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Abstract This work is devoted to the analysis of multi-physics dynamical systems stemming from automated modeling processes in system simulation software. The multi-physical model consists of (simple connected) networks of different or the same physical type (liquid flow, electric, gas flow, heat flow) which are connected via interfaces or coupling conditions. Since the individual networks result in differential algebraic equations (DAEs), the combination of them gives rise to a system of DAEs. While for the individual networks existence and uniqueness results, including the formulation of index reduced systems, is available through the techniques of *modified nodal analysis* or *topological based index analysis*, topological results for coupled system are not available so far. We present an approach for the application of topological based index analysis for coupled systems of the same physical type and give the outline of this approach for coupled liquid flow networks. Exploring the network structure via graph theoretical approaches allows to develop topological criteria for the existence of solutions of the coupled systems. The conditions imposed on the coupled network are illustrated via various examples. Those results can be interpreted as a natural extensions of the topological existence and index criteria provided by the topological analysis for single connected circuits.

Keywords differential-algebraic equation · topological index criteria · hydraulic network · coupled system · modified nodal analysis

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1 Introduction

Increasingly demanding emissions legislation specifies the performance requirements for the next generation of products from vehicle manufacturers. Conversely, the increasingly stringent emissions legislation is coupled with the trend in increased power, drivability and safety expectations from the consumer market. Promising approaches to meet these requirements are downsizing the internal combustion engines (ICE), the application of turbochargers, variable valve timing, advanced combustion systems or comprehensive exhaust aftertreatment but also different variants of combinations of the ICE with an electrical engine in terms of hybridization or even a purely electric propulsion. The challenges in the development of future powertrains do not only lie in the design of individual components but in the assessment of the powertrain as a whole. On a system engineering level it is required to optimize individual components globally and to balance the interaction of different sub-systems. A typical system engineering model comprises several sub-systems. For instance in case of a hybrid propulsion these can be the vehicle chassis, the drive line, the air path of the ICE including combustion and exhaust aftertreatment, the cooling and lubrication system of the ICE and battery packs, the electrical propulsion system including the engine and a battery pack, the air conditioning and passenger cabin models, waste heat recovery and finally according control systems.

State-of-the-art modeling and simulation packages such as Dymola[1], OpenModelica[2], Matlab/Simulink[3], Flowmaster[4], Amesim[5], SimulationX[6], or Cruise M[7] offer many concepts for the automatic generation of dynamic system models. Modeling is done in a modularized way, based on a network of subsystems which again consists of simple standardized sub-components. The automated modeling process allows the usage of various advanced libraries for different subcomponents of the system from possibly different physical domains. The connections between those subcomponents are typically based on physical coupling conditions or predefined controller interfaces. Furthermore the network structure (topology) carries the core information of the network properties and therefore is predestinated to be exploited for the analysis and numerical simulation of those. In the application of vehicle system simulation the equations of the subsystems are differential-algebraic equations (DAEs) of higher index. Hence, this type of modeling leads systems of coupled large DAEs-systems. Consequently the analysis of existence and uniqueness of solutions for both, the individual physical subsystems and the full coupled system of DAE-systems, is a delicate issue.

Topology based index analysis for networks connects the research fields of *Analysis for DAEs* [22] and *Graph Theory* [5] in order to provide the appropriate base to analyze DAEs stemming from automatic generated system models. So far it has been established for various types of networks, including electric circuits [25]

[1] http://www.dynasim.com
[2] http://www.openmodelica.org
[3] http://www.mathworks.com
[4] http://www.mentor.com
[5] http://www.plm.automation.siemens.com
[6] http://www.iti.de
[7] http://www.avl.com

(*Modified Nodal Analysis*), gas supply networks [7], thermal liquid flow networks [1,2] and water supply networks [8,9,23]. Although all those networks share some similarities, an individual investigation is required due to their different physical nature. Recently, a unified modeling approach for different types of flow networks has been introduced in [10], aiming for a unified topology based index analysis for the different physical domains on an abstract level. In the mentioned approaches, the analysis of the different physical domains is always restricted to a simple connected network of one physical type. Anyhow, all the approaches have in common, that they provide an index reduced (d-index 1 or s-index 0) formulation of the original DAE, which is suitable for numerical integration.

Due to the increasing complexity in vehicle system simulation the interchangeability of submodels is gaining increasing importance. Submodels are exchanged between different simulation environments in terms of white-box or black-box libraries describing a set of DAEs. The interconnection to the system of physical based DAEs is again established by predefined controller interface or physical coupling conditions. The individual subnetworks are assumed to be of index reduced form (d-index 1 or s-index 0). This can be achieved by the *Topological index analysis* or *Modified Nodal Analysis*. It is well known [21], that the combination of d-index 1 DAEs may not form a d-index 1 DAEs.

Furthermore the Functional Mock-up Interface (FMI)[8] provides a tool independent standard to support model exchange of subsystems. On the one hand those black box approaches promote the possibility for hiding intellectual property and guarantee platform independence, but on the other hand they raise the challenge to incorporate those systems in the automated modeling and simulation process of multi-physics dynamical systems.

During the development phase of a multi-physics model, different abstraction levels may be obtained:

1. Combination of networks of the same physical type.
2. Combination of physical networks with black box models of the same physical types.
3. Combination of networks of different physical type.
4. Combination of physical networks with black box models of unknown physical types.
5. Combination of networks and controller elements.

Especially scenario (1) and (2) are of special interest in the context of *Topology based index analysis* or *Modified Nodal Analysis*, since they allow to extract additional information due to the knowledge of the underlying physics. In the following we address those two cases and explore the physical properties (e.g. conservation laws) of the system to derive topological based index and solvability conditions. For the treatment of (3)–(5) we refer to other approaches, which do not rely on physical properties but on purely structural properties like the Signature method [20] or the Pantelides algorithm [18] with algebraic regularization techniques, e.g. [21].

(ad 1) Combination of circles of the same physical type The artificial coupling of circles of the same physical type via (defined) physical coupling conditions within one simulation package might appear superfluous, since the circuit could be modeled

[8] http://fmi-standard.org/

all at once. Due to increasing complexity also within one physical domain, the modeling of subcircuits is distributed among high specialized teams and finally combined to the complete circuit. Using physical coupling conditions allows to combine the subcircuits to a single circuit without modifying the developed submodels. For the case of DAEs of higher index, this is of special importance, since the set of feasible initial conditions is often defined by structural properties (e.g. chord sets or spanning trees) and they might change in a coupling process. Due to integrity of the overall modeling process, this type of change should be avoided. Typically the physical coupling conditions are defined to ensure that certain conservation laws are satisfied, e.g. conservation of mass in liquid flow networks or conservation of charge in electric systems. Consequently an appropriate treatment of those coupling conditions is a delicate issue.

(ad 2) Combination of physical circles with black box models of the same physical types
This scenario extends the previous one. If the protection of intellectual property of a specific subcircuit model is of high priority, the specific part can be incorporated in a black box model. Although the actual physical content is not known, educated guesses based on the offered connection points allow to apply physical based rules to the coupling interface. Therein it is assumed that the black box model offers a suitable pair of ports, which allows to build up a feasible connection to the coupling interface. Examples for black box model with user defined content, but framework defined physical connections can be found, e.g., in Cruise M[9].

