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NOTES ON TRACTABILITY CONDITIONS FOR LINEAR MULTIVARIATE PROBLEMS

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Abstract. We study approximations of compact linear multivariate operators defined over Hilbert spaces. We provide necessary and sufficient conditions on various notions of tractability. These conditions are mainly given in terms of sums of certain functions depending on the singular values of the multivariate problem. They do not require the ordering of these singular values which in many cases is difficult to achieve.

1. Introduction

Tractability of multivariate problems has become a popular research subject in the last 25 years. In this paper we study tractability in the worst case setting and for algorithms that use finitely many arbitrary continuous linear functionals. The information complexity of a d-variate compact linear operator S_d is defined as the minimal number $n(\varepsilon, S_d)$ of such linear functionals which is needed to find an ε approximation. There are various notions of tractability which may be summarized by the algebraic and exponential cases. For the algebraic case, we want to verify that the information complexity $n(\varepsilon, S_d)$ is bounded by certain functions of d and $\max(1, \varepsilon^{-1})$ which are, in particular, polynomial or *not* exponential in some powers of d and $\max(1, \varepsilon^{-1})$. For the exponential case, we replace the pair $(d, \max(1, \varepsilon^{-1}))$ by $(d, 1 + \ln \max(1, \varepsilon^{-1}))$, and consider the same notions of tractability as before.

The algebraic case has been studied in many papers, and necessary and sufficient conditions on various notions of tractability are known in terms of sums of the singular values of S_d. The exponential case has been studied in a relatively small number of papers, and the corresponding necessary and sufficient conditions on tractability are provided in this paper.

The information complexity requires to order the singular values of S_d. This is usually a difficult combinatorial problem. This problem is eliminated by the necessary and sufficient conditions on the singular values since they are given by sums which are invariant with respect to the ordering of the singular values.

For the reader's convenience we provide all conditions for both algebraic and exponential cases for such notions of tractability as strong polynomial, polynomial, quasi-polynomial, various weak tractabilities, and uniform weak tractability. Furthermore, we do this for the absolute and normalized error criteria. The results are presented in five tables.

In this paper we study general compact linear multivariate problems. In the next paper we illustrate the results of this paper for tensor product problems for which the singular values of a d-variate problem are given as products of the singular values of univariate problems.

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2. Preliminaries

Consider two sequences of Hilbert spaces $\{\mathcal{H}_d\}_{d\in\mathbb{N}}$ and $\{\mathcal{G}_d\}_{d\in\mathbb{N}}$, and a sequence of compact linear solution operators

$$\mathcal{S} = \{S_d : \mathcal{H}_d \to \mathcal{G}_d\}_{d\in\mathbb{N}}.$$

Here, we denote by $\mathbb{N}$ the set of positive integers, whereas $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Our aim is to determine tractability conditions of the problem of finding approximations to $\{S_d(f)\}$ for f from the unit ball of $\mathcal{H}_d$. The approximations are obtained by algorithms $\{A_{d,n} : \mathcal{H}_d \to \mathcal{G}_d\}_{d\in\mathbb{N},n\in\mathbb{N}_0}$. For $n = 0$, we set $A_{d,0} := 0$, and for $n \geq 1$, $A_{d,n}(f)$ depends only on n continuous linear functionals $L_1(f), L_2(f), \ldots, L_n(f)$, i.e.,

$$A_{d,n}(f) = \phi_n(L_1(f), L_2(f), \ldots, L_n(f))$$

with $\phi_n : \mathbb{C}^n \to \mathcal{G}_d$ and $L_j \in \mathcal{H}_d^*$. The choice of L_j as well as n can be adaptive, i.e., $L_j = L_j(\cdot; L_1(f), L_2(f), \ldots, L_{j-1}(f))$ and n can be a function of the $L_j(f)$'s, see [6] as well as [2] for details. We consider the worst case setting in which the error of $A_{d,n}$ is given by

$$e(A_{d,n}) = \sup_{\substack{f\in\mathcal{H}_d \\ \|f\|_{\mathcal{H}_d}\leq 1}} \|S_d(f) - A_{d,n}(f)\|_{\mathcal{G}_d}.$$

Let

$$e(n, S_d) = \inf_{A_{d,n}} e(A_{d,n})$$

denote the nth minimal worst case error, where the infimum is extended over all admissible algorithms $A_{d,n}$. Then the information complexity $n(\varepsilon, \mathcal{S}_d)$ is the minimal number n of continuous linear functionals which is needed to find an algorithm $A_{d,n}$ which approximates S_d with error at most ε. More precisely, we consider the absolute (ABS) and normalized (NOR) error criteria in which

$$\begin{aligned} n(\varepsilon, S_d) = n_{\mathrm{ABS}}(\varepsilon, S_d) &= \min\{n : \ e(n, S_d) \leq \varepsilon\}, \\ n(\varepsilon, S_d) = n_{\mathrm{NOR}}(\varepsilon, S_d) &= \min\{n : \ e(n, S_d) \leq \varepsilon \, \|S_d\|\}. \end{aligned}$$

It is known from [6], see also [2], that the information complexity is fully determined by the singular values of S_d, which are the same as the square roots of the eigenvalues of the compact self-adjoint and positive semi-definite linear operator $W_d = S_d^* S_d : \mathcal{H}_d \to \mathcal{H}_d$. We denote these eigenvalues by $\lambda_{d,1}, \lambda_{d,2}, \ldots$, ordered in a non-increasing fashion. Then for $\varepsilon > 0$,

$$n_{\mathrm{ABS}}(\varepsilon, S_d) = \min\{n : \ \lambda_{d,n+1} \leq \varepsilon^2\}, \tag{1}$$

$$n_{\mathrm{NOR}}(\varepsilon, S_d) = \min\{n : \ \lambda_{d,n+1} \leq \varepsilon^2 \lambda_{d,1}\}. \tag{2}$$

Clearly, $n_{\mathrm{ABS}}(\varepsilon, S_d) = 0$ for $\varepsilon \geq \sqrt{\lambda_{d,1}} = \|S_d\|$, and $n_{\mathrm{NOR}}(\varepsilon, S_d) = 0$ for $\varepsilon \geq 1$. Therefore for ABS we can restrict ourselves to $\varepsilon \in (0, \|S_d\|)$, whereas for NOR to $\varepsilon \in (0, 1)$. Since $\|S_d\|$ can be arbitrarily large, to deal simultaneously with ABS and NOR we consider $\varepsilon \in (0, \infty)$. It is known that $n_{\mathrm{ABS/NOR}}(\varepsilon, S_d)$ is finite for all $\varepsilon > 0$ iff S_d is compact, which justifies our assumption about the compactness of S_d.

We study how $n(\varepsilon, S_d)$ depends on ε and d. We compare two types of tractability:

- Tractability with respect to $(d, \max(1, \varepsilon^{-1}))$ which is called algebraic tractability and abbreviated by ALG.

- Tractability with respect to $(d, 1+\ln\,\max(1,\varepsilon^{-1}))$ which is called exponential tractability and abbreviated by EXP.

We now recall various notions of tractability which will be studied in this paper.

- $\mathcal{S}$ **is ALG-SPT-ABS/NOR** (strongly polynomially tractable in the algebraic case for the absolute or normalized error criterion) iff there are non-negative C and p such that for all $d \in \mathbb{N},\ \varepsilon > 0$ we have

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\,(\max(1,\varepsilon^{-1}))^p.$$

  The infimum of p satisfying the bound above is denoted by p^* and called the exponent of ALG-SPT-ABS/NOR.

- $\mathcal{S}$ **is EXP-SPT-ABS/NOR** (strongly polynomially tractable in the exponential case for the absolute or normalized error criterion) iff there are non-negative C and p such that for all $d \in \mathbb{N},\ \varepsilon > 0$ we have

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\,(1+\ln\,\max(1,\varepsilon^{-1}))^p.$$

  The infimum of p satisfying the bound above is denoted by p^* and called the exponent of EXP-SPT-ABS/NOR.

