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Higher regularity for solutions to elliptic systems in divergence form subject to mixed boundary conditions

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RICAM-Report 2018-02

HIGHER REGULARITY FOR SOLUTIONS TO ELLIPTIC SYSTEMS IN DIVERGENCE FORM SUBJECT TO MIXED BOUNDARY CONDITIONS

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Abstract. This note combines some known results from operator- and interpolation theory to show that elliptic systems in divergence form admit maximal elliptic regularity on the Bessel potential scale $\mathrm{H}^s_D(\Omega)$ for $s > 1$ sufficiently close to 1, if the coefficient in the main part satisfies a certain multiplier property on the spaces $\mathrm{H}^{s-1}(\Omega)$. Ellipticity is enforced by assuming a Gårding inequality and the result is established for spaces incorporating mixed boundary conditions with very low regularity requirements for the underlying spatial set. Finally, a phase-field damage model is given as a practical application where higher differentiability results are obtained as a corollary to our findings.

1. Introduction

In this paper, we consider elliptic systems in divergence form for an unknown function $u\colon \Omega \to \mathbb{C}^n$ (or $\mathbb{R}^n$). We consider the weak form of a system of elliptic $(n \times n)$ operators $\mathbb{A}$ in divergence form induced by the tensor A for a function $u\colon \Omega \to \mathbb{C}^n$:

$$\begin{aligned} (\mathbb{A}u)_1 &= -\sum_{\alpha,\beta=1}^{d}\sum_{j=1}^{n}\partial_\alpha\big(A^{1,j}_{\alpha,\beta}\partial_\beta u_j\big) \\ \vdots \quad & \quad \vdots \qquad\qquad \vdots \\ (\mathbb{A}u)_n &= -\sum_{\alpha,\beta=1}^{d}\sum_{j=1}^{n}\partial_\alpha\big(A^{n,j}_{\alpha,\beta}\partial_\beta u_j\big). \end{aligned} \tag{1}$$

Each equation $j \in \{1,\dots,n\}$ in this system is subject to possibly mixed boundary conditions with a Dirichlet boundary part D_j and we set $D := \times_{j=1}^{n} D_j$. Precise assumptions on A, the domain Ω, and the Dirichlet boundary parts D_j are given below in Assumption 7.

As a motivation for our results, let us assume for the moment that the operator $\mathbb{A}$ defines an isomorphism between the Sobolev Hilbert spaces $\mathbb{H}^1_D(\Omega) \to \mathbb{H}^{-1}_D(\Omega)$, as it can be asserted by the notorious Lax-Milgram theorem in many cases. It is known under mild assumptions on Ω and D that if $A^{i,j}_{\alpha,\beta} \in L^\infty(\Omega)$, then $\mathbb{A}$ remains an isomorphism for small perturbations in the integrability scale, i.e., there exists

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1991 *Mathematics Subject Classification.* Primary: 35B65, Secondary: 35J57, 35J25.
Key words and phrases. maximal elliptic regularity, non Lipschitz coefficients, second order divergence operators, elliptic system, mixed boundary conditions, phase-field damage.

$\varepsilon > 0$ such that the mapping

$$\mathbb{A}\colon \mathbb{H}^{1,p}_D(\Omega) \to \mathbb{H}^{-1,p}_D(\Omega)$$

remains an isomorphism for all p satisfying $2 - \varepsilon \leq p \leq 2 + \varepsilon$, see e.g. [11].

We will show that such a result is also true on the *differentiability* scale: Indeed, if A is a multiplier on $\mathbb{H}^\varepsilon$ for some $0 < \varepsilon < \frac{1}{2}$, then we prove that there exists $\bar{\theta} > 0$ such that the mapping

$$\mathbb{A}\colon \mathbb{H}^{1+\theta}_D(\Omega) \to \mathbb{H}^{\theta-1}_D(\Omega)$$

is still an isomorphism for any θ satisfying $-\bar{\theta} \leq \theta \leq \bar{\theta}$. The multiplier property is in particular satisfied if A is σ-Hölder-continuous for $\sigma > \varepsilon$. In this case, the norm of the inverse of $\mathbb{A}$ is uniform in its coercivity constant and the bound on the multiplier norm; in particular, it does not depend explicitly on the actual multiplier at hand. Similar results have been obtained by Jochmann in [12] for the case of a scalar elliptic problem with piecewise smooth boundary. Our work can therefore be seen as an extension to elliptic systems, thereby permitting much less regular geometries for Ω and the boundary parts D_j.

Such results are interesting, firstly because they provide a sharp maximal elliptic regularity result for the abstract equation $\mathbb{A}u = f$. Further, they are of interest if compactness properties in the space $\mathbb{H}^1_D(\Omega)$ are needed, for instance if weakly converging data $f_k \rightharpoonup f$ in $\mathbb{H}^{\theta-1}_D(\Omega)$ needs to give rise to strongly convergent states $u_k \to u$ in $\mathbb{H}^1_D(\Omega)$. Such a property is particularly useful in the analysis of optimization problems, where typically only weak convergence of the data is available. Moreover, in the analysis of discretization errors for such equations, certain convergence rates can be obtained only if a gap in differentiability is present. Finally, the fact that the norm of the inverse of $\mathbb{A}$ is uniform for all multipliers with a certain coercivity constant and multiplier norm makes the result attractive to use in a nonlinear setting, e.g. for fixed-point techniques.

Throughout the paper, the considered Banach spaces are in general complex vector spaces. By $\cong$ we understand that two normed spaces are equal up to equivalent norms. Moreover, The restriction of $f\colon U \to \mathbb{C}$ to Λ ($U \supseteq \Lambda$) will be denoted by $f_{\restriction \Lambda}$ and we use $B_r(\mathrm{x})$ for the ball of radius r around x in $\mathbb{R}^d$.

The rest of the paper is structured as follows: We will start by stating our main result in Section 2 and will properly introduce the notation of the subsequent sections. In Section 3, we will give the details on the assumed regularity of the domain: we assume that (the closure of) the non-Dirichlet boundary parts admit bi-Lipschitz boundary charts and allow the Dirichlet parts of the domain to be $(d-1)$-sets. In Section 4, we will define the Bessel potential function spaces needed in the statement of our result. The collection of preliminaries ends in Section 5, where we briefly introduce the concept of a multiplier space and provide some more accessible examples for when a coefficient function is in fact a multiplier. After these preparations, we come to the proof of the main result in Section 6. We conclude the paper by an application of our results to a phase-field fracture/damage model in Section 7.

