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Laurent Series Solutions of Algebraic Ordinary Differential Equations

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This paper concerns Laurent series solutions of algebraic ordinary differential equations (AODEs). We first present several approaches to compute formal power series solutions of a given AODE. Then we determine a bound for orders of its Laurent series solutions. Using the order bound, one can transform a given AODE into a new one whose Laurent series solutions are only formal power series. The idea is basically inherited from the Frobenious method for linear ordinary differential equations. As applications, new algorithms are presented for determining all particular polynomial and rational solutions of certain classes of AODEs.

1 Introduction

An algebraic ordinary differential equation (AODE) is of the following form

$$F(x, y, y', \ldots, y^{(n)}) = 0,$$

where F is a polynomial in $y, y', \ldots, y^{(n)}$ with coefficients in $\mathbb{K}(x)$, the field of rational functions over an algebraically closed field $\mathbb{K}$ of characteristic zero, and $n \in \mathbb{N}$. For instance, $\mathbb{K}$ can be the field of complex numbers, or the field of algebraic numbers. Many problems from applications (such as physics, combinatorics and statistics) can be characterized in terms of AODEs. Therefore, determining (closed form) solutions of an AODE is one of the central problems in mathematics and computer science.

Although linear ODEs [10] have been intensively studied, there are still many challenging problems for solving (nonlinear) AODEs. As far as we know, approaches for solving AODEs are only available for very specific subclasses. For example, Riccati equations,

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which have the form $y' = f_0(x) + f_1(x)y + f_2(x)y^2$ for some $f_0, f_1, f_2 \in \mathbb{K}(x)$, can be considered as the simplest form of nonlinear AODEs. In [13], Kovacic gives a complete algorithm for determining Liouvillian solutions of a Riccati equation with rational function coefficients. The study of general solutions without movable singularities can be found in [7, 15, 18] for first-order, and in [4, 10] for higher order AODEs.

Since the problem of solving an arbitrary AODE is very difficult, it is natural to ask whether the given AODE admits some special kinds of solutions, such as polynomial, rational function, formal power solutions. During the last two decades, an algebraic-geometric approach for finding symbolic solutions of AODEs have been developed. The work by Feng and Gao in [5, 6] for computing rational general solutions of first-order autonomous AODEs can be considered as the starting point. In [16, 8, 23, 22], the authors developed methods for finding different kinds of solutions of non-autonomous, higher order AODEs. For formal power series solutions, we refer to [3, 19].

As far as we know, there is little reference concerning Laurent series solutions of AODEs. In this paper, we gives a method for determining such solutions. The approach is an analogue of the Frobenius method for linear ODEs. Our results generalize the work of Vo, Winkler and Grasegger in [9], Behloul and Cheng in [1], Krushelnitskij in [14]. The main contribution of our work is to derive a bound (Theorem 3.2) for orders of Laurent series solutions of a given AODE. Once the order bound is given, one can transform the given AODE into a new one whose Laurent series solutions are always formal power series. In Section 2, we present several approaches (Theorem 2.5, Proposition 2.8 and 2.11) to calculate formal power series solutions of a given AODE. Theorem 3.2 has two applications: (i) In Section 4, we give a necessary condition for an AODE admitting a degree bound for its polynomial solutions. An AODE satisfying this condition is called *noncritical* (Definition 4.1). For noncritical AODEs, we always determine such a degree bound and then compute all polynomial solutions if there is any. We also show in Proposition 4.6 and 4.7 that two important classes of AODEs in applications are indeed noncritical. (ii) In Section 5, we prove in Theorem 5.3 that a class of AODEs having the property that the poles of their rational solutions are recognizable from their "highest" coefficients. Differential equations of this type are called *maximally comparable* (Definition 5.2). An algorithm for determining all rational solutions of a maximally comparable AODE is then followed. In Section 6, by doing statistical investigation, we show that all AODEs (around 834 examples) from Kamke's collection [11] are noncritical, and around 78.54% of them are maximally comparable.

2 Formal power series solutions

In this section, we focus on formal power series solutions of AODEs around the origin, as a point in $\mathbb{K}$ can always be translated to the origin. Firstly, we describe the algebraic structure of the set of all formal power series solutions for AODEs at the origin. In particular, we give a simplified proof of one specific case of [3, Lemma 2.3]. We also present one approach to calculate singular formal power series solutions of first-order AODEs. At last, we show that formal power series solutions at infinity of an AODE are equivalent to that at the origin of another AODE.

By $\mathbb{K}$ we mean an algebraically closed field of characteristic zero with the zero derivation. Let $\mathbb{K}[[x]]$ be the ring of formal power series with respect to x. For $f \in \mathbb{K}[[x]]$ and $k \in \mathbb{N}$, we use the notation $[x^k]f$ to refer the coefficient of x^k in f. Besides, we use $f^{(k)}$ to denote the k-th usual formal derivative [12, section 2.2] of f. A direct calculation implies that:

Lemma 2.1. *Let $f \in \mathbb{K}[[x]]$ and $k \in \mathbb{N}$. Then $[x^k]f = [x^0]\left(\frac{1}{k!}f^{(k)}\right)$.*

As a matter of notation, we use $\mathbb{K}[x]\{y\}$, (respectively $\mathbb{K}(x)\{y\}$), to denote the ring of differential polynomials in y with coefficients in the ring of polynomials $\mathbb{K}[x]$ (respectively the field of rational functions $\mathbb{K}(x)$), where the derivation of x is 1. A differential polynomial is of order $n \geq 0$ if the n-th derivative $y^{(n)}$ of y is the highest derivative appearing in it. Next, we recall a lemma in [21].

Lemma 2.2. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \geq 0$. Then for each $k \geq 1$, there exists a differential polynomial $R_k \in \mathbb{K}[x]\{y\}$ of order at most $n+k-1$ such that*

$$F^{(k)} = S_F \cdot y^{(n+k)} + R_k, \tag{1}$$

where $S_F := \frac{\partial F}{\partial y^{(n)}}$ is the the separant of F.

Consider an AODE $F(y) = 0$. If we substitute a formal power series $z \in \mathbb{K}[[x]]$ into $F(y)$, then $F(z)$ is still a formal power series with respect to x. The following proposition gives the coefficients of $F(z)$.

Proposition 2.3. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \geq 0$ and $z = \sum_{i=0}^{\infty} \frac{c_i}{i!}x^i \in \mathbb{K}[[x]]$. Then the following claims hold:*

(i) $[x^0]F(x, z, \ldots, z^{(n)}) = F(0, c_0, \ldots, c_n)$.

(ii) For each $k \geq 1$, we have

$$[x^k]F(x, z, \ldots, z^{(n)}) = \frac{1}{k!}\left(S_F(0, c_0, \ldots, c_n)c_{n+k} + R_k(0, c_0, \ldots, c_{n+k-1})\right),$$

where R_k is specified in (1).

Proof. (i) It is straightforward to see it by definition.

