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Improved Bounds for Pencils of Lines

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RICAM-Report 2018-20

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May 28, 2018

Abstract

We consider a question raised by Rudnev: given four pencils of n concurrent lines in $\mathbb{R}^2$, with the four centres of the pencils non-collinear, what is the maximum possible size of the set of points where four lines meet? Our main result states that the number of such points is $O(n^{11/6})$, improving a result of Chang and Solymosi [2].

We also consider constructions for this problem. Alon, Ruzsa and Solymosi [1] constructed an arrangement of four non-collinear n-pencils which determine $\Omega(n^{3/2})$ four-rich points. We give a construction to show that this is not tight, improving this lower bound by a logarithmic factor. We also give a construction of a set of m n-pencils, whose centres are in general position, that determine $\Omega_m(n^{3/2})$ m-rich points.

1 Introduction

An n-pencil with centre $p \in P^2(\mathbb{R})$ is defined to be a set of n concurrent lines passing through p. Given m n-pencils, a point is said to be m-rich if one line from each of the pencils passes through it. The question we study in this paper is the following: what is the maximum possible size of the set of m-rich points determined by m n-pencils?

The first interesting case is when $m = 4$. For $m = 2, 3$ there are natural constructions giving $\Omega(n^2)$ m-rich points, which is certainly maximal.[1] Furthermore, when $m = 4$ and the centres of the

Mathematics Subject Classification (2010) - 52C10, 11B30

[1]For $m = 2$, any two n-pencils with distinct directions determine exactly n^2 crossing points. For $m = 3$, one can

four pencils are collinear, it is still possible[2] to give a construction generating $\Omega(n^2)$ 4-rich points. With these degenerate cases dismissed, we arrive at the following two questions of Rudnev.

Problem 1. *Given four n-pencils whose centres do not lie on a single line, what is the maximum possible size of the set of* 4*-rich points they determine?*

Problem 2. *Given four n-pencils whose centres are in general position (i.e. no three of the centres are collinear), what is the maximum possible size of the set of* 4*-rich points they determine?*

It is possible that the answers to these two questions are the same.

Some progress on the first problem was given in a recent paper of Alon, Ruzsa and Solymosi [1]. They gave a construction of four n-pencils with non-collinear centres which determine $\Omega(n^{3/2})$ 4-rich points. From the other side, a result of Chang and Solymosi [2] implies that for any four n-pencils with non-collinear centres, the number of 4-rich points is $O(n^{2-\delta})$. Their proof gives the value $\delta = 1/24$.

The main results of this paper are the following two theorems, which give improved upper and lower bounds respectively for the maximum possible number of 4-rich points.

Theorem 1. *Let P be the set of* 4*-rich points defined by a set of four non-collinear n-pencils. Then we have*

$$|P| = O(n^{11/6}).$$

Theorem 2. *There exist four n-pencils with non-collinear centres which determine $\Omega(n^{3/2}\log^c n)$* 4*-rich points, for some absolute constant $c > 0$.*

The construction given in [1] of four pencils determining $\Omega(n^{3/2})$ had three of the centres on a line, and thus it did not immediately give any progress towards Problem 2. We give a similar construction with no three of the centres on a line.

take two of the centres of the pencils on the line at infinity so that their crossing points give a grid $A \times A$ where A is a geometric progression. Choosing the origin as the centre for the third pencil, $\Omega(n^2)$ of the points of $A \times A$ can be covered by n lines through the origin by using the ratio set as the set of slopes.

[2]One way to see this is by taking the four centre points on the line at infinity. The first two pencils again intersect in a grid $A \times A$, and this time we make $A = \{1, 2, \ldots, n\}$. The second two pencils give a family of lines with slopes 1 and -1 respectively, and both directions give rise to a family of lines of size $2n - 1$ which cover $A \times A$. Thus we have four pencils of size $O(n)$ (with their centres collinear) and n^2 4-rich points.

Theorem 3. *There exist four n-pencils, whose centres are in general position, which determine $\Omega(n^{3/2})$ 4-rich points.*

Furthermore, we generalise this to give a construction of m n-pencils determining many m-rich points.

