www.oeaw.ac.at

Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

Partial inversion of the 2D attenuated Radon transform with data on an arc

K. Sadiq, A. Tamasan

RICAM-Report 2017-41

www.ricam.oeaw.ac.at

PARTIAL INVERSION OF THE 2D ATTENUATED RADON TRANSFORM WITH DATA ON AN ARC

KAMRAN SADIQ AND ALEXANDRU TAMASAN

ABSTRACT. In two dimensions, we consider the problem of inversion of the attenuated X-ray transform of a compactly supported function from data restricted to lines leaning on a given arc. We provide a method to reconstruct the function on the subdomain encompassed by this arc. The attenuation is assumed known in this subdomain. The method of proof uses the range characterization in terms of a Hilbert transform associated with A-analytic functions in the sense of Bukhgeim.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain and Λ be an arc of its boundary $\varGamma$. The chord L joining the endpoints of the arc Λ partitions the domain in two subdomains $\Omega^{\pm}$, where Ω^+ denotes the domain enclosed by $\Lambda \cup L$. For a function f compactly supported in Ω, we provide a method to reconstruct $f|_{\Omega^+}$ from its attenuated X-ray transform over lines leaning on Λ; see Figure 1. The attenuation a is assumed known in Ω^+. Different partial data problems were studied in [2, 24].

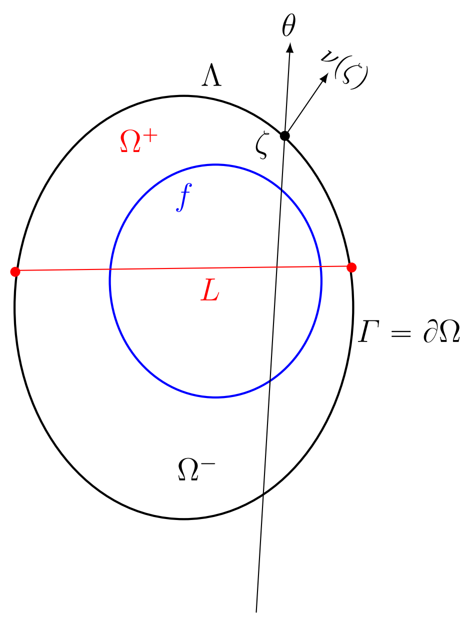

FIGURE 1. Geometric setup; $\Lambda = \overline{\Omega^+} \cap \partial\Omega$

Date: November 29, 2017.

2010 *Mathematics Subject Classification.* Primary 35J56, 30E20; Secondary 45E05.

Key words and phrases. Attenuated X-ray transform, Attenuated Radon transform, partial data, A-analytic maps, Hilbert transform.

For a real valued function $a \in L^1(\mathbb{R}^2)$, the attenuated X-ray transform of f is given by,

$$X_a f(z,\theta) := \int_{-\infty}^{\infty} f(z+s\theta) e^{-Da(z+s\theta,\theta)} ds, \quad (z,\theta) \in \Omega \times \mathbf{S}^1, \tag{1}$$

where

$$Da(z,\theta) := \int_0^{\infty} a(z+t\theta) dt \tag{2}$$

is the divergence beam transform of a. For the non attenuated case $a \equiv 0$ we use the notation Xf.

We approach the inversion problem through the known equivalence between the attenuated X-ray transform and the boundary value problems for the transport equation: Let $\varGamma_\pm := \{(\zeta,\theta) \in \varGamma \times \mathbf{S}^1 : \pm\nu(\zeta)\cdot\theta > 0\}$ denote the outgoing $(+)$, respectively incoming $(-)$ submanifolds of the unit tangent bundle of $\varGamma$, with $\nu(\zeta)$ being the outer normal at $\zeta \in \varGamma$ and θ is a direction in the unit sphere $\mathbf{S}^1$. If $u(z,\theta)$ is the unique solution to

$$\theta \cdot \nabla u(z,\theta) + a(z) u(z,\theta) = f(z) \quad (z,\theta) \in \Omega \times \mathbf{S}^1, \tag{3a}$$

$$u|_{\varGamma_-} = 0, \tag{3b}$$

then its trace on $\varGamma_+$ satisfies

$$u|_{\varGamma_+}(\zeta,\theta) = e^{Da(\zeta,\theta)} X_a f(\zeta,\theta), \quad (\zeta,\theta) \in \varGamma_+. \tag{4}$$

Let $\Lambda_\pm \subset \varGamma_\pm$ be the unit tangent subbundle on the arc Λ,

$$\Lambda_\pm := \{(\zeta,\theta) \in \Lambda \times \mathbf{S}^1 : \pm\nu(\zeta)\cdot\theta > 0\}. \tag{5}$$

In here, the data $X_a f$ is only available on Λ_+, which by (4) determines $u|_{\Lambda_+}$. Our main result (proven in Section 4.1) is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^2 boundary $\varGamma$ of strictly positive curvature, and $\Lambda \subset \varGamma$ be an arc of its boundary. Let $\Omega^+ \subset \Omega$ be the subset enclosed by Λ and by the chord L joining the endpoints of Λ. Let $\alpha > 1/2$, and $a \in C^{2,\alpha}(\overline{\Omega})$, be a real valued function known in Ω^+. Let $f \in C_0^{2,\alpha}(\Omega)$ be an unknown real valued function.*

Then the restriction of f in Ω^+ is reconstructed from the partial data $X_a f|_{\Lambda_+}$; see (36).