(ii) By Lemma 2.1, we have that

$$[x^k]F(x, z, \ldots, z^{(n)}) = [x^0]\left(\frac{1}{k!}F^{(k)}(x, z, \ldots, z^{(n+k)})\right). \tag{2}$$

On the other hand, it follows from Lemma 2.2 that

$$[x^0]F^{(k)}(x, z, \ldots, z^{(n+k)}) = S_F(0, c_0, \ldots, c_n)c_{n+k} + R_k(0, c_0, \ldots, c_{n+k-1}). \tag{3}$$

Above all, we conclude from (2) and (3) that the claim holds. □

As a matter of notation, we set $C = \mathbb{K}[c_0, c_1, \dots]$ to be the ring of commutative polynomials, where $c_0, c_1, \dots$ are algebraically independent variables.

Definition 2.4. *Let $F \in \mathbb{K}[x]\{y\} \setminus \{0\}$ be of order $n \geq 0$ and $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i \in C[[x]]$. For each $m \in \mathbb{N}$, we call*

$$J_m(F) = \langle [x^0]F(0, z, \dots, z^{(n)}), \dots, [x^m]F(0, z, \dots, z^{(n)}) \rangle$$

the m-th jet ideal *of F in C. When F is clear from context, we simply denote $J_m(F)$ by J_m.*

The above definition is compatible with [17, Definition 2.5]. Proposition 2.3 gives the explicit formula for generators of each jet ideal of a given differential polynomial. The following theorem presents the structure of formal power series solutions of an AODE at the origin.

Theorem 2.5. *Let $F \in \mathbb{K}[x]\{y\} \setminus \{0\}$ be of order $n \geq 0$. Then the following claims hold:*

(i) the sequence $\{J_m\}_{m=0}^{\infty}$ is an ascending chain.

(ii) the set of formal power series solutions of F at the origin is in bijection with the zero set of $\cup_{m=0}^{\infty} J_m$.

(iii) If $(c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ satisfies $F(0, c_0, \dots, c_n) = 0$ and $S_F(0, c_0, \dots, c_n) \neq 0$, then there is a unique formal power series solution $z \in \mathbb{K}[[x]]$ of F such that

$$z \equiv c_0 + c_1 x + \cdots + c_n x^n \quad \textit{mod} \quad x^{n+1}.$$

Proof. (i) By Definition 2.4, it is straightforward to see that $J_m \subset J_{m+1}$ for $m \in \mathbb{N}$.

(ii) Assume that $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$ is solution of $F(0, z, \dots, z^{(n)}) = 0$, which means

$$[x^k]F(0, z, \dots, z^{(n)}) = 0 \quad \text{for each} \quad k \in \mathbb{N}.$$

It is equivalent to that $(c_0, c_1, \dots) \in \mathbb{K}^{\mathbb{N}}$ is a zero of $\cup_{i=0}^{\infty} J_m$.

(iii) For each $k \geq 1$, since $S_F(c_0, \dots, c_n) \neq 0$, we set

$$c_{n+k} = -\frac{R_k(0, c_0, \dots, c_{n+k-1})}{S_F(0, c_0, \dots, c_n)},$$

where R_k is specified in (1). Set $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$. By Proposition 2.3 (ii), we have

$$[x^k]F(x, z, \dots, z^{(n)}) = 0.$$

On the other hand, it follows from Proposition 2.3 (i) and assumption that

$$[x^0]F(x, z, \dots, z^{(n)}) = 0.$$

Thus, we conclude that z is a solution of $F(x, y, \dots, y^{(n)}) = 0$. The uniqueness is clear from Proposition 2.3 (ii). □

Item (iii) of the above theorem is a generalization of Implicit Function Theorem [12, Theorem 2.9] for algebraic equations. It implies that if the separant of a given differential polynomial does not vanish at some initial values of a given formal power series, then the zero of m-th jet ideal is uniquely determined by that of the 0-th jet ideal, where $m \in \mathbb{N}$. We can use it to describe the algebraic structure of formal power series solutions for a class of AODEs.

Example 2.6. *Consider the following AODE*

$$y^{(n)} = P(x, y, \dots, y^{(n-1)}), \tag{4}$$

where $n \in \mathbb{N}$, P is a polynomial in n variables. The separant of (4) is 1. Therefore, it follows from Theorem 2.5 (ii), (iii) that the set of formal power series solutions of (4) at the origin is in bijection with the hypersurface

$$\{(c_0, c_1, \dots, c_n) \in \mathbb{K}^{n+1} \mid c_n = P(0, c_0, \dots, c_{n-1})\}.$$

When the separant of a given AODE vanishes at some initial values of a given formal power series, various cases [12, Page 119] will happen.

For algebraic equations, the set of formal power series solutions can be determined by computing their Puiseux series solutions [24]. The corresponding implementation is available by the command `"gfun[algeqtoseries]"` in `Maple`.

For linear ODEs, we can compute formal power series solutions [10] by solving a system of linear equations and a P-recursive equation. There are also algorithms [2, 20] to calculate formal power series solutions of linear PDEs with finite-dimensional solution spaces.

Definition 2.7. *Let $F \in \mathbb{K}[x]\{y\} \setminus \{0\}$ be of order $n \geq 0$. A solution $y(x)$ of the AODE $F(y) = 0$ is called a* regular solution *if $S_F(x, y(x), \dots, y^{(n)}(x)) \neq 0$. Otherwise, it is called a* singular solution*.*

Item (iii) of Theorem 2.5 gives one method for computing a class of regular formal power series solutions of a given AODE. In the literature, we do not find suitable papers concerning singular formal power series of an AODE. Next, we present an approach to calculate singular formal power series solutions of first-order AODEs.

Proposition 2.8. *Let F be an irreducible polynomial in $\mathbb{K}[x, y, y'] \setminus \mathbb{K}[x, y]$. Then there exists a finite set $G \subset \mathbb{K}[x, y]$ such that z is a singular solution of $F(x, y, y') = 0$ if and only if $g(x, z) = 0$ for some $g \in G$.*

Proof. We first construct a finite set $G \subset \mathbb{K}[x, y]$ and then prove that G satisfies the property as claimed above.

Set $m(x, y) = \text{Res}_{y'}(F, S_F)$, which is the resultant of F and S_F with respect to y'. Since F is irreducible, $m(x, y)$ is a nonzero polynomial in $\mathbb{K}[x, y]$. Without loss of generality, we may assume that $\deg_y(m) > 0$ (otherwise, we set $G = \{m\}$). Let $m_1, \dots, m_k$ be the distinct irreducible factors of m in $\mathbb{K}[x, y]$ with positive degree in y.

For each $i \in \{1, \ldots, k\}$, we set $m_{i1} = \frac{\partial m_i}{\partial x}$ and $m_{i2} = \frac{\partial m_i}{\partial y}$. Since m_i is irreducible, there exists $u_i, v_i \in \mathbb{K}(x)[y]$ such that

$$u_i m_i + v_i m_{i2} = 1.$$

Set $F_i = F(x, y, -v_i m_{i1})$ and $S_{F,i} = S_F(x, y, -v_i m_{i1})$. Let $g_i = \gcd(m_i, F_i, S_{F,i})$ in $\mathbb{K}(x)[y]$. By clearing denominators, we may further assume that $g_i \in \mathbb{K}[x, y]$. Set $G = \{g_1, \ldots, g_k\}$.