- $\mathcal{S}$ **is ALG-PT-ABS/NOR** (polynomially tractable in the algebraic case for the absolute or normalized error criterion) iff there are non-negative C, p, and q such that for all $d \in \mathbb{N},\ \varepsilon > 0$ we have

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\,d^{\,q}\,(\max(1,\varepsilon^{-1}))^p.$$

- $\mathcal{S}$ **is EXP-PT-ABS/NOR** (polynomially tractable in the exponential case for the absolute or normalized error criterion) iff there are non-negative C, p, and q such that for all $d \in \mathbb{N},\ \varepsilon > 0$ we have

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\,d^{\,q}\,(1+\ln\,\max(1,\varepsilon^{-1}))^p.$$

- $\mathcal{S}$ **is ALG-QPT-ABS/NOR** (quasi-polynomially tractable in the algebraic case for the absolute or normalized error criterion) iff there are non-negative C and p such that for all $d \in \mathbb{N},\ \varepsilon > 0$ we have

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\,\exp\left(p\,(1+\ln\,d)(1+\ln\,\max(1,\varepsilon^{-1}))\right).$$

  The infimum of p satisfying the bound above is denoted by p^* and called the exponent of ALG-QPT-ABS/NOR.

- $\mathcal{S}$ **is EXP-QPT-ABS/NOR** (quasi-polynomially tractable in the exponential case for the absolute or normalized error criterion) iff there are non-negative C and p such that for all $d \in \mathbb{N},\ \varepsilon > 0$ we have

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\,\exp\left(p\,(1+\ln\,d)(1+\ln(1+\ln\,\max(1,\varepsilon^{-1})))\right).$$

  The infimum of p satisfying the bound above is denoted by p^* and called the exponent of EXP-QPT-ABS/NOR.

- $\mathcal{S}$ **is ALG-(s,t)-WT-ABS/NOR** ((s,t)-weakly tractable in the algebraic case for the absolute or normalized error criterion) for positive s and t iff

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln\,\max(1, n_{\rm ABS/NOR}(\varepsilon, S_d))}{d^{\,t}+(\max(1,\varepsilon^{-1}))^s} \;=\; 0.$$

- $\mathcal{S}$ **is EXP-(s,t)-WT-ABS/NOR** ((s,t)-weakly tractable in the exponential case for the absolute or normalized error criterion) for positive s and t iff

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln\,\max(1, n_{\rm ABS/NOR}(\varepsilon, S_d))}{d^{\,t}+(1+\ln\,\max(1,\varepsilon^{-1}))^s} \;=\; 0.$$

- $\mathcal{S}$ **is ALG-UWT-ABS/NOR** (uniformly weakly tractable in the algebraic case for the absolute or normalized error criterion) iff $\mathcal{S}$ **is ALG-(s,t)-WT-ABS/NOR** for all positive s and t.

- $\mathcal{S}$ **is EXP-UWT-ABS/NOR** (uniformly weakly tractable in the exponential case for the absolute or normalized error criterion) iff $\mathcal{S}$ **is EXP-(s,t)-WT-ABS/NOR** for all positive s and t.

For the algebraic case, necessary and sufficient conditions on the eigenvalues $\lambda_{d,n}$'s of W_d for various notions of tractability as well as the formulas for the exponents of tractability can be found in [2]–[4] for ALG-SPT, ALG-PT, ALG-QPT, and in [7] for ALG-(s,t)-WT. ALG-UWT was defined in [5], and conditions on tractability in this case can be easily obtained by combining conditions on ALG-(s,t)-WT as will be done in this paper. For the exponential case, corresponding necessary and sufficient conditions on $\lambda_{d,n}$'s as well as the formulas and bounds for the exponents of tractability will be derived in this paper.

A few words of comment on these tractability definitions are in order. Note that the tractability notions are defined in terms of $\max(1,\varepsilon^{-1})$ and $1+\ln\,\max(1,\varepsilon^{-1})$. Before, this was usually done in terms of ε^{-1} and $\ln\,\varepsilon^{-1}$ with an extra assumption that $\varepsilon\in(0,1)$. Since we want to consider arbitrary positive ε, the term ε^{-1} is arbitrarily small for large ε, and then the term $\ln\,\varepsilon^{-1}$ is arbitrarily close to $-\infty$. These undesired properties disappear if we consider $\max(1,\varepsilon^{-1})$ instead of ε^{-1}, and $1+\ln\,\max(1,\varepsilon^{-1})$ instead of $\ln\,\varepsilon^{-1}$, and they tend to 1 as ε becomes large.

We stress that we did not define the exponents of polynomial tractability. The reason is that in this case the pair (p,q) is usually *not* uniquely defined and we may decrease, say, p at the expense of q and vice versa. Obviously, we would be interested in finding the smallest possible p and q for a given problem $\mathcal{S}$.

Modulo UWT, we listed the tractability notions from the most demanding to the most lenient ones. Obviously, we have

$$\begin{array}{rcl} \text{ALG/EXP-SPT-ABS/NOR} & \Longrightarrow & \text{ALG/EXP- PT-ABS/NOR} \;\Longrightarrow \\ \text{ALG/EXP-QPT-ABS/NOR} & \Longrightarrow & \text{ALG/EXP-}(s,t)\text{-WT-ABS/NOR}\ \forall\, s,t>0. \end{array}$$

Furthermore, for all $s_1\ge s_2$ and $t_1\ge t_2$

$$\text{ALG/EXP-}(s_2,t_2)\text{-WT-ABS/NOR} \quad\Longrightarrow\quad \text{ALG/EXP-}(s_1,t_1)\text{-WT-ABS/NOR}.$$

3. Overview of previous and new results

We summarize previous and newly found conditions for the various tractability notions in the Tables 1–5.

Table 1. **SPT**

$\mathcal{S}$ is ALG-SPT-ABS iff

$\exists\ \tau > 0$ and $\widetilde{C} \in \mathbb{N}$ such that

$$\sup_{d\in\mathbb{N}} \sum_{j=\widetilde{C}}^{\infty} \lambda_{d,j}^{\tau} < \infty.$$

The exponent $p^* = \inf\{2\,\tau :\ \tau \text{ satisfies the bound above}\}$.

$\mathcal{S}$ is EXP-SPT-ABS iff

$\exists\ \tau > 0$ and $\widetilde{C} \in \mathbb{N}$ such that

$$\sup_{d\in\mathbb{N}} \sum_{j=\widetilde{C}}^{\infty} \lambda_{d,j}^{j^{-\tau}} < \infty.$$

The exponent $p^* = \inf\{1/\tau :\ \tau \text{ satisfies the bound above}\}$.

$\mathcal{S}$ is ALG-SPT-NOR iff

$\exists\ \tau > 0$ such that

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{\tau} < \infty.$$

The exponent $p^* = \inf\{2\,\tau :\ \tau \text{ satisfies the bound above}\}$.

$\mathcal{S}$ is EXP-SPT-NOR iff

$\exists\ \tau > 0$ such that

$$\sup_{d\in\mathbb{N}} \sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{j^{-\tau}} < \infty.$$

The exponent $p^* = \inf\{1/\tau :\ \tau \text{ satisfies the bound above}\}$.

We stress that for SPT-ABS the values of finitely many largest eigenvalues do not matter and they may be arbitrarily large. For SPT-NOR, the eigenvalues are normalized and their quotients are at most 1. However, the multiplicity of the largest eigenvalue must be uniformly bounded in d to achieve SPT.

Table 2. **PT**

$\mathcal{S}$ is ALG-PT-ABS iff

$\exists\ \tau_1, \tau_3 \ge 0$ and τ_2, $\widetilde{C} > 0$ such that

$$\sup_{d\in\mathbb{N}}\ d^{-\tau_1} \sum_{j=\lceil \widetilde{C} d^{\tau_3}\rceil}^{\infty} \lambda_{d,j}^{\tau_2} < \infty.$$

$\mathcal{S}$ is EXP-PT-ABS iff

$\exists\ \tau_1, \tau_3 \ge 0$ and τ_2, $\widetilde{C} > 0$ such that

$$\sup_{d\in\mathbb{N}}\ d^{-\tau_1} \sum_{j=\lceil \widetilde{C} d^{\tau_3}\rceil}^{\infty} \lambda_{d,j}^{j^{-\tau_2}} < \infty.$$

$\mathcal{S}$ is ALG-PT-NOR iff

$\exists\ \tau_1 \ge 0$ and $\tau_2 > 0$ such that

$$\sup_{d\in\mathbb{N}}\ d^{-\tau_1} \sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{\tau_2} < \infty.$$

$\mathcal{S}$ is EXP-PT-NOR iff

$\exists\ \tau_1 \ge 0$ and $\tau_2 > 0$ such that

$$\sup_{d\in\mathbb{N}}\ d^{-\tau_1} \sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{j^{-\tau_2}} < \infty.$$

We stress that for PT-ABS the values of polynomially many largest eigenvalues are irrelevant. Again, for PT-NOR all of them matter and the multiplicity of the largest eigenvalue must be polynomially bounded in d.