In the non-attenuated case, the uniqueness of $f|_{\Omega^+}$ follows as a consequence of the Helgason support theorem [8]. In the attenuated case, however, the uniqueness does not follow directly from Boman and Quinto [4] support theorem as the measure e^{-Da} need not be real analytic in the spatial variable.

The method of proof of Theorem 1.1 uses the range characterization in terms of a Hilbert transform [25, 26, 27] associated with A-analytic functions in the sense of Bukhgeim [5].

In the Euclidean setting, range characterization of the Radon transform in terms of the moment conditions has been long known [7, 8, 11]. Extension to non-Euclidean settings [28, 29] have fairly recent found range characterization in terms of the scattering relation in [22, 30, 21]. Of particular interest, the works in [13, 14] connect the moment conditions to the characterization given in terms of the scattering relation. For breakthrough works on the inversion of the attenuated X-ray transform in the Euclidean setting we refer to [1, 19]; see other approaches in [18, 3, 2].

2. A NEW PROPERTY OF A-ANALYTIC MAPS

For $z = x + iy$, let $\overline{\partial} = (\partial_x + i\partial_y)/2$, and $\partial = (\partial_x - i\partial_y)/2$ be the Cauchy-Riemann operators. For $0 < \alpha < 1$, $k = 1, 2$, and S some part of the boundary $\varGamma$, we use the Banach spaces:

$$
\begin{aligned}
l^{1,k}_{\infty}(S) &:= \left\{ \mathbf{u} = \langle u_{-1}, u_{-2}, ... \rangle : \sup_{\xi \in S} \sum_{j=1}^{\infty} j^k |u_{-j}(\xi)| < \infty \right\}, \\
Y_\alpha(S) &:= \left\{ \mathbf{u} \in l^{1,2}_{\infty}(S) : \sup_{\substack{\xi,\mu \in S \\ \xi \neq \mu}} \sum_{j=1}^{\infty} j \frac{|u_{-j}(\xi) - u_{-j}(\mu)|}{|\xi - \mu|^\alpha} < \infty \right\}, \qquad (6) \\
C^\alpha(S; l_1) &:= \left\{ \mathbf{u} : \sup_{\xi \in S} \|\mathbf{u}(\xi)\|_{l_1} + \sup_{\substack{\xi,\eta \in S \\ \xi \neq \eta}} \frac{\|\mathbf{u}(\xi) - \mathbf{u}(\eta)\|_{l_1}}{|\xi - \eta|^\alpha} < \infty \right\},
\end{aligned}
$$

where l_∞, l_1 are the space of bounded, respectively summable sequences. We similarly consider $C^\alpha(\overline{\Omega}; l_1)$, respectively, $C^\alpha(\overline{\Omega}; l_\infty)$.

The following regularity result in [25, Proposition 4.1] is needed.

Proposition 2.1. [25, Proposition 4.1] *Let $\alpha > 1/2$ and $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ... \rangle$ be the sequence valued map of non-positive Fourier modes of u.*
(i) If $u \in C^\alpha(\varGamma; C^{1,\alpha}(\mathbf{S}^1))$, then $\mathbf{u} \in l^{1,1}_{\infty}(\varGamma) \cap C^\alpha(\varGamma; l_1)$.
(ii) If $u \in C^\alpha(\varGamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C(\varGamma; C^{2,\alpha}(\mathbf{S}^1))$, then $\mathbf{u} \in Y_\alpha(\varGamma)$.

A sequence valued map $z \mapsto \mathbf{u}(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), ... \rangle$ is called $\mathcal{L}$-*analytic*, if $\mathbf{u} \in C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$ and

$$
\overline{\partial}\mathbf{u}(z) + \mathcal{L}\partial\mathbf{u}(z) = 0, \quad z \in \Omega, \qquad (7)
$$

where $\mathcal{L} : l_\infty \to l_\infty$ is the left shift: $\mathcal{L}\langle u_0, u_{-1}, u_{-2}, ... \rangle = \langle u_{-1}, u_{-2}, ... \rangle$.

