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Truncation in Average and Worst Case Settings for Special Classes of ∞-Variate Functions

Peter Kritzer*, Friedrich Pillichshammer†, G. W. Wasilkowski

Abstract

The paper considers truncation errors for functions of the form $f(x_1, x_2, \dots) = g(\sum_{j=1}^{\infty} x_j\, \xi_j)$, i.e., errors of approximating f by $f_k(x_1, \dots, x_k) = g(\sum_{j=1}^{k} x_j\, \xi_j)$, where the numbers ξ_j converge to zero sufficiently fast and x_j's are i.i.d. random variables. As explained in the introduction, functions f of the form above appear in a number of important applications. To have positive results for possibly large classes of such functions, the paper provides sharp bounds on truncation errors in both the average and worst case settings. In the former case, the functions g are from a Hilbert space G endowed with a zero mean probability measure with a given covariance kernel. In the latter case, the functions g are from a reproducing kernel Hilbert space, or a space of functions satisfying a Hölder condition.

Keywords: Dimension truncation, Average case error, Worst case error, Covariance kernel, Reproducing kernel
MSC 2000: 65D30, 65Y20, 41A55, 41A63

1 Introduction

In this paper, we are interested in problems that require computation of the expectation of $g(\boldsymbol{X}(t))$, where $\boldsymbol{X}(t)$ is the value at time t of a stochastic process $\boldsymbol{X}$, and g is a function from a given function space G.

Such a situation may, for example, occur in the context of mathematical finance, or when studying PDEs with random coefficients; the latter topic has attracted much interest recently in the field of quasi-Monte Carlo (QMC) methods. To be more precise, the term $g(\boldsymbol{X}(t))$, for a given and fixed time t, could be a quantity of interest obtained from the solution of a PDE in which one of the coefficients is modeled as a random field. We refer to [7] for a recent and detailed overview.

Let us in the following assume that $\boldsymbol{X}$ can be expressed in terms of its Karhunen-Loève (cf. [8]) expansion,

$$\boldsymbol{X}(t) = \sum_{j=1}^{\infty} x_j\, \psi_j(t),$$

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where $(\psi_j)_{j\geq 1}$ form an orthonormal basis and $(x_j)_{j\geq 1}$ are i.i.d. random variables with the corresponding probability measure denoted by ω. In this case, the expectation problem reduces to the integration of

$$f(\boldsymbol{x}) = g\left(\sum_{j=1}^{\infty} x_j\, \xi_j\right) \quad \text{with} \quad \xi_j = \psi_j(t)$$

with respect to $\omega^{\mathbb{N}}$, the countable product of ω.

As in [2, 5], the main focus of the paper is on the truncation errors, i.e., errors caused by replacing the infinite sum $\sum_{j=1}^{\infty} x_j\, \xi_j$ with the truncated sum $\sum_{j=1}^{k} x_j\, \xi_j$. Here we study how the truncation errors depend on k in the *average case* and *worst case* settings with respect to functions g.

Throughout this paper we assume that

$$\sum_{j=1}^{\infty} |\xi_j| \,<\, \infty.$$

2 Average and Worst Case Settings

We consider two settings: the *average* and *worst case settings* for spaces G of functions

$$g \,:\, D \to \mathbb{R},$$

where D is an interval (possibly unbounded) in $\mathbb{R}$. In the former setting, G is a Hilbert space endowed with a zero mean probability measure μ whose covariance kernel is denoted by K_μ^{cov}. In the latter setting, the space G is either a reproducing kernel Hilbert space whose reproducing kernel is denoted by K^{rep}, or a normed space of functions satisfying a Hölder condition.

Recall that the covariance kernel of a measure μ on G is defined by

$$K_\mu^{\mathrm{cov}}(x,y) \,=\, \mathbb{E}_\mu(g(x)\, g(y)) \,=\, \int_G g(x)\, g(y)\, \mu(\mathrm{d}g),$$

and a reproducing kernel K^{rep} satisfies the following: $K^{\mathrm{rep}}(\cdot, x) \in G$ for any $x \in D$ and

$$g(x) \,=\, \langle g, K^{\mathrm{rep}}(\cdot, x)\rangle_G \quad \text{for any } x \in D \text{ and any } g \in G.$$

Finally, in what we call the Hölder condition case, we assume that there are constants $C > 0$ and $\beta \in (0, 1]$ such that for any points x and y and any function g from G we have

$$|g(x) - g(y)| \,\leq\, C\, \|g\|_G\, |x - y|^\beta.$$

Let ω denote the probability measure related to the random variables x_j. To simplify the notation, we will often use

$$Y_k \,=\, Y_k(\boldsymbol{x}) \,:=\, \sum_{j=1}^{k} x_j\, \xi_j \quad \text{and} \quad Y_\infty \,=\, Y_\infty(\boldsymbol{x}) \,:=\, \sum_{j=1}^{\infty} x_j\, \xi_j,$$

where $\boldsymbol{x} = (x_j)_{j\geq 1}$. With this notation we have

$$Y_\infty - Y_k \;=\; \sum_{j=k+1}^{\infty} x_j\,\xi_j,$$

a quantity that plays a crucial role in the following considerations.

2.1 Average Case Setting

We assume that Fubini's theorem holds, i.e.,

$$\mathbb{E}_\mu\,\mathbb{E}_{\omega^{\mathbb{N}}} \;=\; \mathbb{E}_{\omega^{\mathbb{N}}}\,\mathbb{E}_\mu.$$

We would like to estimate the square average error of approximating the expectation of $g(Y_\infty)$ by the expectation of $g(Y_k)$ over G as well as the expected square average error of approximating $g(Y_\infty)$ by $g(Y_k)$. The former error is given by

$$\begin{aligned} e_1^{\mathrm{trnc}}(k;K_\mu^{\mathrm{cov}},\omega) \;&:=\; \left[\mathbb{E}_\mu\left((\mathbb{E}_{\omega^{\mathbb{N}}}(g(Y_\infty)) - \mathbb{E}_{\omega^{\mathbb{N}}}(g(Y_k)))^2\right)\right]^{1/2} \\ &=\; \left[\int_G \left(\int_{\mathbb{R}^{\mathbb{N}}} (g(Y_\infty(\boldsymbol{x})) - g(Y_k(\boldsymbol{x})))\,\omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x})\right)^2 \mu(\mathrm{d}g)\right]^{1/2}, \end{aligned}$$

and the latter by

$$\begin{aligned} e_2^{\mathrm{trnc}}(k;K_\mu^{\mathrm{cov}},\omega) \;&:=\; \left[\mathbb{E}_{\omega^{\mathbb{N}}}\,\mathbb{E}_\mu\left((g(Y_\infty) - g(Y_k))^2\right)\right]^{1/2} \\ &=\; \left[\int_{\mathbb{R}^{\mathbb{N}}}\int_G (g(Y_\infty(\boldsymbol{x})) - g(Y_k(\boldsymbol{x})))^2\,\mu(\mathrm{d}g)\,\omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x})\right]^{1/2}. \end{aligned} \tag{1}$$

e_errineq **Remark 1** Applying the Cauchy-Schwarz inequality to the innermost integral in e_1^{trnc}, it is easy to see that

$$e_1^{\mathrm{trnc}}(k;K_\mu^{\mathrm{cov}},\omega) \;\leq\; e_2^{\mathrm{trnc}}(k;K_\mu^{\mathrm{cov}},\omega).$$

Hence, in the following we will mainly concentrate on $e_2^{\mathrm{trnc}}(k;K_\mu^{\mathrm{cov}},\omega)$. Upper bounds on $e_2^{\mathrm{trnc}}(k;K_\mu^{\mathrm{cov}},\omega)$ also apply to $e_1^{\mathrm{trnc}}(k;K_\mu^{\mathrm{cov}},\omega)$.