The only difference between SPT and PT is that the corresponding sums of some powers of the eigenvalues must be bounded in the SPT case whereas in the PT case they may polynomially increase with d.

Table 3. **QPT**

$\mathcal{S}$ is ALG-QPT-ABS iff

$\exists\ \tau_1 \geq 0$, and τ_2, $\widetilde{C} > 0$ such that

$$\sup_{d\in\mathbb{N}} d^{-2} \left(\sum_{j=\lceil \widetilde{C} d^{\tau_1} \rceil}^{\infty} \lambda_{d,j}^{\tau_2(1+\ln\, d)} \right)^{1/\tau_2} < \infty.$$

The exponent $p^* = \inf\{\max(\tau_1, 2\tau_2):\ \tau_1, \tau_2$ satisfy the bound above$\}$.

$\mathcal{S}$ is EXP-QPT-ABS iff

$\exists\ \tau > 0$ such that

$$\sup_{d\in\mathbb{N}} d^{-\tau} \sum_{j=1}^{\infty} \left[1 + \tfrac{1}{2}\ \ln\, \max\left(1, \frac{1}{\lambda_{d,j}}\right)\right]^{-\tau(1+\ln\, d)} < \infty.$$

The exponent $p^* = \inf\{\tau:\ \tau$ satisfies the bound above$\}$.

$\mathcal{S}$ is ALG-QPT-NOR iff

$\exists\ \tau > 0$ such that

$$\sup_{d\in\mathbb{N}} d^{-2} \left(\sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}} \right)^{\tau(1+\ln\, d)} \right)^{1/\tau} < \infty.$$

The exponent $p^* = \inf\{2\tau:\ \tau$ satisfies the bound above$\}$.

$\mathcal{S}$ is EXP-QPT-NOR iff

$\exists\ \tau > 0$ such that

$$\sup_{d\in\mathbb{N}} d^{-\tau} \sum_{j=1}^{\infty} \left[1 + \tfrac{1}{2} \ln \frac{\lambda_{d,1}}{\lambda_{d,j}} \right]^{-\tau(1+\ln\, d)} < \infty.$$

The exponent $p^* = \inf\{\tau:\ \tau$ satisfies the bound above$\}$.

Table 4. (s,t)-**WT**

$\mathcal{S}$ is ALG-(s,t)-WT-ABS iff

$$\sup_{d\in\mathbb{N}} \exp(-cd^{\,t}) \sum_{j=1}^{\infty} \exp\left(-c\left(\frac{1}{\lambda_{d,j}}\right)^{s/2}\right) < \infty \quad \text{for all } c>0.$$

$\mathcal{S}$ is EXP-(s,t)-WT-ABS iff

$$\sup_{d\in\mathbb{N}} \exp\left(-cd^{\,t}\right) \sum_{j=1}^{\infty} \exp\left(-c\left[1+\ln\left(2\,\max\left(1,\frac{1}{\lambda_{d,j}}\right)\right)\right]^{s}\right) < \infty \quad \text{for all } c>0.$$

$\mathcal{S}$ is ALG-(s,t)-WT-NOR iff

$$\sup_{d\in\mathbb{N}} \exp\left(-cd^{\,t}\right) \sum_{j=1}^{\infty} \exp\left(-c\left(\frac{\lambda_{d,1}}{\lambda_{d,j}}\right)^{s/2}\right) < \infty \quad \text{for all } c>0.$$

$\mathcal{S}$ is EXP-(s,t)-WT-NOR iff

$$\sup_{d\in\mathbb{N}} \exp\left(-cd^{\,t}\right) \sum_{j=1}^{\infty} \exp\left(-c\left[1+\ln\frac{2\lambda_{d,1}}{\lambda_{d,j}}\right]^{s}\right) < \infty \quad \text{for all } c>0.$$

For the case of ALG, we need to guarantee the convergence of the series depending on $\lambda_{d,j}^{-s/2}$ or $(\lambda_{d,1}/\lambda_{d,j})^{s/2}$, whereas for the case of EXP, the corresponding series now depends on the logarithms of $\lambda_{d,j}^{-1}$ or $(\lambda_{d,1}/\lambda_{d,j})$ raised to the power s. Furthermore, in both cases, the convergent series for a fixed d must be at most of order $\exp(cd^{\,t})$ and this must hold for all positive c.

Note that for ABS, the number of eigenvalues $\lambda_{d,j}\ge 1$ must be of order $\exp(o(d^{\,t}))$, whereas for NOR, the multiplicity of the largest eigenvalues $\lambda_{d,1}$ must be of order $\exp(o(d^{\,t}))$.

TABLE 5. **UWT**

$\mathcal{S}$ **is ALG-UWT-ABS iff**

$$\lim_{n\to\infty} \inf_{d\in\mathbb{N}:\ d\le[\ln\, n]^k} \frac{\ln \frac{1}{\lambda_{d,n}}}{\ln\,\ln\, n} = \infty \quad \text{for all } k\in\mathbb{N},$$

$\mathcal{S}$ **is EXP-UWT-ABS iff**

$$\lim_{n\to\infty} \inf_{d\in\mathbb{N}:\ d\le[\ln\, n]^k} \frac{\ln\left(\max\left(1, \ln \frac{1}{\lambda_{d,n}}\right)\right)}{\ln\,\ln\, n} = \infty \quad \text{for all } k\in\mathbb{N}.$$

$\mathcal{S}$ **is ALG-UWT-NOR iff**

$$\lim_{n\to\infty} \inf_{d\in\mathbb{N}:\ d\le[\ln\, n]^k} \frac{\ln \frac{\lambda_{d,1}}{\lambda_{d,n}}}{\ln\,\ln\, n} = \infty \quad \text{for all } k\in\mathbb{N},$$

$\mathcal{S}$ **is EXP-UWT-NOR iff**

$$\lim_{n\to\infty} \inf_{d\in\mathbb{N}:\ d\le[\ln\, n]^k} \frac{\ln\left(\max\left(1, \ln \frac{\lambda_{d,1}}{\lambda_{d,n}}\right)\right)}{\ln\,\ln\, n} = \infty \quad \text{for all } k\in\mathbb{N}.$$

This is the only table which depends on the ordered eigenvalues $\lambda_{d,n}$. We obtain UWT if $\lambda_{d,n}$'s go to zero sufficiently fast. Note that the case of ALG requires the single logarithm of $1/\lambda_{d,n}$ or $\lambda_{d,1}/\lambda_{d,n}$, whereas the case of EXP requires the double logarithms of the same expressions. This quantifies how much harder the case of EXP is as compared to the case of ALG.

For example, take $\lambda_{d,n} = n^{-\alpha}$ for an arbitrary $\alpha > 0$ for all $n, d \in \mathbb{N}$. Then ABS=NOR and we obtain ALG-UWT-ABS/NOR, however EXP-UWT-ABS/NOR does not hold. Hence, polynomial decay of the eigenvalues $\lambda_{d,n}$ is enough for ALG-UWT-ABS/NOR, and not enough for EXP-UWT-ABS/NOR. On the other hand, we obtain EXP-UWT-ABS/NOR if, say, $\lambda_{d,n} = \exp(-n^{\alpha})$ for an arbitrary $\alpha > 0$ for all $n, d \in \mathbb{N}$.

The dependence on d is only through the infimum of $d \le [\ln\, n]^k$. Note that for large n or k, we need to consider more d's and even the smallest quotient with respect to d must be sufficiently large for large n.

4. Proofs

In this section we are ready to prove necessary and sufficient conditions on the eigenvalues $\lambda_{d,j}$'s for tractability in the exponential case presented in the tables above. The subsequent subsections will address these conditions for various notions of tractability.

It turns out that the proofs for the absolute and normalized error criteria are similar. Therefore we combine them by using the abbreviation

$$\mathrm{CRI}_d = \begin{cases} 1 & \text{for ABS,} \\ \lambda_{d,1} & \text{for NOR.} \end{cases}$$

4.1. **Strong Polynomial and Polynomial Tractability.**

Theorem 1 (EXP-SPT/PT-ABS/NOR)**.**

$\mathcal{S}$ is EXP-SPT/PT-ABS/NOR iff there exist $\tau_1, \tau_3 \ge 0$ and $\tau_2, \widetilde{C} > 0$ such that

$$M := \sup_{d\in\mathbb{N}} d^{-\tau_1} \sum_{j=\lceil \widetilde{C}\, d^{\tau_3}\rceil}^{\infty} \left(\frac{\lambda_{d,j}}{\mathrm{CRI}_d}\right)^{j^{-\tau_2}} < \infty. \tag{3}$$

For SPT, we have $\tau_1 = \tau_3 = 0$, and for NOR we have $\widetilde{C} = 1$ and $\tau_3 = 0$.