Analogous to the analytic maps, the $\mathcal{L}$-analytic maps also enjoy a Cauchy integral formula [5]. For $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ... \rangle \in l^{1,1}_{\infty}(\varGamma) \cap C^\alpha(\varGamma; l_1)$, the Cauchy- Bukhgeim operator $\mathcal{B}$ acting on $\mathbf{u}$ is defined component-wise for $n \leqslant 0$ by

$$
(\mathcal{B}\mathbf{u})_n(z) := \frac{1}{2\pi i} \int_\varGamma \frac{u_n(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_\varGamma \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta - z}} \right\} \sum_{j=1}^{\infty} u_{n-j}(\zeta) \left(\frac{\overline{\zeta - z}}{\zeta - z} \right)^j, \ z \in \Omega; \qquad (8)
$$

The explicit form above follows [6].

Associated with the traces of $\mathcal{L}$-analytic maps, we recall the Hilbert-Bukhgeim transform $\mathcal{H}$ acting on $\mathbf{u} \in l^{1,1}_{\infty}(\varGamma) \cap C^\alpha(\varGamma; l_1)$ defined in [25] component-wise for $n \leqslant 0$ by

$$
(\mathcal{H}\mathbf{u})_n(\xi) = \frac{1}{\pi} \int_\varGamma \frac{u_n(\zeta)}{\zeta - \xi} d\zeta + \frac{1}{\pi} \int_\varGamma \left\{ \frac{d\zeta}{\zeta - \xi} - \frac{d\overline{\zeta}}{\overline{\zeta - \xi}} \right\} \sum_{j=1}^{\infty} u_{n-j}(\zeta) \left(\frac{\overline{\zeta - \xi}}{\zeta - \xi} \right)^j, \ \xi \in \varGamma. \qquad (9)
$$

The following result recalls the necessary and sufficient conditions for a sufficiently regular map to be the boundary value of an $\mathcal{L}$-analytic function.

Theorem 2.1. *Let $0 < \alpha < 1$. Let $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ... \rangle$ be a sequence valued map defined at the boundary $\varGamma$ and $\mathcal{B}$ be the Cauchy-Bukhgeim operator acting on $\mathbf{u}$ as in* (8).
(i) If $\mathbf{u} \in l^{1,1}_{\infty}(\varGamma) \cap C^\alpha(\varGamma; l_1)$, then $\mathcal{B}\mathbf{u} \in C^{1,\alpha}(\Omega; l_\infty) \cap C(\overline{\Omega}; l_\infty)$ is $\mathcal{L}$-analytic in Ω.
(ii) If $\mathbf{u} \in Y_\alpha$, for $\alpha > 1/2$, then $\mathcal{B}\mathbf{u} \in C^{1,\alpha}(\Omega; l_1) \cap C^\alpha(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty)$ is $\mathcal{L}$-analytic in Ω.

Moreover, for $\mathbf{u}$ *to be the boundary value of an* $\mathcal{L}$*-analytic function it is necessary and sufficient that*

$$(I + i\mathcal{H})\mathbf{u} = \mathbf{0}, \tag{10}$$

where $\mathcal{H}$ *is the Hilbert-Bukhgeim transform in* (9).

For the proof of part (i) and (ii) we refer to [26, Theorem 2.2 and Proposition 2.3]. For the proof of the last statement of the Theorem we refer to [25, Theorem 3.2 and Corollary 4.1].

Let $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle$ be the boundary value of an $\mathcal{L}$-analytic function in Ω^+. The new ingredient (Theorem 2.2) allows to recover the trace $\mathbf{u}|_L$ on L from $\mathbf{u}$ on Λ. The method of proof recovers separately the projections $P^{\pm}(u_n|_L)$, where

$$P^{\pm}g := \frac{1}{2}[I \pm iH]g \tag{11}$$

are the projections of $g \in C^{\alpha}(L)$ associated with the Hilbert transform

$$Hg(x) = \frac{1}{\pi}\int_L \frac{g(s)}{x-s}ds. \tag{12}$$

To simplify the statement of the result, let us introduce the functions $F_n(z)$ defined on Λ for each $n \leqslant 0$, by

$$F_n(z) := \pi i u_n(z) - \int_\Lambda \frac{u_n(\zeta)}{\zeta - z}d\zeta - \int_\Lambda \left\{\frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}}\right\}\sum_{j=1}^{\infty} u_{n-j}(\zeta)\left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^j, \; z \in \Lambda, \tag{13}$$

where the first integral is in the sense of principal value. Note that F_n is known since $u_n|_\Lambda$ are known for all $n \leqslant 0$. In the proof of the result below we shall see that F_n is the trace on Λ of an analytic function in the upper half plane. The arc Λ and the chord L are considered without their endpoints.

Theorem 2.2. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with* C^2 *boundary* Γ *of strictly positive curvature, and* $\Lambda \subset \Gamma$ *be an arc of its boundary. Let* $\Omega^+ \subset \Omega$ *be the subset enclosed by* Λ *and by the chord* L *joining the endpoints of* Λ *and* $0 < \alpha < 1$.