Proposition 2 *We have*

$$\begin{aligned} e_1^{\mathrm{trnc}}(k;K_\mu^{\mathrm{cov}},\omega) \;=\; \Big[\int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}} \big[&K_\mu^{\mathrm{cov}}(Y_\infty(\boldsymbol{x}),Y_\infty(\boldsymbol{z})) - 2\,K_\mu^{\mathrm{cov}}(Y_\infty(\boldsymbol{x}),Y_k(\boldsymbol{z})) \\ &+ K_\mu^{\mathrm{cov}}(Y_k(\boldsymbol{x}),Y_k(\boldsymbol{z}))\big]\,\omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{z})\,\omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x})\Big]^{1/2}. \end{aligned} \tag{2}$$

and

$$\begin{aligned} e_2^{\mathrm{trnc}}(k;K_\mu^{\mathrm{cov}},\omega) \;=\; \Big[\int_{\mathbb{R}^{\mathbb{N}}} \big[&K_\mu^{\mathrm{cov}}(Y_\infty(\boldsymbol{x}),Y_\infty(\boldsymbol{x})) - 2\,K_\mu^{\mathrm{cov}}(Y_\infty(\boldsymbol{x}),Y_k(\boldsymbol{x})) \\ &+ K_\mu^{\mathrm{cov}}(Y_k(\boldsymbol{x}),Y_k(\boldsymbol{x}))\big]\,\omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x})\Big]^{1/2}. \end{aligned} \tag{3}$$

Proof. We have

$$\begin{aligned}
&\left(e_1^{\mathrm{trnc}}(k; K_\mu^{\mathrm{cov}}, \omega)\right)^2 \\
&\quad = \mathbb{E}_\mu \left[(\mathbb{E}_{\omega^{\mathbb{N}}}(g(Y_\infty)))^2 - 2\,\mathbb{E}_{\omega^{\mathbb{N}}}(g(Y_\infty))\,\mathbb{E}_{\omega^{\mathbb{N}}}(g(Y_k)) + (\mathbb{E}_{\omega^{\mathbb{N}}}(g(Y_k)))^2 \right] \\
&\quad = \int_{\mathbb{R}^{\mathbb{N}}} \int_{\mathbb{R}^{\mathbb{N}}} \left[K_\mu^{\mathrm{cov}}(Y_\infty(\boldsymbol{x}), Y_\infty(\boldsymbol{z})) - 2\, K_\mu^{\mathrm{cov}}(Y_\infty(\boldsymbol{x}), Y_k(\boldsymbol{z})) \right. \\
&\qquad\qquad\qquad \left. + K_\mu^{\mathrm{cov}}(Y_k(\boldsymbol{x}), Y_k(\boldsymbol{z})) \right] \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{z})\, \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x})
\end{aligned}$$

and

$$\begin{aligned}
&\left(e_2^{\mathrm{trnc}}(k; K_\mu^{\mathrm{cov}}, \omega)\right)^2 \\
&\quad = \mathbb{E}_{\omega^{\mathbb{N}}}\, \mathbb{E}_\mu \left(g(Y_\infty)\, g(Y_\infty) - 2\, g(Y_\infty)\, g(Y_k) + g(Y_k)\, g(Y_k) \right) \\
&\quad = \mathbb{E}_{\omega^{\mathbb{N}}} \left(K_\mu^{\mathrm{cov}}(Y_\infty, Y_\infty) - 2\, K_\mu^{\mathrm{cov}}(Y_\infty, Y_k) + K_\mu^{\mathrm{cov}}(Y_k, Y_k) \right) \\
&\quad = \int_{\mathbb{R}^{\mathbb{N}}} \left(K_\mu^{\mathrm{cov}}(Y_\infty(\boldsymbol{x}), Y_\infty(\boldsymbol{x})) - 2\, K_\mu^{\mathrm{cov}}(Y_\infty(\boldsymbol{x}), Y_k(\boldsymbol{x})) + K_\mu^{\mathrm{cov}}(Y_k(\boldsymbol{x}), Y_k(\boldsymbol{x})) \right) \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x}).
\end{aligned}$$

□

2.2 Worst Case Setting

In the worst case setting, we are interested in the worst case truncation error defined by

$$\sup_{\|g\|_G \le 1} \left[\mathbb{E}_{\omega^{\mathbb{N}}} \left(g(Y_\infty) - g(Y_k) \right)^2 \right]^{1/2}.$$

In the reproducing kernel Hilbert space setting, we will denote the above truncation error by

$$e_3^{\mathrm{trnc}}(k; K^{\mathrm{rep}}, \omega),$$

and in the Hölder's condition setting we will denote the error by

$$e_3^{\mathrm{trnc}}(k; G, \omega).$$

2.2.1 Reproducing Kernel Setting

From the reproducing kernel property and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|g(Y_\infty) - g(Y_k)| &= |\langle g, K^{\mathrm{rep}}(\cdot, Y_\infty) - K^{\mathrm{rep}}(\cdot, Y_k) \rangle_G| \\
&\le \|g\|_G\, \|K^{\mathrm{rep}}(\cdot, Y_\infty) - K^{\mathrm{rep}}(\cdot, Y_k)\|_G
\end{aligned} \tag{4}$$

and

$$\|K^{\mathrm{rep}}(\cdot, Y_\infty) - K^{\mathrm{rep}}(\cdot, Y_k)\|_G^2 = K^{\mathrm{rep}}(Y_\infty, Y_\infty) + K^{\mathrm{rep}}(Y_k, Y_k) - 2\, K^{\mathrm{rep}}(Y_\infty, Y_k).$$

Since the inequality (4) is sharp, we have the following proposition.

