C^{p-1}$ is automatically provided.

Approximation properties of $u_h$ are analysed by studying convergence of the error $e = u - u_h$ measured in terms of different norms. The first norm is defined in (4.2) and the second one is

$$\|e\|^2 := \|\nabla_x e\|_Q^2 + \|e\|_{T,h}^2.$$

The norm $\|e\|^2$ is controlled by the majorant (see, e.g., [54])

$$M_1(u_h, y_h) := (1 + \beta) \|y_h - \nabla_x u_h\|_Q^2 + (1 + \beta C^2_{\Phi}) \|\text{div}_x y_h + f - \partial_t u_h\|_Q^2$$

$$= (1 + \beta) m_{d,1}^{l,2} + (1 + \beta) C^2_{\Phi} m_{\text{eq}}^{l,2},$$

where $\beta > 0$ and $y_h \in Y_h \subset H^{\text{div},0}(\Omega)$. The space

$$Y_h \equiv S_h^q := \{\psi_{h,i} := \otimes^{d+1} S_h^{q,i} \circ \Phi^{-1}\}$$

is generated by the push-forward of $\otimes^{d+1} S_h^{qi}$, where $S_h^{qi}$ is the space of splines of the degree $q$ used to approximate components of $y_h = (y_h^{(1)}, \ldots, y_h^{(d+1)})^T$. The sharpest estimate is obtained by the minimisation of $\bar{M}_1(u_h, y_h)$ w.r.t.

$$y_h(x,t) = y_h(x_1, \ldots, x_{d+1}) = \sum_{i \in I^{d+1}} y_{h,i} \psi_{h,i}(x_1, \ldots, x_{d+1}).$$
Here, $y_h := [y_{h,i}]_{i \in I} \in \mathbb{R}^{(d+1)|I|}$ (i.e., it is a vector of the dimension $(d + 1)|I|$) is defined by the linear system

$$
\left( C_F^2 \text{Div}_h + \beta M_h \right) y_h = -C_F^2 z_h + \beta g_h,
$$

where

$$
\text{Div}_h := \left[ \left( \text{div} \psi_{h,i}, \text{div} \psi_{h,j} \right) Q \right]_{i,j=1}^{(d+1)|I|}, \quad z_h := \left[ \left( f - v_i, \text{div} \psi_{h,j} \right) Q \right]_{j=1}^{(d+1)|I|},
$$

$$
M_h := \left[ \left( \psi_{h,i}, \psi_{h,j} \right) Q \right]_{i,j=1}^{(d+1)|I|}, \quad g_h := \left[ \left( \nabla_x \psi_{h,j} \right) Q \right]_{j=1}^{(d+1)|I|}.
$$

The optimal value for $\beta$ reads as $\beta := C_F \tilde{m}_{eq}^h / \tilde{m}_{eq}^l$. According to numerical results obtained in [34, 47, 43], the most efficient majorant reconstruction is obtained with spline degree $q \gg p$. At the same time, the approximation $u_h$ is reconstructed on the mesh $K_h$, whereas a coarser mesh $K_{Mh}$, $M \in \mathbb{N}^+$, is used to recover $y_h$. This helps to minimise the number of d.o.f. for the latter one. The initial mesh $K_h^0$ and corresponding basis functions are assumed to be given via the geometry representation of the computational domain. Throughout the set of numerical examples, $K_h^0$ are generated by $N_{ref,0}$ initial uniform refinements before actual testing. In our implementation, (5.2) is solved by the sparse direct LDL$^T$ Cholesky factorisations.

In addition to $\tilde{M}^l$, [54] provides an advanced form of the majorant $\tilde{M}^l(u_h, y_h, w_h)$, i.e.,

$$
\| \nabla_x e \|^2_Q \leq \tilde{M}^l(u_h, w_h) \quad := \| w_h - u_h \|^2_{\Sigma^l} + 2 F(u_h, w_h) + (1 + \beta) \| r^l \|^2_Q + C_F^2 (1 + \frac{1}{F}) \| r_{eq} \|^2_Q,
$$

where

$$
F(u_h, w_h) := (\nabla_x u_h, \nabla_x (w_h - u_h)) + (\partial_t u_h - f, w_h - u_h),
$$

$$
r^l(u_h, y_h, w_h) := y_h + \nabla_x w_h - 2 \nabla_x u_h, \quad \text{and}
$$

$$
r_{eq}(y_h, w_h) := \text{div}_x y_h + f - \partial_t w_h.
$$

Here, $w_h$ is the solution to (4.1) on the approximation space

$$
W_{0h} := W_h \cap V_0^{\lambda_{x-1}}, \quad \text{with} \quad W_h \equiv S^r_h := \left\{ \chi_{h,i} := \hat{S}^r_h \circ \chi^{-1} \right\},
$$

where $\hat{S}^r_h$ is the space of degree $r$. The function $w_h$ can be represented by

$$
w_h(x,t) = w_h(x_1, \ldots, x_{d+1}) := \sum_{i \in I} w_{h,i} \chi_{h,i}.
$$

Here, $w_h := [w_{h,i}]_{i \in I} \in \mathbb{R}^{|I|}$ is the vector of control points of $w_h$ defined by the linear system $K_h^{(r)} w_h = f^{(r)}$, where $K_h^{(r)} := [a_{loc,h}(\chi_{h,i}, \chi_{h,j})]_{i,j \in I}$.
Adaptive Space-Time IgA of Parabolic Evolution Problems

\[ f_h^{(r)} := \left[ I_{loc,h}(\chi_{h,i}) \right]_{i \in \mathcal{I}}. \]