Theorem 4. *For any $m \in \mathbb{N}$, there exist m n-pencils whose centres are in general position which determine $\Omega_m(n^{3/2})$ m-rich points.*

For a precise version of this result with the dependence on m made explicit, see the forthcoming Proposition 1.

1.1 Notation

Throughout this paper, the standard notation $\ll, \gg$ and O, Ω is applied to positive quantities in the usual way. $X \gg Y$, $Y \ll X$, $X = \Omega(Y)$ and $Y = O(X)$ all mean that $X \geq cY$, for some absolute constant $c > 0$.

2 Connection with the sum-product problem

The construction relating to Problem 1 given in [1] arose from some surprising constructions for the sum-product problem restricted to graphs. For a finite set $A \subseteq \mathbb{R}$, define the sum and product set as

$$A + A = \{a + b : a, b \in A\}$$

$$A \cdot A = \{ab : a, b \in A\}.$$

We can also define the difference and ratio set in an analogous way. The famous Erdős - Szemerédi conjecture states that for all $\epsilon > 0$, there exists an absolute constant $c(\epsilon)$ such that for all finite $A \subset \mathbb{Z}$

$$\max\{|A + A|, |AA|\} \geq c(\epsilon)|A|^{2-\epsilon}.$$

Erdős and Szemerédi also considered taking sums and products restricted to a specified subset of $A \times A$, as follows. Let G be a bipartite graph with vertices being two distinct copies of A, and let

$E(G) \subseteq A \times A$ be the edges of G. We define the sumset of A along G to be

$$A +_G A = \{a + b : (a, b) \in E(G)\}.$$

In more generality, for A and B two finite subsets of $\mathbb{R}$, we take a set of edges $E(G) \subseteq A \times B$, and define the sum set

$$A +_G B = \{a + b : (a, b) \in E(G)\}.$$

The restricted product set, ratio set etc. are defined in the same way. Erdős and Szemerédi also gave a stronger version of their conjecture in this restricted setting, essentially saying that for sufficiently dense graphs $G \subset A \times A$, at least one of $|A +_G A|$ or $|A \cdot_G A|$ is close to $|G|$. In [1], the authors gave several constructions to show that this stronger conjecture, and variants thereof, do not hold. One such result was the following.

Theorem 5 (Alon, Ruzsa, Solymosi)**.** *For arbitrarily large n, there exists $A \subseteq \mathbb{R}$ finite with $|A| = \Theta(n)$, and a subset $S \subseteq A \times A$ with $|S| = \Omega(n^{3/2})$, such that S is the set of edges of a graph G with*

$$|A +_G A| + |A/_G A| = O(n).$$

Both the sumset and the ratio set are at most linear in size, but the graph has many edges. The construction used in this theorem is then converted, via a projective transformation, into a construction of a set of four n-pencils of lines, with non-collinear centres, that determine $\Omega(n^{3/2})$ 4-rich points.

Similarly, our results in Theorems 1, 2 and 3 follow from considering sum-product type problems restricted to graphs. The sum-product problem that is most relevant to this paper is that of showing that if the product set of A is small, then the product set of a shift of A must be large. In this direction, it was proven by Garaev and Shen [5], that for any finite $A, B, C \in \mathbb{R}$ and any non-zero $x \in \mathbb{R}$,

$$|AB|, |(A + x)C| \gg |A|^{3/4}|B|^{1/4}|C|^{1/4}. \tag{1}$$

This result and its proof closely follow the seminal work of Elekes [3] in which the Szemerédi-Trotter Theorem was first used to prove sum-product results.

In the process of proving Theorems 1, 2, and 3, we obtain some results about this version of the sum-product problem restricted to graphs which may be of independent interest. For example, we prove the following result.

Theorem 6. *For arbitrarily large n, there exists $A, B \subseteq \mathbb{Q}$ with $|A|, |B| \gg n$, and a subset $S \subseteq A \times B$ with $|S| = \Omega(n^{3/2} \log(n)^{\frac{43}{1000}})$, such that S is the set of edges of a graph G with*

$$|A/_G B| + |(A+1)/_G B| + |(A+2)/_G B| \ll n.$$

In the above $A/_G B := \{a/b : (a,b) \in E(G)\}$. More generally, for any $x, y \in \mathbb{R}$,

$$(A+x)/_G(B+y) := \left\{ \frac{a+x}{b+y} : (a,b) \in E(G) \right\}.$$

Finally, since we will use the Szemerédi-Trotter Theorem in the forthcoming section, we state it below.