Proposition 3 *We have*

$$e_3^{\mathrm{trnc}}(k; K^{\mathrm{rep}}, \omega) \;=\; \left[\mathbb{E}_{\omega^{\mathbb{N}}}\left(K^{\mathrm{rep}}(Y_\infty, Y_\infty) + K^{\mathrm{rep}}(Y_k, Y_k) - 2\,K^{\mathrm{rep}}(Y_\infty, Y_k)\right)\right]^{1/2} \tag{5}$$

$$= \left[\int_{\mathbb{R}^{\mathbb{N}}} \left(K^{\mathrm{rep}}(Y_\infty(\boldsymbol{x}), Y_\infty(\boldsymbol{x})) + K^{\mathrm{rep}}(Y_k(\boldsymbol{x}), Y_k(\boldsymbol{x})) - 2\,K^{\mathrm{rep}}(Y_\infty(\boldsymbol{x}), Y_k(\boldsymbol{x}))\right)\,\omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x})\right]^{1/2}.$$

Remark 4 Observe that the dependence of e_2^{trnc} on the covariance kernel K_μ^{cov}, see (3) is the same as the dependence of e_3^{trnc} on the reproducing kernel K^{rep}, see (5). Moreover, any covariance kernel is also a reproducing kernel. This is why we will estimate the truncation errors

$$e^{\mathrm{trnc}}(k; K, \omega) \;=\; \left[\mathbb{E}_{\omega^{\mathbb{N}}}(K(Y_\infty, Y_\infty) + K(Y_k, Y_k) - 2\,K(Y_\infty, Y_k))\right]^{1/2}, \tag{6}$$

errall

for different kernels K representing either covariance kernels of probability measures μ or reproducing kernels of the spaces G generated by those kernels.

2.2.2 Hölder Condition Setting

Due to the assumption of a Hölder condition, we immediately get

$$e_3^{\mathrm{trnc}}(k; G, \omega) \;\le\; C\,\left[\mathbb{E}_{\omega^{\mathbb{N}}}(|Y_\infty - Y_k|^{2\beta})\right]^{1/2} \;=\; C\,\left[\mathbb{E}_{\omega^{\mathbb{N}}}\left(\sum_{j=k+1}^{\infty} x_j\,\xi_j\right)^{2\beta}\right]^{1/2}. \tag{7}$$

eq:Hoelderc

A primary example of such spaces is provided by the following. For $p \in (1, \infty]$, let $G = G_p$ be the space of functions g on $D = [0, T]$ that are absolutely continuous with $g' \in L_p$. The norm in the space G_p is defined by

$$\|g\|_{G_p} \;=\; \left(|g(0)|^p + \|g'\|_{L_p}^p\right)^{1/p}.$$

Here T can be any positive number or $T = \infty$. In the latter case $D = \mathbb{R}_+ = [0, \infty)$. Note that for $p = 2$ the subspace of G_2 with $g(0) = 0$ is the reproducing kernel Hilbert space with $K^{\mathrm{rep}}(x, y) = \min(x, y)$. It is considered in the next section.

Since $g(x) = g(0) + \int_0^x g'(t)\,\mathrm{d}t$ for any $g \in G$, we have for any $x, y \in D$ with $x \ge y$ that

$$|g(x) - g(y)| = \left|\int_D g'(t)\left((x-t)_+^0 - (y-t)_+^0\right)\,\mathrm{d}t\right| \le \|g'\|_{L_p}\,(x-y)^{1/p^*}.$$

Here p^* is the conjugate of p and, in particular, $p^* = 1$ if $p = \infty$. Since the Hölder inequality used above is sharp, we conclude that functions from G_p satisfy a Hölder condition with $C = 1$ and $\beta = 1/p^*$.

Of course, the same holds if the domain $D = [-T, T]$ or if it is any interval containing 0. Then the subspace of G_2 with $g(0) = 0$ is the reproducing kernel Hilbert space with kernel $K^{\mathrm{rep}}(x, y) = \frac{1}{2}(|x| + |y| - |x - y|)$.

3 Estimates of the expectation of $|Y_\infty - Y_k|^M$

We now elaborate on estimating the expectation of $|Y_\infty - Y_k|^M$ with respect to $\omega^{\mathbb{N}}$. Estimates of this particular expectation are required in order to find good bounds on $e_3^{\mathrm{trnc}}(k; G, \omega)$ via (7). We will see in Section 4 that such estimates will be also helpful in obtaining good bounds on $e^{\mathrm{trnc}}(k; K, \omega)$ in (6).

In the following let

$$m_r := \mathbb{E}_\omega(|x|^r) = \int_{\mathbb{R}} |x|^r \, \omega(\mathrm{d}x), \quad \text{for } r \in \mathbb{N}. \tag{8}$$ def_mr

First we consider the case $M = 2\beta$ for $\beta \in (0, 1]$.

prop:frct **Proposition 5** *For $\beta \le 1/2$ and any $k \in \mathbb{N}_0$ we have*

$$\mathbb{E}_{\omega^{\mathbb{N}}}\left(|Y_\infty - Y_k|^{2\beta}\right) \le \left(m_1 \sum_{j=k+1}^{\infty} |\xi_j|\right)^{2\beta}. \tag{9}$$ eq:fr1

In general, for any $\beta \in (0, 1]$ and any $k \in \mathbb{N}_0$ we have

$$\mathbb{E}_{\omega^{\mathbb{N}}}\left(|Y_\infty - Y_k|^{2\beta}\right) \le \left(\left(\mathbb{E}_\omega(x_1) \sum_{j=k+1}^{\infty} \xi_j\right)^2 + \mathrm{Var}_\omega(x_1) \sum_{j=k+1}^{\infty} \xi_j^2\right)^{\beta},$$

where $\mathrm{Var}_\omega(x_1) = \mathbb{E}_\omega(x_1^2) - (\mathbb{E}_\omega(x_1))^2$. *Moreover, if x_1 is a zero-mean random variable, i.e., $\mathbb{E}_\omega(x_1) = 0$, then*

$$\mathbb{E}_{\omega^{\mathbb{N}}}\left(|Y_\infty - Y_k|^{2\beta}\right) \le \left(\mathbb{E}_\omega(x_1^2) \sum_{j=k+1}^{\infty} \xi_j^2\right)^{\beta}. \tag{10}$$ eq:0-mean

Proof. If $2\beta \le 1$ then, using Hölder's inequality with $p = 1/(2\beta)$, we get

$$\begin{aligned} \mathbb{E}_{\omega^{\mathbb{N}}}\left(\left|\sum_{j=k+1}^{\infty} x_j \, \xi_j\right|^{2\beta}\right) &\le \left(\mathbb{E}_{\omega^{\mathbb{N}}}\left|\sum_{j=k+1}^{\infty} x_j \, \xi_j\right|^{2\beta p}\right)^{1/p} \left(\mathbb{E}_{\omega^{\mathbb{N}}} 1^{p^*}\right)^{1/p^*} \\ &= \left(\mathbb{E}_{\omega^{\mathbb{N}}}\left|\sum_{j=k+1}^{\infty} x_j \, \xi_j\right|\right)^{2\beta} \le \left(m_1 \sum_{j=k+1}^{\infty} |\xi_j|\right)^{2\beta}, \end{aligned}$$

as needed. In general (for $\beta \in (0, 1]$) we use Hölder's inequality with $p = 1/\beta$ and get

$$\mathbb{E}_{\omega^{\mathbb{N}}}\left(\left|\sum_{j=k+1}^{\infty} x_j \, \xi_j\right|^{2\beta}\right) \le \left(\mathbb{E}_{\omega^{\mathbb{N}}}\left(\sum_{j=k+1}^{\infty} x_j \, \xi_j\right)^2\right)^{\beta}.$$