Since \( \partial_t w_h \) is approximated by a richer space, the term 
\[ \| r_{eq}^I(y_h, w_h) \|^2_Q \]

is expected to be smaller than 
\[ \| r_{eq}^I(y_h, u_h) \|^2_Q. \]

Therefore, the value of the error bound \( M_{II} \) must be improved. The optimal parameter \( \beta \) is calculated by

\[ \beta := C_{II} \| r_{eq}^I \|_Q / \| r_{eq}^I \|_Q. \]

The last error norm is generated by the solution operator \( \mathcal{L} := \partial_t - \Delta_x \), i.e.,

\[ \| e \|_L^2 := \| \Delta_x e \|^2_Q + \| \partial_t e \|^2_Q + \| \nabla_x e \|^2_{\Sigma_T}. \]

It is controlled by the error identity \[2\]

\[ \mathcal{E}_{II}^2(u_h) := \| \nabla_x (u_0 - u_h) \|^2_{\Sigma_0} + \| \Delta_x u_h + f - \partial_t u_h \|^2_Q. \]

Marking of the elements in \( \mathcal{K}_h \) is driven by the bulk marking criterion (also known as Dörfler’s marking \[12\]) denoted by \( M_{\text{BULK}}(\sigma), \sigma \in [0,1] \). Finally, the effectiveness of the error estimators is evaluated by efficiency indices, i.e.,

\[ I_{\text{eff}}(M_{\text{II}}) := \frac{M_{\text{II}}}{\| e \|_Q}, \quad I_{\text{eff}}(\mathcal{E}_{II}) := \frac{\mathcal{E}_{II}}{\| e \|_L} = 1. \]

Below we study the behaviour of the above discussed error control tools within a series of benchmark examples. We begin with a rather simple example, which is intended to demonstrate important properties of the numerical scheme. More complicated problems with non-trivial geometries and singular solutions are considered at the end of the section. The implementation was carried out using the open-source C++ library G+Smo \[25\].

### 5.1 Example 1: polynomial solution

First, we consider a simple example, where the solution of (2.1) is a polynomial function

\[ u(x, t) = (1 - x) x^2 (1 - t) t, \quad (x, t) \in \overline{Q} := [0, 1]^2, \]

and generated by its RHS

\[ f(x, t) = -(1 - x) x^2 (1 - 2 t) - (2 - 6 x) (1 - t)^2, \quad (x, t) \in Q := (0, 1)^2. \]

\( u(x, t) \) satisfies homogeneous Dirichlet boundary and initial conditions on \( \Sigma := \partial \Omega \times (0, 1) \) and \( \Sigma_0 \), respectively.

The initial mesh is obtained by one global refinement \( (N_{\text{ref}} = 1) \). Further, refinements are done with eight steps (hence \( N_{\text{ref}} = 8 \)). The approximation space for \( u_h \) is \( S_h^2 \). For the auxiliary functions, we assume that \( y_h \in \oplus^2 S_{3h}^3 \), and \( w_h \in S_{3h}^3 \). Such a choice of discretisation spaces saves computational efforts in reconstruction of the error estimates considerably. Table 2 illustrates the ratio between the time spent for approximating \( u_h \) to the time spent for its error estimation, i.e., \( t_{\text{appr.}} / t_{\text{est.}} \), along with total time needed for assembling and solving systems generating d.o.f. of \( u_h, y_h, \) and
Table 1: Example 1. Efficiency of \( \overline{M}^1 \), \( \overline{M}^l \), and \( \mathcal{K} \) and for \( u_h \in \mathcal{S}_h^2 \), \( y_h \in \oplus^2 \mathcal{S}_h^3 \), and \( w_h \in \mathcal{S}_h^3 \), and order of convergence for \( \| e \|_{loc,h} \) and \( \| e \|_{L} \) (\( N_{ref,0} = 1 \)).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\# ref. & \| \nabla \chi \|_Q & \| e \|_{loc,h} \|_{\overline{M}^1} & \| e \|_{loc,h} \|_{\overline{M}^l} & \| e \|_L & \text{e.o.c.} (\| e \|_{loc,h}) & \text{e.o.c.} (\| e \|_L) \\
\hline
(a) \sigma = 0.4 & & & & & & \\
2 & 2.5516e-03 & 1.07 & 1.03 & 2.5520e-03 & 7.9057e-02 & 1.00 & 3.43 & 1.71 \\
4 & 2.2743e-04 & 1.41 & 1.19 & 2.2745e-04 & 2.1712e-02 & 1.00 & 2.36 & 1.37 \\
6 & 2.9936e-05 & 1.09 & 1.02 & 2.9936e-05 & 7.9512e-03 & 1.00 & 2.71 & 1.28 \\
8 & 4.9501e-06 & 1.12 & 1.05 & 4.9501e-06 & 3.1138e-03 & 1.00 & 1.51 & 0.93 \\
\hline
(b) \sigma = 0.6 & & & & & & \\
2 & 2.5516e-03 & 1.07 & 1.03 & 2.5520e-03 & 7.9057e-02 & 1.00 & 3.43 & 1.71 \\
4 & 3.3298e-04 & 1.30 & 1.11 & 3.3305e-04 & 2.5410e-02 & 1.00 & 1.80 & 1.22 \\
6 & 5.9048e-05 & 1.34 & 1.14 & 5.9050e-05 & 1.0976e-02 & 1.00 & 3.18 & 1.60 \\
8 & 2.3071e-05 & 1.25 & 1.10 & 2.3072e-05 & 6.7335e-03 & 1.00 & 2.06 & 1.41 \\
\hline
\end{array}
\]