2. Main result

We first give our main result. All occurring spaces and the notion of a multiplier are formally introduced and defined below (cf. Definitions 9, 11 and 13).

Assumption 1. For $i,j \in \{1,\dots,n\}$, each matrix $A^{i,j}$ is a *real* $(d\times d)$ matrix satisfying $(A^{i,j})^\top = A^{j,i}$ with $A^{i,j}_{\alpha,\beta} \in \mathrm{L}^\infty(\Omega)$ for $\alpha,\beta \in \{1,\dots,d\}$.

To formulate the weak form of the elliptic system operator (1), let

$$\mathbb{H}^1_D(\Omega) := \prod_{j=1}^n \mathrm{H}^1_{D_j}(\Omega),$$

and let $\mathbb{H}^{-1}_D(\Omega)$ be the anti-dual space of $\mathbb{H}^1_D(\Omega)$. For a tensor A satisfying Assumption 1, we define the form $a\colon \mathbb{H}^1_D(\Omega)\times\mathbb{H}^1_D(\Omega)\to\mathbb{C}$ and the divergence-gradient system operator $-\nabla\cdot A\nabla\colon \mathbb{H}^1_D(\Omega)\to\mathbb{H}^{-1}_D(\Omega)$ by

$$\begin{aligned}\big\langle -\nabla\cdot A\nabla u, v\big\rangle &:= a(u,v)\\ &:= \sum_{i,j=1}^n \int_\Omega \big(A^{i,j}\nabla u_j\big)\cdot\nabla\overline{v_i}\,\mathrm{dx}\quad \text{for } u,v\in\mathbb{H}^1_D(\Omega)\end{aligned}. \tag{2}$$

We extend this slightly by defining $-\nabla\cdot A\nabla+\gamma\colon \mathbb{H}^1_D(\Omega)\to\mathbb{H}^{-1}_D(\Omega)$ for $\gamma\ge 0$ by

$$\big\langle\big(-\nabla\cdot A\nabla+\gamma\big)u,v\big\rangle := \big\langle -\nabla\cdot A\nabla u,v\big\rangle + \sum_{j=1}^n\int_\Omega \gamma u_j\overline{v_j}\,\mathrm{dx}$$

and formulate our main result as follows:

Theorem 2. *Let Assumptions 1 and 7 be satisfied and suppose that the system* (1) *is elliptic in the sense that it satisfies a* Gårding inequality, *i.e., there exist $\lambda>0$ and $\mu\ge 0$ such that*

$$\mathrm{Re}\,\big(a(u,u)\big) \ge \sum_{i=1}^n \lambda\|\nabla u_i\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^n)} - \mu\|u_i\|^2_{\mathrm{L}^2(\Omega)}\quad \textit{for all } u\in\mathbb{H}^1_D(\Omega).$$

Assume further that each matrix $A^{i,j}$ is a multiplier on $\mathrm{H}^\varepsilon(\Omega)^d$ for some $0\le\varepsilon<\frac12$. Then there exist $\gamma\ge 0$ large enough and $0<\delta\le\varepsilon$ such that

$$-\nabla\cdot A\nabla+\gamma \in \mathcal{L}_{\mathrm{iso}}\big(\mathbb{H}^{\theta+1}_D(\Omega);\mathbb{H}^{\theta-1}_D(\Omega)\big)\quad \textit{for all } |\theta|<\delta, \tag{3}$$

i.e., $-\nabla\cdot A\nabla+\gamma$ is a topological isomorphism between $\mathbb{H}^{\theta+1}_D(\Omega)$ and $\mathbb{H}^{\theta-1}_D(\Omega)$ for every $-\delta<\theta<\delta$.

Remark 3. (i) The need for the perturbation $\gamma\ge 0$ in Theorem 2 is due to the possibility that 0 might be an eigenvalue of $\mathbb{A}$. If this is not the case, $\gamma=0$ can be chosen. In particular, $\gamma=0$ is allowed if $\mu=0$ and if a Poincaré inequality holds true for $\mathbb{H}^1_D(\Omega)$. The latter is already satisfied for $D\neq\emptyset$ in our geometric setting as given in Section 3 below, cf. [1, Rem. 3.4].

(ii) We give sufficient conditions for the matrix functions $A^{i,j}$ to be multipliers on $\mathrm{H}^\varepsilon(\Omega)^d$ in Lemma 14 below. A particular case is when $A^{i,j}_{\alpha\beta}\in\mathrm{C}^\sigma(\Omega)$ for $\varepsilon<\sigma<1$ for all $\alpha,\beta\in\{1,\dots,n\}$, where $\mathrm{C}^\sigma(\Omega)$ is the space of Hölder continuous functions on Ω. This also implies that $\mathrm{C}^{\frac12}(\Omega)$ is always a suitable multiplier space for Theorem 2.

(iii) We consider the Gårding inequality as the adequate abstract tool to enforce coercivity in our context since it is known that if A satisfies the *Legendre-Hadamard condition* and the coefficient functions are uniformly continuous (cf. the previous point), then the Gårding inequality is indeed satisfied at least for $D=\emptyset$ (see [6, Ch. 3.4.3]). Coercivity of system operators $-\nabla\cdot A\nabla$

in the setting $D \neq \emptyset$ without a very strong ellipticity assumption in the form of a *Legendre condition* is both an interesting and (very) difficult topic, see e.g. [16, 20] and the references therein.