From here the remaining results follow easily. $\square$

Example 6 We now illustrate the bounds (9) and (10) using uniform distribution on $[-1/2, 1/2]$ and standard normal distribution on $\mathbb{R}$ for ω, and

$$|\xi_j| \le j^{-a} \quad \text{for } a > 1.$$

Note that for $a > 1$ we have

$$\frac{1}{(a-1)\,(k+1)^{a-1}} \le \sum_{j=k+1}^{\infty} \frac{1}{j^a} \le \frac{1}{(a-1)\,(k+1/2)^{a-1}}. \tag{11}$$

bd_weights_

Clearly, $m_1 = 1/4$ and $m_2 = 1/12$ for uniform distribution, and $m_1 = \sqrt{2/\pi}$, $m_2 = 1$ for the normal distribution, and in both cases x_1 is zero-mean. The estimates (7) and (9) together with (11) give the bound

$$e_3^{\mathrm{trnc}}(k; G, \omega) \le C\, m_1^{\beta} \frac{1}{(a-1)^{\beta}\,(k+1/2)^{\beta(a-1)}},$$

and (7) and (10) together with (11) give

$$e_3^{\mathrm{trnc}}(k; G, \omega) \le C\, m_2^{\beta/2} \frac{1}{(2a-1)^{\beta/2}\,(k+1/2)^{\beta(a-1/2)}},$$

where C is as in (7). Note that the second bound is slightly better with respect to the order of convergence in k.

Now we estimate the expectation of $|Y_\infty - Y_k|^M$ for positive integer exponents M.

em:powerM

Proposition 7 *For a positive integer M, define*

$$C(M, \omega) := \max\left\{ \prod_{j=1}^{\ell} m_{r_j} \;:\; r_j \in \mathbb{N},\ \sum_{j=1}^{\ell} r_j = M \quad \text{and} \quad \ell \in \{1, 2, \dots, M\} \right\}, \tag{12}$$

eq:CM

where m_r is as in (8). Then for any $k \in \mathbb{N}_0$ we have

$$\mathbb{E}_{\omega^{\mathbb{N}}}(|Y_\infty - Y_k|^M) \le C(M, \omega) \left(\sum_{j=k+1}^{\infty} |\xi_j| \right)^M.$$

In particular, for $k = 0$, we have $\mathbb{E}_{\omega^{\mathbb{N}}}(|Y_\infty^M|) \le C(M, \omega) \left(\sum_{j=1}^{\infty} |\xi_j|\right)^M$.

Proof. We have

$$\begin{aligned} \mathbb{E}_{\omega^{\mathbb{N}}}(|Y_\infty - Y_k|^M) &= \int_{\mathbb{R}^{\mathbb{N}}} \left| \sum_{j=k+1}^{\infty} \xi_j\, x_j \right|^M \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x}) \\ &\le \sum_{j_1=k+1}^{\infty} \cdots \sum_{j_M=k+1}^{\infty} |\xi_{j_1} \cdots \xi_{j_M}| \int_{\mathbb{R}^{\mathbb{N}}} |x_{j_1} \cdots x_{j_M}|\, \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x}) \end{aligned}$$

$$\leq \left(\sum_{j=k+1}^{\infty} |\xi_j| \right)^M \max_{(j_1,\dots,j_M)} \int_{\mathbb{R}^{\mathbb{N}}} |x_{j_1} \cdots x_{j_M}| \, \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x}),$$

where the maximum is extended over all $(j_1, j_2, \dots, j_M) \in \{k+1, k+2, \dots\}^M$.

For a fixed $(j_1, j_2, \dots, j_M) \in \{k+1, k+2, \dots\}^M$ let $v_1, v_2, \dots, v_\ell$ be the different j_i's such that v_1 appears r_1 times, v_2 appears r_2 times, $\dots$, v_ℓ appears r_ℓ times. Of course, $r_1 + r_2 + \cdots + r_\ell = M$ and $\ell = \ell(j_1, \dots, j_M) \in \{1, 2, \dots, M\}$. Then we have

$$\int_{\mathbb{R}^{\mathbb{N}}} |x_{j_1} \cdots x_{j_M}| \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x}) \;=\; \int_{\mathbb{R}^{\mathbb{N}}} |x_{v_1}^{r_1}| \cdots |x_{v_\ell}^{r_\ell}| \, \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x}) \;=\; m_{r_1} \cdots m_{r_\ell}.$$

Hence,

$$\max_{(j_1,\dots,j_M)} \int_{\mathbb{R}^{\mathbb{N}}} |x_{j_1} \cdots x_{j_M}| \, \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x}) \leq C(M, \omega)$$

and this concludes the proof. □

We now provide the values of (or bounds on) $C(M, \omega)$ for a number of measures ω.

p:Cvalues **Lemma 8** *(i) If ω is the uniform measure on $[0,1]$, then*

$$C(M, \omega) \;=\; \frac{1}{M+1}.$$

(ii) If ω is the uniform measure on $[-1/2, 1/2]$, then

$$C(M, \omega) \;=\; \frac{1}{2^M \, (M+1)}.$$

(iii) If ω is the exponential measure on $[0, \infty)$ with density $\frac{1}{\lambda} \, \mathrm{e}^{-x/\lambda}$ for $\lambda > 0$, then

$$C(M, \omega) \;=\; \lambda^M \, M!.$$

(iv) If ω is the logistic measure on $\mathbb{R}$ with density $\frac{1}{\lambda \, (1 + \mathrm{e}^{-x/\lambda})^2} \, \mathrm{e}^{-x/\lambda}$ for $\lambda > 0$, then

$$\frac{1}{2} \, \lambda^M \, M! \;<\; C(M, \omega) \;<\; 2 \, \lambda^M \, M!.$$

(v) If ω is the zero-mean Gaussian measure on $\mathbb{R}$ with density $\frac{1}{\sqrt{2 \, \pi \, \sigma^2}} \, \mathrm{e}^{-x^2/(2\sigma^2)}$ with variance $\sigma^2 > 0$, then

$$C(M, \omega) \;\leq\; \sigma^M \, (M-1)!!,$$

where, for $k \in \mathbb{N}_0$,

$$k!! := \prod_{j=0}^{\lceil k/2 \rceil - 1} (k - 2j)$$

is the double factorial of k.

Proof. For the cases (i) and (ii), $m_r = 1/(r+1)$ and $m_r = 2^{-r}/(r+1)$, respectively. Hence in both cases the maximum in the definition of $C(M,\omega)$ is attained for $\ell = 1$.

For the case (iii), $m_r = \lambda^r\, r!$ and again the maximum is attained at $\ell = 1$.

For the case (iv),

$$m_r = \lambda^r\, 2 \int_0^\infty t^r \frac{\mathrm{e}^{-t}}{(1+\mathrm{e}^{-t})^2}\,\mathrm{d}t \;<\; \lambda^r\, 2 \int_0^\infty t^r\, \mathrm{e}^{-t}\,\mathrm{d}t \;=\; 2\,\lambda^r\, r!,$$

and, on the other hand, $m_r > \lambda^r\, r!/2$ which gives the bounds for $C(M,\omega)$.