In [8] a unified modeling approach for different types of flow networks (electric circuits, water and gas networks) has been stated. One specific part of this classifications are the boundary conditions, that prescribe a certain pressure or potential for node elements and flow sources. In the case of electric networks, those elements are voltage sources and current sources. In the case of gas and liquid flow networks, those are reservoirs and demand branches. Those boundary conditions provide the starting point for defining appropriate coupling and interface conditions. As an example we explore the coupling for the case of two liquid flow networks via reservoirs and demand branches. Providing pressure controlled flow sources and flow controlled pressures establishes a strong coupling of the individual liquid flow networks.

The structure of this work is the following. In Section 2 we state a simple model for an incompressible liquid flow network and summarize the existence and uniqueness results as well as DAE index results, that have been obtained in [2], in Section 3. Therein we especially focus on the methods, that are used to derive the index and existence results and provide a descriptive explanation in the context of linear algebra and graph theory. In Section 4 we state a coupled model of incompressible flow networks. The challenges arising for these kind of models are described via a set of characteristic examples. An analysis for the coupled flow network is presented in Section 5. The analysis is specialized to some specific configurations, where topological conditions for the coupled flow networks can be obtained. Finally, Section 6 provides an overview of the addressed issues. Therein another major focus is put on the description of open topics and further research requirements.

[9] http://www.avl.com

2 A network model for incompressible flow networks

We consider a network

$$\mathcal{N} = \{\mathcal{PI}, \mathcal{PU}, \mathcal{DE}, \mathcal{JC}, \mathcal{RE}\}, \tag{1}$$

that is composed of pipes $\mathcal{PI}$, pumps $\mathcal{PU}$, demands $\mathcal{DE}$, junctions $\mathcal{JC}$ and reservoirs $\mathcal{RE}$. The network $\mathcal{N}$ is represented as a linear graph, see e.g. [3,5,24]. The pipes, pumps and demands correspond to the edges of the graph while the junctions and reservoirs correspond to its vertices, cf. Figure 1.

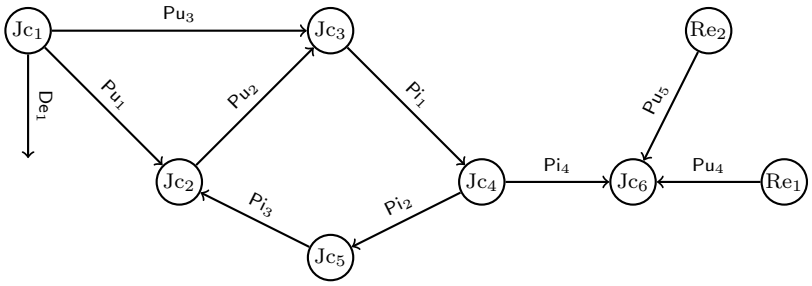

Fig. 1 An example of a graph $\mathcal{G}$ for a network $\mathcal{N}$.

Each network element comes with its characteristic equation. In a pipe Pi_j, $j = 1, ..., n_{\mathrm{Pi}}$, directed from node j_1 to node j_2, the mass flow $q_{\mathrm{Pi},j}$ is specified by the transient momentum equation

$$\dot{q}_{\mathrm{Pi},j} = c_{1,j}\Delta p_j + c_{2,j}|q_{\mathrm{Pi},j}|q_{\mathrm{Pi},j} + c_{3,j} \tag{2a}$$

depending on the pressure difference $\Delta p_j = p_{j_1} - p_{j_2}$ between the adjacent nodes j_1, j_2 and constants $c_{i,j}$ depending, e.g., on the pipe diameter, length, inclination angle, and other physical properties. In a pump Pu_j, $j = 1, ..., n_{\mathrm{Pu}}$, directed from node j_1 to node j_2, the mass flow $q_{\mathrm{Pu},j}$ is specified algebraically by the pressure drop $\Delta p_j = p_{j_1} - p_{j_2}$, i.e.,

$$\Delta p_j = f_{\mathrm{Pu}_j}(q_{\mathrm{Pu},j}). \tag{2b}$$

The function f_{Pu_j} is given by specialized pump models, cp. e.g., [6]. Due to mass conservation, in a junction Jc_i, $i = 1, ..., n_{\mathrm{Jc}}$, the amount of mass entering and leaving Jc_i is equal. Summarizing the indices of pipes, pumps and demand branches that are incident to Jc_i in the set $\hat{J}_i$, we thus get that

$$\sum_{j \in \hat{J}_i} q_j = 0. \tag{2c}$$

In a demand branch De_j, $j = 1, ..., n_{\mathrm{De}}$, the mass flow $q_{\mathrm{De},j}$ is specified by a given function $\bar{q}_{\mathrm{De},j}$, i.e.,

$$q_j = \bar{q}_{\mathrm{De},j}. \tag{2d}$$

Similarly, in a reservoir Re_i, $i = 1, ..., n_{\mathrm{Re}}$, the pressure $p_{\mathrm{Re},i}$ is specified by a given function $\bar{p}_{\mathrm{Re},i}$, i.e.,

$$p_i = \bar{p}_{\mathrm{Re},i}. \tag{2e}$$

The connection structure of the network $\mathcal{N}$ is described by the incidence matrix $A_{\mathcal{N}} = (a_{ij})$, which is defined as, cp. e.g. [3,5,24],

$$a_{ij} = \begin{cases} 1, & \text{if the branch } j \text{ leaves the node } i, \\ -1, & \text{if the branch } j \text{ enters the node } i, \\ 0, & \text{else.} \end{cases}$$

Sorting the rows and columns of $A_{\mathcal{N}}$ according to the different element types, we obtain the incidence matrix as

$$A_{\mathcal{N}} = \begin{bmatrix} A_{\mathrm{Jc,Pi}} & A_{\mathrm{Jc,Pu}} & A_{\mathrm{Jc,De}} \\ A_{\mathrm{Re,Pi}} & A_{\mathrm{Re,Pu}} & A_{\mathrm{Re,De}} \end{bmatrix}.$$

Accordingly, the flows and pressures are summarized as

$$q = \begin{bmatrix} q_{\mathrm{Pi}} \\ q_{\mathrm{Pu}} \\ q_{\mathrm{De}} \end{bmatrix}, \quad p = \begin{bmatrix} p_{\mathrm{Jc}} \\ p_{\mathrm{Re}} \end{bmatrix}.$$