Assume that z is a singular solution of $F(x, y, y') = 0$. Since $m(x, y)$ is equal to $\operatorname{Res}_{y'}(F, S_F)$, we have $m(x, z) = 0$. Therefore, there exists $i \in \{1, \ldots, k\}$ such that $m_i(x, z) = 0$. By the argument in [12, Page 117], we have that

$$z' = -v(x, z) m_{i1}(x, z). \tag{5}$$

Since z is a singular solution of $F(y) = 0$, it follows that

$$F(x, z, z') = 0 \quad \text{and} \quad S_F(x, z, z') = 0. \tag{6}$$

Substituting (5) into (6), we have

$$F_i(x, z) = 0 \quad \text{and} \quad S_{F,i}(x, z) = 0.$$

Since $g_i = \gcd(m_i, F_i, S_{F,i})$ in $\mathbb{K}(x)[y]$, we conclude from Bézout's identity that

$$g_i(x, z) = 0.$$

Conversely, assume that z is a solution of $g_i(x, y) = 0$ for some $i \in \{1, \ldots, k\}$. Since $g_i = \gcd(m_i, F_i, S_{F,i})$ in $\mathbb{K}(x)[y]$, we have that $m_i(x, z) = 0$. By the argument in [12, Page 117], we have that

$$z' = -v(x, z) m_{i1}(x, z). \tag{7}$$

Since $g_i = \gcd(m_i, F_i, S_{F,i})$ in $\mathbb{K}(x)[y]$, it follows from definitions of F_i and $S_{F,i}$ that

$$F(x, z, -v_i(x, z) m_{i1}(x, z)) = 0 \quad \text{and} \quad S_F(x, z, -v_i(x, z) m_{i1}(x, z)) = 0. \tag{8}$$

Substituting (7) into (8), we have that

$$F(x, z, z') = 0 \quad \text{and} \quad S_F(x, z, z') = 0. \qquad \square$$

Since we can calculate formal power series solutions of algebraic equations by using Implicit Function Theorem or computing Puiseux series solutions, the above proposition gives rise to an algorithm to compute singular formal power series solutions of a given first-order AODE.

Example 2.9. *Consider the following first-order AODE [21, Example 5.4.3]:*

$$F(x, y, y') = (y')^3 - x y^4 y' - y^5 = 0.$$

By computation, we find that $G = \{y, 4x^3y^2 - 27\}$ satisfies the properties in Proposition 2.8. It is straightforward to see that $z_1 = 0$ is the solution of $y = 0$. Furthermore, we find that $z_2 = \frac{3\sqrt{3}}{2} x^{-\frac{3}{2}}$ and $z_3 = -\frac{3\sqrt{3}}{2} x^{-\frac{3}{2}}$ are solutions of $4x^3y^2 - 27 = 0$. By Proposition 2.8, we conclude that $F(y) = 0$ has only one singular formal power series solution z_1.

Example 2.10. *Consider the following first-order AODE [11, Example 1.537]:*

$$F(x, y, y') = (xy' - y)^3 + x^6 y' - 2x^5 y = 0.$$

By computation, we find that $G = \{1\}$ satisfies the properties in Proposition 2.8. Therefore, we conclude from Proposition 2.8 that $F(y) = 0$ has no singular solution.

In the literature, we do not find suitable papers to cover the formal power series solutions of a given AODE at infinity. Next, we show that it can be turned into the problem of finding formal power series solutions at the origin.

Proposition 2.11. *Let $F \in \mathbb{K}[x]\{y\} \setminus \{0\}$ be of order $n \geq 0$. Then there exists another differential polynomial $\bar{F} \in \mathbb{K}[x]\{y\} \setminus \{0\}$ of order n such that $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^{-i}$ is a solution of $F(y) = 0$ if and only if $t = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i$ is a solution of $\bar{F}(y) = 0$.*

Proof. We first construct a nonzero differential polynomial $\bar{F}$ of order n and then prove that $\bar{F}$ satisfies the property as claimed above.

Set $\bar{x} = x^{-1}$. Assume that $z(x) = t(\bar{x})$ is a solution of $F(y) = 0$, where $t(x)$ is a differentiable function in x. By the chain rule, we have that

$$z^{(k)}(x) = (-1)^k x^{-2k} t^{(k)}(\bar{x}) + R_k(\bar{x}, t(\bar{x}), \ldots, t^{(k-1)}(\bar{x})), \tag{9}$$

where R_k is a polynomial in k variables, $0 \leq k \leq n$. Substituting (9) into $F(z)$, we have

$$F(z) = F(x, t(\bar{x}), \ldots, (-1)^n x^{-2n} t^{(n)}(\bar{x}) + R_n(\bar{x}, t(\bar{x}), \ldots, t^{(n-1)}(\bar{x}))).$$

Let $m \in \mathbb{Z}$ be the largest exponent of x in the right side of the above equation. Then we can write $x^{-m} F(z)$ in the following form:

$$x^{-m} F(z) = \bar{F}(\bar{x}, t(\bar{x}), \ldots, t^{(n)}(\bar{x})),$$

where $\bar{F}$ is a nonzero differential polynomial of order n.

Assume that $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^{-i}$ is a solution of $F(y) = 0$. Set $t = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i$. Then we have $z(x) = t(\bar{x})$. It follows from the construction of $\bar{F}$ that

$$\bar{F}(\bar{x}, t(\bar{x}), \ldots, t^{(n)}(\bar{x})) = 0.$$

Therefore, $t(x)$ is a solution of $\bar{F}(y) = 0$.

Conversely, assume that $t = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i$ is a solution of $\bar{F}(y) = 0$. Set $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^{-i}$. Then we have $t(x) = z(\bar{x})$. Similarly as in the construction of $\bar{F}$, we conclude that $z(x)$ is a solution of $F(y) = 0$.

□

Example 2.12. *Consider the following first-order AODE in Example 2.9:*

$$F(x, y, y') = (y')^3 - xy^4 y' - y^5 = 0.$$

Assume that $z(x) = t(\bar{x})$ is a solution of the above equation, where $\bar{x} = x^{-1}$. Following the construction in Proposition 2.11, we find that $t(x)$ satisfies the following first-order AODE:

$$\bar{F}(x, y, y') = x^6(y')^3 - xy^4y' + y^5.$$

By Proposition 2.11, we conclude that $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^{-i}$ is a solution of $F(y) = 0$ if and only if $t = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i$ is a solution of $\bar{F}(y) = 0$.

Example 2.13. *Consider the following first-order AODE in Example 2.10:*

$$F(x, y, y') = (xy' - y)^3 + x^6y' - 2x^5y = 0.$$

Assume that $z(x) = t(\bar{x})$ is a solution of the above equation, where $\bar{x} = x^{-1}$. Following the construction in Proposition 2.11, we find that $t(x)$ satisfies the following first-order AODE:

$$\bar{F}(x, y, y') = x^5(xy' + y)^3 + xy' + 2y.$$

By Proposition 2.11, we conclude that $z = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^{-i}$ is a solution of $F(y) = 0$ if and only if $t = \sum_{i=0}^{\infty} \frac{c_i}{i!} x^i$ is a solution of $\bar{F}(y) = 0$.

3 Laurent series solutions

Given an AODE, we show in Theorem 3.2 that the orders of its Laurent series solutions can be bounded in an algorithmic way. Whenever the bound is computed, one can transform the given AODE into a new one whose Laurent series solutions are only formal power series. Therefore, the results in Section 2 are applicable.