5.2 Example 2: parameterized solution

Next, we discuss an example with the parameterized exact solution. Let \( Q = (0, 1)^2 \) be a unit square, and let the exact solution, the RHS, and the Dirichlet boundary condition be chosen as follows:

\[
\begin{align*}
  u(x, t) &= \sin k_1 \pi x \sin k_2 \pi t \quad (x, t) \in \overline{Q} = [0, 1]^2, \\
  f(x, t) &= \sin k_1 \pi x \left( k_2 \pi \cos k_2 \pi t + k_1^2 \pi^2 \sin k_2 \pi t \right) \quad (x, t) \in Q = (0, 1)^2, \\
  u_0(x, t) &= 0, \quad (x, t) \in \Sigma_0, \\
  u_D(x, t) &= 0, \quad (x, t) \in \Sigma := \partial \Omega \times (0, 1).
\end{align*}
\]

First, we set \( k_1 = k_2 = 1 \) (Example 2-1). We consider eight adaptive refinement steps (\( N_{ref} = 8 \)) preceded by three global refinements (\( N_{ref,0} = 3 \)) to generate the initial mesh.
Adaptive Space-Time IgA of Parabolic Evolution Problems

21
d.o.f. $u_h$ $y_h$ $w_h$ $t_{as}$ $t_{sol}$ $t_{appr.}$ $t_{ref.est.}$

<table>
<thead>
<tr>
<th># ref.</th>
<th>$u_h$</th>
<th>$y_h$</th>
<th>$w_h$</th>
<th>$t_{as}$</th>
<th>$t_{sol}$</th>
<th>$t_{appr.}$</th>
<th>$t_{ref.est.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>240</td>
<td>50</td>
<td>25</td>
<td>2.64e-01</td>
<td>1.44e-02</td>
<td>1.08e-02</td>
<td>4.01e-03</td>
</tr>
<tr>
<td>6</td>
<td>2027</td>
<td>50</td>
<td>25</td>
<td>2.42e+00</td>
<td>1.81e-02</td>
<td>1.61e-02</td>
<td>1.94e-01</td>
</tr>
<tr>
<td>8</td>
<td>11512</td>
<td>152</td>
<td>76</td>
<td>1.35e+01</td>
<td>1.97e-01</td>
<td>1.68e-01</td>
<td>3.39e+00</td>
</tr>
<tr>
<td>(a) $\sigma = 0.4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>206</td>
<td>50</td>
<td>25</td>
<td>2.27e-01</td>
<td>1.94e-02</td>
<td>1.54e-02</td>
<td>3.35e-03</td>
</tr>
<tr>
<td>6</td>
<td>896</td>
<td>50</td>
<td>25</td>
<td>1.07e+00</td>
<td>1.20e-02</td>
<td>1.96e-02</td>
<td>4.17e-02</td>
</tr>
<tr>
<td>8</td>
<td>2706</td>
<td>158</td>
<td>79</td>
<td>3.44e+00</td>
<td>1.65e-01</td>
<td>1.37e-01</td>
<td>2.69e-01</td>
</tr>
<tr>
<td>(b) $\sigma = 0.6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>25.20</td>
<td>1.21</td>
<td>1.00</td>
<td>367.33</td>
<td>0.92</td>
<td>1.00</td>
<td>12.25</td>
</tr>
</tbody>
</table>

Table 2: Example 1. Assembling and solving time (in seconds) spent for the systems defining d.o.f. of $u_h \in S^2_h$, $y_h \in \oplus^2 S^3_{5h}$, and $w_h \in S^3_{5h}$ ($N_{ref,0} = 1$).

Figure 1: Example 1. Comparison of the error and majorant $\overline{M}^1$ order of convergence for $y_h \in \oplus^2 S^3_{7h}$ and for $y_h \in \oplus^2 S^3_{h}$.