Combining the element equations with the connection structure, the dynamic of the network ist described by the DAE

$$\dot{q}_{\mathrm{Pi}} = C_1(A_{\mathrm{Jc,Pi}}^T p_{\mathrm{Jc}} + A_{\mathrm{Re,Pi}}^T p_{\mathrm{Re}}) + C_2 \operatorname{diag}(|q_{\mathrm{Pi},j}|)\, q_{\mathrm{Pi}} + C_3 \tag{3a}$$

$$0 = A_{\mathrm{Jc,Pu}}^T p_{\mathrm{Jc}} + A_{\mathrm{Re,Pu}}^T p_{\mathrm{Re}} - f_{\mathrm{Pu}}(q_{\mathrm{Pu}}) \tag{3b}$$

$$0 = A_{\mathrm{Jc,Pi}} q_{\mathrm{Pi}} + A_{\mathrm{Jc,Pu}} q_{\mathrm{Pu}} + A_{\mathrm{Jc,De}} q_{\mathrm{De}} \tag{3c}$$

$$q_{\mathrm{De}} = \bar{q}_{\mathrm{De}} \tag{3d}$$

$$p_{\mathrm{Re}} = \bar{p}_{\mathrm{Re}} \tag{3e}$$

where $C_I := \operatorname{diag}\left(c_{I,j}\right)_{j=1,\dots,n_{\mathrm{Pi}}}$ for $I = 1, 2, 3$ and $f_{\mathrm{Pu}} := [f_{\mathrm{Pu}_j}]_{j=1,\dots,n_{\mathrm{Pu}}}$. The unknowns are given by $q(t)$ and $p(t)$. The system is square with size $n_{\mathrm{Pi}} + n_{\mathrm{Pu}} + n_{\mathrm{De}} + n_{\mathrm{Re}} + n_{\mathrm{Jc}}$.

3 Topology based index analysis of a single network

To analyze the solvability of the DAE (3), we impose the following assumptions on the connection structure of the network $\mathcal{N}$.

Assumption 1 *Consider a network $\mathcal{N}$ as in* (1).

(N1) Two junctions are connected at most by one pipe or one pump.
(N2) Each pipe, pump and demand has an assigned direction.
(N3) The network is connected, i.e., every pair of junctions and/or reservoirs can be reached by a sequence of pipes and pumps.
(N4) Every junction is adjacent to at most one demand branch. Every reservoir is connected at most to one pipe or pump.

Under Assumption 1 the network graph is simple (N1), oriented (N2) and connected (N3). Assigning a direction to each pipe, pump and demand, allows to speak of a positive or negative mass flow. Note that this orientation of the pipes and pumps is arbitrary and only serves as a reference condition, it is not necessarily related with the true or expected direction of the fluid flow. As the reservoirs are end vertices and the demands are connected to junctions only, cf. (N4), implies that no reservoir is connected to a demand branch and hence the corresponding sub-matrix of the incidence matrix is zero, i.e. $A_{\mathrm{Re,De}} = 0$.

Graphtheoretical prerequisites In the following, we use graph theoretical concepts like paths, spanning trees, cycles, connected components, etc. A comprehensive introduction to this topic can be found, e.g., in [3,5,24].

For our purposes, we need these concepts for subsets describing the connection structure of two specific element types. Asking, e.g., for the solvability of the pump equations (3b), we are interested in the connection structure of the junction and pump subset $\{\mathcal{JC}, \mathcal{PU}\}$. This set is not necessarily a subgraph as it might contain isolated pumps (corresponding to a pump connecting two reservoirs), isolated junctions (corresponding to a junction connected to pipes and demands only) or loose edges (corresponding to a pump connecting a junction and a reservoir). Consequently, the connection matrix $A_{\mathrm{Jc,Pu}}$ does not have the usual entry pattern of an incidence matrix. Still, the ideas of trees, cycles, etc. and their correspondence to fundamental subspaces of the connection matrix can be easily extended, see [2].

Looking at the junction and pump subset $\mathcal{G}_{\mathrm{Jc,Pu}} := \{\mathcal{JC}, \mathcal{PU}\}$, we are interested in particular in the following substructures.

Substructure 1 *Substructure 1 of $\mathcal{G}_{Jc,Pu}$.*

a) Paths of pumps connecting two reservoirs.
b) Cycles of pumps.

An example of Substructure 1 is given in Figure 2. On each of the substructures of Substructure 1, the pressure difference is fixed. On a path of pumps between two reservoirs, the pressure drop across the path is fixed by the two reservoirs. On a cycle of pumps, the pressure difference vanishes as the path is closed. Regarding the solvability of the DAE, this means that on Substructure 1, the pumps have to work against their usual mode of operation. Instead of returning a pressure drop for a given mass flow, they have to adjust the mass flow to a given pressure. This means that the pump characteristic has to be invertible. Algebraically, the pumps of Substructure 1 corresponds to the kernel of the connection matrix $A_{\mathrm{Jc,Pu}}$ of the set $\mathcal{G}_{\mathrm{Jc,Pu}}$. If $V_2 \in \mathbb{R}^{n_{\mathrm{Pu}} \times n_{V_2}}$ selects the paths of pumps between reservoirs as well as the cycles of pumps, then $\mathrm{span}(V_2) = \ker(A_{\mathrm{Jc,Pu}})$ [2].

Substructure 2 *Substructure 2 of $\mathcal{G}_{Jc,Pu}$.*

a) Connected components of junctions and pumps without loose pumps.
b) Isolated junctions.

An example of Substructure 1 is given in Figure 2. On a connected component of junctions and pumps *without* loose pumps, only the pressure difference is fixed by the pumps. The absolute value cannot be specified as connected component misses a connection to a reservoir giving a reference value. On isolated junctions, the pressure is naturally not fixed by pumps but by the incident pipes. Regarding the solvability of the DAE, this means that on Junctions of Substructure 2, the (absolute) value of the pressure is not specified by the pump equation. Instead these pressures are specified by the hidden constraints, see Theorem 2. Algebraically, the junctions of Substructure 2 correspond to the left kernel of the connection matrix $A_{\mathrm{Jc,Pu}}$ of the set $\mathcal{G}_{\mathrm{Jc,Pu}}$. If $[U_{2,1}^T, U_{2,2}^T]^T \in \mathbb{R}^{n_{\mathrm{Jc}} \times n_{U_2}}$ is such that $U_{2,1}$ selects the isolated junctions in $\mathcal{G}_{\mathrm{Jc,Pu}}$ and $U_{2,2}$ selects the junctions belonging to the connected components of $\mathcal{G}_{\mathrm{Jc,Pu}}$, then $\ker(A_{\mathrm{Jc,Pu}}) = \mathrm{span}(U_2)$ [2]. Graphically, the action of $U_{2,2}$ on $A_{\mathrm{Jc,Pu}}$ corresponds to the *vertex identification* of the connected components $\mathcal{G}_{\mathrm{Jc,Pu};1}, ..., \mathcal{G}_{\mathrm{Jc,Pu};n_k}$ of $\mathcal{G}_{\mathrm{Jc,Pu}}$, i.e., we melt every connected component of pumps and junctions into a single junction

$$\overline{\mathrm{Jc}}_k := \bigcup_{i\colon \mathrm{Jc}_i \in \mathcal{G}_{\mathrm{Jc,Pu};k}} \mathrm{Jc}_i, \tag{4}$$

for $k = 1, ..., n_k$. An example of the vertex identification (4) as well as of the graph $\mathcal{G}_{\overline{\mathrm{Jc}},\mathrm{Pi}}$ is given in Figure 3.