Let $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle \in l^{1,1}_{\infty}(\partial\Omega^+) \cap C^{\alpha}(\partial\Omega^+; l_1)$ *be the boundary value of an* $\mathcal{L}$*-analytic function in* Ω^+*. Then the trace* $\mathbf{u}|_L$ *on* L *is recovered from the trace* $\mathbf{u}|_\Lambda$ *on* Λ *as follows:*

(i) For each $n \leqslant 0$*, the projection* $P^-(u_n|_L)$ *is recovered pointwise by*

$$P^-(u_n)(x) = \frac{i}{2\pi}\int_\Lambda \frac{u_n(\zeta)}{x-\zeta}d\zeta - \frac{i}{2\pi}\int_\Lambda \left\{\frac{d\zeta}{\zeta - x} - \frac{d\overline{\zeta}}{\overline{\zeta} - x}\right\}\sum_{j=1}^{\infty} u_{n-j}(\zeta)\left(\frac{\overline{\zeta} - x}{\zeta - x}\right)^j, \; x \in L. \tag{14}$$

(ii) For each $n \leqslant 0$*, the projection* $g_n := P^+(u_n|_L)$ *is the unique solution in the range* $P^+(C^{\alpha}(L))$ *to the integral equation*

$$\int_L \frac{g_n(s)}{s-z}ds = F_n(z) - \int_L \frac{P^-(u_n)(s)}{s-z}ds, \quad z \in \Lambda, \tag{15}$$

where the right hand side is determined by the data from (13) *and* (14).

Proof. Upon a rotation and translation of the domain Ω, we may assume without loss of generality that the arc Λ lies in the upper half plane with the endpoints on the real axis. In particular, $\Omega \cap \{\mathbb{I}\mathrm{m}\, z = 0\} = L$.

Since $\mathbf{u} \in l^{1,1}_{\infty}(\partial\Omega^+) \cap C^{\alpha}(\partial\Omega^+; l_1)$ is the boundary value of an $\mathcal{L}$-analytic function in Ω^+, then Theorem 2.1 (i) yields

$$[I + i\mathcal{H}](\mathbf{u}|_{\partial\Omega^+}) = 0, \tag{16}$$

where $\mathcal{H}$ is the Hilbert-Bukhgeim transform in (9).

To prove the formula (14) we consider (16) on L, where for each $x \in L$ and $n \leqslant 0$, the n-th component yields

$$\begin{aligned} u_n(x) - \frac{i}{\pi}\int_L \frac{u_n(s)}{x-s}ds = \frac{i}{\pi}\int_\Lambda \frac{u_n(\zeta)}{x-\zeta}d\zeta - \frac{i}{\pi}\int_\Lambda \left\{\frac{d\zeta}{\zeta - x} - \frac{d\overline{\zeta}}{\overline{\zeta}-x}\right\}\sum_{j=1}^{\infty} u_{n-j}(\zeta)\left(\frac{\overline{\zeta}-x}{\zeta-x}\right)^j \\ - \frac{i}{\pi}\int_L \left\{\frac{d\zeta}{\zeta - x} - \frac{d\overline{\zeta}}{\overline{\zeta}-x}\right\}\sum_{j=1}^{\infty} u_{n-j}(\zeta)\left(\frac{\overline{\zeta}-x}{\zeta-x}\right)^j. \end{aligned} \tag{17}$$

We note that the last integral in (17) ranges over the reals, and thus it vanishes. The remaining integrals on the right hand side of (17) now depend only on the data on Λ, while the left hand side is $2P^-(u_n)$. The expression (14) follows.

Let us consider now (16) on Λ. With F_n as in (13), $n \leqslant 0$, the n-th component of (16) reads

$$F_n(z) = \int_L \frac{u_n(s)}{s-z}ds = \int_L \frac{P^+(u_n)(s)}{s-z}ds + \int_L \frac{P^-(u_n)(s)}{s-z}ds, \; z \in \Lambda. \tag{18}$$

This shows that $g_n := P^+(u_n)$ is indeed a solution to the Fredholm integral equation (15). Moreover, (18) also shows that F_n is the trace on Λ of a function $\widetilde{F}_n$ analytic in the upper half plane, and that

$$\int_L \frac{g_n(s)}{s-x-iy}ds = \widetilde{F}_n(x+iy) - \int_L \frac{P^-(u_n)(s)}{s-x-iy}ds, \quad x \in L, \; y > 0.$$

By letting $y \mapsto 0^+$, and using the Sokhotski-Plemelj formula (e.g., [16]) we obtain

$$P^+(g_n)(x) = \frac{1}{2\pi i}\widetilde{F}_n(x+i0) - P^+P^-(u_n)(x).$$

Moreover, since g_n is in the range of P^+, we get

$$g_n(x) = \frac{1}{2\pi i}\widetilde{F}_n(x+i0), \quad x \in L, \tag{19}$$

which shows uniqueness.