Theorem 7 (Szemerédi-Trotter Theorem)**.** *Let $P \subset \mathbb{R}^2$ be finite and let L be a finite set of lines in $\mathbb{R}^2$. Then*

$$I(P, L) := |\{(p, l) \in P \times L : p \in l\}| \ll (|P||L|)^{2/3} + |P| + |L|.$$

3 Proof of Theorem 1

We begin by giving a way to translate a question concerning pencils into a question concerning ratio and sum sets. The setup here is similar to that of Chang and Solymosi [2].

We take four non-collinear pencils $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$, with $|\mathcal{L}_i| = n$ for each i. As they are non-collinear, there exists a pair (say $\mathcal{L}_1$ and $\mathcal{L}_2$) such that the line connecting the centres of these pencils does not contain the centre of $\mathcal{L}_3$ or $\mathcal{L}_4$. We apply a projective transformation to send the centres of $\mathcal{L}_1$ and $\mathcal{L}_2$ to the projective coordinates $(1; 0; 0)$ and $(0; 1; 0)$ respectively. $\mathcal{L}_1$ now consists of horizontal lines, and $\mathcal{L}_2$ of vertical lines. By the choice we made, both the pencils $\mathcal{L}_3$ and $\mathcal{L}_4$ have affine centres.

Pencils $\mathcal{L}_1$ and $\mathcal{L}_2$ define a cartesian product $A \times B$, where $|A|, |B| = n$. Let $S \subseteq A \times B$ be the set of 4-rich points. Let (x_1, y_1) and (x_2, y_2) be the centres of $\mathcal{L}_3$ and $\mathcal{L}_4$ respectively. Both $\mathcal{L}_3$ and $\mathcal{L}_4$ cover S, and by identifying an element λ of $(A - x_1)/_G(B - y_1)$ with its corresponding line of slope λ through (x_1, y_1), we have

$$(A - x_1)/_G(B - y_1) \subseteq \mathcal{L}_3 \implies |(A - x_1)/_G(B - y_1)| \leq n \tag{2}$$

$$(A - x_2)/_G(B - y_2) \subseteq \mathcal{L}_4 \implies |(A - x_2)/_G(B - y_2)| \leq n,$$

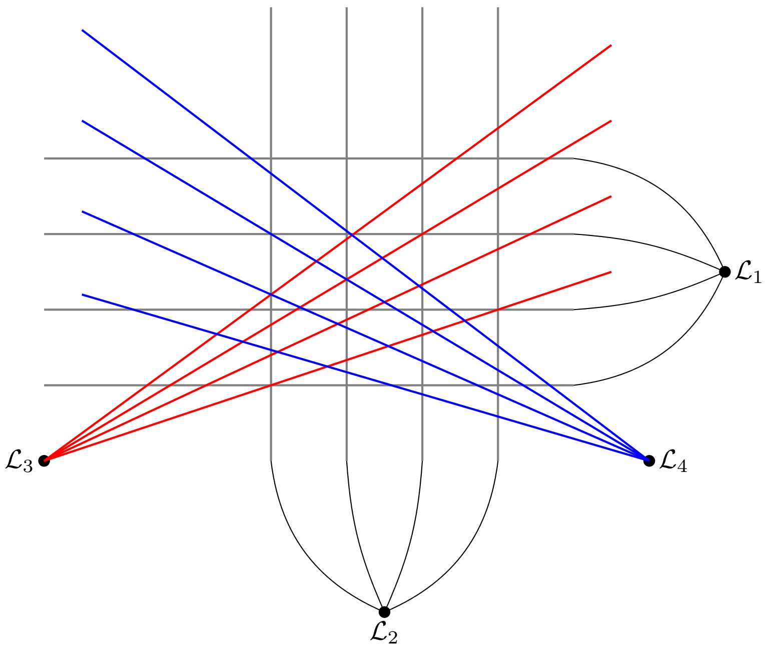

Figure 1: An example of four pencils after a projective transformation.

where G is the bipartite graph on $A \times B$ induced by taking the set of edges to be S. We see that the question now concerns bounding S, the amount of edges of the graph G. We prove the following lemma, which is based on the proof of inequality (1) given in [5]. A similar result can be found in [1], based on the argument of Elekes [3].