If this holds then

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le \lfloor M \mathrm{e} d^{\tau_1} \rfloor + \lceil \widetilde{C}\, d^{\tau_3} \rceil + \lceil \max(0, 2\ln \varepsilon^{-1})^{1/\tau_2} \rceil,$$

and the exponent of EXP-SPT-ABS/NOR is

$$p^* = \inf\{1/\tau_2 : \ \tau_2 \text{ satisfies } (3)\}.$$

Proof. Let us first assume that (3) holds. We then need to show that for some $C, q, p \ge 0$ we have

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\, d^{\,q} (1 + \ln \max(1, \varepsilon^{-1}))^p \quad \text{for all } \varepsilon > 0 \text{ and } d \in \mathbb{N},$$

where $q = 0$ in the case of SPT. To this end, let

$$B_d := \left\{ j \in \mathbb{N} : \ j \ge \lceil \widetilde{C}\, d^{\tau_3} \rceil \quad \text{and} \quad \left(\frac{\lambda_{d,j}}{\mathrm{CRI}_d}\right)^{j^{-\tau_2}} > \frac{1}{\mathrm{e}} \right\}.$$

Since (3) holds, we see that $|B_d| < M \mathrm{e}\, d^{\tau_1}$ and $|B_d| \le \lfloor M \mathrm{e}\, d^{\tau_1} \rfloor$.

Suppose now that $j \ge \lceil \widetilde{C}\, d^{\tau_3} \rceil$ but $j \notin B_d$, which means that

$$\left(\frac{\lambda_{d,j}}{\mathrm{CRI}_d}\right)^{j^{-\tau_2}} \le \frac{1}{\mathrm{e}}, \quad \text{or equivalently,} \quad \frac{\lambda_{d,j}}{\mathrm{CRI}_d} \le \exp\left(-j^{\tau_2}\right).$$

This implies that

$$\lambda_{d,j} \le \varepsilon^2\, \mathrm{CRI}_d \quad \text{if} \quad j \notin B_d \ \text{ and } \ j \ge \max\left(\lceil \widetilde{C}\, d^{\tau_3} \rceil, \lceil \max(0, 2\ln \varepsilon^{-1})^{1/\tau_2} \rceil \right). \tag{4}$$

Due to (1) and (2), our observation regarding $|B_d|$, and (4), it follows that

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le \lfloor M \mathrm{e}\, d^{\tau_1} \rfloor + \lceil \widetilde{C}\, d^{\tau_3} \rceil + \lceil \max(0, 2\ln \varepsilon^{-1})^{1/\tau_2} \rceil,$$

as claimed. This easily implies

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\, d^{\max(\tau_1, \tau_3)} \left(1 + \ln \max(1, \varepsilon^{-1})\right)^{1/\tau_2}$$

for some suitably chosen C. Hence, EXP-SPT/PT-ABS/NOR holds.

For SPT, we have $\tau_1 = \tau_3 = 0$, and $q = \max(\tau_1, \tau_3) = 0$. For the exponent of SPT we have $p^* \le \inf\{1/\tau_2 : \ \tau_2 \text{ satisfies } (3)\}$.

Let us now assume that there are non-negative C, q, and p such that

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\, d^{\,q}\, (1 + \ln\, \max(1, \varepsilon^{-1}))^p$$

holds for all $d \in \mathbb{N}$ and all $\varepsilon > 0$. For SPT we have $q = 0$ and p can be arbitrarily close to p^*, say $p = p^* + \delta$ for some (small) positive δ.

Then

$$\lambda_{d, n_{\mathrm{ABS/NOR}}(\varepsilon, S_d)+1} \le \varepsilon^2\, \mathrm{CRI}_d.$$

The latter inequality holds for all $\varepsilon > 0$, but we will use it only for $\varepsilon \in (0, 1]$. Without loss of generality we may assume that $C \ge 1$.

Since the eigenvalues $\lambda_{d,j}$ are non-increasing, we have

$$\lambda_{d, \lfloor C\, d^{\,q} (1+\ln\, \max(1,\varepsilon^{-1}))^p \rfloor + 1} \le \varepsilon^2\, \mathrm{CRI}_d. \tag{5}$$

Let

$$j = \lfloor C\, d^{\,q} (1 + \ln\, \max(1, \varepsilon^{-1}))^p \rfloor + 1.$$

If we vary $\varepsilon \in (0, 1]$, we see that $j = j_d^*, j_d^* + 1, j_d^* + 2, \dots,$ where

$$j_d^* = \lfloor C\, d^{\,q} \rfloor + 1 \ge 2.$$

Note that

$$j \le C\, d^{\,q} (1 + \ln\, \max(1, \varepsilon^{-1}))^p + 1,$$

or equivalently,

$$\varepsilon \le \exp\left(-\frac{1}{C^{1/p}\, d^{\,q/p}} (j-1)^{1/p} + 1 \right).$$

For $j \ge j_d^* \ge 2$ we have $(j - 1) \ge j/2$ and therefore

$$\varepsilon \le \ \mathrm{e}\, \exp\left(-\frac{1}{(2C)^{1/p}\, d^{\,q/p}}\, j^{1/p} \right).$$

By inserting into (5), we see that

$$\frac{\lambda_{d,j}}{\mathrm{CRI}_d} \le \mathrm{e}^2 \exp\left(-\frac{2}{(2C)^{1/p}\, d^{\,q/p}}\, j^{1/p} \right) \quad \text{for all } j \ge j_d^*. \tag{6}$$

Consequently,

$$\sum_{j=j_d^*}^{\infty} \left(\frac{\lambda_{d,j}}{\mathrm{CRI}_d} \right)^{j^{-\tau_2}} \le \mathrm{e}^2 \sum_{j=j_d^*}^{\infty} \exp\left(-\frac{2}{(2C)^{1/p}\, d^{\,q/p}}\, j^{1/p-\tau_2} \right).$$

Choose $\tau_2 < 1/p$, or equivalently, $1/p - \tau_2 > 0$. Then the terms of the last sum are decreasing in j and

$$\begin{aligned} \sum_{j=j_d^*}^{\infty} \exp\left(-\frac{2}{(2C)^{1/p}\, d^{\,q/p}}\, j^{1/p-\tau_2} \right) &\le \int_{j_d^*-1}^{\infty} \exp\left(-\frac{2}{(2C)^{1/p}\, d^{\,q/p}}\, x^{1/p-\tau_2} \right) \mathrm{d}x \\ &\le \int_{0}^{\infty} \exp\left(-\frac{2}{(2C)^{1/p}\, d^{\,q/p}}\, x^{1/p-\tau_2} \right) \mathrm{d}x. \end{aligned}$$

We now put

$$B := \frac{2}{(2C)^{1/p} d^{\,q/p}}, \quad V := 1/p - \tau_2,$$

so the above integral equals

$$I := \int_0^\infty \exp(-B\,x^V)\,\mathrm{d}x.$$

By substituting t for Bx^V, we obtain

$$I = B^{-1/V}\frac{1}{V}\int_0^\infty t^{1/V-1}\exp(-t)\mathrm{d}t = B^{-1/V}\frac{1}{V}\,\Gamma\left(\frac{1}{V}\right),$$

where

$$\Gamma(s) := \int_0^\infty t^{s-1}\exp(-t)\mathrm{d}t$$

is the Gamma function. We therefore get

$$I = \frac{p}{1-\tau_2 p}2^{\frac{1-p}{1-\tau_2 p}}\,C^{\frac{1}{1-\tau_2 p}}\,d^{\,\frac{q}{1-\tau_2 p}}\,\Gamma\left(\frac{p}{1-\tau_2 p}\right).$$

In summary,

$$\sum_{j=j_d^*}^\infty \left(\frac{\lambda_{d,j}}{\mathrm{CRI}_d}\right)^{j^{-\tau_2}} = \mathcal{O}\left(d^{\,q/(1-\tau_2 p)}\right) \quad \text{with} \quad j_d^* = \mathcal{O}(d^{\,q}),$$

where the last two factors in the big $\mathcal{O}$ notation are independent of d.

Consider now ABS. We see that (3) holds for $\tau_1 = q/(1-\tau_2 p)$, $\tau_2 < 1/p$ and $\tau_3 = q$. For SPT, we have $q = 0$ which implies that $\tau_1 = \tau_3 = 0$, and the exponent of SPT is $\inf\{1/\tau_2 :\ \tau_2 \text{ satisfies (3)}\} = p = p^* + \delta$. Since this holds for all positive δ, together with the previous inequality we conclude that $p^* = \inf\{1/\tau_2 :\ \tau_2 \text{ satisfies (3)}\}$, as claimed.