For the marking criterion, we chose bulk parameter $\sigma = 0.4$. The function $u_h$ is approximated both by $S^2_h$ (case (a)) and $S^3_h$ (case (b)) spaces, whereas corresponding auxiliary functions by $y_h \in \oplus^2 S^4_{7h}$ and $w_h \in S^4_{7h}$ as well as $y_h \in \oplus^2 S^6_{5h}$ and $w_h \in S^6_{5h}$, respectively, see Tables 3–4. Figure 2a illustrates different error orders of convergence for different approximations $u_h \in S^2_h$ and $u_h \in S^3_h$, which perform slightly better than expected rates $O(h^2)$ and $O(h^3)$, respectively.
U. Langer, S. Matculevich and S. Repin

Figure 2: Example 2. The error order of convergence for (a) $k_1 = k_2 = 1$ and (b) $k_1 = 6$, $k_2 = 3$.

(a) $k_1 = k_2 = 1$

(b) $k_1 = 6$, $k_2 = 3$

Table 3: Example 2-1. Efficiency of $\mathcal{M}_I$, $\mathcal{M}_I^I$, $\mathcal{M}_h$, $\mathcal{E}_l$, and order of convergence of $\|e\|_{loc,h}$ and $\|e\|_L$ for $\sigma = 0.4$ ($N_{ref,0} = 3$).

We also demonstrate the quantitative effectiveness of the error indication provided by $\mathcal{M}_I$. In Figure 3, the comparison of the meshes illustrates that the refinement based on local values of $\|\nabla^x e\|_K$ (first row) and the indicator $\mathcal{m}_{d,K}$ (second row) provide similar adaptive patterns.

Next, we set parameters $k_1 = 3$ and $k_2 = 6$. In this case, auxiliary variables are approximated by $y_h \in \oplus^2 S^7_{5h}$ and $w_h \in S^7_{5h}$. Figure 2b illustrates the order of convergence of errors and corresponding majorants (for two different marking strategy $\mathcal{M}_{BULK}(0.4)$ and $\mathcal{M}_{BULK}(0.6)$) and compares these results to the theoretical one $O(h^2)$. It is easy to see from the plot that efficiency of the majorant deteriorates on the
Adaptive Space-Time IgA of Parabolic Evolution Problems

Table 4: Example 2-1. Assembling and solving time (in seconds) spent for the systems generating d.o.f. of $u_h$, $y_h$, and $w_h$ for $\sigma = 0.4$ ($N_{\text{ref,0}} = 3$).

<table>
<thead>
<tr>
<th># ref.</th>
<th>$u_h$</th>
<th>$y_h$</th>
<th>$w_h$</th>
<th>$t_{\text{as}}$</th>
<th>$t_{\text{sol}}$</th>
<th>$t_{\text{app.}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5695</td>
<td>288</td>
<td>144</td>
<td>7.89e+00</td>
<td>3.84e-01</td>
<td>2.36e-03</td>
</tr>
<tr>
<td>6</td>
<td>12935</td>
<td>288</td>
<td>144</td>
<td>1.55e+00</td>
<td>3.83e-01</td>
<td>2.30e-03</td>
</tr>
<tr>
<td>7</td>
<td>34037</td>
<td>288</td>
<td>144</td>
<td>4.90e+00</td>
<td>3.73e-01</td>
<td>3.36e-03</td>
</tr>
<tr>
<td>8</td>
<td>61258</td>
<td>288</td>
<td>144</td>
<td>9.37e+00</td>
<td>3.62e-01</td>
<td>1.83e-03</td>
</tr>
</tbody>
</table>

Table 5: Example 2-2. Efficiency of $\overline{M}$, $\overline{M}_h$, and the order of convergence of $\|e\|_{L}$ for $u_h \in S_h^2$, $y_h \in \oplus^2 S^7_{5h}$, and $w_h \in S^7_{5h}$ ($N_{\text{ref,0}} = 3$).

<table>
<thead>
<tr>
<th># ref.</th>
<th>$|\nabla x e|_Q$</th>
<th>$I_{\text{eff}}(\overline{M})$</th>
<th>$I_{\text{eff}}(\overline{M}_h)$</th>
<th>$|e|_{loc,h}$</th>
<th>$|e|_L$</th>
<th>$e_{\text{o.c.}}(|e|_{loc,h})$</th>
<th>$e_{\text{o.c.}}(|e|_L)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5.7161e-01</td>
<td>2.11</td>
<td>1.38</td>
<td>5.7163e-01</td>
<td>6.2371e+01</td>
<td>1.00</td>
<td>2.99</td>
</tr>
<tr>
<td>3</td>
<td>1.3927e-01</td>
<td>5.77</td>
<td>2.20</td>
<td>1.3928e-01</td>
<td>3.1026e+01</td>
<td>1.00</td>
<td>2.30</td>
</tr>
<tr>
<td>7</td>
<td>4.8735e-03</td>
<td>1.43</td>
<td>1.15</td>
<td>4.8736e-03</td>
<td>4.7350e+00</td>
<td>1.00</td>
<td>0.68</td>
</tr>
<tr>
<td>8</td>
<td>1.2298e-03</td>
<td>1.44</td>
<td>1.16</td>
<td>1.2298e-03</td>
<td>2.6917e+00</td>
<td>1.00</td>
<td>5.60</td>
</tr>
</tbody>
</table>

first refinement steps, but it improves drastically on the last refinements. Tables 5 and 6 compare numerical results obtained for different marking parameters $\sigma = 0.4$ (part (a)) and $\sigma = 0.6$ (part (b)). Finally, Figure 4 demonstrates the evolution of meshes associated with the refinement steps 4–6 for the same cases.
Figure 3: Example 2-1. Meshes obtained in the adaptive procedure based on indicator $\bar{m}_{d,K}$ (top row) and based on the exact error $\|\nabla_x e\|_K$ (bottom row) w.r.t. refinement steps 5–7.