Finally, for (v),

$$m_{2k} \;=\; \sigma^{2k}\,(2k-1)!! \quad \text{and} \quad m_{2k+1} \;=\; \sigma^{2k+1}\sqrt{\frac{2}{\pi}}\,(2k)!! \;\le\; \sigma^{2k+1}\,(2k)!!$$

which yields the bound on $C(M,\omega)$. $\square$

4 Applications

sec:appl

In this section we provide several concrete examples.

4.1 Fractional Wiener Kernel

Consider functions g defined on $D = \mathbb{R}$ with the (covariance or reproducing) kernel

$$K_\beta(x,y) \;=\; \frac{|x|^{2\beta} + |y|^{2\beta} - |x-y|^{2\beta}}{2}, \quad \text{where } \beta \in (0,1). \tag{13}$$

defCKWiener

The zero-mean Gaussian measure with the covariance kernel given by K_β is the *fractional Wiener* measure, see, e.g., [10]. Moreover, for $\beta = 1/2$, it is the classical Wiener measure. This is why we call K_β the fractional Wiener kernel.

From (6) we obtain

$$\begin{aligned}\left(e^{\mathrm{trnc}}(k;K_\beta,\omega)\right)^2 \;&=\; \mathbb{E}_{\omega^{\mathbb{N}}}\left(|Y_\infty|^{2\beta} - 2\,\frac{|Y_\infty|^{2\beta} + |Y_k|^{2\beta} - |Y_\infty - Y_k|^{2\beta}}{2} + |Y_k|^{2\beta}\right)\\ &=\; \mathbb{E}_{\omega^{\mathbb{N}}}\left(|Y_\infty - Y_k|^{2\beta}\right).\end{aligned}$$

Hence the estimates from Proposition 5 apply.

4.2 r-folded Wiener Kernel

ec:rfoldWK

Let $D = \mathbb{R}_+$ be the domain of functions g and consider

$$K_r(x,y) \;=\; \int_0^\infty \frac{(x-t)_+^{r-1}\,(y-t)_+^{r-1}}{((r-1)!)^2}\,\mathrm{d}t \tag{14}$$

defrfoldWK

for $r = 2, 3, \ldots$. It is well known that K_r is the covariance kernel of the r-folded Wiener measure. It also generates the Hilbert space G_r of functions g satisfying $g(0) = g^{(1)}(0) = \cdots = g^{(r-1)}(0) = 0$ and the norm in G_r is given by $\|g\|_{G_r} = \|g^{(r)}\|_{L_2(\mathbb{R}_+)}$.

Because the domain of g is $\mathbb{R}_+$, we assume that the random variables x_j take on only non-negative values and $\xi_j \ge 0$.

prop:est1 **Proposition 9** *Let*

$$c_r := \left[\frac{1}{2r-1} + \frac{(r-1)^2}{2r-3}\right]^{1/2} \frac{1}{(r-1)!}. \tag{15}$$ defcr

Suppose that $\|Y_\infty\|_{L_\infty} < \infty$, *then*

$$e^{\mathrm{trnc}}(k; K_r, \omega) \le c_r \, \|Y_\infty\|_{L_\infty}^{r-3/2} \left(\left(\mathbb{E}_\omega(x_1) \sum_{j=k+1}^{\infty} \xi_j\right)^2 + \mathrm{Var}_\omega(x_1) \sum_{j=k+1}^{\infty} \xi_j^2\right)^{1/2}. \tag{16}$$ prop:bounde

For the case where $\|Y_\infty\|_{L_\infty} = \infty$, *but* $\mathbb{E}_{\omega^{\mathbb{N}}}(Y_\infty^{4r-6}) < \infty$, *we have*

$$e^{\mathrm{trnc}}(k; K_r, \omega) \le c_r \, \left(C(4,\omega)\, C(4\,r-6,\omega)\right)^{1/4} \left(\sum_{j=k+1}^{\infty} \xi_j\right) \left(\sum_{i=1}^{\infty} \xi_i\right)^{r-3/2}, \tag{17}$$ prop:unboun

where $C(M,\omega)$ *is defined in* (12).

Remark 10 Note that $c_r \sim (\frac{r}{2})^{1/2} \frac{1}{(r-1)!}$ as $r \to \infty$ and $c_r \le (\frac{2r}{3})^{1/2} \frac{1}{(r-1)!}$ for all $r \ge 2$. For simplicity we will sometimes use this bound on c_r in the following.

Proof. Using (6) we obtain

$$e^{\mathrm{trnc}}(k; K_r, \omega) = \frac{1}{(r-1)!} \left(\int_{\mathbb{R}_+^{\mathbb{N}}} \int_0^\infty \left[(Y_\infty(\boldsymbol{x}) - t)_+^{r-1} - (Y_k(\boldsymbol{x}) - t)_+^{r-1}\right]^2 \mathrm{d}t \, \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x})\right)^{1/2}.$$

Hence we are concerned with

$$E(Y_\infty, Y_k) := \int_0^\infty \left[(Y_\infty - t)_+^{r-1} - (Y_k - t)_+^{r-1}\right]^2 \mathrm{d}t = E_1 + E_2,$$

where

$$E_1 = \int_{Y_k}^{Y_\infty} (Y_\infty - t)^{2(r-1)} \mathrm{d}t = \frac{(Y_\infty - Y_k)^{2r-1}}{2r-1}$$

and

$$\begin{aligned} E_2 &= \int_0^{Y_k} \left[(Y_\infty - t)^{r-1} - (Y_k - t)^{r-1}\right]^2 \mathrm{d}t \\ &= \int_0^{Y_k} \left[(Y_\infty - Y_k) \sum_{j=0}^{r-2} (Y_\infty - t)^j \, (Y_k - t)^{r-2-j}\right]^2 \mathrm{d}t \\ &\le (Y_\infty - Y_k)^2 \, (r-1)^2 \int_0^{Y_k} (Y_\infty - t)^{2r-4} \mathrm{d}t \\ &\le (Y_\infty - Y_k)^2 \, \frac{(r-1)^2}{2r-3} \, Y_\infty^{2r-3}. \end{aligned}$$

Hence

$$E(Y_\infty, Y_k) \le \frac{(Y_\infty - Y_k)^{2r-1}}{2r-1} + (Y_\infty - Y_k)^2 \, \frac{(r-1)^2}{2r-3} \, Y_\infty^{2r-3}$$

$$\le \quad (Y_\infty - Y_k)^2 Y_\infty^{2r-3}\left[\frac{1}{2r-1}+\frac{(r-1)^2}{2r-3}\right].$$

With c_r as in (15) we get

$$e^{\mathrm{trnc}}(k;K_r,\omega)\le c_r\left(\int_{\mathbb{R}_+^{\mathbb{N}}}(Y_\infty - Y_k)^2\,Y_\infty^{2r-3}\,\omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x})\right)^{1/2}. \qquad (18) \quad \boxed{\texttt{aeer}}$$

If $\|Y_\infty\|_{L_\infty}<\infty$ we use (18) and Proposition 5 to obtain the desired result. When $\|Y_\infty\|_{L_\infty}=\infty$, but $\mathbb{E}_{\omega^{\mathbb{N}}}(Y_\infty^{4r-6})<\infty$, we proceed as follows: We have

$$e^{\mathrm{trnc}}(k;K_r,\omega)\ \le\ c_r\left(\mathbb{E}_{\omega^{\mathbb{N}}}(|Y_\infty-Y_k|^4)\right)^{1/4}\left(\mathbb{E}_{\omega^{\mathbb{N}}}(Y_\infty^{4r-6})\right)^{1/4}. \qquad (19) \quad \boxed{\texttt{eq:rfold}}$$

Now Proposition 7 and (19) yield the desired result. $\square$

As in the previous section, consider

$$|\xi_j|\ \le\ j^{-a} \quad \text{for } a>1,$$

and the following two examples of ω.