Lemma 1. *Let A, B be finite sets of real numbers, and let $|A| = |B| = n$. Let (x_1, y_1), (x_2, y_2) be two distinct points in $\mathbb{R}^2$, and let G be a bipartite graph on $A \times B$. Then*

$$|(A - x_1)/_G(B - y_1)| + |(A - x_2)/_G(B - y_2)| \gg \frac{|E(G)|^{3/2}}{n^{7/4}}.$$

Proof. Since the points (x_1, y_1) and (x_2, y_2) are distinct, at least one of $x_1 \neq x_2$ or $y_1 \neq y_2$ holds. We will assume without loss of generality that $x_1 \neq x_2$. We also assume, without loss of generality, that $y_1, y_2 \notin B$, so as to avoid issues with division by zero.

Furthermore, we can assume that $|E(G)| \geq Cn^{3/2}$ for some sufficiently large constant C, as otherwise the result holds for trivial reasons. Indeed, for any $x_1 \in \mathbb{R}$, $y_1 \in \mathbb{R} \setminus B$ and any graph G on $A \times B$ with $|E(G)| \ll n^{3/2}$,

$$|(A - x_1)/_G(B - y_1)| \geq \frac{|E(G)|}{|A|} \gg \frac{|E(G)|^{3/2}}{|A|^{7/4}}.$$

Let $P = (A - x_1)/_G(B - y_1) \times (A - x_2)/_G(B - y_2)$. Define the line l_{b_1,b_2} by the equation $(b_2 - y_2)y = (b_1 - y_1)x + (x_1 - x_2)$, and let $L = \{l_{b_1,b_2} : b_1, b_2 \in B\}$. Since, $x_1 \neq x_2$, all of these lines are distinct, and so $|L| = |B|^2 = n^2$. For each $a \in A$, if $(a, b_1), (a, b_2) \in E(G)$, the pair $\left(\frac{a-x_1}{b_1-y_1}, \frac{a-x_2}{b_2-y_2}\right) \in P$ lies on line l_{b_1,b_2}. For $a \in A$, let $N(a)$ denote the neighbourhood of A in G, that is, $N(a) := \{b \in B : (a, b) \in E(G)\}$. Then we have a bound for the number of incidences:

$$\begin{aligned} I(P, L) &\geq \sum_{a \in A} |N(a)|^2 \\ &\geq \frac{|E(G)|^2}{n} \end{aligned}$$

by Cauchy-Schwarz. We use the Szemerédi-Trotter Theorem to bound on the other side as

$$\frac{|E(G)|^2}{n} \ll |P| + |L| + (|P||L|)^{2/3}.$$

Since $|E(G)| \geq Cn^{3/2}$ and $|L| = n^2$, the middle term here can be dismissed and we have

$$\frac{|E(G)|^2}{n} \ll |P| + (|P||L|)^{2/3}. \tag{3}$$

If the second term on the right-hand side dominates, we get

$$\Big[|(A - x_1)/_G(B - y_1)||(A - x_2)/_G(B - y_2)|\Big]^{2/3} n^{4/3} \gg \frac{|E(G)|^2}{n},$$

and so

$$|(A - x_1)/_G(B - y_1)| + |(A - x_2)/_G(B - y_2)| \gg \frac{|E(G)|^{3/2}}{n^{7/4}}.$$

If, on the other hand, the first term on the right hand side of (3) dominates, we get a stronger inequality than that claimed in the statement of the lemma, and so the proof of Lemma 1 is complete.

□

Continuing with our four pencils from before, we had the information from the inequalities (2), which when we combine with Lemma 1 gives

$$n \gg |(A - x_1)/_G(B - y_1)| + |(A - x_2)/_G(B - y_2)| \gg \frac{|E(G)|^{3/2}}{n^{7/4}}$$

so that the number of edges, and thus the number of four-rich points, satisfies

$$|E(G)| \ll n^{11/6}.$$

This concludes the proof of Theorem 1. □

This argument can be repeated to give similar results in other fields by using a suitable replacement for the Szemerédi-Trotter Theorem. In the complex setting we can use a result of Toth [7] (see also Zahl [8]), obtaining the same results as above. Over $\mathbb{F}_p$ we can use an incidence theorem for cartesian products due to Stevens and de Zeeuw [6]. We calculated that this gives an upper bound $O(n^{2-\frac{1}{8}})$ for the number of 4-rich points.