Given $x_0 \in \mathbb{K} \cup \{\infty\}$, a Laurent series f at $x = x_0$ has the form

$$\begin{array}{ll} \sum\limits_{k=m}^{\infty} c_k (x - x_0)^k & \text{if} \;\; x_0 \in \mathbb{K}, \\ \sum\limits_{k=m}^{\infty} c_k x^{-k} & \text{if} \;\; x_0 = \infty, \end{array}$$

where $c_k \in \mathbb{K}, c_m \neq 0$ and $m \in \mathbb{Z}$. We call $-m$ *the order* of f (at $x = x_0$), and denote it by $\mathrm{ord}_{x_0}(f)$. The coefficient c_m is called *the lowest coefficient* of f (at $x = x_0$), and denoted by $c_{x_0}(f)$. Then we can rewrite f as follows:

$$\begin{array}{llll} c_{x_0}(f)(x - x_0)^{-\mathrm{ord}_{x_0}(f)} & + & \text{higher terms in } (x - x_0) & \text{if} \;\; x_0 \in \mathbb{K}, \\ c_{\infty}(f) x^{\mathrm{ord}_{x_0}(f)} & + & \text{higher terms in } x & \text{if} \;\; x_0 = \infty. \end{array}$$

For each $I = (i_0, i_1, \ldots, i_n) \in \mathbb{N}^{n+1}$ and $r \in \{0, \ldots, n\}$, we set $||I||_r := i_r + \ldots + i_n$. We simply write $||I||_0$ by $||I||$. Furthermore, the notation $||I||_\infty := i_1 + 2i_2 + \ldots + ni_n$ will be also used frequently.

Let $F(y) = \sum\limits_{I \in \mathbb{N}^{n+1}} f_I(x) y^{i_0} (y')^{i_1} \cdots (y^{(n)})^{i_n} \in \mathbb{K}(x)\{y\}$ be a differential polynomial of order n. We will use the following notations:

$$\begin{aligned} \mathcal{E}(F) &:= \{I \in \mathbb{N}^{n+1} \,|\, f_I \neq 0\}, \\ d(F) &:= \max\{||I|| \,|\, I \in \mathcal{E}(F)\}, \\ \mathcal{D}(F) &:= \{I \in \mathcal{E}(F) \,|\, ||I|| = d(F)\}. \end{aligned}$$

Moreover, for each $x_0 \in \mathbb{K}$, we denote

$$\begin{aligned} m_{x_0}(F) &:= \max\{\operatorname{ord}_{x_0} f_I + ||I||_\infty \mid I \in \mathcal{D}(F)\}, \\ \mathcal{M}_{x_0}(F) &:= \{I \in \mathcal{D}(F) \mid \operatorname{ord}_{x_0} f_I + ||I||_\infty = m_{x_0}(F)\}, \\ \mathcal{P}_{x_0,F}(t) &:= \sum_{I \in \mathcal{M}_{x_0}(F)} c_{x_0}(f_I) \cdot \prod_{r=0}^{n-1} (-t-r)^{||I||_{r+1}}, \end{aligned}$$

and if $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$, we set

$$b_{x_0}(F) := \max\left\{ \frac{\operatorname{ord}_{x_0} f_I + ||I||_\infty - m_{x_0}(F)}{d(F) - ||I||} \mid I \in \mathcal{E}(F) \setminus \mathcal{D}(F) \right\}.$$

In case that $x_0 = \infty$, we also denote

$$\begin{aligned} m_\infty(F) &:= \max\{\operatorname{ord}_\infty f_I - ||I||_\infty \mid I \in \mathcal{D}(F)\}, \\ \mathcal{M}_\infty(F) &:= \{I \in \mathcal{D}(F) \mid \operatorname{ord}_\infty f_I - ||I||_\infty = m_\infty(F)\}, \\ \mathcal{P}_{\infty,F}(t) &:= \sum_{I \in \mathcal{M}_\infty(F)} c_\infty(f_I) \cdot \prod_{r=0}^{n-1} (t-r)^{||I||_{r+1}}, \end{aligned}$$

and

$$b_\infty(F) := \max\left\{ \frac{\operatorname{ord}_\infty f_I - ||I||_\infty - m_\infty(F)}{d(F) - ||I||} \mid I \in \mathcal{E}(F) \setminus \mathcal{D}(F) \right\}$$

if $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$.

Definition 3.1. *Let $F(y) \in \mathbb{K}(x)\{y\}$ be a differential polynomial of order n. For each $x_0 \in \mathbb{K} \cup \{\infty\}$. We call $\mathcal{P}_{x_0,F}$ the* indicial polynomial *of F at $x = x_0$.*

Note that the above definition is a generalization of the usual indicial polynomials [2, 10] of linear ODEs.

Theorem 3.2. *Given an AODE $F(y) = 0$, and $x_0 \in \mathbb{K} \cup \{\infty\}$. If $r \geq 1$ is the order of a Laurent series solution of $F(y) = 0$ at $x = x_0$, then one of the following claims hold:*

(i) $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$, and $r \leq b_{x_0}(F)$;

(ii) r is a positive integer root of $\mathcal{P}_{x_0,F}(t)$.

Proof. Let $F(y) = \sum_{I \in \mathbb{N}^{n+1}} f_I(x) y^{i_0} (y')^{i_1} \ldots (y^{(n)})^{i_n} \in \mathbb{K}(x)\{y\}$ be a differential polynomial of order n. Let $x_0 \in \mathbb{K}$ and $z \in \mathbb{K}((x - x_0)) \setminus \mathbb{K}$ be a Laurent series solution of $F(y) = 0$ of order $r \geq 1$. Then $z^{(k)}$ is of order $k + r$ for each $k \in \mathbb{N}$. For each $I \in \mathcal{E}(F)$, we may write the coefficient f_I in the following form:

$$f_I = \frac{c_{x_0}(f_I)}{(x - x_0)^{\operatorname{ord}_{x_0} f_I}} + h_I,$$

where $h_I \in \mathbb{K}((x))$ and $\operatorname{ord}_{x_0} h_I < \operatorname{ord}_{x_0} f_I$. Since z is a solution of $F(y) = 0$, we have

$$\begin{aligned} 0 &= F(z) \\ &= S_1 + S_2 + S_3 + S_4, \end{aligned}$$

where

$$S_1 = \sum_{I \in \mathcal{M}_{x_0}(F)} \frac{c_{x_0}(f_I)}{(x-x_0)^{\mathrm{ord}_{x_0} f_I}} \cdot z^{i_0}(z')^{i_1} \cdots (z^{(n)})^{i_n}, \qquad S_2 = \sum_{I \in \mathcal{M}_{x_0}(F)} h_I \cdot z^{i_0}(z')^{i_1} \cdots (z^{(n)})^{i_n},$$

$$S_3 = \sum_{I \in \mathcal{D}(F) \setminus \mathcal{M}_{x_0}(F)} f_I \cdot z^{i_0}(z')^{i_1} \cdots (z^{(n)})^{i_n}, \qquad S_4 = \sum_{I \in \mathcal{E}(F) \setminus \mathcal{D}(F)} f_I \cdot z^{i_0}(z')^{i_1} \cdots (z^{(n)})^{i_n}.$$

The orders of terms in S_1 are equal to $D := d(F)r + m_{x_0}(F)$, which is strictly larger than that of terms in S_2 and S_3. One of the two following cases will happen:

Case 1: The order of S_1 is equal to D. Then the term of order D in S_1 must be killed by terms of S_4. In this case, we have $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$. By comparing with the orders of terms in S_4, we obtain

$$D \leq \max \{ ||I|| \cdot r + ||I||_\infty + \mathrm{ord}_{x_0} f_I \mid I \in \mathcal{E}(F) \setminus \mathcal{D}(F) \} .$$

On the other hand, since $D = d(F)r + m_{x_0}(F)$, we conclude that

$$r \leq \max \left\{ \frac{||I||_\infty + \mathrm{ord}_{x_0} f_I - m_{x_0}(F)}{d(F) - ||I||} \mid I \in \mathcal{E}(F) \setminus \mathcal{D}(F) \right\} .$$

i. e. , $r \leq b_{x_0}(F)$.