Theorem 2 yields the following corollary:

Corollary 4. *In the situation of Theorem 2, let* $f \in \mathbb{H}_D^{\theta-1}(\Omega)$ *for some* $0 < \theta < \delta$. *Then the elliptic system*

$$-\nabla \cdot A\nabla u + \gamma u = f \quad \text{in } \mathbb{H}_D^{\theta-1}(\Omega) \tag{4}$$

has a unique solution $u \in \mathbb{H}_D^{\theta+1}(\Omega)$ *satisfying*

$$\|u\|_{\mathbb{H}_D^{\theta+1}(\Omega)} \leq C\|f\|_{\mathbb{H}_D^{\theta-1}(\Omega)}$$

for some constant $C \geq 0$ *independent of* f. *Moreover, for all* $0 < \eta < \theta$ *there exist* $p > 2$ *and* $C^{\bullet} \geq 0$ *such that* $u \in \mathbb{H}_D^{1+\eta,p}(\Omega)$ *and*

$$\|u\|_{\mathbb{H}_D^{1+\eta,p}(\Omega)} \leq C^{\bullet}\|f\|_{\mathbb{H}_D^{\theta-1}(\Omega)}.$$

Remark 5. There exist qualitative estimates on the size of δ in Theorem 2. These show e.g. that δ is uniform in the multiplier norm of the matrices $A^{i,j}$ and the constants from Gårding's inequality together with γ. The same is true for the norm of the inverse of $-\nabla \cdot A\nabla + \gamma$ (and thus the constant C in Corollary 4); in particular, the norm does not depend on the actual multiplier at hand. We refer to [4, Ch. 1.3.5] and Remark 16 below.

3. Assumptions on the domain

We formulate the assumptions on the spatial domain $\Omega \subset \mathbb{R}^d$ and its boundary. As part of the assumptions on Theorem 2, these are supposed to be valid in all of the following. A preliminary definition we need is the following:

Definition 6 (($d-1$)-set)**.** Let $F \subset \mathbb{R}^d$ be a Borel set. We say that F is a $(d-1)$-*set* or that F satisfies the *Ahlfors-David condition* if there is $c \geq 1$ such that

$$c^{-1}r^{d-1} \leq \mathcal{H}^{d-1}\big(F \cap B_r(\mathrm{x})\big) \leq cr^{d-1} \quad \text{for all } \mathrm{x} \in F,\ 0 < r \leq 1,$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure and $B_r(\mathrm{x})$ the ball of radius r around x.

The assumptions on Ω and D_j for $j \in \{1, \dots, n\}$ are then as follows, where we set $\mathfrak{D} := \cap_{j=1}^n D_j$:

Assumption 7. The set $\Omega \subset \mathbb{R}^d$ is a bounded domain and each $D_j \subseteq \partial\Omega$, where $j \in \{1, \dots, n\}$, is either empty or a closed $(d-1)$-set. For every point $\mathrm{x} \in \overline{\partial\Omega \setminus \mathfrak{D}}$ there are Lipschitz boundary charts available, that is, there exists an open neighborhood U_x of x and a bi-Lipschitz map $\phi_\mathrm{x} \colon U_\mathrm{x} \to (-1,1)^d$ such that $\phi_\mathrm{x}(\mathrm{x}) = 0$ and

$$\phi_\mathrm{x}(U_\mathrm{x} \cap \Omega) = \big\{\mathrm{x} \in (-1,1)^d \colon \mathrm{x}_d < 0\big\},$$
$$\phi_\mathrm{x}(U_\mathrm{x} \cap \partial\Omega) = \big\{\mathrm{x} \in (-1,1)^d \colon \mathrm{x}_d = 0\big\}.$$

Remark 8. (i) For $\mathfrak{D} = \emptyset$, the assumptions on Ω fall back to that of a classical Lipschitz domain (cf. [8]). On the other side of the spectrum, for $\mathfrak{D} = \partial\Omega$, so pure Dirichlet conditions for every equation in the system (1), we do not require local descriptions of $\partial\Omega$ by boundary charts *at all.*

(ii) If $\Omega \cup D_j$ is regular in the sense of Gröger (cf. [9, 10]) for some $j \in \{1, \dots, n\}$, then Assumption 7 is already satisfied. Indeed, in this case D_j is already a $(d-1)$-set, and there are already bi-Lipschitz charts available for the *whole* $\partial\Omega$, so Ω is again a Lipschitz domain. This follows from the facts that the concept of Gröger requires that $D_j \supseteq \mathfrak{D}$ is also described by local bi-Lipschitz charts as $\overline{\partial\Omega \setminus \mathfrak{D}}$ is in Assumption 7, that such a local bi-Lipschitz description of D_j implies that D_j is a $(d-1)$-set by [13, Ch. II.1.1, Ex. 1], and that finite unions of $(d-1)$-sets are again $(d-1)$-sets. Clearly, Assumption 7 is also satisfied if $\Omega \cup D_j$ is regular in the sense of Gröger for *every* $j \in \{1, \dots, n\}$.

(iii) With the same argument as in the previous point, we find that under Assumption 7, the whole boundary $\partial\Omega$ is always a $(d-1)$-set.

4. Definitions and basics

We move to formal definitions of the fundamental function spaces. Here, we mostly work only with the scalar-valued spaces $\mathrm{H}^{s,p}_F(\Omega)$ for $(d-1)$-sets F satisfying $\mathfrak{D} \subseteq F \subseteq \partial\Omega$ since their properties translate to n-fold products of such spaces immediately. Note that under Assumption 7, every D_j is a valid choice for such F, as is $\partial\Omega$ by Remark 8 iii.

Definition 9 (Bessel potential spaces)**.** For $-\infty < t < \infty$ and $1 < p < \infty$, let $\mathrm{H}^{t,p}(\mathbb{R}^d)$ be the classical Bessel potential spaces with $\mathrm{H}^t(\mathbb{R}^d) := \mathrm{H}^{t,2}(\mathbb{R}^d)$, cf. [17, Ch. 2.3.1/Thm. 2.3.3]. Consider $\frac{1}{2} < s < \frac{3}{2}$ and a $(d-1)$-set F such that $\mathfrak{D} \subseteq F \subseteq \partial\Omega$. Then we define as follows:

(i) Set

$$\mathrm{H}^{s,p}_F(\mathbb{R}^d) := \Big\{ f \in \mathrm{H}^{s,p}(\mathbb{R}^d) \colon \lim_{r \searrow 0} \frac{1}{|B_r(\mathrm{x})|} \int_{B_r(\mathrm{x})} f(\mathrm{y}) \,\mathrm{dy} = 0 \text{ for } \mathcal{H}^{d-1}\text{-a.e. } \mathrm{x} \in F \Big\}$$

with $\mathrm{H}^s_F(\mathbb{R}^d) := \mathrm{H}^{s,2}_F(\mathbb{R}^d)$ and $\|\cdot\|_{\mathrm{H}^{s,p}_F(\mathbb{R}^d)} = \|\cdot\|_{\mathrm{H}^{s,p}(\mathbb{R}^d)}$.