4 Proof of Theorem 2

In order to prove Theorem 2, we will first prove Theorem 6. We will then show this sum-product construction implies a construction with four pencils determining many 4-rich points.

We make use of the following theorem due to Ford [4] concerning the product set of the first n integers.

Theorem 8. *Let $A(n)$ be the number of positive integers $m \leq n$ which can be written as a product $m = m_1 m_2$, where m_1, $m_2 \in \{1, 2, ..., \lfloor\sqrt{n}\rfloor\}$. Then*

$$A(n) \sim \frac{n}{(\log n)^{\delta}(\log\log n)^{3/2}}$$

where $\delta = 1 - \frac{1+\log\log 2}{\log 2} = 0.086071\ldots$.

As a corollary, we re-write this theorem in the language of product sets.

Corollary 1. *Let $A = \{1, 2, ..., n\}$. Then the product set AA has size*

$$|AA| \ll \frac{n^2}{(\log n)^{\frac{43}{500}}}.$$

Here we have absorbed the $\log\log$ factor by slightly reducing the exponent of the log factor, for simplicity of the forthcoming calculations. We now have the tools to prove Theorem 6.

Proof of Theorem 6. Let $d > 0$ be some parameter to be chosen later. Define the sets

$$A = \left\{ \frac{i}{j} : i, j \in \mathbb{Z},\ (i,j) = 1,\ 1 \leq i, j \leq \frac{\sqrt{n}}{(\log n)^d},\ j \geq \frac{\sqrt{n}}{2(\log n)^d} \right\} \tag{4}$$

$$B = \left\{ \frac{1}{l} : l \in \mathbb{Z},\ 1 \leq l \leq \frac{n}{(\log n)^d} \right\}. \tag{5}$$

Note that we have the size of A being

$$|A| \sim \frac{n}{(\log n)^{2d}}.$$

Indeed, the number of coprime pairs of integers less than some parameter x is asymptotically equal to $\frac{6}{\pi^2}x^2$, and so

$$|A| \geq \frac{6}{\pi^2}\left(\frac{\sqrt{n}}{(\log n)^d}\right)^2 - \frac{6}{\pi^2}\left(\frac{\sqrt{n}}{(2\log n)^d}\right)^2 + \text{ lower order terms } \gg \frac{n}{(\log n)^{2d}}.$$

We define a bipartite graph on $A \times B$, where the edges $E(G)$ are defined by the following.

$$E(G) = \left\{\left(\frac{i}{j}, \frac{1}{l}\right) \in A \times B : j|l\right\}.$$

The number of edges is given by the formula

$$|E(G)| = \sum_j \left|\left\{i : (i,j) = 1\right\}\right| \left|\left\{k \in \mathbb{Z} : 1 \leq kj \leq \frac{n}{(\log n)^d}\right\}\right|.$$

The size of the set $\left\{k \in \mathbb{Z} : 1 \leq kj \leq \frac{n}{(\log n)^d}\right\}$ gives the amount of multiples of j up to $\frac{n}{(\log n)^d}$. As $j \leq \frac{\sqrt{n}}{(\log n)^d}$, a lower bound for the amount of these multiples is $\sqrt{n}$. We can thus move this outside of the sum over j, obtaining

$$|E(G)| \geq \sqrt{n}\sum_j \left|\left\{i : (i,j) = 1\right\}\right| = \sqrt{n}|A| \gg \frac{n^{3/2}}{(\log n)^{2d}}.$$

The ratio set $A/_G B$ consists of the elements

$$\begin{aligned} A/_G B &= \left\{\frac{il}{j} \text{ such that } \frac{i}{j} \in A, \frac{1}{l} \in B,\ j|l\right\} \\ &\subseteq \left\{il' : 1 \leq i \leq \frac{\sqrt{n}}{(\log n)^d},\ 1 \leq l' \leq 2\sqrt{n}\right\} \\ &\subseteq CC \end{aligned}$$

where $C = \{1, 2, ..., 2\sqrt{n}\}$. Thus we have[3] by Corollary 1

$$|A/_G B| \ll \frac{n}{(\log n)^{\frac{43}{500}}}.$$

[3]It is possible to be more careful here, and use an analogue of Ford's result for an asymmetric multiplication table, in order to make a saving in the exponent of the logarithmic factor in Theorem 6 and thus in turn Theorem 2. In order to simplify the calculations we do not pursue this improvement.