5.3 Example 3: Gaussian distribution

As the next test case, we consider the exact solution defined by a sharp local Gaussian distribution $u(x, t) = (x^2 - x)(t^2 - t) e^{-100 |(x, t) - (0.8, 0.05)|}$, $(x, t) \in \bar{Q} := [0, 1]^2$, where the peak is located in the point $(x, t) = (0.8, 0.05)$. Then $f$ is computed by substituting $u$ into (2.1). The Dirichlet boundary condition is obviously homogeneous.

For the discretisation spaces, we use the standard configuration, i.e., $u_h \in S^2_h$ for the approximate solution, as well as $y_h \in \oplus^2 S^3_h$ and $w_h \in S^3_h$ for the auxiliary functions. We start with four initial global refinements ($N_{\text{ref},0} = 4$), and continue with seven adaptive steps ($N_{\text{ref}} = 7$). As the marking criteria, we choose $M_{\text{BULK}}(0.6)$. The error order of convergence is illustrated in Figure 5. It confirms that majorants reconstructed with $y_h \in \oplus^2 S^6_{2h}$, and $w_h \in S^6_{2h}$ are as efficient as the one reconstructed with $y_h \in \oplus^2 S^3_h$, and $w_h \in S^3_h$. They also drastically improve the convergence order on the first refinement steps.

Numbers exposed in Table 7 demonstrate the efficiency of the majorants and the error identity in terms of error estimation and show that $\bar{M}^\parallel$ is twice sharper than $\bar{M}^\perp$. 


Adaptive Space-Time IgA of Parabolic Evolution Problems

### Table 6: Example 2-2

<table>
<thead>
<tr>
<th># ref.</th>
<th>u_h</th>
<th>y_h</th>
<th>w_h</th>
<th>d.o.f.</th>
<th>t_as</th>
<th>t_sol</th>
<th>t_app.</th>
<th>t_est.</th>
<th>e.o.c.</th>
<th>e.o.c.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>u_h</td>
<td>y_h</td>
<td>w_h</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>11426</td>
<td>450</td>
<td>225</td>
<td>1.50e+01</td>
<td>2.28e+00</td>
<td>2.92e+00</td>
<td>1.20e+00</td>
<td>7.89e-03</td>
<td>3.37e-03</td>
<td>3.11</td>
</tr>
<tr>
<td>6</td>
<td>30101</td>
<td>450</td>
<td>225</td>
<td>5.99e+01</td>
<td>2.29e+00</td>
<td>2.92e+00</td>
<td>3.57e+00</td>
<td>8.52e-03</td>
<td>4.33e-03</td>
<td>12.14</td>
</tr>
<tr>
<td>7</td>
<td>86849</td>
<td>1058</td>
<td>529</td>
<td>3.57e+02</td>
<td>9.30e+00</td>
<td>9.41e+00</td>
<td>1.11e+01</td>
<td>5.19e-02</td>
<td>3.47e-02</td>
<td>19.58</td>
</tr>
<tr>
<td>8</td>
<td>141987</td>
<td>2850</td>
<td>1425</td>
<td>6.36e+02</td>
<td>6.50e+01</td>
<td>5.91e+01</td>
<td>2.56e+01</td>
<td>3.00e-01</td>
<td>1.29e-01</td>
<td>5.31</td>
</tr>
</tbody>
</table>

### Table 7: Example 3

<table>
<thead>
<tr>
<th># ref.</th>
<th>∥∇_x e∥_Q</th>
<th>J_{eff}(M^1)</th>
<th>J_{eff}(M^2)</th>
<th>∥e∥_{loc,h}</th>
<th>∥e∥_L</th>
<th>I_{eff}(Id)</th>
<th>e.o.c.</th>
<th>e.o.c.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.1311e-04</td>
<td>2.85</td>
<td>1.55</td>
<td>3.1335e-04</td>
<td>5.6510e-02</td>
<td>1.00</td>
<td>17.71</td>
<td>8.64</td>
</tr>
<tr>
<td>3</td>
<td>1.0915e-04</td>
<td>3.93</td>
<td>1.73</td>
<td>1.0944e-04</td>
<td>3.1506e-02</td>
<td>1.00</td>
<td>6.49</td>
<td>3.60</td>
</tr>
<tr>
<td>5</td>
<td>2.2033e-05</td>
<td>2.22</td>
<td>1.36</td>
<td>2.2042e-05</td>
<td>1.4796e-02</td>
<td>1.00</td>
<td>5.87</td>
<td>3.59</td>
</tr>
<tr>
<td>7</td>
<td>5.2517e-06</td>
<td>2.38</td>
<td>1.22</td>
<td>5.2526e-06</td>
<td>7.2473e-03</td>
<td>1.00</td>
<td>2.41</td>
<td>1.27</td>
</tr>
</tbody>
</table>

whereas the error identity, as expected, reflects the error ∥e∥_L exactly. In Table 8, we see that the assembling of matrices for the y_h and w_h is 3 times more time-consuming in comparison to assembling the system for u_h.