Finally, for NOR we can take $\widetilde{C} = 1$ and $\tau_3 = 0$ and use the fact that

$$\sum_{j=1}^\infty \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{j^{-\tau_2}} \le j_d^* + \sum_{j=j_d^*}^\infty \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{j^{-\tau_2}} = \mathcal{O}\left(d^{\,q} + d^{\,q/(1-\tau_2 p)}\right),$$

and (3) holds with $\tau_1 = q/(1-\tau_2 p)$ for all $\tau_2 < 1/p$. The rest is done as for ABS. This completes the proof. $\square$

4.2. **Quasi-polynomial tractability.**

Theorem 2 (EXP-QPT-ABS/NOR)**.**
$\mathcal{S}$ is EXP-QPT-ABS/NOR iff there exists $\tau > 0$ such that

$$M := \sup_{d\in\mathbb{N}}\ d^{-\tau}\sum_{j=1}^\infty \left[1 + \tfrac{1}{2}\,\ln\,\max\left(1, \frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right]^{-\tau(1+\ln\,d)} < \infty. \tag{7}$$

If this holds then

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le 1 + M\,d^\tau + M\,d^{\,\tau}\left[\max(0, 1+\ln\,\varepsilon^{-1})\right]^{\tau(1+\ln\,d)},$$

and the exponent of EXP-QPT-ABS/NOR is

$$p^* = \inf\{\,\tau\ :\ \tau\ \ \textit{satisfies}\ (7)\}.$$

Proof. Let us first assume that (7) holds. We then need to show that for some $C, p > 0$ we have

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\exp[\,p\,(1+\ln\,d)\,(1+\ln(1+\ln\,\max(1,\varepsilon^{-1})))]$$

for all $\varepsilon > 0$ and $d \in \mathbb{N}$. Let

$$j_1^*(d) = |\{\, j \in \mathbb{N} \,:\, \mathrm{CRI}_d/\lambda_{d,j} < 1\,\}|.$$

Note that $j_1^*(d) = 0$ for NOR, whereas $j_1^*(d)$ may be positive for ABS.

From (7) we conclude that

$$M \geq d^{-\tau} \sum_{j=1}^{j_1^*(d)} 1 = d^{-\tau}\, j_1^*(d).$$

Hence,

$$j_1^*(d) \leq M\, d^{\tau} \quad \text{for both ABS and NOR.}$$

For $j > j_1^*(d)$ we have $\mathrm{CRI}_d/\lambda_{d,j} \geq 1$ and

$$1 + \tfrac{1}{2}\, \ln\, \max\left(1, \ln\, \frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right) = 1 + \tfrac{1}{2}\, \ln\, \frac{\mathrm{CRI}_d}{\lambda_{d,j}} \geq 1.$$

Again due to (7), we have

$$\sum_{j=j_1^*(d)+1}^{\infty} \left[1 + \tfrac{1}{2}\, \ln\, \frac{\mathrm{CRI_d}}{\lambda_{d,j}}\right]^{-\tau(1+\ln\, d)} \leq M\, d^{\tau}.$$

Note that the terms of the last sum are non-increasing. Therefore

$$(n - j_1^*(d)) \left[1 + \tfrac{1}{2}\, \ln\, \frac{\mathrm{CRI_d}}{\lambda_{d,n}}\right]^{-\tau(1+\ln\, d)} \leq M\, d^{\tau}.$$

After simple algebraic manipulations we conclude that

$$\sqrt{\frac{\lambda_{d,n}}{\mathrm{CRI_d}}} \leq \mathrm{e}\, \exp\left(-\left[\frac{n - j_1^*(d)}{M\, d^{\tau}}\right]^{1/(\tau(1+\ln\, d))}\right).$$

We now assume that $\varepsilon \in (0, \mathrm{e})$. Hence, the right-hand side of the last inequality is at most ε for

$$n \geq j_1^*(d) + M\, d^{\tau}\, \left[1 + \ln\, \varepsilon^{-1})\right]^{\tau(1+\ln\, d)}.$$

Using the estimate for $j_1^*(d)$, this means that

$$n := n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \leq 1 + M\, d^{\tau} + M\, d^{\tau}\, \left[1 + \ln\, \varepsilon^{-1}\right]^{\tau(1+\ln\, d)},$$

as claimed.

This can be slightly overestimated by

$$n \leq (1 + M) \exp(\tau\, \ln\, d) + M \exp\left(\tau(1 + \ln\, d)(1 + \ln(1 + \ln\, \varepsilon^{-1}))\right).$$

It is easy to check that

$$\ln\, \left(1 + \ln\, \varepsilon^{-1}\right) \leq 1 + \ln(1 + \ln\, \max(1, \varepsilon^{-1}))$$

and therefore

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \leq (1 + 2M)\, \exp\left(\tau(1 + \ln\, d)(1 + \ln(1 + \ln\, \max(1, \varepsilon^{-1})))\right).$$

This means that EXP-QPT-ABS/NOR holds. Furthermore, the exponent of EXP-QPT-ABS/NOR is at most $\inf\{\, \tau \,:\, \tau \text{ satisfies (7)}\,\}$.

Assume now that EXP-QPT-ABS/NOR holds, i.e., for some $C \geq 1$ and positive p we have

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \leq C \exp\left[p\, (1 + \ln\, d)\, (1 + \ln(1 + \ln\, \max(1, \varepsilon^{-1})))\right]$$

holds for all $d \in \mathbb{N}$ and all $\varepsilon > 0$. This can be rewritten as

$$n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le C\mathrm{e}^p d^p [1 + \ln\,\max(1, \varepsilon^{-1})]^{p(1+\ln\,d)}.$$

We have

$$\lambda_{d, n_{\mathrm{ABS/NOR}}(\varepsilon, S_d)+1} \le \varepsilon^2 \mathrm{CRI}_d.$$

Since the eigenvalues $\lambda_{d,j}$ are non-increasing, we have

$$\lambda_{d, \lfloor C\mathrm{e}^p d^p [1+\ln\,\max(1,\varepsilon^{-1})]^{p(1+\ln\,d)} \rfloor + 1} \le \varepsilon^2 \mathrm{CRI}_d. \tag{8}$$

Although the estimate of $n_{\mathrm{ABS/NOR}}(\varepsilon, S_d)$ holds for all $\varepsilon > 0$, we assume that $\varepsilon \in (0, 1]$.

If we vary $\varepsilon \in (0, 1]$, we see that

$$j = \lfloor C\mathrm{e}^p d^p [1 + \ln\,\varepsilon^{-1}]^{p(1+\ln\,d)} \rfloor + 1$$

attains the values $j = j_d, j_d + 1, \dots,$ where

$$j_d = \lfloor C\mathrm{e}^p d^p \rfloor + 1 \ge 2.$$

Furthermore, we have

$$j \le C\mathrm{e}^p d^p [1 + \ln\,\varepsilon^{-1}]^{p(1+\ln\,d)} + 1$$

or equivalently,

$$\varepsilon \le \exp\left(- \left(\frac{j-1}{C\mathrm{e}^p d^p} \right)^{1/(p(1+\ln\,d))} + 1 \right).$$

Inserting this into (8) we conclude that

$$1 + \tfrac{1}{2}\,\ln\,\frac{\mathrm{CRI}_d}{\lambda_{d,j}} \ge \left(\frac{j-1}{C\,\mathrm{e}^p\,d^p} \right)^{1/(p(1+\ln\,d)} \qquad \text{for all} \quad j \ge j_d.$$

Therefore

$$\left(1 + \tfrac{1}{2}\,\ln\,\frac{\mathrm{CRI}_d}{\lambda_{d,j}} \right)^{-\tau(1+\ln\,d)} \le C^{\tau/p}\,\mathrm{e}^{\tau}\,d^{\tau}\,(j-1)^{-\tau/p} \quad \text{for all} \quad j \ge j_d.$$

Finally,

$$\sum_{j=1}^{\infty} \left[1 + \tfrac{1}{2}\,\ln\,\max\left(1, \frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right) \right]^{-\tau(1+\ln\,d)} \le j_d + C^{\tau/p}\,\mathrm{e}^{\tau}\,d^{\tau} \sum_{j=j_d+1}^{\infty} (j-1)^{-\tau/p}.$$

The last series is finite if we take $\tau > p$. Therefore

$$M = \sup_{d \in \mathbb{N}} d^{-\tau} \sum_{j=1}^{\infty} \left[1 + \tfrac{1}{2}\,\ln\,\max\left(1, \frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right) \right]^{-\tau(1+\ln\,d)} < \infty,$$

as claimed.