When we apply a shift of 1 to A and calculate the ratio set $(A+1)/_G B$, we get the same result.

$$\begin{aligned}(A+1)/_G B &= \left\{\frac{(i+j)l}{j} : \frac{i}{j} \in A, \frac{1}{l} \in B,\ j|l\right\} \\ &\subseteq \left\{(i+j)l' : 1 \le i \le \frac{\sqrt{n}}{(\log n)^d},\ \frac{\sqrt{n}}{2(\log n)^d} \le j \le \frac{\sqrt{n}}{(\log n)^d}\ , 1 \le l' \le 2\sqrt{n}\right\} \\ &\subseteq \left\{kl' : 1 \le k \le \frac{2\sqrt{n}}{(\log n)^d},\ 1 \le l' \le 2\sqrt{n}\right\} \subseteq CC.\end{aligned}$$

For $(A+2)/_G B$ we find an extra constant, but we still have the same result. We now have the sum

$$|A/_G B| + |(A+1)/_G B| + |(A+2)/_G B| \ll \frac{n}{(\log n)^{\frac{43}{500}}}$$

where the amount of edges on G is

$$|E(G)| \gg \frac{n^{3/2}}{(\log n)^{2d}}.$$

We now set $d = \frac{43}{1000}$, and let $m = \frac{n}{(\log n)^{\frac{43}{500}}}$. This gives us the following;

$$|B| \gg |A| \gg \frac{n}{(\log n)^{\frac{43}{500}}} = m$$

$$|A/_G B| + |(A+1)/_G B| + |(A+2)/_G B| \ll \frac{n}{(\log n)^{\frac{43}{500}}} = m$$

$$|E(G)| \gg \frac{n^{3/2}}{(\log n)^{2d}} \gg m^{3/2}(\log m)^{\frac{43}{1000}},$$

thus completing the proof. □

We can immediately use this result to create a set of four pencils with many 4-rich points.

Proof of Theorem 2. We consider our construction from Theorem 6. The edges of the graph correspond to a set $S \subseteq A \times B \subset \mathbb{R}^2$. The amount of elements of $A/_G B$ and the two shifts are exactly the amount of lines needed to cover S through either the origin for $A/_G B$, the point $(-1, 0)$ for $(A+1)/_G B$ or $(-2, 0)$ for $(A+2)/_G B$. These are our first three pencils, which we already know have cardinality $O(m)$. Our fourth pencil will have its centre on the line at infinity, and will consist of vertical lines covering S. The amount needed is precisely $|A| = O(m)$. The amount of 4-rich points is at least the size of S, since each pencil covers S. Thus we have at least $m^{3/2}(\log m)^{\frac{43}{1000}}$ 4-rich points.

Note also that the centres of the four pencils we have chosen are non-collinear. The point at infinity met by the line connecting $(0, 0)$, $(-1, 0)$ and $(-2, 0)$ is not the equal to the point corresponding to the centre of the fourth pencil. □

5 Constructions with arbitrarily many pencils

We give a construction of a set where the sum-set, ratio set, an additive shift of the ratio set, and the difference set are all linear when we restrict to a graph, where the graph has many edges. We also show using shifts of ratio sets that there are sets of m n-pencils of lines that determine $\Omega_m(n^{3/2})$ m-rich points.