(ii) Further, set $\mathrm{H}^{s,p}_F(\Omega) := \big\{ f_{\restriction\Omega} \colon f \in \mathrm{H}^{s,p}_F(\mathbb{R}^d) \big\}$, equipped with the factor space norm

$$\|f\|_{\mathrm{H}^{s,p}_F(\Omega)} := \inf\big\{ \|g\|_{\mathrm{H}^{s,p}(\mathbb{R}^d)} \colon g \in \mathrm{H}^{s,p}_F(\mathbb{R}^d),\ g_{\restriction\Omega} = f \big\}.$$

We set, again, $\mathrm{H}^s_F(\Omega) := \mathrm{H}^{s,2}_F(\Omega)$, and for $F = \emptyset$, we write $\mathrm{H}^{s,p}(\Omega) := \mathrm{H}^{s,p}_\emptyset(\Omega)$.

(iii) Denote by $\mathrm{H}^{-s}_F(\mathbb{R}^d)$ and $\mathrm{H}^{-s}_F(\Omega)$ the space of antilinear continuous functionals acting on $\mathrm{H}^s_F(\mathbb{R}^d)$ and $\mathrm{H}^s_F(\Omega)$, respectively. We agree that the convention $\mathrm{H}^{-s}(\Omega) := \mathrm{H}^{-s}_\emptyset(\Omega)$ still applies.

(iv) Finally, for $\Lambda \in \{\Omega, \mathbb{R}^d\}$ and D_j from Assumption 7, set $\mathbb{H}^{s,p}_D(\Lambda) := \prod_{j=1}^n \mathrm{H}^{s,p}_{D_j}(\Lambda)$, with all the previous conventions for $p = 2$, and let $\mathbb{H}^{-s}_D(\Lambda)$ be the space of continuous antilinear functionals on $\mathbb{H}^s_D(\Lambda)$, so $\mathbb{H}^{-s}_D(\Lambda) := \prod_{j=1}^n \mathrm{H}^{-s}_{D_j}(\Lambda)$.

Remark 10. (i) For $1 \leq s < \frac{3}{2}$, it is easy to see that $\mathrm{H}^{s,p}_F(\mathbb{R}^d) = \mathrm{H}^{1,p}_F(\mathbb{R}^d) \cap \mathrm{H}^{s,p}(\mathbb{R}^d)$ and $\mathrm{H}^{s,p}_F(\Omega) \subseteq \mathrm{H}^{1,p}_F(\Omega) \cap \mathrm{H}^{s,p}(\Omega)$. If there exists an operator E which maps $\mathrm{H}^{1,p}_F(\Omega)$ into $\mathrm{H}^{1,p}_F(\mathbb{R}^d)$ and $\mathrm{H}^{s,p}(\Omega)$ into $\mathrm{H}^{s,p}(\mathbb{R}^d)$ at the same time such that $Ef_{\restriction\Omega} = f$, then the reverse inclusion and thus

$$\mathrm{H}^{s,p}_F(\Omega) = \mathrm{H}^{1,p}_F(\Omega) \cap \mathrm{H}^{s,p}(\Omega)$$

follows. A particular case in which this extension property for Ω is satisfied is when $\Omega \cup D_j$ is regular in the sense of Gröger for some $j \in \{1, \dots, n\}$ (cf. Remark 8 ii) because Ω is then a Lipschitz domain for which the $\mathrm{H}^{s,p}$-extension property is classical ([7, Thm. 7.25]), and the preservation of the zero trace on F for the $\mathrm{H}^{1,p}$-extension follows as in [4, Cor. 2.2.13].

(ii) Many authors commonly use $\mathrm{H}^s_0(\Omega)$ instead of $\mathrm{H}^s_{\partial\Omega}(\Omega)$ and $\mathrm{H}^{-1}(\Omega)$ instead of $\mathrm{H}^{-1}_{\partial\Omega}(\Omega)$. We feel that while this is adequate as long as only one fixed part of the boundary, e.g. $F = \partial\Omega$, is considered, a more careful notation is needed in view of the importance of both the sets D_j *and* $\partial\Omega$.

The rather abstract definition of $\mathrm{H}^1_F(\Omega)$ turns out to be equivalent to the nowadays classical Sobolev space with partially vanishing trace $\mathrm{W}^{1,2}_F(\Omega)$ which we formally define as follows.

Definition 11 (Sobolev spaces with partially vanishing trace)**.** Let F be a $(d-1)$-set satisfying $\mathfrak{D} \subseteq F \subseteq \partial\Omega$ and let $\Lambda \subseteq \mathbb{R}^d$ be a domain. Then we set

$$\mathrm{C}^\infty_F(\Lambda) := \Big\{ f_{\restriction\Lambda} \colon f \in \mathrm{C}^\infty_c(\mathbb{R}^d),\ \operatorname{supp} f \cap F = \emptyset \Big\}$$

and

$$\mathrm{W}^{1,2}_F(\Lambda) := \overline{\mathrm{C}^\infty_F(\Lambda)}^{\|\cdot\|_{\mathrm{W}^{1,2}(\Lambda)}}$$

for

$$\|f\|_{\mathrm{W}^{1,2}(\Lambda)} := \left(\int_\Lambda |f|^2 + \|\nabla f\|_2^2 \,\mathrm{dx} \right)^{\frac{1}{2}}.$$

Proposition 12 ([5, Cor. 3.8])**.** *Let F be a $(d-1)$-set satisfying $\mathfrak{D} \subseteq F \subseteq \partial\Omega$. Then there holds $\mathrm{W}^{1,2}_F(\Omega) \cong \mathrm{H}^1_F(\Omega)$.*

Using Proposition 12, we easily verify that $-\nabla \cdot A\nabla$ as in (2) is indeed well defined as an operator from $\mathbb{H}^1_D(\Omega)$ to $\mathbb{H}^{-1}_D(\Omega)$.

5. Multipliers

We finally turn to the notion of a multiplier.

Definition 13 (Multiplier)**.** Let X and Y be Banach spaces whose elements are functions on a common domain of definition Λ. We say that Y is a *multiplier space* of X if for every $\rho \in Y$ the pointwise multiplication operator T_ρ defined by $(T_\rho f)(\mathrm{x}) := \rho(\mathrm{x}) f(\mathrm{x})$ for $\mathrm{x} \in \Lambda$ is a continuous linear operator from X into itself. In this case, the functions $\rho \in Y$ are called *multipliers* for X.