### 5.4 Example 4: solution with singularity w.r.t. t-coordinate

For Example 4, we consider the solution with the singularity w.r.t. time coordinate, i.e., we take

\[ u(x, t) = \sin \pi x (1 - t)^\lambda, \quad (x, t) \in Q = (0, 1) \times (0, 2), \]

where parameter \( \lambda = \{ 3/2, 1, 1/2 \} \) (see Figure 7 with u for different \( \lambda \)). The RHS \( f(x, t) \)
Figure 4: Example 2-2. Meshes obtained by marking criteria $M_{\text{BULK}}(0.6)$ (top row) and $M_{\text{BULK}}(\sigma = 0.4)$ (bottom row) w.r.t. the refinement steps 4–6.

Figure 5: Example 3. The majorant and e.o.c. for $u_h \in S_h^2$ and two different setting of auxiliary functions (a) $y_h \in \bigoplus S_h^3$, and $w_h \in S_h^3$ and (b) $y_h \in \bigoplus S_{2h}^6$, and $w_h \in S_{2h}^6$.

follows from the substitution of $u$ into (2.1), and the Dirichlet boundary condition is defined as $u_D = u$ on $\Sigma$. 

Adaptive Space-Time IgA of Parabolic Evolution Problems

Table 8: Example 3. Assembling and solving time (in seconds) spent for the systems generating d.o.f. of $u_h \in S_2^h$, $y_h \in \oplus^2 S_3^h$, and $w_h \in S_3^h$ for the bulk marking parameter $\sigma = 0.6$ for ($N_{ref,0} = 4$).

<table>
<thead>
<tr>
<th># ref</th>
<th>$u_h$</th>
<th>$y_h$</th>
<th>$w_h$</th>
<th>$u_h$</th>
<th>$y_h$</th>
<th>$w_h$</th>
<th>$u_h$</th>
<th>$y_h$</th>
<th>$w_h$</th>
<th>$t_{as}$</th>
<th>$t_{sol}$</th>
<th>$t_{appr.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>520</td>
<td>1088</td>
<td>544</td>
<td>6.50e-01</td>
<td>2.39e+00</td>
<td>1.80e+00</td>
<td>1.09e-02</td>
<td>8.60e-03</td>
<td>1.91e-02</td>
<td>0.27</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1232</td>
<td>2474</td>
<td>1237</td>
<td>1.72e+00</td>
<td>6.26e+00</td>
<td>4.71e+00</td>
<td>6.12e-02</td>
<td>8.25e-02</td>
<td>1.26e-01</td>
<td>0.28</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4368</td>
<td>8492</td>
<td>4246</td>
<td>6.20e+00</td>
<td>3.04e+01</td>
<td>1.87e+01</td>
<td>6.38e-01</td>
<td>6.05e-01</td>
<td>5.26e-01</td>
<td>0.22</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 6: Example 3. Meshes obtained on the refinement steps 3–6, $\sigma = 0.6$ ($N_{ref,0} = 4$) for $u_h \in S_2^h$, $y_h \in \oplus^2 S_3^h$, and $w_h \in S_3^h$.

Figure 7: Example 4. Exact solution $u(x, t) = \sin \pi x (1 - t)^{\lambda}$.

The solution $u(x, t)$ is smooth w.r.t. to spatial coordinates, while the regularity in time depends on the parameter $\lambda$. In particular, $p$ satisfies the following inequality
\[ \lambda = p - \frac{1}{2} + \varepsilon, \text{ where } \varepsilon > 0 \text{ is a small number.} \]

Then, the expected convergence in the term \( h^{1/2} \| \partial_t (u - u_h) \|_Q \) is \( O(h^{p-1}) \cdot O(h^{1/2}) \) (see [28]).

Theoretical convergence for each \( \lambda \) were tested in [43] (for \( p = 2 \)). Table 9 illustrates an improved error order of convergence for \( \lambda = \left\{ \frac{1}{2}, 1, \frac{3}{2} \right\} \). The same behaviour can be observed from Figure 7 for different parameters. It presents meshes obtained on the adaptive refinement steps 5–7 and re-confirms that functional error estimates detect the local singularities rather efficiently. For the case \( \lambda = \frac{1}{2} \), we leave the class of solutions \( V^{\Delta_{x_{1}}}:= \{ u \in V^{\Delta_{x_{1}}}: u = u_D \text{ on } \Sigma \} \), and, as consequence, are not able to recover the theoretical error order of convergence, since \( \| \partial_t u \|_{L^2} \) explodes. Nevertheless, the singularity at \( t = 1 \) is captured and very well represented by the error indicator and resulting adaptive mesh. Moreover, for the rest of the times, i.e., \( (0, 1) \cup (1, 2) \), where the solution is smooth, the mesh is not over-refined.