□

We remark that the recovery of $P^-(u_n|_L)$ is well-posed, whereas the recovery of $P^+(u_n|_L)$ is ill-posed. However, in recovering $P^+(u_n|_L)$ we do not use the unique continuation formula (19) but rather solve the Fredholm integral equation of first kind (15). Since the kernel is infinitely smoothing, this amounts to inverting a compact operator, which is an ill posed problem. In practice some regularization is necessary.

3. Inversion in the non-attenuated case

The reconstruction ideas can be easily understood in the non-attenuated case, $a \equiv 0$, which we treat separately.

Let us define (the negative divergence beam in the direction $-\theta$)

$$v(z,\theta) := \int_{-\infty}^{0} f(z+s\theta)ds, \quad (z,\theta) \in \overline{\Omega} \times \mathbf{S}^1, \tag{20}$$

so that it satisfies the transport equation

$$\theta \cdot \nabla_z v(z,\theta) = f(z), \quad z \in \Omega. \tag{21}$$

Since f is supported in Ω, the trace $v|_{\Lambda_+}$ yields the X-ray transform of f on lines intersecting Λ,

$$v(\zeta,\theta) = Xf(\zeta,\theta) = \int_{-\infty}^{\infty} f(\zeta+s\theta)ds, \quad (\zeta,\theta) \in \Lambda_+, \tag{22}$$

whereas the trace $v|_{\Lambda_-} = 0$.

Let $v(z,\theta) = v_{even}(z,\theta) + v_{odd}(z,\theta)$ be the decomposition of $\theta \mapsto v(z,\theta)$ into the even and odd part with respect to θ, for each $z \in \overline{\Omega}$. Since the trace of v on the incoming unit tangent bundle Γ_- is zero, so are the traces of v_{even} and v_{odd}. Moreover, since the right hand side of (21) is an even (constant) function of θ, then so is the left hand side, yielding $\theta \cdot \nabla_z v_{even}(z,\theta) = 0$. As v_{even} also vanish on the incoming tangent bundle it must be that $v_{even} \equiv 0$. Therefore $v = v_{odd}$ and f depends only on the odd Fourier modes of $v(z,\cdot)$. Furthermore, as f is real valued, then so is v, and

$$v(z,\theta(\varphi)) = 2\,\mathbb{R}\mathrm{e}\left\{ \sum_{n\leqslant -1} v_{2n+1}(z) e^{i(2n+1)\varphi} \right\}. \tag{23}$$

For v as in (20), let us consider the sequence valued map

$$\overline{\Omega} \ni z \mapsto \mathbf{v}(z) := \langle v_{-1}(z), v_{-3}(z), ...\rangle \tag{24}$$

of its odd negative Fourier modes.

The connection between the Radon transform and the theory of $\mathcal{L}$-analytic functions comes from the fact that $\mathbf{v}$ defined above is an $\mathcal{L}$-analytic map in Ω, see [5].

To reconstruct f in Ω^+ in this non-attenuated case, we apply Theorem 2.2 to the sequence valued map $\mathbf{v}$ in (24) playing the role of $\mathbf{u}$. i.e.,

$$u_n = v_{2n-1}, \quad n \leqslant 0.$$

As a result we now know $\mathbf{v}$ on $\Lambda \cup L$.

By using the Cauchy-Bukhgeim integral formula (8) we extend the Fourier mode v_{-1} from $\Lambda \cup L$ to Ω^+. From the uniqueness of an $\mathcal{L}$-analytic map with a given trace, we recover $v_{-1} = \mathcal{B}(\mathbf{v}|_{\Lambda\cup L})_{-1}$. The restriction of f in Ω^+ is then determined as in [5] by

$$f|_{\Omega^+}(z) = 2\,\mathbb{R}\mathrm{e}\{\partial v_{-1}(z)\}, \quad z \in \Omega^+. \tag{25}$$

We note that recovery of $\mathbf{v}$ on L requires inversion of a compact operator (15), which is an ill posed problem. This ill-posedness is natural, since the support theorems of both Helgason (in even dimensions) and Boman-Quinto are based on an unique analytic continuation argument.

4. Reconstruction in the attenuated case

We assume an attenuation $a \in C^{2,\alpha}(\overline{\Omega}),\ \alpha > 1/2$. As in [25] we start by the reduction to the non-attenuated case via the special integrating factor e^{-h}, where h is explicitly defined in terms of a by

$$h(z,\theta) := Da(z,\theta) - \frac{1}{2}\left(I - iH\right) Ra(z\cdot\theta^{\perp},\theta), \tag{26}$$

where $\theta^{\perp}$ is orthogonal to θ, $Ra(s,\theta) = \int_{-\infty}^{\infty} a\left(s\theta^{\perp} + t\theta\right) dt$ is the Radon transform of the attenuation, and the Hilbert transform H is taken in the first variable and evaluated at $s = z\cdot\theta^{\perp}$. The function $h(z,\cdot)$ extends analytically from $\mathbf{S}^1$ inside the unit disk as noted by Natterrer in [17]; see also [6, 3]. We use the properties of h summarized in the two lemmas below.