We give a sufficient condition on when a matrix function is in fact a multiplier on spaces of the type $\mathrm{H}^\varepsilon(\Omega)^d$ for $0 \le \varepsilon < \frac{1}{2}$, as required in Theorem 2. We do so using Besov spaces of (non-standard) type $\mathrm{B}^s_{\infty,q}(\Omega)$, which however for $0 < s < 1$ and $q = \infty$ coincide with the Hölder spaces; see [18] or [15] for definitions and more.

Lemma 14. *Let $0 \le \varepsilon < \frac{1}{2}$ be given and let $S \colon \Omega \to \mathbb{R}^{d\times d}$ be a matrix-valued function. Then the following conditions are sufficient for S to be a multiplier on $\mathrm{H}^\varepsilon(\Omega)^d$:*

(i) There exists $1 \le q \le 2$ such that $S_{\alpha,\beta} \in \mathrm{B}^\varepsilon_{\infty,q}(\Omega)$ for every $\alpha, \beta \in \{1, \dots, d\}$.
(ii) There exists $\delta > \varepsilon$ and $1 \le q \le \infty$ such that $S_{\alpha,\beta} \in \mathrm{B}^\delta_{\infty,q}(\Omega)$ for every $\alpha, \beta \in \{1, \dots, d\}$.
(iii) There exists $\varepsilon < \delta < 1$ such that $S_{\alpha,\beta} \in \mathrm{C}^\delta(\Omega)$ for every $\alpha, \beta \in \{1, \dots, d\}$.

Here, $\mathrm{C}^\delta(\Omega)$ *is the space of Hölder continuous functions on* Ω*. In particular,* $\mathrm{C}^{\frac{1}{2}}(\Omega)$ *is always a multiplier on* $\mathrm{H}^\varepsilon(\Omega)$ *for* $0 \le \varepsilon < \frac{1}{2}$*.*

Proof. Note that the results from [15] and [18] in the following proof are originally stated only for function spaces on $\mathbb{R}^d$. The occurring function spaces on Ω are defined as restrictions of the ones on $\mathbb{R}^d$ (cf. Definition 9) which however allows to transfer the results from $\mathbb{R}^d$ to Ω by considering functions in the function spaces on $\mathbb{R}^d$ whose restriction is the function of interest defined on Ω.

The multiplier property for $\mathrm{B}^\varepsilon_{\infty,2}(\Omega)$ on $\mathrm{H}^\varepsilon(\Omega)$ is stated in [15, Ch. 4.7.1] (note that $\mathrm{H}^s(\Omega) = \mathrm{W}^{s,2}(\Omega) = \mathrm{B}^s_{2,2}(\Omega)$). The first assertion now follows from the embedding

$$\mathrm{B}^\varepsilon_{\infty,q}(\Omega) \hookrightarrow \mathrm{B}^\varepsilon_{\infty,2}(\Omega) \quad \text{for } 1 \le q \le 2,$$

cf. [18, p. 78], whereas the second assertion is a consequence of the foregoing embedding and

$$\mathrm{B}^\delta_{\infty,q}(\Omega) \hookrightarrow \mathrm{B}^\delta_{\infty,\infty}(\Omega) \hookrightarrow \mathrm{B}^\varepsilon_{\infty,1}(\Omega) \quad \text{for } 1 \le q \le \infty \text{ and } \delta > \varepsilon.$$

Note that the last embedding is not explicitly stated in [18], but follows immediately from the definition of the Besov space there, see [18, Def. 1]. Finally, from [18, Thm. 4], we have

$$\mathrm{C}^\delta(\Omega) \cong \mathrm{B}^\delta_{\infty,\infty}(\Omega) \quad \text{for } 0 < \delta < 1,$$

which then together with the previously established embeddings gives the claim. □

See also [12, Lem. 2] for a similar multiplier result.

6. Proof of the main results

The proof of Theorem 2 rests on the following fundamental theorem by Šneĭberg [19], cf. also [4, Ch. 1.3.5]. For the notions from interpolation theory we refer to [17, Ch. 1.2, 1.9].

Theorem 15 (Stability theorem)**.** *Let* (X_0, X_1) *and* (Y_0, Y_1) *be interpolation couples of Banach spaces and let* T *be a continuous linear operator compatible with that interpolation couple. Then the set*

$$\Big\{\theta \in (0,1) \colon T \in \mathcal{L}_{\mathrm{iso}}\big([X_0, X_1]_\theta; [Y_0, Y_1]_\theta\big)\Big\} \tag{5}$$

is open.

Remark 16. Given a number ϑ which is an element of the set (5) in Theorem 15, there exist estimates on the size of the open set (5), see [4, Ch. 1.3.5]. These show that the size depends on the operator norms of T as a linear operator from X_i to Y_i for $i = 1, 2$, and the operator norm of T^{-1} between $[Y_0, Y_1]_\vartheta$ and $[X_0, X_1]_\vartheta$. This is in fact the connection to the claim about the norm of the inverses of $-\nabla \cdot A\nabla + \gamma$ being uniform in the multiplier norms in Remark 5.

In order to use Theorem 15 we need to have a suitable interpolation scale at hand. For this, we rely on [5, Ch. 7] from which we cite

Theorem 17 ([5, Thm. 7.1])**.** *Let* F *be a* $(d-1)$*-set satisfying* $\mathfrak{D} \subseteq F \subseteq \partial\Omega$*. Let further* $0 < \theta < 1$ *and* $\frac{1}{2} < s_0, s_1 < \frac{3}{2}$ *and put* $s_\theta := (1-\theta)s_0 + \theta s_1$*. Then*

$$\big[\mathrm{H}^{s_0}_F(\Omega), \mathrm{H}^{s_1}_F(\Omega)\big]_\theta = \mathrm{H}^{s_\theta}_F(\Omega)$$

and

$$\big[\mathrm{L}^2(\Omega),\mathrm{H}^1_F(\Omega)\big]_\theta = \begin{cases}\mathrm{H}^\theta_F(\Omega) & \text{if } \theta > \frac{1}{2},\\ \mathrm{H}^\theta(\Omega) & \text{if } \theta < \frac{1}{2}.\end{cases}$$

Before we prove our main result, we establish a few preparatory lemmas building upon Theorem 17.