Furthermore, the infimum of τ satisfying (7) is at most p and p can be arbitrarily close to the exponent of EXP-QPT-ABS/NOR. Hence, $p^* = \inf\{\,\tau : \tau \text{ satisfies (7)}\,\}/$. This completes the proof. □

4.3. (s,t)**-weak tractability.**

Theorem 3 (EXP-(s,t)-WT-ABS/NOR)**.**
$\mathcal{S}$ is EXP-(s,t)-WT-ABS/NOR iff

$$\mu(c,s,t) := \sup_{d\in\mathbb{N}} \sigma(c,d,s)\exp(-cd^{\,t}) < \infty \quad \forall c>0, \tag{9}$$

where

$$\sigma(c,d,s) := \sum_{j=1}^{\infty} \exp\left(-c\left[1+\ln\left(2\,\max\left(1,\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right)\right]^s\right).$$

Proof. First of all, note that (9) combines the formulas in Table 4 for EXP-(s,t)-WT-ABS/NOR. Indeed, for ABS, we have $\mathrm{CRI}_d = 1$ and

$$1+\ln\left(2\,\max\left(1,\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right) = 1+\ln\left(2\,\max\left(1,\frac{1}{\lambda_{d,j}}\right)\right),$$

whereas for NOR, we have $\mathrm{CRI}_d = \lambda_{d,1}$ and $\mathrm{CRI}_d/\lambda_{d,j} \geq 1$. This yields

$$1+\ln\left(2\,\max\left(1,\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right) = 1+\ln\left(2\,\frac{\lambda_{d,1}}{\lambda_{d,j}}\right).$$

Let us first assume that (9) holds. We then need to show

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln\max(1,n_{\mathrm{ABS/NOR}}(\varepsilon,S_d))}{d^{\,t}+(1+\ln\,\max(1,\varepsilon^{-1}))^s} = 0.$$

The terms in $\sigma(c,d,s)$ are non-increasing, so we have

$$\exp(-cd^{\,t})\,j\,\exp\left(-c\left[1+\ln\left(2\,\max\left(1,\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right)\right]^s\right) \leq \mu(c,s,t).$$

Equivalently,

$$\exp\left(c\left[1+\ln\left(2\,\max\left(1,\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right)\right]^s\right) \geq \frac{j}{\mu(c,s,t)\exp(cd^{\,t})}.$$

In particular, for $j > \mu(c,s,t)\exp(cd^{\,t})$ we obtain

$$1+\ln\left(2\,\max\left(1,\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right) \geq \left(\frac{\ln(j/(\mu(c,s,t)\exp(cd^{\,t}))}{c}\right)^{1/s},$$

or, equivalently,

$$\min\left(1,\frac{\lambda_{d,j}}{\mathrm{CRI}_d}\right) \leq 2\exp\left(1-\left(\frac{\ln(j/(\mu(c,s,t)\exp(cd^{\,t}))}{c}\right)^{1/s}\right).$$

Let now $\varepsilon > 0$. We have

$$2\exp\left(1-\left(\frac{\ln(j/(\mu(c,s,t)\exp(cd^{\,t}))}{c}\right)^{1/s}\right) \leq \varepsilon^2$$

iff

$$j \geq \mu(c,s,t)\exp\left(c\left(\left[\max\left(0,1+\ln\frac{2}{\varepsilon^2}\right)\right]^s + d^{\,t}\right)\right).$$

Therefore, if

$$j_{\varepsilon,d} := \left\lceil \max(1,\mu(c,s,t))\exp\left(c\left(\left[\max\left(0,1+\ln\frac{2}{\varepsilon^2}\right)\right]^s + d^{\,t}\right)\right)\right\rceil \tag{10}$$

we have

$$\min\left(1, \frac{\lambda_{d,j_{\varepsilon,d}}}{\mathrm{CRI_d}}\right) \le \varepsilon^2.$$

We now estimate $j_{\varepsilon,d}$. Since $\max(1,\mu(c,s,t)) \ge 1$, the argument of the ceiling function in the right-hand side of (10) is also at least 1 and we can use $\lceil x \rceil \le 2x$ for all $x \ge 1$, so that

$$j_{\varepsilon,d} \le 2\max(1,\mu(c,s,t))\exp\left(c\left(\left[\max\left(0,1+\ln\frac{2}{\varepsilon^2}\right)\right]^s + d^{\,t}\right)\right).$$

It is easy to check that

$$\max\left(0,1+\ln\frac{2}{\varepsilon^2}\right) \le 2\left(1+\ln\,\max(1,\varepsilon^{-1})\right) \quad \text{for all } \varepsilon > 0.$$

Hence,

$$j_{\varepsilon,d} \le 2\max(1,\mu(c,s,t))\,\exp\left(2^s c\left(\left(1+\ln\,\max(1,\varepsilon^{-1})\right)^s + d^{\,t}\right)\right)$$

which can be abbreviated as

$$j_{\varepsilon,d} = \mathcal{O}\left(\exp\left(2^s c\left(1+\ln\,\max(1,\varepsilon^{-1})\right)^s + d^{\,t}\right)\right),$$

where the factor in the big $\mathcal{O}$ notation is independent of ε^{-1} and d.

For NOR, we have

$$\min\left(1, \frac{\lambda_{d,j_{\varepsilon,d}}}{\mathrm{CRI_d}}\right) = \frac{\lambda_{d,j_{\varepsilon,d}}}{\lambda_{d,1}} \le \varepsilon^2,$$

and therefore

$$n_{\mathrm{NOR}}(\varepsilon,S_d) \le j_{\varepsilon,d} = \mathcal{O}\left(\exp\left(2^s c\,\left(1+\ln\,\max(1,\varepsilon^{-1})\right)^s + d^{\,t}\right)\right).$$

Since this holds for all $c > 0$, we obtain EXP-(s,t)-WT-NOR.

For ABS, let

$$j_1^*(d) = |\{\, j \in \mathbb{N} : \ \lambda_{d,j} > 1 \,\}|.$$

Then

$$\mu(c,s,t) \ge \exp(-cd^{\,t}) \sum_{j=1}^{j_1^*(d)} \exp(-c(1+\ln\,2)^s) = \exp\left(-c(d^{\,t} + (1+\ln 2)^s)\right)\, j_1^*(d).$$

Hence,

$$j_1^*(d) \le \mu(c,s,t)\,\exp\left(c((1+\ln 2)^s + d^{\,t})\right) = \mathcal{O}\left(\exp(cd^{\,t})\right),$$

again with the factor in the big $\mathcal{O}$ notation independent of d. Note that

$$\max\left(j_1^*(d), j_{\varepsilon,d}\right) = \mathcal{O}\left(\exp\left(2^s c\left(\left(1+\ln\,\max(1+\varepsilon^{-1})\right)^s + d^{\,t}\right)\right)\right).$$

For $j = \max\left(j_1^*(d)+1, j_{\varepsilon,d}\right)$ we have

$$\min\left(1, \frac{\lambda_{d,j}}{\mathrm{CRI_d}}\right) = \lambda_{d,j} \le \varepsilon^2.$$

Therefore,

$$n_{\mathrm{ABS}}(\varepsilon,S_d) \le j = \mathcal{O}\left(\exp(2^s c\,(1+\ln\,\max(1,\varepsilon^{-1}))^s + d^{\,t})\right).$$

Since this holds for all choices of $c > 0$, we obtain EXP-(s,t)-WT-ABS.

Let us now assume that we have EXP-(s,t)-WT-ABS/NOR, i.e.,

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln\max(1, n_{\mathrm{ABS/NOR}}(\varepsilon, S_d))}{d^{\,t} + (1+\ln\,\max(1,\varepsilon^{-1}))^s} = 0.$$

Then for any $c>0$ there exists an integer $C = C(c,s,t)$ such that

$$n := n_{\mathrm{ABS/NOR}}(\varepsilon, S_d) \le \left\lfloor \exp\left(c\left(\left[1+\ln\,\max(1,\varepsilon^{-1})\right]^s + d^{\,t}\right)\right)\right\rfloor$$

for all choices of $\varepsilon^{-1} + d \ge C$.