We summarize the junctions $\overline{\mathrm{Jc}}_1, ..., \overline{\mathrm{Jc}}_{n_k}$ arising from the vertex identification (4) as well as the remaining, i.e., isolated, junctions in the set

$$\overline{\mathcal{JC}} := \left\{\overline{\mathrm{Jc}}_1, ..., \overline{\mathrm{Jc}}_{n_k}\right\} \cup \mathcal{JC} \setminus \{\mathrm{Jc}_i | \exists k\colon \mathrm{Jc}_i \in \overline{\mathrm{Jc}}_k\}.$$

and consider the set $\mathcal{G}_{\overline{\mathrm{Jc}},\mathrm{Pi}} := \{\overline{\mathcal{JC}}, \mathcal{PI}\}$. The connection matrix of $\mathcal{G}_{\overline{\mathrm{Jc}},\mathrm{Pi}}$ is given by

$$A_{\overline{\mathrm{Jc}},\mathrm{Pu}} = U_2^T A_{\mathrm{Jc,Pu}}.$$

For $\mathcal{G}_{\overline{\mathrm{Jc}},\mathrm{Pi}}$, we consider the following substructures.

Substructure 3 *Substructure 3 of* $\mathcal{G}_{\overline{Jc},Pi}$.

a) A spanning tree, i.e. the largest subgraph without cycles.
b) The chord set belonging to the spanning tree, i.e., pipes that close a cycle.

An example of Substructure 3 is given in Figure 4. On a spanning tree, the pressure difference across the edges is well-defined. The chord set refers to those edges that destroy this well-definiteness as they close a cycle. Regarding the solvability of the DAE, this means that on Substructure 3 a), the pressure drop across the pipes is well-defined. Algebraically, Substructure 3 corresponds to a permutation $[\Pi_1 \Pi_2]$, where Π_1 selects the edges on the spanning tree and Π_2 the edges on the chord set. Then, $\mathrm{corange}(A_{\overline{\mathrm{Jc}},\mathrm{Pi}}) = \mathrm{span}(\Pi_1)$ [2].

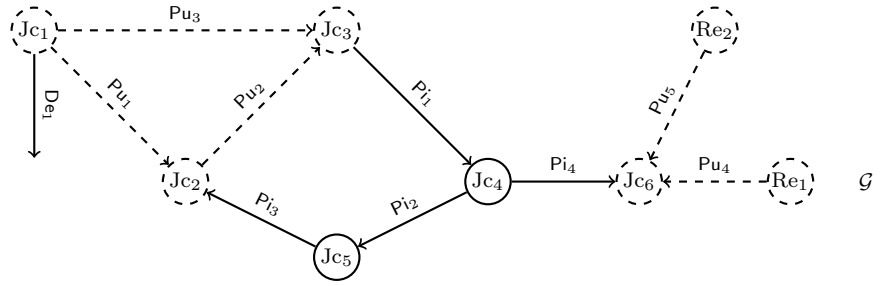

Fig. 2 Example of Substructure 1 and Substructure 2. The pumps $\mathrm{Pu}_4, \mathrm{Pu}_5$ form a path of pumps between the two reservoirs $\mathrm{Re}_1, \mathrm{Re}_2$, while the pumps $\mathrm{Pu}_1, \mathrm{Pu}_2, \mathrm{Pu}_3$ form a cycle of pumps. Together with junction Jc_6 as well as with the junctions $\mathrm{Jc}_1, \mathrm{Jc}_2, \mathrm{Jc}_3$, these pumps form the connected component of $\mathcal{G}_{\mathrm{Jc,Pu}}$. Junction $\mathrm{Jc}_4, \mathrm{Jc}_5$ are isolated in $\mathcal{G}_{\mathrm{Jc,Pu}}$ as they are not incident to any pump.

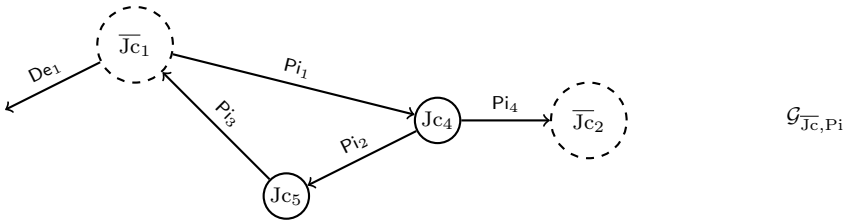

Fig. 3 Example of the vertex identification (4). Considering the graph $\mathcal{G}$ of Figure 2, the vertex identification of the connected components $\{\mathrm{Pu}_4, \mathrm{Pu}_5; \mathrm{Jc}_6\}$ and $\{\mathrm{Pu}_1, \mathrm{Pu}_2, \mathrm{Pu}_3; \mathrm{Jc}_1, \mathrm{Jc}_2, \mathrm{Jc}_3\}$ is given by $\overline{\mathrm{Jc}}_1$ and $\overline{\mathrm{Jc}}_2$, respectively. The resulting graph is $\mathcal{G}_{\overline{\mathrm{Jc}},\mathrm{Pi}}$.

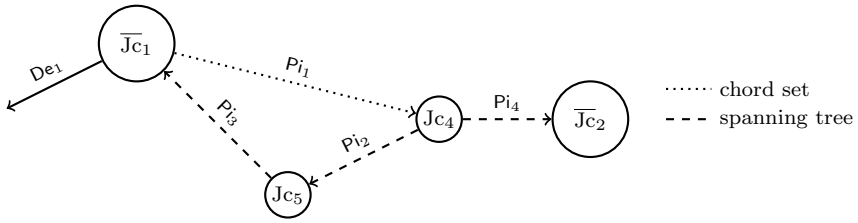

Fig. 4 Example of Substructure 3. For the graph $\mathcal{G}_{\overline{\mathrm{Jc}},\mathrm{Pi}}$ of Figure 3, a spanning tree is given by the pipes $\mathrm{Pi}_2, \mathrm{Pi}_3$ and Pi_4. The associated chord set is given by Pi_1.

Now, we derive solvability conditions for the network DAE. Starting from the DAE (3), we define the network function $F_{\mathcal{N}} \colon \mathbb{D} \to \mathbb{R}^n$, $\mathbb{D} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ with

$$F_{\mathcal{N},1}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}) = \dot{q}_{\mathrm{Pi}} - f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}) \tag{5a}$$

$$F_{\mathcal{N},2}(q_{\mathrm{Pu}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}) = A_{\mathrm{Jc,Pu}}^T p_{\mathrm{Jc}} + A_{\mathrm{Re,Pu}}^T p_{\mathrm{Re}} - f_{\mathrm{Pu}}(q_{\mathrm{Pu}}) \tag{5b}$$

$$F_{\mathcal{N},3}(q_{\mathrm{Pi}}, q_{\mathrm{Pu}}, q_{\mathrm{De}}) = A_{\mathrm{Jc,Pi}} q_{\mathrm{Pi}} + A_{\mathrm{Jc,Pu}} q_{\mathrm{Pu}} + A_{\mathrm{Jc,De}} q_{\mathrm{De}} \tag{5c}$$

$$F_{\mathcal{N},4}(q_{\mathrm{De}}) = q_{\mathrm{De}} - \bar{q}_{\mathrm{De}} \tag{5d}$$

$$F_{\mathcal{N},5}(p_{\mathrm{Re}}) = p_{\mathrm{Re}} - \bar{p}_{\mathrm{Re}}, \tag{5e}$$

where

$$f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}) := C_1(A_{\mathrm{Jc,Pi}}^T p_{\mathrm{Jc}} + A_{\mathrm{Re,Pi}}^T p_{\mathrm{Re}}) + C_2 \operatorname{diag}(|q_{\mathrm{Pi},j}|)\, q_{\mathrm{Pi}} + C_3.$$