Lemma 18. *In the situation of Theorem 17, we also have*

$$\big[\mathrm{H}^{-s_0}_F(\Omega),\mathrm{H}^{-s_1}_F(\Omega)\big]_\theta = \mathrm{H}^{-s_\theta}_F(\Omega)$$

and

$$\big[\mathrm{L}^2(\Omega),\mathrm{H}^{-1}_F(\Omega)\big]_\theta = \begin{cases}\mathrm{H}^{-\theta}_F(\Omega) & \text{if } \theta > \frac{1}{2},\\ \mathrm{H}^{-\theta}(\Omega) & \text{if } \theta < \frac{1}{2}.\end{cases}$$

Proof. This follows quite immediately from the result that the $\mathrm{H}^s_F(\Omega)$ spaces are reflexive [5, Cor. 5.3] and general interpolation duality properties [17, Ch. 1.11.3]. Here, density of $\mathrm{H}^{s_0}_F(\Omega)\cap\mathrm{H}^{s_1}_F(\Omega) = \mathrm{H}^{\max(s_0,s_1)}_F(\Omega)$ in $\mathrm{H}^{s_0}_F(\Omega)$ and $\mathrm{H}^{s_1}_F(\Omega)$ follows from density of $\mathrm{H}^{\max(s_0,s_1)}(\mathbb{R}^d)$ in $\mathrm{H}^{s_0}(\mathbb{R}^d)$ and $\mathrm{H}^{s_1}(\mathbb{R}^d)$ and the characterization $\mathrm{H}^s_F(\mathbb{R}^d) = P_F\mathrm{H}^s(\mathbb{R}^d)$ for a bounded linear projection P_F as proven in [5, Cor. 3.5]. □

Now it only remains to set the stage for the extension of $-\nabla\cdot A\nabla$ to $\mathbb{H}^s_D(\Omega)$ for $s\neq 1$ before we can give the proof of the main results.

Lemma 19. *Let F be a $(d-1)$-set satisfying $\mathfrak{D}\subseteq F\subseteq\partial\Omega$ and let $0\le\sigma<\frac{1}{2}$. Then the weak gradient $\nabla\in\mathcal{L}(\mathrm{H}^1_F(\Omega);\mathrm{L}^2(\Omega)^d)$ maps $\mathrm{H}^{\sigma+1}_F(\Omega)$ continuously into $\mathrm{H}^\sigma(\Omega)^d$ and admits a unique continuous linear extension $\nabla\colon\mathrm{H}^{1-\sigma}_F(\Omega)\to\mathrm{H}^{-\sigma}(\Omega)^d$.*

Proof. The first assertion follows from the corresponding property of $\mathrm{H}^{\sigma+1}(\mathbb{R}^d)$ and the definition of the $\mathrm{H}^{\sigma+1}_F(\Omega)$ spaces. For the second assertion, observe that the distributional gradient $G\colon\mathrm{L}^2(\Omega)\to\mathrm{H}^{-1}_{\partial\Omega}(\Omega)^d$ is a continuous linear operator, as (recall Proposition 12)

$$\big|\langle G\varphi,\xi\rangle\big| := \left|-\int_\Omega\varphi\operatorname{div}\xi\,\mathrm{dx}\right| \le C\|\varphi\|_{\mathrm{L}^2(\Omega)}\|\xi\|_{\mathrm{H}^1(\Omega)^d}\quad\text{for all }\xi\in\mathrm{C}^\infty_c(\Omega)^d.$$

Moreover, the distributional gradient G restricted to $\mathrm{H}^1(\Omega)$ agrees exactly with the weak gradient ∇ on $\mathrm{H}^1(\Omega)$ per partial integration and the fundamental lemma of the calculus of variations. Hence, we are able to interpolate the operator (which we agree to call ∇ from now on) which by Theorem 17 and Lemma 18 yields that

$$\nabla\in\mathcal{L}\Big(\big[\mathrm{L}^2(\Omega),\mathrm{H}^1_F(\Omega)\big]_{1-\sigma};\big[\mathrm{H}^{-1}_{\partial\Omega}(\Omega)^d,\mathrm{L}^2(\Omega)^d\big]_{1-\sigma}\Big) = \mathcal{L}\big(\mathrm{H}^{1-\sigma}_F(\Omega);\mathrm{H}^{-\sigma}(\Omega)^d\big).$$

Here, we have used coordinate-wise interpolation in the second component (cf. [4, Cor. 1.3.8]) and the fundamental interpolation property $[X_0,X_1]_\theta = [X_1,X_0]_{1-\theta}$ for any interpolation couple (X_0,X_1) and $0<\theta<1$, see [17, Thm. 1.9.3 b)]. □

We finally prove the main theorem.

Proof of Theorem 2. We had already noted below Proposition 12 that the operators

$$\mathrm{H}^1_{D_j}(\Omega)\times\mathrm{H}^1_{D_i}(\Omega)\ni(\varphi,\xi)\mapsto\langle-\nabla\cdot A^{i,j}\nabla\varphi,\xi\rangle := \big(A^{i,j}\nabla\varphi,\nabla\xi\big)_{\mathrm{L}^2(\Omega)}$$

are continuous for $i,j \in \{1,\dots,n\}$. We extend them to $\mathrm{H}^{\varepsilon+1}_{D_j}(\Omega) \times \mathrm{H}^{1-\varepsilon}_{D_i}(\Omega)$ using Lemma 19, thereby also extending $-\nabla\cdot A\nabla$ to a continuous operator from $\mathbb{H}^{\varepsilon+1}_D(\Omega)$ to $\mathbb{H}^{\varepsilon-1}_D(\Omega)$, cf. (2).