Theorem 9. *For arbitrarily large n, there exists a set A with $|A| = \Theta(n)$, and a graph G on $A \times A$ with $\Omega(n^{3/2})$ edges, such that*

$$|A +_G A| + |A/_G A| + |(A+1)/_G(A+1)| + |A -_G A| \ll n.$$

Proof. Let

$$A := \left\{ \frac{i}{j} : (i,j) = 1,\ 1 \leq i, j \leq \sqrt{n} \right\}$$

The size of A is the amount of coprime pairs from 1 to $\sqrt{n}$; therefore $|A| = \Theta(n)$. We define a bipartite graph G with vertex set $A \times A$ and

$$E(G) = \left\{ \left(\frac{i}{j}, \frac{k}{j} \right) : 1, \leq i, j, k \leq \sqrt{n}, (i,j) = 1 = (k,j) \right\}.$$

With this definition, we have $|E(G)| \gg n^{3/2}$. Indeed,

$$\begin{aligned} |E(G)| &= \sum_{1 \leq j \leq \sqrt{n}} |\{(i,k) : 1 \leq i, k \leq \sqrt{n}, (i,j) = 1 = (k,j)\}| \\ &= \sum_{1 \leq j \leq \sqrt{n}} |\{i : 1 \leq i \leq \sqrt{n}, (i,j) = 1\}|^2, \end{aligned}$$

and so by the Cauchy-Schwarz inequality,

$$\begin{aligned} n^2 &\ll \left(\sum_{1 \leq j \leq \sqrt{n}} |\{i : 1 \leq i \leq \sqrt{n}, (i,j) = 1\}| \right)^2 \\ &\leq \sqrt{n} \sum_{1 \leq j \leq \sqrt{n}} |\{i : 1 \leq i \leq \sqrt{n}, (i,j) = 1\}|^2 = \sqrt{n}|E(G)|, \end{aligned}$$

as claimed.

- The sum set restricted to G is $A +_G A \subseteq \left\{ \frac{i+k}{j} : i, j, k \in [\sqrt{n}] \right\}$. The numerator ranges from 1 to $2\sqrt{n}$, and the denominator from 1 to $\sqrt{n}$, thus $|A +_G A| \ll n$.

- The ratio set is $A/_G A \subseteq \{\frac{i}{k} : i,k \in [\sqrt{n}]\} = A$, so $|A/_G A| \ll n$.
- The shifted ratio set is $(A+1)/_G(A+1) \subseteq \left\{\frac{i+j}{k+j} : i,j,k \in [\sqrt{n}]\right\}$ and so $|(A+1)/_G(A+1)| \ll n$.
- Finally, the difference set is $A -_G A \subseteq \left\{\frac{i-k}{j} : i,j,k \in [\sqrt{n}]\right\}$, so $|A -_G A| \ll n$.

Therefore the sum of the sizes of these four sets is $\ll n$. □

Using the same construction, we may consider only ratio sets to generalise this to any number of pencils. We may arbitrarily shift the ratio set by any $(x,y) \in \mathbb{Z}^2$ and keep its size linear in n;

$$(A+x)/_G(A+y) \subseteq \left\{\frac{i+xj}{k+yj} : i,j,k \in [\sqrt{n}]\right\}$$
$$\implies |(A+x)/_G(A+y)| \leq (\sqrt{n} + x\sqrt{n})(\sqrt{n} + y\sqrt{n}) \ll xyn,$$

which gives a construction to prove the following proposition, a more precise version of Theorem 4.

Proposition 1. *For any $m \in \mathbb{N}$, there exists a set of m pencils of lines, with any three centres of pencils non-collinear, such that each pencil contains N lines, and the amount of m-rich points is $\Omega(N^{3/2}/m^3)$.*

Proof. To get the best possible dependence on m in this statement, we need to choose a set of m centres which are in general position, and so that their coordinates are as small as possible. It is possible to construct such a set of size m in the lattice $[m] \times [m]$. We take P to be this set of centres.

Let A and G be defined as above. Form $(A+x)/_G(A+y)$ for $(x,y) \in P$. The centres are non-collinear, each pencil contains $\ll m^2 n := N$ lines, and the amount of m-rich points is at least the amount of edges, thus $\Omega(n^{3/2}) = \Omega(N^{3/2}/m^3)$. □

Finally, note that by taking $m = 4$ in the previous proposition, we obtain Theorem 3.

Acknowledgements

Both authors were supported by the Austrian Science Fund (FWF) Project P 30405-N32. We thank Mehdi Makhul and Misha Rudnev for helpful conversations.

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