Furthermore, we define the set of consistent initial values

$$\mathcal{C}_{IV} := \{(t_0, q_0, p_0) \in \mathcal{I} \times \mathbb{R}^{n_\mathcal{E}} \times \mathbb{R}^{n_\mathcal{V}} \mid \exists\, \dot{q}_0, \dot{p}_0 \colon F_\mathcal{N}(t_0, q_0, p_0, \dot{q}_0, \dot{p}_0) = 0\}.$$

Hence, the states q_0 and p_0 are consistent, if there exist vectors $\dot{q}_0$ and $\dot{p}_0$, such that the DAE (3) is algebraically satisfied. Usually, one needs more conditions on the set $\mathcal{C}_{IV}$, see [16]. In our setting, however, the DAE (5) is of s-index $\mu = 1$, see Theorem 1.

Combining the concept of derivative arrays [4] and the strangeness index as developed in [11,12,14,15] with graph theoretical results, the unique solvability of the DAE model (3) can be characterized.

Theorem 1 ([2]) *Let $\mathcal{N}$ be a network given by* (1) *that satisfies Assumptions 1 and let $F_\mathcal{N} \in C^2(\mathbb{D}, \mathbb{R}^n)$. Let $n_{Re} > 0$ and let $V_2^T \operatorname{D} f_{Pu} V_2$ be pointwise nonsingular for* $\operatorname{span}(V_2) = \ker(A_{Jc,Pu})$. *Then,*

1. *The DAE* (3) *has regular s-index $\mu = 1$ (d-index 2).*
2. *The DAE* (3) *is uniquely solvable for every $(t_0, q_0, p_0) \in \mathcal{C}_{IV}$ and the solution is $(q, p) \in C^1(\mathcal{I}, \mathbb{R}^n)$.*

Translated as conditions on the network structure and its elements, the solvability conditions of Theorem 1 mean the following. As the transfer elements (the pipes and pumps) only specify the pressure difference, at least one reservoir is needed to specify a reference value for the pressure in the junctions. On structures as defined in Substructure 1, i.e., paths of pumps between reservoirs or cycles of pumps. By construction, the matrix V_2 selects pumps lying on paths of pumps between reservoirs or cycles of pumps, i.e., structures on which the pressure difference is fixed, cp. Substructure 1. So instead of returning a pressure difference for given mass flow, pumps lying in $\operatorname{span}(V_2)$ must adjust their mass flow to a given pressure difference. Mathematically, this means that the corresponding pump function must be invertible, i.e., the matrix $V_2^T \operatorname{D} f_{\mathrm{Pu}} V_2$ must be pointwise nonsingular.

As the solvability conditions of Theorem 1 are formulated on the connection structure and the element functions, the plausibility of the network can be checked in a preprocessing step *before* the DAE is actually handed to a solver. If the solvability conditions are violated, the critical structures can be located in the network and advice can be given how to modify the model to obtain a physically reasonable system.

We can avoid the nonsingularity check of the matrix $V_2^T \operatorname{D} f_{\mathrm{Pu}} V_2$ by assuming that in every cycle of pumps and in every path of pumps between two reservoirs, there is at least one pipe.

Lemma 1 ([2]) *Let $\mathcal{N}$ be a network given by* (1) *that satisfies Assumptions 1. If on each path between two reservoirs and on each fundamental cycle there is at least one pipe, then* $\ker(A_{Jc,Pu}) = \{0\}$.

Lemma 1 gives a structural condition on the pumps in the network that is independent of the specific element functions. Stated as simple topological criteria, the assumption of Lemma 1 provides a very cheap and reliable preprocessing test for the solvability of the model under consideration.

So we can either impose a solvability condition on element level and check if $V_2^T \, \mathrm{D}f_{\mathrm{Pu}} V_2$ is pointwise nonsingular for a given pump specification or, to make sure that the model works for every kind of pump characteristics, impose the solvability condition on the structural level via the assumptions stated in Lemma 1.

The condition on element level, i.e., the non-singularity of $V_2^T \, \mathrm{D}f_{\mathrm{Pu}} V_2$, can be easily checked for not-to complicated pump constellations, allowing to use a broader class of pump functions. In some cases, the pump characteristic is a strictly monotone function and hence invertible.

The condition on structural level, i.e., the assumptions of Lemma 1, are useful for complex pump constellations and/or applications where the pump characteristics often change.

So depending on the topology of the network and the specific characteristic of the individual pumps, there are two options to ensure the global solvability.

Remark 1 Modeling single, smaller sized networks by hand, cycles of pumps or paths of pumps between reservoirs typically occur if serial or parallel pump constellations are considered. Furthermore, the characteristic pump equation (2b) is also representative for the class of quasi-stationary pipes. Quasi-stationary pipes are used if the transient behavior is negligible and consequently (2a) reduces to

$$c_{1,j} \Delta p_j = c_{2,j} |q_{\mathrm{Pi},j}| q_{\mathrm{Pi},j} + c_{3,j}.$$

Hence, considering networks consisting of transient pipes, quasi-stationary pipes, pumps, demand branches and reservoirs, the critical structures are paths of pumps and quasi-stationary pipes between reservoirs as well as cycles of pumps and quasi-stationary pipes. Indeed, this constellation occurs frequently in automatic modeling procedures.

Surrogate model Since the DAE (3) is of higher index, it is not suitable for a numerical simulation. Being assembled by simply glueing together the single elements, the DAE (3) contains hidden constraints, i.e., equations that every solution has to satisfy but which are not explicitly given in the representation (3). A simple example of such a hidden equation is given in Example 1. The hidden constraints might reduce the order of the method, might lead to drift of the numerical solution and creates problems in the initialization, see e.g., [16,13,19]. Exploiting again the topology, we can locate these constraints in the network and assemble a surrogate model with better numerical performance.