### 5.5 Example 5: quarter-annulus domain extended in time

In the last example, we test the problem defined in the three-dimensional space-time cylinder \( Q = \Omega_{\infty} \times (0, T) \), where \( \Omega_{\infty} \) is represented by a quarter-annulus, which extended form \( t = 0 \) till \( t = T = 1 \). The exact solution is defined by

\[ u(x, y, t) = (1 - x) x^2 (1 - y) y^2 (1 - t) t^2, \quad (x, y, t) \in \overline{Q} := \overline{\Omega_{\infty}} \times [0, 1]. \]

The RHS \( f(x, y, t), (x, y, t) \in Q := \Omega_{\infty} \times (0, 1) \), is computed based on the substitution of \( u \) into the equation (2.1) and the Dirichlet boundary condition is defined as \( u_D = u \) on \( \Sigma \).

The initial mesh for the test is generated by one uniform refinement \( N_{\text{ref}, 0} = 1 \), the bulk marking parameter is set to \( \sigma = 0.4 \). The error order of convergence is illustrated in Figure 12, which corresponds to the theoretical expectation. We start the analysis from Table 10. It is easy to see that all majorants have adequate performance, taking
Adaptive Space-Time IGa of Parabolic Evolution Problems

<table>
<thead>
<tr>
<th># ref.</th>
<th>( | \nabla_x e |_Q )</th>
<th>( I_{\text{eff}}(\mathcal{M}) )</th>
<th>( I_{\text{eff}}(\mathcal{M}^1) )</th>
<th>( | e |_{L^2} )</th>
<th>e.o.c. (( | e |_{L^2} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( \lambda = \frac{1}{2} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( u_h \in S_h^2, y_h \in \oplus^2 S_h^3, ) and ( w_h \in S_h^3 ), theoretical e.o.c. ( O(h^{1/2}) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5.7560e-02</td>
<td>5.94</td>
<td>1.42</td>
<td>5.8807e-02</td>
<td>1.32</td>
</tr>
<tr>
<td>7</td>
<td>4.0317e-02</td>
<td>8.66</td>
<td>1.70</td>
<td>4.0749e-02</td>
<td>1.23</td>
</tr>
<tr>
<td>8</td>
<td>3.2498e-02</td>
<td>10.80</td>
<td>1.94</td>
<td>3.2703e-02</td>
<td>0.80</td>
</tr>
<tr>
<td>(b) ( \lambda = 1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( u_h \in S_h^3, y_h \in \oplus^2 S_h^3, ) and ( w_h \in S_h^3 ), theoretical e.o.c. ( O(h) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3.0549e-02</td>
<td>9.78</td>
<td>1.83</td>
<td>3.3397e-02</td>
<td>1.37</td>
</tr>
<tr>
<td>7</td>
<td>2.8038e-02</td>
<td>10.51</td>
<td>1.85</td>
<td>2.9693e-02</td>
<td>1.23</td>
</tr>
<tr>
<td>9</td>
<td>1.8447e-02</td>
<td>15.74</td>
<td>2.17</td>
<td>1.8700e-02</td>
<td>1.37</td>
</tr>
<tr>
<td>(c) ( \lambda = \frac{3}{2} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( u_h \in S_h^3, y_h \in \oplus^2 S_h^3, ) and ( w_h \in S_h^3 ), theoretical e.o.c. ( O(h^{3/2}) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.1955e-02</td>
<td>4.47</td>
<td>1.44</td>
<td>1.2104e-02</td>
<td>1.85</td>
</tr>
<tr>
<td>7</td>
<td>5.3797e-03</td>
<td>6.84</td>
<td>1.70</td>
<td>5.4167e-03</td>
<td>1.83</td>
</tr>
<tr>
<td>8</td>
<td>2.9478e-03</td>
<td>9.37</td>
<td>2.01</td>
<td>2.9602e-03</td>
<td>1.73</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># ref.</th>
<th>( | e |_{loc,h} )</th>
<th>e.o.c.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( \lambda = \frac{1}{2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5.8807e-02</td>
<td>( O(h^{1/2}) )</td>
</tr>
<tr>
<td>7</td>
<td>4.0749e-02</td>
<td>( O(h^{1/2}) )</td>
</tr>
<tr>
<td>8</td>
<td>3.2703e-02</td>
<td>( O(h^{1/2}) )</td>
</tr>
<tr>
<td>(b) ( \lambda = 1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3.3397e-02</td>
<td>( O(h) )</td>
</tr>
<tr>
<td>7</td>
<td>2.9693e-02</td>
<td>( O(h) )</td>
</tr>
<tr>
<td>9</td>
<td>1.8700e-02</td>
<td>( O(h) )</td>
</tr>
<tr>
<td>(c) ( \lambda = \frac{3}{2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.2104e-02</td>
<td>( O(h^{3/2}) )</td>
</tr>
<tr>
<td>7</td>
<td>5.4167e-03</td>
<td>( O(h^{3/2}) )</td>
</tr>
<tr>
<td>8</td>
<td>2.9602e-03</td>
<td>( O(h^{3/2}) )</td>
</tr>
</tbody>
</table>

Table 9: Example 4. Efficiency of \( \mathcal{M}^1 \), \( \mathcal{M}^2 \), \( \mathcal{M}^3 \), and order of convergence of \( \| e \|_{loc,h} \) and \( \| e \|_L \) for \( \sigma = 0.4 \) \((N_{\text{ref},0} = 1)\).

into account that the auxiliary functions \( y_h \in \oplus^2 S_{3h}^5 \) and \( w_h \in S_{3h}^5 \). Table 11 confirms that assembling and solving of the systems reconstructing d.o.f. of \( u_h \) requires more time than assembling and solving routines for the systems generating \( y_h \) and \( w_h \).