So, let $i,j \in \{1,\dots,n\}$ be given and denote by $M_{i,j}$ the norm of $A^{i,j}$ when the latter is considered as a multiplier acting on $\mathrm{H}^\varepsilon(\Omega)^d$. Since $\mathrm{H}^\varepsilon(\Omega)^d$ is dense in $\mathrm{L}^2(\Omega)^d$, we estimate

$$\begin{aligned}\bigl|\bigl\langle -\nabla\cdot A^{i,j}\nabla\varphi,\xi\bigr\rangle\bigr| &= \bigl|\bigl(A^{i,j}\nabla\varphi,\nabla\xi\bigr)_{\mathrm{L}^2(\Omega)^d}\bigr| \le \|A^{i,j}\nabla\varphi\|_{\mathrm{H}^\varepsilon(\Omega)^d}\|\nabla\xi\|_{\mathrm{H}^{-\varepsilon}(\Omega)^d}\\ &\le M_{i,j}\|\nabla\varphi\|_{\mathrm{H}^\varepsilon(\Omega)^d}\|\nabla\xi\|_{\mathrm{H}^{-\varepsilon}(\Omega)^d} \le CM_{i,j}\|\varphi\|_{\mathrm{H}^{\varepsilon+1}_{D_j}(\Omega)}\|\xi\|_{\mathrm{H}^{1-\varepsilon}_{D_i}(\Omega)}\end{aligned}$$

for all $\varphi \in \mathrm{H}^{\varepsilon+1}_{D_j}(\Omega)$ and $\xi \in \mathrm{H}^1_{D_i}(\Omega)$ using Lemma 19. As $\mathrm{H}^1_{D_i}(\Omega)$ is again dense in $\mathrm{H}^{1-\varepsilon}_{D_i}(\Omega)$, we obtain a unique continuous linear extension of $-\nabla\cdot A^{i,j}\nabla$ to a mapping from $\mathrm{H}^{\varepsilon+1}_{D_j}(\Omega)$ to $\mathrm{H}^{\varepsilon-1}_{D_i}(\Omega)$. By definition (see (2)), this also gives a unique continuous linear extension of $-\nabla\cdot A\nabla$ to a mapping from $\mathbb{H}^{\varepsilon+1}_D(\Omega)$ to $\mathbb{H}^{\varepsilon-1}_D(\Omega)$.

From the assumption $(A^{i,j})^\top = A^{j,i}$ and due to the matrices $A^{i,j}$ being *real*, we further find that the adjoint operator $(-\nabla\cdot A\nabla)^\star$ is a continuous linear extension of $-\nabla\cdot A\nabla$ to an operator $\mathbb{H}^{1-\varepsilon}_D(\Omega) \to \mathbb{H}^{-1-\varepsilon}_D(\Omega)$. Hence the operator is compatible with the interpolation couples $(\mathbb{H}^{1+\varepsilon}_D(\Omega),\mathbb{H}^{1-\varepsilon}_D(\Omega))$ and $(\mathbb{H}^{\varepsilon-1}_D(\Omega),\mathbb{H}^{-1-\varepsilon}_D(\Omega))$ which is then clearly also true for $-\nabla\cdot A\nabla+\gamma$ for any $\gamma\ge 0$.

Now observe that $-\nabla\cdot A\nabla+\gamma \in \mathcal{L}_{\mathrm{iso}}(\mathbb{H}^1_D(\Omega);\mathbb{H}^{-1}_D(\Omega))$ for $\gamma > \mu$ by the Gårding inequality assumption and the Lax-Milgram lemma[1], and that

$$\bigl[\mathbb{H}^{1+\varepsilon}_D(\Omega),\mathbb{H}^{1-\varepsilon}_D(\Omega)\bigr]_{\frac12} = \mathbb{H}^1_D(\Omega) \quad\text{and}\quad \bigl[\mathbb{H}^{\varepsilon-1}_D(\Omega),\mathbb{H}^{-1-\varepsilon}_D(\Omega)\bigr]_{\frac12} = \mathbb{H}^{-1}_D(\Omega)$$

due to Theorem 17 and Lemma 18 (and again coordinate-wise interpolation, see [4, Cor. 1.3.8]). But then the stability result of Šneĭberg as in Theorem 15 tells us that there exists $0<\delta\le\varepsilon$ such that $-\nabla\cdot A\nabla+\gamma \in \mathcal{L}_{\mathrm{iso}}(\mathbb{H}^{\theta+1}_D(\Omega);\mathbb{H}^{\theta-1}_D(\Omega))$ for all $|\theta|<\delta$. This was the claim. □

Proof of Corollary 4. It is a mere reformulation of assertion (3) in Theorem 2 that for every $f \in \mathbb{H}^{\theta-1}_D(\Omega)$ there exists a unique $u \in \mathbb{H}^{\theta+1}_D(\Omega)$ satisfying the elliptic system equation (4) with $\|u\|_{\mathbb{H}^{\theta+1}_D(\Omega)} \le C\|f\|_{\mathbb{H}^{\theta-1}_D(\Omega)}$, where C is independent of f.

Now let $\eta\ge 0$ and $p\ge 2$ be such that $\theta \ge \eta + d(\frac12-\frac1p)$, and consider $j\in\{1,\dots,n\}$. Then, for every function $U_j \in \mathrm{H}^{\theta+1}_{D_j}(\mathbb{R}^d)$ with the property that $(U_j)_{\restriction\Omega} = u_j$ we use the well known (generalized) Sobolev embeddings (cf. [17, Ch. 2.8.1]) as follows:

$$\|u_j\|_{\mathrm{H}^{1+\eta,p}_{D_j}(\Omega)} \le \|U_j\|_{\mathrm{H}^{1+\eta,p}_{D_j}(\mathbb{R}^d)} \le C^\star\|U_j\|_{\mathrm{H}^{\theta+1}_{D_j}(\mathbb{R}^d)}.$$

But this implies that $\|u_j\|_{\mathrm{H}^{1+\eta,p}_{D_j}(\Omega)} \le C^\star\|u_j\|_{\mathrm{H}^{\theta+1}_{D_j}(\Omega)}$ and of course accordingly $\|u\|_{\mathbb{H}^{1+\eta,p}_D(\Omega)} \le C^\star\|u\|_{\mathbb{H}^{\theta+1}_D(\Omega)}$, so the claim follows by observing that if we choose $0<\eta<\theta$, then we are also allowed to choose $p>2$ while still obeying the inequality $\theta\ge\eta+d(\frac12-\frac1p)$. □

[1] Note that if $D\ne\emptyset$, then $\gamma=\mu$ is also allowed due to the Poincaré inequality, cf. Remark 3.