Theorem 2 ([2]) *Let $\mathcal{N}$ be a network given by* (1) *that satisfies Assumptions 1 and let $F_{\mathcal{N}} \in C^2(\mathbb{D}, \mathbb{R}^n)$. Let $n_{Re} > 0$ and let $V_2^T \, \mathrm{D}f_{Pu} V_2$ be pointwise nonsingular for*

$\operatorname{span}(V_2) = \ker(A_{Jc,Pu})$. *The* s-free model *of* $\mathcal{N}$ *is given by*

$$\begin{aligned}
\Pi_2^T \dot{q}_{Pi} &= \Pi_2^T f_{Pi}(q_{Pi}, p_{Jc}, p_{Re}) && \text{(6a)}\\
0 &= U_2^T A_{Jc,Pi} f_{Pi}(q_{Pi}, p_{Jc}, p_{Re}) + U_2^T A_{Jc,De} \dot{\bar{q}}_{De} && \text{(6b)}\\
0 &= A_{Jc,Pu}^T p_{Jc} + A_{Re,Pu}^T p_{Re} - f_{Pu}(q_{Pu}) && \text{(6c)}\\
0 &= A_{Jc,Pi} q_{Pi} + A_{Jc,Pu} q_{Pu} + A_{Jc,De} q_{De} && \text{(6d)}\\
q_{De} &= \bar{q}_{De} && \text{(6e)}\\
p_{Re} &= \bar{p}_{Re} && \text{(6f)}
\end{aligned}$$

where U_2 *and* Π_1 *are such that* $\operatorname{span}(U_2) = \operatorname{coker}(A_{Jc,Pu})$ *and* $\operatorname{corange}(U^T A_{Jc,Pu}) = \operatorname{span}(\Pi_1)$.

1. *The* s-free model *has regular s-index* $\mu = 0$ *(d-index 1).*
2. *A function* $(q, p) \in C^1(\mathcal{I}, \mathbb{R}^{n_\mathcal{E}} \times \mathbb{R}^{n_\mathcal{V}})$ *solves* (3) *if and only if it solves* (6).

The surrogate model (6) can be assembled based on network information only. The matrix U_2 selects the junctions of the connected components in $\mathcal{G}_{\mathrm{Jc,Pu}}$ and performs the vertex identification to construct the graph $\mathcal{G}_{\overline{\mathrm{Jc,Pi}}}$ of which Π_1 selects a spanning tree. Thus, the surrogate model (6) can be directly constructed from the network information, there is no need to compute (6) from (3) by symbolic or numerical manipulation, as it is necessary for example in a general modeling language like Modelica. In a simulation, this saves computational time as the system-to-solve (6) can be assembled directly from the network. Furthermore, the physical meaning of the equations and the states is preserved, i.e., in the DAE (6), each equation and each variable still has a physical counterpart. This is of special importance for the freely choosable initial conditions. Due to Theorem 2, the set of feasible initial conditions is determined by the chord set of $\mathcal{G}_{\overline{\mathrm{Jc,Pi}}}$. This means, that in model assembled from a modular system simulation tool, only those elements are allowed to accept user defined initial conditions. The remaining ones are derived from the algebraic equation (6b)–(6f). At that point it is also clear that the set of feasible initial condition is not unique, since the choice of a spanning tree may not be unique. Thus, errors in the initialization or the simulation can be located in the network, allowing constructive error detection and handling.

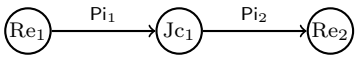

Fig. 5 Network model of Example 1.

Example 1 We consider two pipes $\mathrm{Pi}_1, \mathrm{Pi}_2$ that are coupled by a junction Jc_1, cp. Figure 5. For simplicity, we assume that the pipes are connected to two reservoirs Re_1 and Re_2. Then, we obtain the network DAE

$$\begin{aligned}
\dot{q}_{\mathrm{Pi},1} &= f_{\mathrm{Pi},1}(q_{\mathrm{Pi},1}, p_{\mathrm{Re},1} - p_{\mathrm{Jc},1}), \quad q_{\mathrm{Pi},1}(t_0) = q_{\mathrm{Pi},1,0},\\
\dot{q}_{\mathrm{Pi},2} &= f_{\mathrm{Pi},2}(q_{\mathrm{Pi},2}, p_{\mathrm{Jc},1} - p_{\mathrm{Re},2}), \quad q_{\mathrm{Pi},2}(t_0) = q_{\mathrm{Pi},2,0},\\
q_{\mathrm{Pi},1} &= q_{\mathrm{Pi},2}.
\end{aligned} \tag{7}$$

The pipes specify the mass flows differentially, while the junction relates the flows algebraically. Consequently, only one mass flow evolves dynamically, the other one is fixed algebraically by the mass balance. In particular, only one initial value can be chosen. The pressure only occurs implicitly in the differential equations. Differentiating the algebraic equation and inserting the pipe equations for the derivatives of the mass flows, however, we discover the algebraic equation

$$f_{\mathrm{Pi},1}(q_{\mathrm{Pi},1}, \bar{p}_{\mathrm{Re},1} - p_{\mathrm{Jc},1}) = f_{\mathrm{Pi},2}(q_{\mathrm{Pi},2}, p_{\mathrm{Jc},1} - \bar{p}_{\mathrm{Re},2}). \tag{8}$$

As $\mathrm{D}_2(f_{\mathrm{Pi},2} - f_{\mathrm{Pi},1}) = c_{1,1} + c_{1,2}$ is nonsingular, (8) can be solved for the pressure $p_{\mathrm{Jc},1}$ and (7) is uniquely solvable. Hence, coupling two pipes by a junction, the network model (7) contains a hidden algebraic equation that is needed to specify the pressure in the coupling junction. Also, (7) does not correctly reflect the number of differential and algebraic variables as only one mass flow evolves dynamically. Thus, we consider the surrogate model

$$\begin{aligned} \dot{q}_{\mathrm{Pi},1} &= f_{\mathrm{Pi},1}(q_{\mathrm{Pi},1}, p_{\mathrm{Re},1} - p_{\mathrm{Jc},1}), \qquad q_{\mathrm{Pi},1}(t_0) = q_{\mathrm{Pi},1,0}, \\ f_{\mathrm{Pi},1}(q_{\mathrm{Pi},1}, p_{\mathrm{Re},1} - p_{\mathrm{Jc},1}) &= f_{\mathrm{Pi},2}(q_{\mathrm{Pi},2}, p_{\mathrm{Jc},1} - p_{\mathrm{Re},2}), \\ q_{\mathrm{Pi},1} &= q_{\mathrm{Pi},2}. \end{aligned}$$

which corresponds to the strangeness free representation of equation (6).

4 A model for coupled flow networks

In this section we consider multiple networks as defined in Section 2 and analyzed in Section 3 and couple them via defined coupling conditions. All individual networks are assumed to fulfill Assumption 1 and that Theorem 1 as well as Theorem 2 are applicable. An example of a coupled network is given in Figure 6.

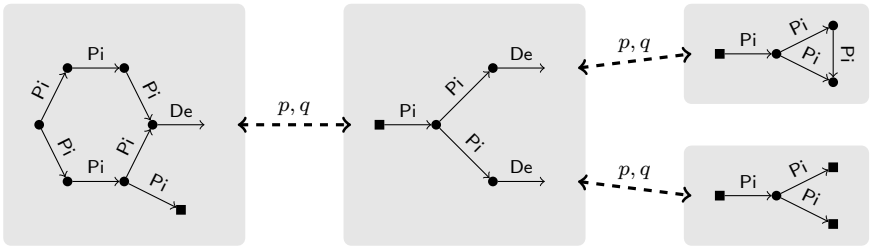

Fig. 6 Example of a coupled network consisting of four liquid flow networks.