Case 2: The order of S_1 is strictly smaller than D. For each $k \in \mathbb{N}$, a direct computation implies that the lowest coefficient $z^{(k)}$ at $x = x_0$ is

$$c_{x_0}(z^{(k)}) = c_{x_0}(z) \prod_{s=1}^{k} (r - s + 1).$$

Therefore, the lowest coefficient of the term indexed by $I \in \mathcal{M}_{x_0}(F)$ in S_1 is

$$c_{x_0}(f_I) \cdot \prod_{k=0}^{n} \left(c_{x_0}(z) \prod_{s=1}^{k} (r - s + 1) \right)^{i_k} = c_{x_0}(f_I) c_{x_0}(y)^{||I||} \prod_{s=1}^{n} (r - s + 1)^{||I||_s}.$$

Since the orders of terms in S_1 are the same and they are strictly larger than that of S_1, the sum of those lowest coefficients must be zero. In other words, we have

$$\sum_{I \in \mathcal{M}_{x_0}(F)} c_{x_0}(f_I) c_{x_0}(y)^{||I||} \prod_{s=1}^{n} (r - s + 1)^{||I||_s} = 0.$$

The left side of the above equality is exactly $c_{x_0}(y)^{d(F)} \cdot \mathcal{P}_{x_0,F}(r)$. Hence, r is a positive integer root of $\mathcal{P}_{x_0,F}(r)$.

The case that $x_0 = \infty$ can be proved in a similar way. □

For a linear homogeneous ordinary differential equation $F(y) = 0$, item (i) of the above theorem will never happen because $\mathcal{E}(F) = \mathcal{D}(F)$.

Example 3.3. *Consider the following first-order AODE:*

$$F(y) = xy' + x^2y^2 + y - 1 = 0. \tag{10}$$

We calculate the order bound for Laurent series solutions of the above equation at the origin. The following table is the list of the exponents of terms of F and related information.

$I \in \mathcal{E}(F)$	$\|\|I\|\|$	$\|\|I\|\|_\infty$	f_I	$\mathrm{ord}_0\, f_I$
$(2,0)$	2	0	x^2	-2
$(0,1)$	1	1	x	-1
$(1,0)$	1	0	1	0
$(0,0)$	0	0	-1	0

Based on the above table, a direct computation implies that $\mathcal{P}_{0,F}(t) = 1$ and $\mathrm{b}_0(F) = 2$. According to Theorem 3.2, the order bound at the origin is 2.

Assume that $y = \frac{1}{x^2}z$, where $z = \sum_{i=0}^{\infty} \frac{c_i}{i!}x^i \in \mathbb{K}[[x]]$ with $c_0 \neq 0$. Substituting this ansatz into (10), we have

$$G(z) = xz' + z^2 - z - x^2 = 0.$$

Since the separant of G is x, we can not apply item (iii) of Theorem 2.5. Nevertheless, we observe from induction that

$$G^{(k)} = xz^{(k+1)} + (2z + k - 1)z^{(k)} + R_{k-1}, \tag{11}$$

where R_{k-1} is a differential polynomial in $\mathbb{K}[x]\{z\}$ of order $k-1$, $k \geq 1$. From $[x^0]G = 0$, we have that $c_0 = 1$. Using (11) and Lemma 2.1, we conclude that $F(y) = 0$ has only one Laurent series solution of order 2 at the origin, which has the following form:

$$y = \frac{1}{x^2} + \frac{1}{x}\left(\sum_{i=1}^{\infty} \frac{c_i}{i!}x^i\right),$$

where $c_k = \frac{1}{k+1}R_{k-1}(0, c_0, \ldots, c_{k-1})$, $k \geq 1$.

Assume that $y = \frac{1}{x}w$, where $w = \sum_{i=0}^{\infty} \frac{c_i}{i!}x^i \in \mathbb{K}[[x]]$ with $c_0 \neq 0$. Substituting this ansatz into (10), we have

$$H(w) = w' + w^2 - 1 = 0.$$

Since the separant of G is 1, we conclude from Example 2.6 that the set of Laurent series solutions of $F(y) = 0$ of order 1 at the origin is in bijection with the set

$$\{(c_0, c_1) \in \mathbb{K}^2 \setminus \{(0,1)\} \mid c_1 = -(c_0)^2 + 1\}.$$

In the above example, the order bound provided by Theorem 3.2 is sharp. However, the following example shows that, in general, it is not true.

Example 3.4. *Consider the following linear ODE:*

$$F(y) = x^2 y'' + 4xy' + (2+x)y = 0. \tag{12}$$

We compute the order bound for Laurent series solutions of the above equation at the origin. The following table is the list of the exponents of terms of F and related information.

$I \in \mathcal{E}(F)$	$\|\|I\|\|$	$\|\|I\|\|_\infty$	f_I	$\mathrm{ord}_0 f_I$
$(0,0,1)$	1	2	x^2	-2
$(0,1,0)$	1	1	$4x$	-1
$(1,0,0)$	1	0	$2+x$	0

Based on the above table, we find that $\mathcal{P}_{0,F}(t) = (t-1)(t-2)$*. According to Theorem 3.2, the order bound at the origin is* 2*.*

Assume that $y = \sum_{i=-\infty}^{\infty} c_i x^i \in \mathbb{K}((x))$ *and substitute it into (12), we get*

$$(1+i)(2+i)c_i + c_{i-1} = 0 \quad \textit{for each} \quad i \in \mathbb{Z}. \tag{13}$$

The above recurrence equation implies that if $F(y) = 0$ *has Laurent series solution at the origin, then the order must be* 1 *or* 2*.*

Substituting $i = -1$ *into (13), we have* $c_{-2} = 0$*. So,* $F(y) = 0$ *does not have Laurent series solution at the origin of order* 2*.*

Assume that $c_{-1} = 1$*, we conclude from (13) that* $F(y) = 0$ *has a Laurent series solution of order* 1 *with the following form:*

$$\sum_{i=-1}^{\infty} (-1)^{i+1} \frac{x^i}{(1+i)!(2+i)!}.$$

Consider an AODE $F(y) = 0$. If $\mathcal{E}(F) = \mathcal{D}(F)$ and the indicial polynomial $\mathcal{P}_{x_0,F}(t)$ is zero, then Theorem 3.2 does not give information for the order bound of Laurent series solution of $F(y) = 0$ at $x = x_0$. In the next section, we will give an example (Example 4.5) that the order can be arbitrary high in this case.