Example 11 Consider the uniform probability measure on $[0,1]$ for ω. Then $Y_\infty(\boldsymbol{x})\le\sum_{j=1}^\infty j^{-a}=\zeta(a)$ is finite and equal to the Riemann Zeta-Function, and (16) together with (11) yields

$$\begin{aligned}
e^{\mathrm{trnc}}(k;K_r,\omega)\ &\le\ \left(\frac{2r}{3}\right)^{1/2}\frac{\zeta(a)^{r-3/2}}{(r-1)!}\left[\frac{1}{4}\left(\sum_{j=k+1}^{\infty}\frac{1}{j^a}\right)^2+\frac{1}{12}\sum_{j=k+1}^{\infty}\frac{1}{j^{2a}}\right]^{1/2}\\
&\le\ \left(\frac{r}{6}\right)^{1/2}\frac{\zeta(a)^{r-3/2}}{(r-1)!}\,\frac{1}{(k+1/2)^{a-1}}\left[\frac{1}{(a-1)^2}+\frac{1}{3(2a-1)}\,\frac{1}{k+1/2}\right]^{1/2}\\
&\le\ c_{r,a}\,\frac{1}{(k+1/2)^{a-1}},
\end{aligned}$$

where

$$c_{r,a}=\left(\frac{r}{6}\right)^{1/2}\frac{\zeta(a)^{r-3/2}}{(r-1)!}\left[\frac{1}{(a-1)^2}+\frac{2}{9(2a-1)}\right]^{1/2}.$$

Example 12 Consider now the exponential probability measure with variance $\lambda>0$ for ω. From Lemma 8 we know that $C(M,\omega)\ =\ \lambda^M\,M!$, and, by (17) and (11),

$$e^{\mathrm{trnc}}(k;K_r,\omega)\le c_{r,\lambda}\,\frac{1}{(k+1/2)^{a-1}},$$

where

$$c_{r,\lambda}=2r^{1/2}\lambda^{r-1/2}\,\frac{((4\,r-6)!)^{1/4}\,\zeta(a)^{r-3/2}}{(r-1)!\,(a-1)}.$$

4.3 Two-Sided r-Folded Wiener Kernel

Let $\mathbb{R}$ be the domain of functions g and consider

$$K_{r,\pm}(x,y) \;=\; \begin{cases} \int_0^\infty \frac{(|x|-t)_+^{r-1}\,(|y|-t)_+^{r-1}}{((r-1)!)^2}\,\mathrm{d}t & \text{if } x\,y \ge 0,\\ 0 & \text{if } x\,y < 0, \end{cases}$$

for $r = 2, 3, \ldots$.

We obtain the following analogue to Proposition 9.

ided2sided

Proposition 13 *Let*

$$Y_\infty^{\mathrm{abs}} = \sum_{j=1}^{\infty} |x_j \xi_j|. \tag{20}$$

def:Yabs

Suppose that $\|Y_\infty^{\mathrm{abs}}\|_{L_\infty} < \infty$*, then*

$$e^{\mathrm{trnc}}(k; K_{r,\pm}, \omega) \;\le\; c_r\, \|Y_\infty^{\mathrm{abs}}\|_{L_\infty}^{r-3/2} \left(\left(\mathbb{E}(x_1) \sum_{j=k+1}^{\infty} \xi_j \right)^2 + \mathrm{Var}_\omega(x_1) \sum_{j=k+1}^{\infty} \xi_j^2 \right)^{1/2}.$$

For the case where $\|Y_\infty^{\mathrm{abs}}\|_{L_\infty} = \infty$*, but* $\mathbb{E}_{\omega^{\mathbb{N}}}((Y_\infty^{\mathrm{abs}})^{4r-6}) < \infty$*, we have*

$$e^{\mathrm{trnc}}(k; K_{r,\pm}, \omega) \;\le\; c_r\, \left(C(4,\omega)\, C(4\,r-6,\omega)\right)^{1/4} \left(\sum_{j=k+1}^{\infty} \xi_j \right) \left(\sum_{i=1}^{\infty} \xi_i \right)^{r-3/2},$$

where c_r is defined in (15) *and $C(M,\omega)$ is defined in* (12)*.*

Proof. Analogously to the proof of Proposition 9, we would like to find an upper bound on

$$e^{\mathrm{trnc}}(k; K_{r,\pm}, \omega) \;=\; \frac{1}{(r-1)!} \left(\int_{\mathbb{R}^{\mathbb{N}}} E_{r,\pm}(Y_\infty, Y_k)\, \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x}) \right)^{1/2},$$

where

$$E_{r,\pm}(Y_\infty, Y_k) \;=\; ((r-1)!)^2\, \left(K_{r,\pm}(Y_\infty, Y_\infty) - 2\,K_{r,\pm}(Y_\infty, Y_k) + K_{r,\pm}(Y_k, Y_k)\right).$$

In the two cases when Y_∞ and Y_k are of the same sign, $E_{r,\pm}(Y_\infty, Y_k)$ can be estimated as in the previous section, so we obtain

$$E_{r,\pm}(Y_\infty, Y_k) \;\le\; |Y_\infty - Y_k|^2 |Y_\infty|^{2r-3} \left[\frac{1}{2r-1} + \frac{(r-1)^2}{2r-3} \right].$$

In the case when Y_∞ and Y_k have different signs we have $K_{r,\pm}(Y_\infty, Y_k) = 0$ and

$$\begin{aligned} E_{r,\pm}(Y_\infty, Y_k) &= \int_0^{|Y_\infty|} (|Y_\infty| - t)^{2(r-1)}\mathrm{d}t + \int_0^{|Y_k|} (|Y_k| - t)^{2(r-1)}\mathrm{d}t \\ &= \frac{1}{2r-1}\, \left(|Y_\infty|^{2r-1} + |Y_k|^{2r-1}\right) \le \frac{1}{2r-1}\, \left(|Y_\infty| + |Y_k|\right)^{2r-1} \end{aligned}$$

$$= \frac{|Y_\infty - Y_k|^{2r-1}}{2r-1}.$$

In any case we have

$$\begin{aligned}
E_{r,\pm}(Y_\infty, Y_k) &\leq |Y_\infty - Y_k|^2 \max(|Y_\infty|, |Y_\infty - Y_k|)^{2r-3} \left[\frac{1}{2r-1} + \frac{(r-1)^2}{2r-3}\right] \\
&\leq |Y_\infty - Y_k|^2 \, (Y_\infty^{\mathrm{abs}})^{2r-3} \left[\frac{1}{2r-1} + \frac{(r-1)^2}{2r-3}\right].
\end{aligned}$$