We start by presentation some examples of coupled liquid flow network in order to point out the difficulties, that arise when dealing with such kind of problems. In all the shown cases one of the assumptions imposed in Theorem 1 or Theorem 2 is not satisfied for the coupled system. Therein the coupling is represented based on the network structure, cf. Figure 7. The boundary condition imposed on the state (■) and the boundary condition imposed on the flow (De) are melt together to a junction (•) via a cycling coupling of the flow q and the state p.

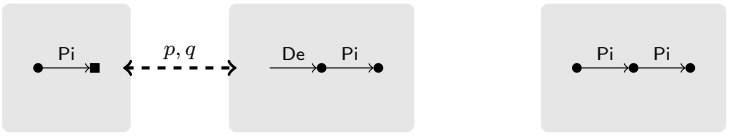

Fig. 7 Definition of the coupling of two networks (left) and the coupled equivalent network (right).

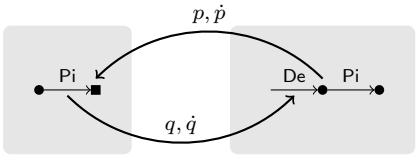

Fig. 8 Definition of the coupling of two networks via a directed databus connection.

In practical applications the coupling as defined in Figure 7 is realize via directed information databusses, see Figure 8. Eliminating the trivial relations leads to the equivalent representation of Figure 7. Hence for the analysis, the representation of Figure 7 is sufficient. At that point we also mention, that one important part of the coupling in Figure 8 is the availability of the derivatives of the coupling variables p and q. This means, that not only p and q are communicated via databusses, but also their derivatives with respect to time $\dot{p}$ and $\dot{q}$. This requirement is automatically fulfilled via the representation in Figure 7.

Example 2 (Missing reference pressure) Consider the network of coupled liquid flow networks as displayed in Figure 9. Clearly, both subnetworks are unique solvable. But the coupled network is not unique solvable, since the reference pressure is lost through the coupling procedure.

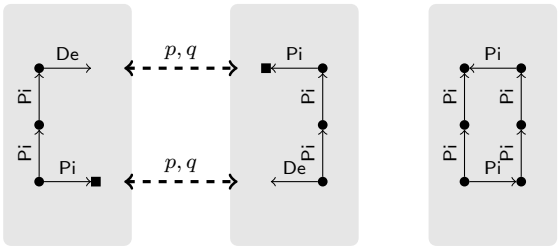

Fig. 9 Example of a coupled network consisting of two liquid flow networks (left) and the equivalent network (right). The coupled network is not uniquely solvable, since there remains no reservoir in the coupled network.

Example 3 (Cycle of pumps) Consider the network of coupled liquid flow networks as displayed in Figure 10. In contrast to Example 2 we replace some pipes by pumps and add an additional reservoir in one of the subnetworks. Clearly, both

subnetworks are unique solvable. But the coupled network may not be solvable at all, since a cycle consising solely of pumps is obtained through the coupling procedure.

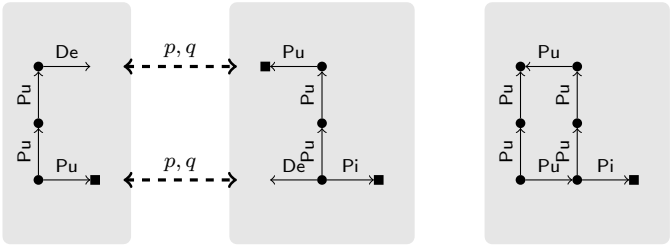

Fig. 10 Example of a coupled network consisting of two liquid flow networks (left) and the equivalent network (right). In contrast to Example 2, there remains a reservoir in the coupled network. Anyhow, the solvability of the coupled network cannot be guaranteed, since there arises a cycle of pumps.

Example 4 (Spanning tree) Consider the network of coupled liquid flow networks as displayed in Figure 11. In contrast to Example 2 we add an additional reservoir in one of the subnetworks. Clearly, both subnetworks are unique solvable and also the coupled network is uniquely solvable. Determining the spanning trees of the subnetworks and the combined networks, we observe, that the spanning tree of the combined network does not form a proper spanning tree of the new network (since it is not a tree). It can easily be seen, that another choice of the spanning tree in the subnetworks leads to a valid combined result.

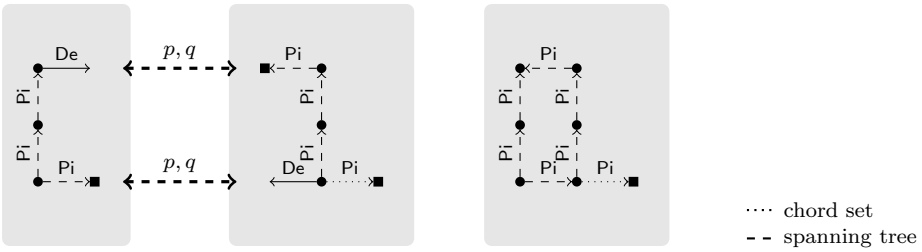

Fig. 11 Example of a coupled network consisting of two liquid flow networks (left) and the equivalent network (right). Both subnetworks as well as the coupled network are uniquely solvable. The surrogate model for the coupled network cannot be derived straight forward by combining the surrogate models of the subnetworks. Indeed, the combination of the spanning trees of the subnetworks does not form a proper spanning tree for the coupled network.

In the next section, the coupling addressed in Figure 7 is defined algebraically. Based on this definition an analysis is established, that gives answers to the issues raised in Example 2–4.

5 Topology based index analysis for coupled flow networks

We consider a set of networks $\mathcal{N}^1, ..., \mathcal{N}^K$ with graphs $\mathcal{G}^1, ..., \mathcal{G}^K$ and network functions $F_{\mathcal{N}}^1, ..., F_{\mathcal{N}}^K$. For $k = 1, ..., K$, we assume that $\mathcal{N}^k$ satisfies the Assumptions 1 as well as the solvability assumptions of Theorem 1. Then, the DAE (3) modeling the dynamics of $\mathcal{N}^k$ has regular s-index $\mu = 1$ and is uniquely solvable for every consistent initial value.

The networks $\mathcal{N}^1, ..., \mathcal{N}^K$ are connected to one large network $\mathcal{N}$ with graph $\mathcal{G}$ and network function $F_{\mathcal{N}}$. The coupling of the networks is performed via the boundary conditions, by the reservoirs and demands. Before we specify the coupling procedure, we point out the issues we are interested in.

Coupling these networks in a physically reasonable way to one large network $\mathcal{N}$, we want to answer the following questions:

1. Under which conditions does the coupled network $\mathcal{N}$ satisfy the Assumptions 1?
2. Can we assemble the network function $F_{\mathcal{N}}$ of the coupled network $\mathcal{N}$ from the individual network functions $F_{\mathcal{N}}^k$?
3. Under which conditions does the coupled network $\mathcal{N}$ satisfy the solvability assumptions of Theorem 1?
4. Can we deduce the index reduced DAE of the coupled system from the index reduced DAE of the subsystems?
5. How do we specify the consistent initial values of the coupled system from the consistent initial values of the subsystems?