Lemma 4.1. [27, Lemma 4.1] *Assume $a \in C^{p,\alpha}(\overline{\Omega})$, $p = 1,2$, $\alpha > 1/2$, and h defined in* (26). *Then $h \in C^{p,\alpha}(\overline{\Omega}\times\mathbf{S}^1)$ and the following hold*

(i) h satisfies

$$\theta\cdot\nabla h(z,\theta) = -a(z), \quad (z,\theta)\in\Omega\times\mathbf{S}^1. \tag{27}$$

(ii) h has vanishing negative Fourier modes yielding the expansions

$$e^{-h(z,\theta)} := \sum_{k=0}^{\infty}\alpha_k(z)e^{ik\varphi},\ \ e^{h(z,\theta)} := \sum_{k=0}^{\infty}\beta_k(z)e^{ik\varphi},\ (z,\theta)\in\overline{\Omega}\times\mathbf{S}^1, \tag{28}$$

with (iii)

$$z \mapsto \langle\alpha_1(z),\alpha_2(z),\alpha_3(z),...,\rangle \in C^{p,\alpha}(\Omega;l_1)\cap C(\overline{\Omega};l_1),$$
$$z \mapsto \langle\beta_1(z),\beta_2(z),\beta_3(z),...,\rangle \in C^{p,\alpha}(\Omega;l_1)\cap C(\overline{\Omega};l_1).$$

In our problem a is assumed known in Ω^+, which yields h is known in $\overline{\Omega^+}\times\mathbf{S}^1$. In particular, α_n and β_n are known in $\overline{\Omega^+}$ for all $n \geqslant 0$.

From (27) it is easy to see that u is the unique solution of (3) if and only if

$$v(z,\theta) = e^{-h(z,\theta)}u(z,\theta), \quad (z,\theta)\in\Omega\times\mathbf{S}^1, \tag{29}$$

is the unique solution of

$$\theta\cdot\nabla v(z,\theta) = f(z)e^{-h(z,\theta)} \quad (z,\theta)\in\Omega\times\mathbf{S}^1, \tag{30a}$$

$$v|_{\Gamma_-} = 0. \tag{30b}$$

If $u(z,\theta) = \sum_{n=-\infty}^{\infty} u_n(z)e^{in\varphi}$ solves (3), then its Fourier modes satisfy

$$\overline{\partial}u_1(z) + \partial u_{-1}(z) + a(z)u_0(z) = f(z), \tag{31}$$

$$\overline{\partial}u_n(z) + \partial u_{n-2}(z) + a(z)u_{n-1}(z) = 0, \quad n \leqslant 0. \tag{32}$$

If $v(z,\theta) = \sum_{n=-\infty}^{\infty} v_n(z)e^{in\varphi}$ solves (30), then its Fourier modes satisfy

$$\overline{\partial}v_1(z) + \partial v_{-1}(z) = \alpha_0(z)f(z),$$
$$\overline{\partial}v_n(z) + \partial v_{n-2}(z) = 0, \quad n \leqslant 0, \tag{33}$$

where α_0 is the Fourier mode in (28).

The connection between (32) and (33) is intrinsic to negative Fourier modes only:

Lemma 4.2. [27, Lemma 4.2] *Assume* $a \in C^{1,\alpha}(\overline{\Omega}), \alpha > 1/2$.

(i) Let $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, ...\rangle \in C^1(\Omega, l_1)$ *satisfy* (33)*, and* $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle$ *be defined componentwise by the convolution*

$$u_n := \sum_{j=0}^{\infty} \beta_j v_{n-j}, \quad n \leqslant 0. \tag{34}$$

where $\beta_j' s$ *are the Fourier modes in* (28)*. Then* $\mathbf{u} \in C^1(\Omega, l_1)$ *solves* (32)*.*

(ii) Conversely, let $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle \in C^1(\Omega, l_1)$ *satisfy* (32)*, and* $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, ...\rangle$ *be defined componentwise by the convolution*

$$v_n := \sum_{j=0}^{\infty} \alpha_j u_{n-j}, \quad n \leqslant 0. \tag{35}$$

where $\alpha_j' s$ *are the Fourier modes in* (28)*. Then* $\mathbf{v} \in C^1(\Omega, l_1)$ *solves* (33)*.*

4.1. **Proof of the Theorem 1.1.** Recall that

$$u(z, \theta) = \sum_{n=-\infty}^{\infty} u_n(z) e^{in\varphi} \quad \text{and} \quad v(z, \theta) = \sum_{n=-\infty}^{\infty} v_n(z) e^{in\varphi}$$

are the solution in $\Omega \times \mathbf{S}^1$ to the boundary value problem (3), respectively (30), with $v = e^{-h}u$. Let $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle$ be the sequence valued map of non-positive Fourier modes of u, and let $\mathbf{v}^{odd} = \langle v_{-1}, v_{-3}, v_{-5}, ...\rangle$, $\mathbf{v}^{even} = \langle v_0, v_{-2}, v_{-4}, ...\rangle$ be the subsequences of negative odd, respectively, even Fourier modes of v.