4 Polynomial Solutions

Theorem 3.2 suggests an answer for the problem of computing all polynomial solutions of an AODE. In fact, the degree of a polynomial is equal to the order of its Laurent series expansion at infinity. Therefore, by applying Theorem 3.2, we can give a degree bound for polynomial solutions of a given AODE. Once then degree bound is given, one can compute all polynomial solutions by making an ansatz and solving the corresponding algebraic equations.

In [14], Krushel'nitskij discusses the properties of the degree of a polynomial solution for a given AODE. However, no full algorithm for computing all polynomial solutions for a given AODE is available so far. Note that not every AODE admits a degree bound for its polynomial solutions (see Example 4.5). Based on the indicial polynomial (Definition 3.1) at infinity, we give a necessary condition for an AODE to have polynomial solutions with bounded degrees.

Definition 4.1. *An AODE $F(y) = 0$ is called* noncritical *if $\mathcal{P}_{\infty,F}(t) \neq 0$.*

Corollary 4.2. *If an AODE $F(y) = 0$ is noncritical, then there exists a bound for the degree of its polynomial solutions.*

Proof. Straightforward from Theorem 3.2. □

Algorithm 4.3. *Given a noncritical AODE $F(y) = 0$, compute all its polynomial solutions.*

(1) Compute $\mathcal{P}_{\infty,F}(t)$. If $\mathcal{P}_{\infty,F}(t)$ has integer roots, then set r_1 to be the largest integer root. Otherwise, set $r_1 = 0$.

(2) Compute $r_2 = \lfloor b_\infty(F) \rfloor$ if $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$. Otherwise set $r_2 = 0$.

(3) Set $r = \max\{r_1, r_2, 0\}$. Make an ansatz $z = \sum_{i=0}^{r} c_i x^i$, where c_i's are unknown. Substitute z into $F(y) = 0$ and solve the corresponding algebraic equations by using Gröbner bases.

(4) Return the solutions in the above step.

The termination of Algorithm 4.3 is obvious. The correctness follows from Theorem 3.2.

Example 4.4 (Kamke 6.234 [11]). *Consider the differential equation*

$$F(y) = a^2 y^2 y''^2 - 2a^2 y y'^2 y'' + a^2 y'^4 - b^2 y''^2 - y'^2 = 0, \tag{14}$$

where $a, b \in \mathbb{K}$ and $a \neq 0$. The following table is a list of the exponents of terms of F and related information.

$I \in \mathcal{E}(F)$	$\|\|I\|\|$	$\|\|I\|\|_\infty$	f_I
$(2,0,2)$	4	4	a^2
$(1,2,1)$	4	4	$-2a^2$
$(0,4,0)$	4	4	a^2
$(0,0,2)$	2	4	$-b^2$
$(0,2,0)$	2	2	-1

From the above table we see that $\mathcal{D}(F)$ is the set of exponents in the first three lines, and $\mathcal{E}(F) \setminus \mathcal{D}(F)$ is the set of exponents in the last two lines. A direct computation shows that $m_\infty(F) = -4$, $\mathcal{M}_\infty(F) = \mathcal{D}(F)$, and $\mathcal{P}_{\infty,F}(t) = a^2 t^2 \neq 0$. Therefore, the differential equation (14) is noncritical. Furthermore, we find that $b_\infty(F) = 1$.

By Theorem 3.2, every polynomial solutions of (14) has degree at most 1. By making an ansatz and solving the corresponding algebraic equations, we obtain all polynomial solutions, which are c, $c + \frac{x}{a}$, and $c - \frac{x}{a}$, where c is an arbitrary constant in $\mathbb{K}$.

Through our investigation, almost all AODEs we see in literature are noncritical (see Section 6). Only few of them are not noncritical. Below is one example for critical AODEs.

Example 4.5. *Consider the differential equation* $xyy'' - xy'^2 + yy' = 0$*. By computation, we find that its indicial polynomial is zero. So, it is a critical AODE. Actually, it has polynomial solutions* $z = cx^n$ *for arbitrary* $c \in \mathbb{K}$ *and* $n \in \mathbb{N}$*.*

We end this section by proving that two important classes of AODEs are noncritical.

Proposition 4.6. *Let* $\mathcal{L} \in \mathbb{K}(x)\left[\frac{\partial}{\partial x}\right]$ *be a differential operator, and* $P(x, y, z) \in \mathbb{K}(x)[y, z]$ *a polynomial in two variables with coefficients in* $\mathbb{K}(x)$*. Then for each* $n > 0$*, the differential equation* $\mathcal{L}(y) + P(x, y, y^{(n)}) = 0$ *is noncritical.*

In particular, linear AODEs, first-order AODEs, which have the form $F(x, y, y') = 0$ *for some* $F \in \mathbb{K}(x)[y, y']$*, and quasi-linear second-order AODEs, which have the form* $y'' + G(x, y, y') = 0$ *for some* $G \in \mathbb{K}(x)[y, y']$*, are noncritical.*

Proof. Let $F(y) := \mathcal{L}(y) + P(x, y, y^{(n)})$. We prove that $\mathcal{P}_{\infty,F}$ is nonzero.

First, we consider the case that P is a linear polynomial in y and z. Then F is a linear differential polynomial, say

$$F(y) = f_{I_{-1}} + f_{I_0} y + \cdots + f_{I_m} y^{(m)},$$

where $f_{I_i} \in \mathbb{K}(x)$ and $f_{I_m} \neq 0$ and $m \in \mathbb{N}$, . A direct computation shows that the indicial polynomial of F at infinity is of the form

$$\mathcal{P}_{\infty,F}(t) = \sum_{\substack{i=0,\dots,m \\ I_i \in \mathcal{M}_\infty(F)}} c_\infty(f_{I_i}) \cdot \prod_{s=1}^{i} (t - s + 1),$$

which is a nonzero polynomial. Therefore, linear AODEs are noncritical.

Next, assume that P is of total degree at least 2. Then we have $\mathcal{D}(F) = \mathcal{D}(P(x, y, y^{(n)}))$ and $\mathcal{M}_\infty(F) = \mathcal{M}_\infty(P(x, y, y^{(n)}))$. We write $P(x, y, y^{(n)})$ as the form

$$P(x, y, y^{(n)}) = \sum_{(i,j)\in\mathbb{N}^2} f_{i,j}(x) y^i (y^{(n)})^j.$$

Then $\mathcal{M}_\infty(F)$ consists elements of the form $e_{i,j} = (i, 0, \dots, 0, j) \in \mathbb{N}^{n+1}$. A direct calculation implies that

$$\mathcal{P}_{\infty,F}(t) = \sum_{\substack{j=1,\dots,n \\ e_{i,j} \in \mathcal{M}_\infty(F)}} c_\infty(f_{i,j}) \cdot [t(t-1)\cdots(t-n+1)]^j.$$

The indicial polynomial $\mathcal{P}_{\infty,F}(t)$ can be viewed as the evaluation of the nonzero univariate polynomial

$$g(T) = \sum_{\substack{j=1,\dots,n \\ e_{i,j} \in \mathcal{M}_\infty(F)}} c_\infty(f_{i,j}) \cdot T^j \quad \text{at} \quad T = t(t-1)\cdots(t-n+1).$$

On the other hand, since $t(t-1)\cdots(t-n+1)$ is transcendental over $\mathbb{K}$, we conclude that $\mathcal{P}_{\infty,F} \neq 0$. □

Proposition 4.7. *Let $\mathcal{L} \in \mathbb{K}(x)\left[\frac{\partial}{\partial x}\right]$ be a differential operator with coefficients in $\mathbb{K}(x)$, and $Q(y,z,w) \in \mathbb{K}[y,z,w]$ a polynomial in three variables with coefficients in $\mathbb{K}$. Then for each $m, n > 0$, the differential equation $\mathcal{L}(y) + Q(y, y^{(n)}, y^{(m)}) = 0$ is noncritical.*

In particular, autonomous second-order AODEs, which have the form $F(y, y', y'') = 0$ for some $F \in \mathbb{K}[y, y', y'']$, and quasi-linear autonomous third-order AODEs, which have the form $y''' + G(y, y', y'') = 0$ for some $G \in \mathbb{K}[y, y', y'']$, are noncritical.