Figure 13 presents an evolution of the adaptive meshes discretising expanded in time quarter-annulus \( Q \). From the plots presented, we can see that the refinement is
Figure 9: Example 3 (case (a): $\lambda = \frac{1}{2}$). Meshes obtained on the refinement steps 5–7 for $u_h \in S^2_h$, $y_h \in \oplus_2 S^3_h$, and $w_h \in S^3_h$.

Figure 10: Example 3 (case (b): $\lambda = 1$). Meshes obtained on the refinement steps 5–7 for $u_h \in S^2_h$, $y_h \in \oplus_2 S^3_h$, and $w_h \in S^3_h$. 

(a) ref. 5  
(b) ref. 6  
(c) ref. 7
Figure 11: Example 3 (case (c): $\lambda = \frac{3}{2}$). Meshes obtained on the refinement steps 5–7 for $u_h \in S_h^2$, $y_h \in \oplus S_h^3$, and $w_h \in S_h^3$.

Figure 12: Example 5. The error order of convergence for $u \in S_h^2$.

localised in the area close to the lateral surface of the quarter-annulus with the radius two. This happens due to fast changes in the solution appearing close to this ‘outer’ surface.

6 Conclusions

We derived a new locally stabilized space-time IgA schemes for parabolic I-BVPs, where global scaling $h$ in the upwind test functions is replaced by a local scaling that
Table 10: Example 5. Efficiency of $\overline{M}_I$, $\overline{M}_{II}$, and $\mathbb{K}_I$ for the bulk marking parameter $\sigma = 0.4$ for $u_h \in S_{3h}^2$, $y_h \in \oplus S_{3h}^1$, and $w_h \in S_{3h}^5$ ($N_{ref,0} = 4$).

Table 11: Example 5. Assembling and solving time (in seconds) spent for the systems generating d.o.f. of $u_h$, $y_h$, and $w_h$ with bulk parameter $\sigma = 0.4$ ($N_{ref,0} = 4$).

Figure 13: Example 5. Mesh evolution for refinement steps 1–3 for marking parameter $\sigma = 0.6$. 

depends on the local element size $h_K$. Adaptive mesh refinement is based on error indicators generated by functional type a posteriori error estimates, which naturally use specific features and advantages of the IgA method. Since error majorants of the functional type are presented by integrals formed by element-wise contributions, they can efficiently be used for indication of the local errors and subsequent mesh refinement.
Adaptive Space-Time IgA of Parabolic Evolution Problems

We consider a fully unstructured space-time adaptive IgA scheme and use localised THB-splines for the mesh refinement. Finally, we illustrated the reliability and efficiency of the presented a posteriori error estimates in a series of examples exhibiting different features of exact solutions. Numerical tests performed have demonstrated high efficiency of the approach. Moreover, we also made a comparative study of the computational expenses for assembling the systems, finding an approximate solution, and computing a guaranteed and sufficiently accurate error bounds. In the majority of examples, error estimation required much lesser time than the reconstruction of the approximate IgA solution. The last but not least item to be mentioned is that the numerical examples have confirmed high efficiency of the locally stabilized space-time THB-spline-based methods used in combination with suitable error indicators and mesh adaptive procedures. Of course, beside THB-spline, other local spline refinement techniques such as mention in the introduction can also be utilized in this adaptive framework. Adaptive methods should be connected with multigrid or multilevel solvers or preconditioners for the algebraic systems that we have to solve since the adaptive procedure naturally provides a space-time hierarchy of meshes. Preceding experiments on massively parallel computers presented in [44] show that even algebraic multigrid preconditioners in connection with GMRES accelerations result in very efficient solvers for very huge systems with billions of space-time unknowns arising from (3+1)d examples. It is clear that the approach presented can be extended to a wider class of parabolic problems and eddy current problems in electromagnetics.

Acknowledgments. We would like to thank A. Mantzaflaris for his permanent support of the open-source C++ library G+Smo [25] that was used to implement the adaptive space-time IgA schemes and to perform all the numeral tests presented in this work.

Bibliography


**Author information**

Ulrich Langer, RICAM Linz, Johann Radon Institute, Linz, Austria.
E-mail: ulrich.langer@ricam.oeaw.ac.at
Svetlana Matculevich, RICAM Linz, Johann Radon Institute, Linz, Austria.
E-mail: smatculevich@ricam.oead.ac.at

Sergey Repin, University of Jyvaskyla, Jyvaskyla; Peter the Great St.Petersburg Polytechnic University, Polytechnicheskaya, 29, St.Petersburg, Finland; Russia.
E-mail: serepin@jyu.fi