Since $f \in C_0^{2,\alpha}(\Omega)$ and $a \in C^{2,\alpha}(\overline{\Omega})$, then $u, h, v \in C^{2,\alpha}(\overline{\Omega} \times \mathbf{S}^1)$. The traces $u|_{\partial\Omega^+ \times \mathbf{S}^1} \in C^{2,\alpha}(\partial\Omega^+; C^{2,\alpha}(\mathbf{S}^1))$ and $v|_{\partial\Omega^+ \times \mathbf{S}^1} \in C^{2,\alpha}(\partial\Omega^+; C^{2,\alpha}(\mathbf{S}^1))$. By applying Proposition 2.1 (ii) we obtain $\mathbf{u}, \mathbf{v}^{even}, \mathbf{v}^{odd} \in Y_\alpha(\partial\Omega^+)$ where Y_α is defined in (6).

By (4), the attenuated X-ray transform $X_a f$ on Λ_+ determines $\mathbf{u}$ on Λ. By formula (35), $\mathbf{u}|_\Lambda$ determines the traces $\mathbf{v}^{odd}|_\Lambda \in Y_\alpha(\Lambda)$ and $\mathbf{v}^{even}|_\Lambda \in Y_\alpha(\Lambda)$ on Λ.

The equations (33) show that $\mathbf{v}^{odd}$ and $\mathbf{v}^{even}$ are $\mathcal{L}$-analytic in Ω, thus in Ω^+. By applying Theorem 2.2 to $\mathbf{v}^{odd}|_\Lambda \in Y_\alpha(\Lambda)$ we recover the trace $\mathbf{v}^{odd}|_L$ on L. Similarly, we recover the trace $\mathbf{v}^{even}|_L$ on L.

The Cauchy-Bukhgeim integral formula (8) extends $\mathbf{v}^{odd}$ and $\mathbf{v}^{even}$ from $\Lambda \cup L$ to Ω^+ as $\mathcal{L}$-analytic maps. From the uniqueness of an $\mathcal{L}$-analytic map with a given trace, we recovered

$$\mathbf{v}^{even}(z) = \mathcal{B}[\mathbf{v}^{even}|_{\Lambda \cup L}](z), \text{ and } \mathbf{v}^{odd}(z) = \mathcal{B}[\mathbf{v}^{odd}|_{\Lambda \cup L}](z), \ z \in \Omega^+.$$

Thus $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, v_{-3}, ...\rangle$ is recovered in Ω^+.

Now use the convolution formula (34) for $n = 0$ and -1 to recover u_0 and u_{-1} in Ω^+.

Finally, f is reconstructed in Ω^+ by

$$f|_{\Omega^+}(z) = 2\,\mathbb{R}\mathrm{e}\{\partial u_{-1}(z)\} + a(z) u_0(z), \quad z \in \Omega^+. \tag{36}$$

ACKNOWLEDGMENT

We thank Plamen Stefanov for essential discussions on the uniqueness aspect of the problem. This work originated in discussions during the Radon 100 Conference at the Johann Radon Institute of Computational and Applied Mathematics, Linz, Austria, March 27-31, 2017. The second author wishes to thank the organizers for their hospitality. The work of A. Tamasan has been supported in part by the DARPA grant HR0011-16-C-0029.

REFERENCES

[1] E. V. Arzubov, A. L. Bukhgeim and S.G. Kazantsev, *Two-dimensional tomography problems and the theory of A-analytic functions*, Siberian Adv. Math. **8** (1998), 1–20.

[2] G. Bal, *On the attenuated Radon transform with full and partial measurements*, Inverse Problems **20** (2004),399–418.

[3] J. Boman and J.-O. Strömberg, *Novikov's inversion formula for the attenuated Radon transform–a new approach*, J. Geom. Anal. **14** (2004), 185–198.

[4] J. Boman and E. T. Quinto, *Support theorems for real-analytic Radon transforms*, Duke Math. J. **55**(1987), no.4, 943–948.

[5] A. L. Bukhgeim, Inversion Formulas in Inverse Problems, in Linear Operators and Ill-Posed Problems by M. M. Lavrentev and L. Ya. Savalev, Plenum, New York, 1995.

[6] D. V. Finch, *The attenuated x-ray transform: recent developments*, in Inside out: inverse problems and applications, Math. Sci. Res. Inst. Publ., 47, Cambridge Univ. Press, Cambridge, 2003, 47–66.

[7] I. M. Gelfand and M.I. Graev, *Integrals over hyperplanes of basic and generalized functions*, Dokl. Akad. Nauk SSSR **135** (1960), no.6, 1307–1310; English transl., Soviet Math. Dokl. **1** (1960), 1369–1372.