Proof. Let $F(y) := \mathcal{L}(y) + Q(y, y^{(m)}, y^{(n)})$. Without loss of generality, we can assume that $0 < m < n$. As we have seen from the previous proposition, a linear AODE is noncritical. Therefore we can assume further that Q is of total degree at least 2. Then we have $\mathcal{D}(F) = \mathcal{D}(Q(y, y^{(m)}, y^{(n)}))$ and $\mathcal{M}_\infty(F) = \mathcal{M}_\infty(Q(y, y^{(m)}, y^{(n)}))$. Let us write $Q(y, y^{(m)}, y^{(n)})$ as the form

$$Q(y, y^{(m)}, y^{(n)}) = \sum_{(ijk)\in\mathbb{N}^3} f_{ijk} y^i (y^{(m)})^j (y^{(n)})^k.$$

For simplicity, we denote $e_{ijk} = (i, 0, \dots, 0, j, 0, \dots, 0, k) \in \mathbb{N}^{n+1}$, where j is the $(m+1)$-th coordinate. Then $\mathcal{M}_\infty(F)$ consists e_{ijk} such that $i+j+k = d(F)$ and $mj+nk = m_\infty(F)$. A direct computation implies that

$$\mathcal{P}_{\infty,F}(t) = \sum_{\substack{(i,j,k)\in\mathbb{N}^3 \\ e_{ijk}\in\mathcal{M}_\infty(F)}} c_\infty(f_{ijk}) \cdot (t(t-1)\cdots(t-m+1))^{j+k} \cdot ((t-m)\cdots(t-n+1))^k.$$

This polynomial can be rewritten as:

$$\mathcal{P}_{\infty,F}(t) = A^{\frac{m_\infty(F)}{m}} \cdot \sum_{\substack{k=0,\dots,n \\ e_{ijk}\in\mathcal{M}_\infty(F)}} c_\infty(f_{ijk}) \left(\frac{B}{A^{\frac{n-m}{m}}}\right)^k, \tag{15}$$

where $A = t(t-1)\cdots(t-m+1)$ and $B = (t-m)\cdots(t-n+1)$. The sum in (15) can be viewed as the evaluation of the univariate polynomial

$$h(T) = \sum_{\substack{k=0,\dots,n \\ e_{ijk}\in\mathcal{M}_\infty(F)}} c_\infty(f_{ijk}) T^k \quad \text{at} \quad T = \frac{B}{A^{\frac{n-m}{m}}}.$$

Since the projection which maps e_{ijk} to k is injective, we have that $h(T)$ is nonzero. On the other hand, since $\frac{B}{A^{\frac{n-m}{m}}}$ is transcendental over $\mathbb{K}$, we conclude that $\mathcal{P}_{\infty,F}$ is nonzero. □

5 Rational Solutions

As another application of Theorem 3.2, we present a method for computing all rational solutions of an AODE. It is well known [2, 10] that poles of rational solutions of a linear ODE with polynomial coefficients only occur at the zeros of the highest coefficient of the equation. In order to generalize this fact to nonlinear AODEs, we first need to define what is the "highest" coefficient in the nonlinear case. To do so, we equip the set of monomials in y and its derivatives with a suitable partial order (Definition 5.1). In case that the given AODE admits the greatest monomial with respect to this ordering, we show in Theorem 5.3 that poles of its rational solutions can only occur at the zeros of the corresponding coefficient. Using this fact, an algorithm (Algorithm 5.4) for determining all rational solutions of such AODEs will be proposed.

Definition 5.1. *Assume that $n \in \mathbb{N}$. For each $I, J \in \mathbb{N}^{n+1}$, we say that $I \gg J$ if $||I|| \geq ||J||$ and $||I|| + ||I||_\infty > ||J|| + ||J||_\infty$.*

It is straightforward to verify that the order defined as above is a strict partial ordering on $\mathbb{N}^{n+1}$, i. e. the following properties hold for all $I, J, K \in \mathbb{N}^{n+1}$:

(i) irreflexivity: $I \not\gg I$;

(ii) transitivity: if $I \gg J$ and $J \gg K$, then $I \gg K$;

(iii) asymmetry: if $I \gg J$, then $J \not\gg I$.

For $I, J \in \mathbb{N}^{n+1}$, we say that I and J are *comparable* if either $I \gg J$ or $J \gg I$. Otherwise, they are called *incomparable*. It is clear that the order $\gg$ is not a total order on $\mathbb{N}^{n+1}$. For example, $(2, 0)$ and $(0, 1)$ are incomparable. For a given point I in $\mathbb{N}^{n+1}$, it is straightforward to verify that the number of points, that are incomparable to I, is finite.

Let S be a subset of $\mathbb{N}^{n+1}$. An element $I \in S$ is called the *greatest element of S* if $I \gg J$ for every $J \in S \setminus \{I\}$. By the asymmetry property of $\gg$, the set S has at most one greatest element. This motivates the following definition.

Definition 5.2. *An AODE $F(y) = 0$ is called* maximally comparable *if $\mathcal{E}(F)$ admits a greatest element with respect to $\gg$.*

Theorem 5.3. *Let $F(y) = \sum_{I \in \mathbb{N}^{n+1}} f_I y^{i_0} (y')^{i_1} \dots (y^{(n)})^{i_n} \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n > 0$. Assume that $F(y) = 0$ is maximally comparable, and I_0 is the greatest element of $\mathcal{E}(F)$ with respect to $\gg$. Then poles of a rational solution of $F(y) = 0$ only occur at infinity or at the zeros of $f_{I_0}(x)$.*

Proof. We prove the above claim by contradiction. Suppose that there is $x_0 \in \mathbb{K}$ such that x_0 is a pole of order $r \geq 1$ of a rational solution of the AODE $F(y) = 0$, and $f_{I_0}(x_0) \neq 0$. Then $\text{ord}_{x_0} f_{I_0} = 0$.

We first prove that $\mathcal{M}_{x_0}(F) = \{I_0\}$. Since I_0 is the greatest element of $\mathcal{E}(F)$ with respect to $\gg$, we see that $||I_0|| \geq ||J||$ for all $J \in \mathcal{E}(F)$. So $I_0 \in \mathcal{D}(F)$. Now let us fix any $J \in \mathcal{D}(F) \setminus \{I_0\}$. Since $||I_0|| = ||J||$ and $||I_0|| + ||I||_\infty > ||J|| + ||J||_\infty$, we have that $||I_0||_\infty > ||J||_\infty$. Therefore, we conclude that $\text{ord}_{x_0}(f_{I_0}) + ||I_0||_\infty > \text{ord}_{x_0}(f_J) + ||J||_\infty$ because $\text{ord}_{x_0} f_{I_0} = 0 \geq \text{ord}_{x_0}(f_J)$. In other words, I_0 is the only element of $\mathcal{M}_{x_0}(F)$.