7. Application

As an application, we consider a standard phase-field model for brittle fracture as given in [2]. For the following exposition, we consider the formulation given in [14], where the fracture irreversibility is relaxed by a penalty approach. After introduction of a time-discretization, the evolution is given by a sequence of problems associated to each time-step. Namely, for a bounded domain $\Omega \subset \mathbb{R}^2$ satisfying Assumption 7, one searches for a (vector-valued) displacement $u \in \mathbb{H}^1_D(\Omega)$ and a (scalar) phase-field $\phi \in \mathrm{H}^1(\Omega)$ solving the system of equations

$$\begin{aligned} \big(g(\phi)e(u) : e(v)\big) &= \ell(v), \\ \big(\epsilon^{-1}(\phi - 1) + (1-\kappa)\big(\phi e(u) : e(u)\big) & \\ +\gamma[(\phi-\phi^-)^+]^3, \psi\big)_{\mathrm{L}^2(\Omega)} + \ \big\langle -\nabla\cdot\epsilon\nabla\phi, \psi\big\rangle &= 0 \end{aligned} \tag{6}$$

for all $v \in \mathbb{H}^1_D(\Omega)$ and $\psi \in \mathrm{H}^1(\Omega)$, with given loads $\ell \in \mathbb{H}^{\theta_0-1}_D(\Omega)$ for some $\theta_0 > 0$, ϕ^- satisfying $0 \le \phi^- \le 1$, with $0 < \kappa \ll \epsilon \ll 1$ and $g(\phi) = (1-\kappa)\phi^2 + \kappa$ where $e(u)$ and $e(v)$ denotes the symmetric gradient of u and v, respectively. It has been shown in [14] that this problem admits a Hilbert space solution $(u, \varphi) \in \mathbb{H}^1_D(\Omega) \times \mathrm{H}^1(\Omega)$ with the additional regularity $u \in \mathbb{W}^{1,p}(\Omega)$ for some $p > 2$ and $\phi \in \mathrm{L}^\infty(\Omega)$; in fact, $0 \le \phi(\mathrm{x}) \le 1$ holds for almost all $\mathrm{x} \in \Omega$.

With the results obtained in this work, we can now show the following improved differentiability result.

Corollary 20. *There exists $0 < \bar\theta \le \theta_0$ such that the solution $(u, \phi) \in \big(\mathbb{W}^{1,p}(\Omega) \cap \mathbb{H}^1_D(\Omega)\big) \times \big(\mathrm{H}^1(\Omega) \cap \mathrm{L}^\infty(\Omega)\big)$ of (6) admits the additional regularity $u \in \mathbb{H}^{\theta+1}_D(\Omega)$ and $\phi \in \mathrm{H}^{\theta+1}(\Omega)$ for any θ satisfying $0 < \theta \le \bar\theta$. Moreover we obtain the estimate*

$$\|u\|_{\mathbb{H}^{1+\theta}_D(\Omega)} \le C\|\ell\|_{\mathbb{H}^{\theta_0-1}_D(\Omega)}$$

with a constant $C = C(\|\ell\|^2_{\mathbb{H}^{-1,p}_D(\Omega)}, \gamma, \epsilon)$.

Proof. Slightly rewriting the second equation in (6), we see that ϕ satisfies

$$\big(-\nabla\cdot\epsilon\nabla + \epsilon^{-1}\big)\phi = \epsilon^{-1} + (\kappa - 1)\big(\phi e(u) : e(u)\big) - \gamma[(\phi-\phi^-)^+]^3 \quad \text{in } \mathrm{H}^{-1}(\Omega).$$

By the regularity $\phi \in \mathrm{L}^\infty(\Omega)$ and $u \in \mathbb{W}^{1,p}(\Omega)$ it is clear that the right hand side is in fact an element of $\mathrm{L}^{p/2}(\Omega)$. Consequently, by Sobolev embedding, there exists some $\vartheta > 0$ such that it is an element of $\mathrm{H}^{\vartheta-1}(\Omega)$. Theorem 2 then shows that we have $\phi \in \mathrm{H}^{\theta+1}(\Omega)$ for all $0 < \theta \le \bar\vartheta$ for some $\bar\vartheta \le \vartheta$, and standard Sobolev embedding theorems assert that $\phi \in \mathrm{C}^\sigma(\Omega)$ for $\sigma = 1 + \theta - \frac{2}{p}$. Moreover, by [14, Corollary 4.2], we have that $\|\phi e(u) : e(u)\|_{\mathrm{L}^{p/2}(\Omega)} \le c\|\ell\|^2_{\mathbb{H}^{-1,p}_D(\Omega)}$ for some constant $c \ge 0$, and thus

$$\|\phi\|_{\mathrm{C}^\sigma(\Omega)} \le c\big(\|\ell\|^2_{\mathbb{H}^{-1,p}_D(\Omega)} + \gamma + \epsilon^{-1}\big).$$

But then, by definition, $g(\phi) \in \mathrm{C}^\sigma(\Omega)$ too and Lemma 14 (iii) shows that this is indeed a multiplier on $\mathbb{H}^\theta(\Omega)$. Now another application of Theorem 2 to the equation

$$\big(g(\phi)e(u) : e(v)\big) = \ell(v) \quad \text{for all } v \in \mathbb{H}^1_D(\Omega)$$

yields the claimed regularity. For the stability estimate, we utilize the above bound on $\|\phi\|_{\mathrm{C}^\sigma(\Omega)}$ together with Remark 5. □

Remark 21. In the case where the irreversibility of the fracture is not relaxed via a penalization approach, the equation for ϕ becomes an obstacle problem where the term involving $\gamma([(\phi - \phi^-)^+]^3$ is replaced by the requirement $\phi \leq \phi^-$. If the domain is sufficiently regular, then classical $\mathrm{W}^{2,p/2}(\Omega)$-regularity of the obstacle problem, i.e., $\phi \in \mathrm{W}^{2,p/2}(\Omega)$ as long as $\phi^- \in \mathrm{W}^{2,p/2}(\Omega)$, can be used to show that ϕ is again a multiplier (see e.g. [3, Corollary II.3]).

Acknowledgment

The authors express their gratitude to Joachim Rehberg (WIAS Berlin) for valuable discussions. Hannes Meinlschmidt is grateful for support of his former institution TU Darmstadt; Winnifried Wollner acknowledges funding by the DFG priority program 1962.

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