[8] S. Helgason, *An analogue of the Paley-Wiener theorem for the Fourier transform on certain symmetric spaces*, Math. Ann. **165** (1966), 297–308.

[9] S. Helgason, The Radon Transform, Birkhäuser, Boston, 1999.

[10] S. G. Kazantsev and A. A. Bukhgeim, *Inversion of the scalar and vector attenuated X-ray transforms in a unit disc*, J. Inverse Ill-Posed Probl., **15** (2007), 735–765.

[11] D. Ludwig, *The Radon transform on euclidean space*, Comm. Pure Appl. Math. **19** (1966), 49–81.

[12] F. Monard, *Efficient tensor tomography in fan-beam coordinates*, Inverse Probl. Imaging, **10(2)** (2016), 433–459.

[13] F. Monard, *Inversion of the attenuated geodesic X-ray transform over functions and vector fields on simple surfaces*, SIAM J. Math. Anal., **48(2)** (2016), 1155–1177.

[14] F. Monard, *Efficient tensor tomography in fan-beam coordinates. ii: attenuated transforms*, (2017) arXiv: 1704.08294

[15] F. Monard, P. Stefanov, and G. Uhlmann, *The geodesic X-ray transform on Riemannian surfaces with conjugate points*, Comm. Math. Phys., **337** (2015), 1491–1513.

[16] N.I. Muskhelishvili, Singular Integral Equations, Dover, New York, 2008.

[17] F. Natterer, The mathematics of computerized tomography, Wiley, New York, 1986.

[18] F. Natterer, *Inversion of the attenuated Radon transform*, Inverse Problems **17** (2001), 113–119.

[19] R. G. Novikov, *Une formule d'inversion pour la transformation d'un rayonnement X atténué*, C. R. Acad. Sci. Paris Sér. I Math., **332** (2001), 1059–1063.

[20] R. G. Novikov, *On the range characterization for the two-dimensional attenuated x-ray transformation*, Inverse Problems **18** (2002), no. 3, 677–700.

[21] G. P. Paternain, M. Salo, and G. Uhlmann, *Tensor Tomography: Progress and Challenges*, Chin. Ann. Math. Ser. B **35** (2014), no. 3, 399–428.

[22] L. Pestov and G. Uhlmann, *On characterization of the range and inversion formulas for the geodesic X-ray transform*, Int. Math. Res. Not. **80** (2004), 4331–4347.

[23] J. Radon, *Uber die Bestimmung von Funktionen durch ihre Integralwerte langs gewisser Mannigfaltigkeiten*, Ber. Verh. Sachs. Akad. Wiss. Leipzig, Math-Nat., **K** 1 **69** (1917), 262–267.

[24] H. Rullgård, *Stability of the inverse problem for the attenuated Radon transform with* 180° *data*, Inverse Problems **20**(2004), 781–797.

[25] K. Sadiq and A. Tamasan, *On the range of the attenuated Radon transform in strictly convex sets*, Trans. Amer. Math. Soc., **367(8)** (2015), 5375–5398.

[26] K. Sadiq and A. Tamasan, *On the range characterization of the two dimensional attenuated Doppler transform*, SIAM J. Math.Anal., **47(3)** (2015), 2001–2021.

[27] K. Sadiq, O. Scherzer, and A. Tamasan, *On the X-ray transform of planar symmetric 2-tensors*, J. Math. Anal. Appl. **442(1)** (2016), 31–49.

[28] V. A. Sharafutdinov, Integral geometry of tensor fields, VSP, Utrecht, 1994.

[29] V. A. Sharafutdinov, *The finiteness theorem for the ray transform on a Riemannian manifold,* Inverse Problems **11** (1995), pp. 1039-1050.

[30] M. Salo and G. Uhlmann, *The attenuated ray transform on simple surfaces*, J. Differential Geom. **88** (2011), no. 1, 161-187.

[31] P. Stefanov, G. Uhlmann, and A. Vasy, *Local and global boundary rigidity and the geodesic X-ray transform in the normal gauge*, preprint, arXiv:1702.03638v2, (2017).

[32] A. Tamasan, *Optical tomography in weakly anisotropic scattering media*, Contemporary Mathematics **333** (2003), 199–207.

[33] A. Tamasan, *Tomographic reconstruction of vector fields in variable background media* Inverse Problems **23** (2007), 2197–2205.

[34] G. Uhlmann and A. Vasy, *The inverse problem for the local geodesic ray transform*, Invent. Math., **205** (2016), 83–120.

JOHANN RADON INSTITUTE OF COMPUTATIONAL AND APPLIED MATHEMATICS (RICAM), ALTENBERGERSTRASSE 69, 4040 LINZ, AUSTRIA
E-mail address: kamran.sadiq@ricam.oeaw.ac.at

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, 32816 FLORIDA, USA
E-mail address: tamasan@math.ucf.edu