Hence

$$e^{\mathrm{trnc}}(k; K_{r,\pm}, \omega) \;\leq\; c_r \left(\int_{\mathbb{R}^{\mathbb{N}}} |Y_\infty - Y_k|^2 \, (Y_\infty^{\mathrm{abs}})^{2r-3} \, \omega^{\mathbb{N}}(\mathrm{d}\boldsymbol{x})\right)^{1/2}.$$

From here the results follow in the same way as in the proof of Proposition 9, by noting that the proof of Proposition 12 also can be used to bound Y_∞^{abs}. □

eduniform **Example 14** Consider the uniform distribution on $[-1/2, 1/2]$ for ω. Then $\mathbb{E}_\omega(x_1) = 0$ and $\mathrm{Var}_\omega(x_1) = \frac{1}{12}$. Furthermore, $\|Y_\infty^{\mathrm{abs}}\|_{L_\infty} \leq \frac{1}{2} \sum_{j=1}^\infty |\xi_j|$. Then we get from Proposition 13,

$$e^{\mathrm{trnc}}(k; K_{r,\pm}, \omega) \;\leq\; c_r \left(\frac{1}{2} \sum_{j=1}^\infty |\xi_j|\right)^{r-3/2} \left(\frac{1}{12} \sum_{j=k+1}^\infty \xi_j^2\right)^{1/2}.$$

dednormal **Example 15** Consider the zero mean Gaussian measure with $\sigma^2 > 0$ variance for ω. Then we obtain from Proposition 13 and Lemma 8,

$$e^{\mathrm{trnc}}(k; K_{r,\pm}, \omega) \;\leq\; c_r \, \left(\sigma^{4r-2} 3(4r-7)!!\right)^{1/4} \left(\sum_{j=k+1}^\infty \xi_j\right) \left(\sum_{i=1}^\infty \xi_i\right)^{r-3/2}.$$

Remark 16 Note that it is again sufficient to assume $|\xi_j| \leq j^{-a}$ with $a > 1$ to make use of the upper bounds in Examples 14 and 15.

4.4 Korobov Kernel

Let G be the Korobov space of functions g defined on $D = [0, 1]$ generated by the kernel

$$K_r^{\mathrm{kor}}(x, y) = \sum_{h \in \mathbb{Z}} r(h) \, \mathrm{e}^{2\pi \mathrm{i} h(x-y)},$$

where $r : \mathbb{Z} \to (0, \infty)$ is a positive weight function with $r(h) = r(-h)$. Korobov spaces are very well studied in the field of quasi-Monte Carlo methods, see [9, Appendix A.1] for an introduction.

In [9], the function r is such that $r(h)$ is of order $h^{-2\alpha}$, for a nonnegative real α. The parameter α is called the smoothness parameter of the Korobov space, and shows up in the norm of the space G. To be more precise, the norm of $g \in G$ is $\|g\|_{\mathrm{kor}} =$

$\left(\sum_{h\in\mathbb{Z}} r(h)^{-1}\,|\widehat{g}(h)|^2\right)^{1/2}$, where $\widehat{g}(h)$ is the h^{th} Fourier coefficient of g. Hence α reflects the decay of the Fourier coefficients of the elements of G. Another approach, taken in [6], assumes exponentially decaying $r(h)$, resulting in infinitely smooth functions as elements of G.

Due to the symmetry property of r, and since

$$\begin{aligned} \mathrm{e}^{2\pi \mathtt{i} h Y_\infty} - \mathrm{e}^{2\pi \mathtt{i} h Y_k} &= \mathrm{e}^{\pi \mathtt{i} h (Y_\infty + Y_k)} \left(\mathrm{e}^{\pi \mathtt{i} h (Y_\infty - Y_k)} - \mathrm{e}^{-\pi \mathtt{i} h (Y_\infty - Y_k)}\right) \\ &= 2\,\mathtt{i}\mathrm{e}^{\pi \mathtt{i} h (Y_\infty + Y_k)} \sin(\pi h (Y_\infty - Y_k)) \end{aligned}$$

we obtain

$$\begin{aligned} K_r^{\text{kor}}(Y_\infty, Y_\infty) - 2\,K_r^{\text{kor}}(Y_\infty, Y_k) + K_r^{\text{kor}}(Y_k, Y_k) &= 2 \sum_{h=1}^{\infty} r(h)\, \left|\mathrm{e}^{2\pi \mathtt{i} h Y_\infty} - \mathrm{e}^{2\pi \mathtt{i} h Y_k}\right|^2 \\ &= 8 \sum_{h=1}^{\infty} r(h) \sin^2(\pi h (Y_\infty - Y_k)) \\ &\le 8 \sum_{h=1}^{\infty} r(h) \min\left(1\,,\, \pi^2 h^2\,(Y_\infty - Y_k)^2\right) \\ &\le 8\,\pi^2\,|Y_\infty - Y_k|^2 \sum_{h=1}^{\infty} h^2\, r(h). \end{aligned}$$

We assume that r is such that

$$C_r^2 := \sum_{h=1}^{\infty} h^2\, r(h) \;<\; \infty.$$

This assumption is satisfied by choosing the smoothness parameter $\alpha > 3/2$ in [9], and also satisfied for Korobov spaces of infinitely smooth functions studied in [6]. Then, according to (5),

$$e^{\text{trnc}}(k; K^{\text{kor}}, \omega) \;\le\; 2\sqrt{2}\,\pi\, C_r\, (\mathbb{E}_{\omega^{\mathbb{N}}}(|Y_\infty - Y_k|^2))^{1/2}. \tag{21}$$

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The following two examples are similar to Examples 14 and 15, and in particular can be used if $|\xi_j| \le j^{-a}$ for $a > 1$.

Example 17 Consider the uniform distribution on $[-1/2, 1/2]$ for ω. We can then use Proposition 5 with $\beta = 1$ and the fact that $\mathbb{E}_\omega(x_1^2) = 1/12$, and we get from (21) and (10),

$$e^{\text{trnc}}(k; K_r^{\text{kor}}, \omega) \;\le\; \sqrt{\frac{2}{3}}\,\pi\, C_r \left(\sum_{j=k+1}^{\infty} \xi_j^2\right)^{1/2}.$$

Example 18 Consider the zero mean Gaussian measure with $\sigma^2 > 0$ variance for ω. We can then use Proposition 5 with $\beta = 1$ and the fact that $\mathbb{E}_\omega(x_1^2) = \sigma^2$, and we get from (21) and (10),

$$e^{\text{trnc}}(k; K_{r,\pm}, \omega) \;\le\; 2\sqrt{2}\,\pi\, C_r\, \sigma \left(\sum_{j=k+1}^{\infty} \xi_j^2\right)^{1/2}.$$

4.5 Hermite Kernel

Let G be a Hermite space of functions defined on $D = \mathbb{R}$ generated by the reproducing kernel

$$K_r^{\mathrm{H}}(x,y) \;=\; \sum_{\ell=0}^{\infty} r(\ell)\, H_\ell(x)\, H_\ell(y),$$

where H_ℓ is the ℓ^{th} (normalized probabilists') Hermite polynomial

$$H_\ell(x) \;=\; \frac{(-1)^\ell}{\sqrt{\ell!}} \exp(x^2/2)\, \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \exp(-x^2/2), \quad x \in \mathbb{R},$$

and $r : \mathbb{N}_0 \to (0,\infty)$ is a positive weight function. Integration and function approximation over such spaces have been considered in, e.g., [1, 3, 4].