For $d \in \mathbb{N}$, choose $\varepsilon > 0$ such that $\varepsilon^{-1} \ge \max(1, C-d)$. Since the eigenvalues $\lambda_{d,j}$ are non-increasing, we have

$$\lambda_{d,\lfloor \exp(c([1+\ln\,\max(1,\varepsilon^{-1})]^s + d^{\,t}))\rfloor + 1} \le \varepsilon^2 \mathrm{CRI}_d. \tag{11}$$

Let

$$j = \left\lfloor \exp\left(c\left(\left[1+\ln\,\max(1,\varepsilon^{-1})\right]^s + d^{\,t}\right)\right)\right\rfloor + 1,$$

and

$$k_1^*(d) := \left\lfloor \exp\left(c\left(\left[1+\ln\,\max(1,C-d)\right)\right]^s + d^{\,t}\right)\right)\right\rfloor + 1 = \Theta\left(\exp(cd^{\,t})\right)$$

for all d with the factor in the Θ notation independent of d.

If we vary $\varepsilon^{-1} \in [\max(1, C-d), \infty)$, j will attain any integer value greater than or equal to $k_1^*(d)$. Furthermore we have

$$j \le \exp\left(c\left(\left[1+\ln\,\max(1,\varepsilon^{-1})\right]^s + d^{\,t}\right)\right) + 1,$$

or equivalently,

$$\varepsilon \le \exp\left(-\left(\frac{\ln((j-1)/\exp(cd^{\,t}))}{c}\right)^{1/s} + 1\right) \quad \text{for any } j \ge k_1^*(d).$$

Therefore, by inserting into (11), we see that for all

$$j \ge k_1^*(d) = \Theta\left(\exp(cd^{\,t})\right)$$

we have

$$\frac{\lambda_{d,j}}{\mathrm{CRI}_d} \le \exp\left(-2\left(\frac{\ln((j-1)/\exp(cd^{\,t}))}{c}\right)^{1/s} + 2\right).$$

The latter inequality is equivalent to

$$c\left[1 - \frac{1}{2}\ln\frac{\lambda_{d,j}}{\mathrm{CRI}_d}\right]^s \ge \ln((j-1)/\exp(cd^{\,t})),$$

which, in turn, is equivalent to

$$c\left[1 + \frac{1}{2}\ln\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right]^s \ge \ln((j-1)/\exp(cd^{\,t})).$$

The last inequality holds iff

$$\exp\left(-2c\left[1+\frac{1}{2}\ln\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right]^s\right)\exp(-2cd^{\,t}) \le \frac{1}{(j-1)^2}.$$

We are ready to estimate

$$\exp(-2cd^{\,t})\,\sigma(2c,d,s) = \exp(-2cd^{\,t})\sum_{j=1}^{\infty}\exp\left(-2c\left[1+\ln\left(2\,\max\left(1,\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right)\right]^s\right).$$

For NOR, we have $\max(1, \mathrm{CRI}_d/\lambda_{d,j}) = \lambda_{d,1}/\lambda_{d,j}$ and

$$1 + \ln\,\frac{2\lambda_{d,1}}{\lambda_{d,j}} \geq 1 + \tfrac{1}{2}\,\ln\,\frac{\lambda_{d,1}}{\lambda_{d,j}}.$$

Therefore,

$$\begin{aligned}\exp(-2cd^{\,t})\,\sigma(2c,d,s) &\leq \exp(-2cd^{\,t})\sum_{j=1}^{\infty}\exp\left(-2c\left[1+\frac{1}{2}\ln\frac{\lambda_{d,1}}{\lambda_{d,j}}\right]^{s}\right)\\ &\leq \exp(-2cd^{\,t})(k_1^*(d)-1) + \sum_{j=k_1^*(d)}^{\infty}\frac{1}{(j-1)^2}.\end{aligned}$$

Obviously, the latter sum is bounded by $\pi^2/6$. Furthermore,

$$\exp(-2cd^{\,t})(k_1^*(d)-1) = \mathcal{O}\left(\exp(-cd^{\,t})\right).$$

Hence for any $c > 0$ it is true that

$$\mu(2c,s,t) = \sup_{d\in\mathbb{N}} \sigma(2c,d,s)\exp(-2cd^{\,t}) < \infty.$$

By varying the constant c, we see the validity of (9), finishing the proof for NOR.

For ABS, as before, we consider

$$j_1^*(d) = |\{\, j \;:\; \lambda_{d,j} > 1\,\}|.$$

Note that

$$j_1^*(d) \leq n_{\mathrm{ABS}}(1, S_d) \quad \text{for all } d \in \mathbb{N}.$$

Furthermore, for $d \geq C(c,s,t)$, with $C(c,s,t)$ defined as before, we have

$$n_{\mathrm{ABS}}(1,S_d) \leq \exp\left(c\left((1+\ln\,2)^s + d^{\,t}\right)\right) = \mathcal{O}\left(\exp(cd^{\,t})\right).$$

We now estimate

$$\begin{aligned}\sigma(2c,d,s) &= \sum_{j=1}^{\infty}\exp\left(-2c\left[1+\ln\left(2\,\max\left(1,\frac{1}{\lambda_{d,j}}\right)\right)\right]^{s}\right)\\ &= \sum_{j=1}^{j_1^*(d)}\exp\left(-2c(1+\ln\,2)^s\right) + \sum_{j=j_1^*(d)+1}^{\infty}\exp\left(-2c\left(1+\ln\,\frac{2}{\lambda_{d,j}}\right)^{s}\right)\\ &\leq \max(j_1^*(d),k_1^*(d)) + \sum_{j=\max(j_1^*(d),k_1^*(d))+1}^{\infty}\exp\left(-2c\left(1+\ln\,\frac{2}{\lambda_{d,j}}\right)^{s}\right).\end{aligned}$$

Note that for $j \geq \max(j_1^*(d), k_1(d)) + 1$ we have

$$\lambda_{d,j} \leq 1 \quad \text{and} \quad 1 + \ln(2/\lambda_{d,j}) \geq 1 + \tfrac{1}{2}\ln(1/\lambda_{d,j}).$$

Therefore, we conclude as before that

$$\sum_{j=\max(j_1^*(d),k_1^*(d))+1}^{\infty}\exp\left(-2c\left(1+\ln\,\frac{2}{\lambda_{d,j}}\right)^{s}\right) \leq \frac{\pi^2}{6} + \exp\left(cd^{\,t}\right).$$

Hence,

$$\exp(-2cd^{\,t})\,\sigma(2c,d,s) = \mathcal{O}\left(1+\exp(-2cd^{\,t}+cd^{\,t})\right)$$

is uniformly bounded in d, and $\mu(2c,s,t) < \infty$. By varying the constant c, we conclude the proof for ABS. □

4.4. **Uniform weak tractability.**

We stress that we can verify UWT by checking (s,t)-WT for all positive s and t by criteria presented in Table 4. The advantage of this approach is that these criteria are independent of the ordering of the singular values $\lambda_{d,j}$'s.

Table 5 presents necessary and sufficient conditions on the decay of the ordered eigenvalues $\lambda_{d,n}$'s in order to achieve UWT. We need to prove these conditions for both ALG and EXP since the case of ALG has also not yet been considered.

Theorem 4.

- *$\mathcal{S}$ is ALG-UWT-ABS/NOR iff*

$$\lim_{n\to\infty} \inf_{d\le[\ln\, n]^k} \frac{\ln \frac{\mathrm{CRI}_d}{\lambda_{d,n}}}{\ln\,\ln\, n} = \infty \quad \textit{for all } k\in\mathbb{N}. \tag{12}$$

- *$\mathcal{S}$ is EXP-UWT-ABS/NOR iff*

$$\lim_{n\to\infty} \inf_{d\le[\ln\, n]^k} \frac{\ln\left(\max\left(1,\ln \frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)\right)}{\ln\,\ln\, n} = \infty \quad \textit{for all } k\in\mathbb{N}. \tag{13}$$

Proof. We first consider ALG. Assume that we have ALG-UWT-ABS/NOR. We need to show (12). Since $\mathcal{S}$ is ALG-(s,t)-WT-ABS/NOR for all positive s and t, due to Table 4 we have for all positive c,

$$M_{c,s,t} := \sup_{d\in\mathbb{N}} \exp(-cd^{\,t}) \sum_{j=1}^{\infty} \exp\left(-c\left(\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)^{s/2}\right) < \infty.$$

Since the terms $\exp\left(-c(\mathrm{CRI}_d/\lambda_{d,j})^{s/2}\right)$ are non-increasing, we obtain

$$\exp(-cd^{\,t})\, n \exp\left(-c\,(\mathrm{CRI}_d/\lambda_{d,n})^{s/2}\right) \le M_{c,s,t}.$$