Since $\mathcal{M}_{x_0}(F) = \{I_0\}$, the indicial polynomial at $x = x_0$ has the form

$$\mathcal{P}_{x_0,F}(t) = c_{x_0}(f_{I_0}) \cdot \prod_{r=0}^{n-1} (-t-r)^{||I_0||_{r+1}}.$$

It is straightforward to see that $\mathcal{P}_{x_0,F}(t)$ has no positive integer root. Due to Theorem 3.2 and $r \geq 1$, we have $\mathcal{E}(F) \setminus \mathcal{D}(F) \neq \emptyset$ and

$$\begin{aligned}
r \leq b_{x_0}(F) &= \max \left\{ \frac{\mathrm{ord}_{x_0}(f_J) + ||J||_\infty - ||I_0||_\infty}{||I_0|| - ||J||} \mid J \in \mathcal{E}(F) \setminus \mathcal{D}(F) \right\} \\
&= \max \left\{ 1 - \frac{-\mathrm{ord}_{x_0}(f_J) + (||I_0|| + ||I_0||_\infty) - (||J|| + ||J||_\infty)}{||I_0|| - ||J||} \mid J \in \mathcal{E}(F) \setminus \mathcal{D}(F) \right\} \\
&< 1.
\end{aligned}$$

This contradicts the assumption that $r \geq 1$. □

The above theorem gives us a necessary condition for an AODE having no "unexpected" poles. As we have seen from Theorem 3.2, once a candidate for poles of a rational solution is found, we can bound the order at this candidate. Moreover, for maximally comparable AODEs, there are only finitely many candidates for poles of rational function solutions. Combined with the partial fraction decomposition of a rational function, we present the following algorithm for determining all rational solutions of a maximally comparable AODE.

Algorithm 5.4. *Given a maximally comparable AODE $F(y) = 0$, compute all its rational function solutions.*

(1) Compute the greatest element I_0 of $\mathcal{E}(F)$ with respect to $\gg$. Compute distinct roots $x_1, \ldots, x_m$ of $f_{I_0}(x)$ in $\mathbb{K}$.

(2) For $i \in \{1, \ldots, m\}$, compute an order bound r_i for rational solutions of $F(y) = 0$ at $x = x_i$ by Theorem 3.2. Similarly, compute the order bound N for rational solutions of the equation at infinity.

(3) Make an ansatz as the partial fraction decomposition:

$$z = \sum_{i=1}^{m} \sum_{j=1}^{r_i} \frac{c_{ij}}{(x - x_i)^j} + \sum_{k=0}^{N} c_i x^i\,, \tag{16}$$

where c_{ij} and c_i are unknown. Substitute (16) into $F(y) = 0$ and solve the corresponding algebraic equations by using Gröbner bases.

(4) Return the solutions in the above step.

Algorithm 5.4 evidently terminates. The correctness follows from Theorem 3.2 and 5.3.

Example 5.5. *Consider the differential equation*

$$\begin{aligned} F(y) &= x^2(x-1)^2y''^2 + 4x^2(x-1)y'y'' - 4x(x-1)yy'' + \\ & \quad 4x^2y'^2 - 8xyy' + 4y^2 - 2(x-1)y'' \\ &= 0. \end{aligned}$$

We first collect some information about the exponents of terms of $F(y)$.

$I \in \mathcal{E}(F)$	$\|\|I\|\|$	$\|\|I\|\|_\infty$	$\|\|I\|\| + \|\|I\|\|_\infty$	f_I
$(0,0,2)$	2	4	6	$x^2(x-1)^2$
$(0,1,1)$	2	3	5	$4x^2(x-1)$
$(1,0,1)$	2	2	4	$-4x(x-1)$
$(0,2,0)$	2	2	4	$4x^2$
$(1,1,0)$	2	1	3	$-8x$
$(2,0,0)$	2	0	2	4
$(0,1,0)$	1	1	2	$-2(x-1)$

In the above table, $\mathcal{D}(F)$ is the first 6 elements of $\mathcal{E}(F)$, and $\mathrm{d}(F) = 2$. The first one $(0,0,2)$ is the greatest element of $\mathcal{E}(F)$ with respect to $\gg$. By Theorem 5.3, poles of a rational solution of $F(y) = 0$, can only occur at the zeros of the polynomial $x^2(x-1)^2$, which is 0 and 1, and probably at infinity.

A simple computation based on Theorem 3.2 shows that the orders of poles of a rational solution of $F(y) = 0$ at $0, 1$ and infinity is at most $0, 1$ and 1, respectively.

Hence, we make an ansatz of the form:

$$z = \frac{c_1}{x-1} + c_2 + c_3 x \quad \textit{for some} \quad c_1, c_2, c_3 \in \mathbb{K}.$$

Substituting z into $F(y) = 0$ and solving the corresponding algebraic equations, we find that rational solutions of $F(y) = 0$ are $c_3 x$ and $\frac{1}{x-1} + c_3 x$, where c_3 is an arbitrary constant in $\mathbb{K}$.

Through our investigation, most AODEs we see in literature are maximally comparable (see Section 6). Only few of them are not. Below is one example for non-maximally comparable AODEs.

Example 5.6. *Consider the differential equation $F(y) = y' + y^2 = 0$. It is straightforward to see that this equation is not maximally comparable because $(2,0)$ and $(0,1)$ are not comparable with respect to $\gg$. Actually, it has rational function solutions $z = \frac{1}{x-c}$ for arbitrary $c \in \mathbb{K}$.*

6 Experimental results

In Section 4 and 5, we concentrate on the problem of computing all particular polynomial and rational solutions of some classes of AODEs. For a noncritical AODE (Definition 4.1), we bound the degree of its polynomial solutions, therefore determine (Algorithm 4.3) all polynomial solutions if there is any. For the class of maximally comparable

AODEs (Definition 5.2), we propose a method (Algorithm 5.4) for computing all rational solutions. In this section, we do some statistical investigation for noncriticality and the maximal comparability of AODEs from the famous collection of differential equations by Kamke [11]. The corresponding `Maple` worksheet is available in:

https://yzhang1616.github.io/KamkeODEs.mw

There are 834 AODEs in Kamke's collection. All of them are noncritical. It means that our method can be used to determine all polynomial solutions, if there is any, of each AODE from Kamke's collection. Among them, there are 655 maximally comparable AODEs (≈ 78.54 %).

The class of AODEs covers around 79.66 % the entire collection of ODEs. The remaining ODEs have coefficients involving trigonometric functions ($\sin x, \cos x$,...), hyperbolic functions ($\sinh x, \cosh x$, ...), exponential functions e^x, logarithmic functions $\log x$, or power functions with parameters in the exponents (x^α, y^β, ...). For certain choice of parameters, the ODEs will become algebraic. More precisely, there are 35 ODEs containing parameters in the power functions. If the parameters are chosen in a suitable way such that the corresponding ODEs are algebraic, then all of them are noncritical and 21 among them (60 %) are maximally comparable.

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