Since $H_0 \equiv 1$, we have

$$K_r^{\mathrm{H}}(Y_\infty, Y_\infty) - 2\, K_r^{\mathrm{H}}(Y_\infty, Y_k) + K_r^{\mathrm{H}}(Y_k, Y_k) \;=\; \sum_{\ell=1}^{\infty} r(\ell)\, (H_\ell(Y_\infty) - H_\ell(Y_k))^2.$$

By the mean value theorem,

$$|H_\ell(Y_\infty) - H_\ell(Y_k)| \;=\; |H_\ell'(\eta_\ell)|\, |Y_\infty - Y_k|$$

for some $\eta_\ell \in I(Y_k, Y_\infty)$, where $I(Y_k, Y_\infty) = (Y_k, Y_\infty)$ if $Y_k < Y_\infty$ and $I(Y_k, Y_\infty) = (Y_\infty, Y_k)$ if $Y_k > Y_\infty$. The identity $H_\ell' = \ell\, H_{\ell-1}$ yields

$$|H_\ell(Y_\infty) - H_\ell(Y_k)| \;=\; \ell\, |H_{\ell-1}(\eta_\ell)|\, |Y_\infty - Y_k|\,.$$

For $\ell = 1$, this yields $|H_\ell(Y_\infty) - H_\ell(Y_k)| = |Y_\infty - Y_k|$. For $\ell \geq 2$, we use a slightly stronger version of Cramer's bound proved in [1], namely

$$H_{\ell-1}(x) \leq \min\left\{1, \frac{\sqrt{\pi}}{(\ell-1)^{1/12}}\right\} \frac{1}{\sqrt{\phi(x)}} \leq \frac{c}{\ell^{1/12}} \frac{1}{\sqrt{\phi(x)}},$$

where ϕ is the standard normal density function. Thus, for $\eta_\ell \in I(Y_k, Y_\infty)$ we have

$$|H_{\ell-1}(\eta_\ell)| \leq \frac{c}{\ell^{1/12}} \sup_{x \in I(Y_k, Y_\infty)} \sqrt[4]{2\pi}\, \mathrm{e}^{x^2/4} = \frac{c}{\ell^{1/12}} \sqrt[4]{2\pi}\, \max\left(\mathrm{e}^{Y_k^2/4}, \mathrm{e}^{Y_\infty^2/4}\right) \leq \frac{c}{\ell^{1/12}} \sqrt[4]{2\pi}\, \mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2/4},$$

where Y_∞^{abs} is as in (20).

Let us now assume that

$$V := \sum_{\ell=1}^{\infty} r(\ell) \ell^{11/6} < \infty.$$

We remark that this assumption is satisfied for the Hermite spaces considered in [3], and those in [1] if one chooses the parameter $\alpha > 17/6$ in that paper. Then we obtain

$$\sum_{\ell=1}^{\infty} r(\ell)\, \mathbb{E}_{\omega^{\mathbb{N}}} \left((H_\ell(Y_\infty) - H_\ell(Y_k))^2\right) \;\; \leq \;\; c^2 \sqrt{2\pi}\, V\, \mathbb{E}_{\omega^{\mathbb{N}}} \left(\mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2/2} (Y_\infty - Y_k)^2\right),$$

for some suitably chosen $\widetilde{c}$. Hence,

$$e^{\mathrm{trnc}}(k; K_r^{\mathrm{H}}, \omega) \;\le\; c\, \left(\sqrt{2\pi}\, V\, \mathbb{E}_{\omega^{\mathbb{N}}} \left(\mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2/2} (Y_\infty - Y_k)^2\right)\right)^{1/2}. \tag{22}$$

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Suppose that $\|\mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2/2}\|_{L_\infty} < \infty$, then

$$e^{\mathrm{trnc}}(k; K_r^{\mathrm{H}}, \omega) \;\le\; c\, \left(\sqrt{2\pi}\, V\, \|\mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2/2}\|_{L_\infty} \mathbb{E}_{\omega^{\mathbb{N}}} \left((Y_\infty - Y_k)^2\right)\right)^{1/2}.$$

Example 19 Consider the uniform distribution on $[-1/2, 1/2]$ for ω. Then we have

$$\|\mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2/2}\|_{L_\infty} \le \mathrm{e}^{\frac{1}{8}\left(\sum_{j=1}^\infty |\xi_j|\right)^2} < \infty$$

according to our standing assumption that $\sum_{j=1}^\infty |\xi_j| < \infty$. We can then use Proposition 5 with $\beta = 1$, and the fact that $\mathbb{E}_\omega(x_1^2) = 1/12$ and we get from (10)

$$e^{\mathrm{trnc}}(k; K_r^{\mathrm{H}}, \omega) \;\le\; \widetilde{c}\; \mathrm{e}^{\frac{1}{16}\left(\sum_{j=1}^\infty |\xi_j|\right)^2} \left(\sum_{j=k+1}^\infty \xi_j^2\right)^{1/2}.$$

where $\widetilde{c} = c\,(\sqrt{2\pi}\, V/12)^{1/2}$. This bound can be used, for example, if $|\xi_j| \le j^{-a}$ with some $a > 1$. In this case we have

$$e^{\mathrm{trnc}}(k; K_r^{\mathrm{H}}, \omega) \;\le\; \frac{\widetilde{c}\; \mathrm{e}^{\frac{1}{16}\zeta(a)^2}}{\sqrt{2a-1}} \frac{1}{(k+1/2)^{a-1/2}}.$$

Suppose that $\|\mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2/2}\|_{L_\infty} = \infty$, but $\mathbb{E}_{\omega^{\mathbb{N}}}(\mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2}) < \infty$, then

$$\mathbb{E}_{\omega^{\mathbb{N}}} \left(\mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2/2} (Y_\infty - Y_k)^2\right) \le \mathbb{E}_{\omega^{\mathbb{N}}}(\mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2})^{1/2}\, \mathbb{E}_{\omega^{\mathbb{N}}} \left((Y_\infty - Y_k)^4\right)^{1/2}.$$

Hence

$$e^{\mathrm{trnc}}(k; K_r^{\mathrm{H}}, \omega) \;\le\; c\, \left(\sqrt{2\pi}\, V\, \mathbb{E}_{\omega^{\mathbb{N}}}(\mathrm{e}^{(Y_\infty^{\mathrm{abs}})^2})^{1/2}\, \mathbb{E}_{\omega^{\mathbb{N}}} \left((Y_\infty - Y_k)^4\right)^{1/2}\right)^{1/2}.$$

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