Hence,

$$\exp\left(-c\left(\frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)^{s/2}\right) \le \frac{M_{c,s,t}\exp(cd^{\,t})}{n},$$

and by taking the logarithms we conclude

$$\left(\frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)^{s/2} \ge \frac{\ln\, n - \ln(M_{c,s,t}) - cd^{\,t}}{c}.$$

Take now an arbitrary (large) integer k. For this k, we choose $t = 1/(2k)$. Then there exists $n_{c,s,t} \ge 2$ such that for all $n \ge n_{c,s,t}$ and $d \le [\ln\, n]^k$ we have

$$\frac{\ln\, n - \ln(M_{c,s,t}) - cd^{\,t}}{c} \ge \frac{\ln\, n - \ln(M_{c,s,t}) - c(\ln\, n)^{1/2}}{c} \ge (\ln n)^{1/2}.$$

Using this estimate we conclude that

$$\inf_{d\le[\ln\, n]^k} \frac{\mathrm{CRI}_d}{\lambda_{d,n}} \ge (\ln\, n)^{1/s},$$

and by taking again the logarithms

$$\inf_{d\le[\ln\, n]^k} \frac{\ln\left(\frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)}{\ln\,\ln\, n} \ge \frac{1}{s} \quad \text{for all } n \ge n_{c,s,t}.$$

Since s can be arbitrarily small, the left hand side of the last inequality is arbitrarily large for large n. This means that the limit in (12) is infinity, as claimed.

We now assume that (12) holds. We need to prove ALG-(s,t)-WT-ABS/NOR for all positive s and t. Due to Table 4, we need to show that

$$\sup_{d\in\mathbb{N}} \exp(-cd^{\,t}) \sum_{j=1}^{\infty} \exp\left(-c\left(\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)^{s/2}\right) < \infty \quad \text{for all } c>0.$$

Take an arbitrary (small) positive c. From (12) we know that for all $k \in \mathbb{N}$ and $M>0$ there exists an integer $N(k,M)\ge 3$ such that

$$\frac{\ln\left(\frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)}{\ln\,\ln\, n} \ge M \quad \text{for all } n \ge N(k,M) \text{ and for all } d \le [\ln\, n]^k.$$

Note that $d \le [\ln\, n]^k$ iff $n \ge \exp(d^{\,1/k})$. Therefore we can rewrite the last expression as

$$\frac{\mathrm{CRI}_d}{\lambda_{d,n}} \ge (\ln n)^M \quad \text{for all } d\in\mathbb{N} \text{ and } n \ge \max\big(N(k,M),\exp(d^{1/k})\big).$$

Take now $M = 4/s$ and $k > 1/t$, and let

$$N^* = N(k,M,c,d) = \max\Big(N(k,M),\exp(d^{1/k}),\exp(2/c)\Big).$$

Then

$$\alpha := \sum_{n=1}^{\infty} \exp\left(-c\left(\frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)^{s/2}\right) \le N^* - 1 + \sum_{n=N^*}^{\infty} \exp\left(-c(\ln\, n)^2\right).$$

Note that $\exp(-c(\ln\, n)^2) \le 1/n^2$ for $n \ge \exp(2/c)$. Therefore

$$\alpha \le N^* + \sum_{n=N^*}^{\infty} \frac{1}{n^2} \le \max\left(N(k,M),\exp(2/c)\right) + \frac{\pi 2}{6} + \exp\left(d^{\,1/k}\right).$$

Hence,

$$\exp(-cd^{\,t}) \sum_{n=1}^{\infty} \exp\left(-c\left(\frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)^{s/2}\right) = \mathcal{O}\left(\exp\left(-cd^{\,t} + d^{\,1/k}\right)\right)$$

with the factor in the big $\mathcal{O}$ notation independent of d. Since $t > 1/k$, the last expression is uniformly bounded in d, and we have ALG-(s,t)-WT-ABS/NOR for all positive s and t. This means that ALG-UWT-ABS/NOR holds, as claimed.

We now consider the case of EXP. Assume first that we have EXP-UWT-ABS/NOR. We need to prove (13). Since we have EXP-(s,t)-WT-ABS/NOR for all positive s and t, due to Theorem 3 we have for all positive c,

$$M_{c,s,t} := \sup_{d\in\mathbb{N}} \exp(-cd^{\,t}) \sum_{j=1}^{\infty} \exp\left(-c\left[1+\ln\left(2\max\left(1,\frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right)\right]^s\right) < \infty.$$

As for ALG, we conclude that

$$\exp(-cd^{\,t})\, n\, \exp\left(-c\left[1+\ln\left(2\max\left(1,\frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)\right)\right]^s\right) \le M_{c,s,t},$$

which yields

$$\left[1+\ln\left(2\max\left(1,\frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)\right)\right]^s \ge \frac{\ln\, n - \ln(M_{c,s,t}) - cd^{\,t}}{c} \quad \text{for all } n\in\mathbb{N}.$$

Similarly as before, for an arbitrary integer k, we choose $t = 1/(2k)$ and conclude the existence of $n_{c,s,t} \geq 3$ such that for all $n \geq n_{c,s,t}$ and all $d \leq [\ln\, n]^k$ we have

$$\frac{\ln\, n - \ln(M_{c,s,t}) - cd^{\,t}}{c} \geq \frac{\ln\, n - \ln(M_{c,s,t}) - c(\ln\, n)^{1/2}}{c} \geq (\ln\, n)^{1/2}.$$

Hence, by taking the logarithms we conclude

$$\ln\left(1 + \ln\left(2 \max\left(1, \inf_{d \leq [\ln\, n]^k} \frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)\right)\right) \geq \frac{1}{2s}\, \ln\, \ln\, n \quad \text{for all } n \geq n_{c,s,t}.$$

Let

$$x = \inf_{d \leq [\ln\, n]^k} \frac{\mathrm{CRI}_d}{\lambda_{d,n}}.$$

For small s and $n \geq n_{c,s,t}$, we have large x, say, at least equal to $\exp(2)$. It is easy to check that

$$\ln(1 + \ln(2 \max(1, x))) \leq 2 \ln\left(\max(1, \ln\, x)\right) \quad \text{for all } x \geq \exp(2).$$

Therefore, for small s we obtain

$$\inf_{d \leq [\ln\, n]^k} \frac{\ln\left(\max\left(1, \ln\, \frac{\mathrm{CRI}_d}{\lambda_{d,n}}\right)\right)}{\ln\, \ln\, n} \geq \frac{1}{4s}.$$

Since s can be arbitrarily small, the limit of the left hand side is infinity as n goes to infinity, and (13) holds.

We finally assume that (13) holds. We need to prove EXP-UWT-ABS/NOR, or equivalently that EXP-(s,t)-WT-ABS/NOR holds for all positive s and t. This means that we need to prove that for all positive c,

$$M_{c,s,t} := \sup_{d \in \mathbb{N}} \exp(-cd^{\,t}) \sum_{j=1}^{\infty} \exp\left(-c\left[1 + \ln\left(2 \max\left(1, \frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right)\right]^s\right) < \infty.$$

From (13) we know that for all $k \in \mathbb{N}$ and $M > 0$ there exists $N(k, M) \geq 3$ such that

$$\frac{\mathrm{CRI}_d}{\lambda_{d,n}} \geq \exp\left((\ln\, n)^M\right)$$

for all $d \in \mathbb{N}$ and for all $n \geq n^* := \max\left(N(k, M), \exp(d^{\,1/k}), \exp(2/c)\right)$. Let

$$\alpha := \sum_{j=1}^{\infty} \exp\left(-c\left[1 + \ln\left(2 \max\left(1, \frac{\mathrm{CRI}_d}{\lambda_{d,j}}\right)\right)\right]^s\right).$$

Then

$$\alpha \leq n^* - 1 + \sum_{n=n^*}^{\infty} \exp\left(-c(\ln\, n)^{Ms}\right).$$

We now take $M = 2/s$ and use again the fact that $\exp(-c(\ln\, n)^2) \leq 1/n^2$ for $n \geq \exp(2/c)$. Then

$$\alpha \leq n^* + \frac{\pi^2}{6} = \mathcal{O}\left(\exp(d^{\,1/k})\right).$$

Taking $k > 1/t$, we conclude that

$$M_{c,s,t} = \sup_{d \in \mathbb{N}} \mathcal{O}\left(\exp\left(-cd^{\,t} + d^{\,1/k}\right)\right) < \infty.$$

This completes the proof. □

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