For a network $\mathcal{N}^k$, we denote the boundary conditions that serve as coupling points by Re_c^k and De_c^k and summarize them in the sets $\mathcal{RE}_c^k$ and $\mathcal{DE}_c^k$, respectively. The boundary conditions that are not coupled are denoted by $\mathrm{Re}_{\not c}^k$ and $\mathrm{De}_{\not c}^k$ and summarized in the sets $\mathcal{RE}_{\not c}^k$ and $\mathcal{DE}_{\not c}^k$, respectively. Then, $\mathcal{RE}^k = \mathcal{RE}_c^k \cup \mathcal{RE}_{\not c}^k$ and $\mathcal{DE}^k = \mathcal{DE}_c^k \cup \mathcal{DE}_{\not c}^k$. We call the elements of $\mathcal{RE}_c^k$ and $\mathcal{DE}_c^k$ coupling reservoirs and coupling demands. Accordingly, we partition the junctions and edges incident to a coupling boundary condition by Jc_c^k and Pi_c^k, Pu_c^k and summarize them in the sets $\mathcal{JC}_c^k$ and $\mathcal{PI}_c^k$, $\mathcal{PU}_c^k$, respectively. The junctions and edges that are not incident to a coupling boundary condition are denoted by $\mathrm{Jc}_{\not c}^k$ and $\mathrm{Pi}_{\not c}^k$, $\mathrm{Pu}_{\not c}^k$ and summarized in the sets $\mathcal{JC}_{\not c}^k$ and $\mathcal{PI}_{\not c}^k$, $\mathcal{PU}_{\not c}^k$. Then, $\mathcal{JC}^k = \mathcal{JC}_c^k \cup \mathcal{JC}_{\not c}^k$ and $\mathcal{PI}^k = \mathcal{PI}_c^k \cup \mathcal{PI}_{\not c}^k$, $\mathcal{PU}^k = \mathcal{PU}_c^k \cup \mathcal{PU}_{\not c}^k$. We call the elements of $\mathcal{JC}_c^k$ and $\mathcal{PI}_c^k$, $\mathcal{PU}_c^k$ coupling junctions and coupling edges. In the following, we frequently summarize the set of pipes and pumps as $\mathcal{P} := \mathcal{PI} \cup \mathcal{PU}$ and denote its elements by P. The partitioning into coupling and non-coupling elements straightforward extends to $\mathcal{P}$ and its elements. For the coupling edges, we indicate the incident nodes where necessary by, e.g., $\mathrm{P}(\mathrm{Jc}_i, \mathrm{Jc}_j)$ if P is a pump or pipe incident to Jc_i and Jc_j.

As the considered networks satisfy Assumption 1, every reservoir is incident to exactly one pipe or pump, and every junction is incident to at most one demand. This one-to-one correspondence allows to number the coupling elements such that the coupling reservoir $\mathrm{Re}_{c,l}^k$ is incident to the coupling edge $\mathrm{P}_{c,l}^k$ and the coupling demand $\mathrm{De}_{c,m}^k$ is incident to the coupling junction $\mathrm{Jc}_{c,m}^k$.

With this notation, we define the coupling of two networks.

Definition 1 *Consider two networks $\mathcal{N}^1$, $\mathcal{N}^2$ as in (1). Let $Re_c^1 \in \mathcal{RE}^1$ be a coupling reservoir with coupling edge $P_c^1(Jc^1, Re_c^1) \in \mathcal{P}^1$ for $Jc^1 \in \mathcal{JC}^1$. Let $De_c^2 \in \mathcal{DE}^2$ be*

a coupling demand with coupling junction $Jc_c^2 \in \mathcal{JC}^2$*. The coupling of* $\mathcal{N}^1$*,* $\mathcal{N}^2$ *via* (Re_c^1, De_c^2) *is the network*

$$\mathcal{N} = \left(\mathcal{N}^1 \setminus \{Re_c^1, P_c^1(Jc^1, Re_c^1)\}\right) \cup \left(\mathcal{N}^2 \setminus \{De_c^2\}\right) \cup \left\{P_c^1(Jc^1, Jc_c^2)\right\}.$$

Hence, coupling $\mathcal{N}^1, \mathcal{N}^2$ via the pair $(\mathrm{Re}_c^1, \mathrm{De}_c^2)$ means that the coupling boundary conditions $\mathrm{Re}_c^1, \mathrm{De}_c^2$ are removed, while the coupling edge P_c^1 is connected to the coupling junction Jc_c^2. An example of the coupling procedure is given in Figure 12.

The incidence matrix $A_{\mathcal{N}}$ of the coupled network $\mathcal{N}$ reflects this coupling procedure as follows. With $A_{\mathcal{N}}$ given by

$$A_{\mathcal{N}} = \begin{bmatrix} A_{\mathrm{Jc}^1,\mathrm{P}^1_{\bullet}} & A_{\mathrm{Jc}^1,\mathrm{P}^1_c} & 0 & 0 & A_{\mathrm{Jc}^1,\mathrm{De}^1_{\bullet}} & 0 \\ 0 & 0 & A_{\mathrm{Jc}^2_{\bullet},\mathrm{P}^2} & 0 & 0 & A_{\mathrm{Jc}^2_{\bullet},\mathrm{De}^2_c} \\ 0 & A_{\mathrm{Re}^1_c,\mathrm{P}^1_c} & 0 & A_{\mathrm{Jc}^2_c,\mathrm{P}^2} & 0 & 0 \\ A_{\mathrm{Re}^1_{\bullet},\mathrm{P}^1_{\bullet}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{\mathrm{Re}^2_{\bullet},\mathrm{P}^2} & 0 & 0 \end{bmatrix}, \tag{9}$$

we see that the coupling boundary conditions $\mathrm{Re}_c^1, \mathrm{De}_c^2$ are removed, while the connection information of the coupling reservoir, i.e., the block $A_{\mathrm{Re}_c^1,\mathrm{P}_c^1}$, moves to the row of the coupling junction Jc_c^2. If $\mathrm{sgn}(\mathrm{P}_c^1) = \mathrm{sgn}(\mathrm{De}_c^2)$, then $A_{\mathrm{Re}_c^1,\mathrm{P}_c^1} = A_{\mathrm{Jc}_c^2,\mathrm{De}_c^2}$ and we can equivalently move the connection information of the coupling demand, i.e., the block $A_{\mathrm{Jc}_c^2,\mathrm{De}_c^2}$, to the column of the coupling edge P_c^1.

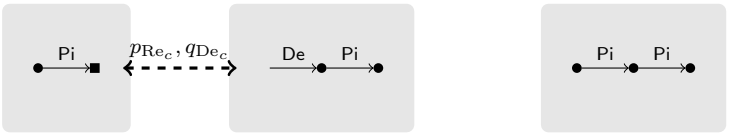

Fig. 12 Example of the coupling procedure defined in Definition 1.

Considering several networks $\mathcal{N}^1, ..., \mathcal{N}^K$, the coupling procedure of Definition 1 is successively applied to couple $\mathcal{N}^1, ..., \mathcal{N}^K$ into a single network. The information how the subnetworks are connected is stored in the *adjacency matrix* $B \in \mathbb{R}^{K \times K}$ defined by

$$B_{kl} = \begin{cases} 1, & k = l, \\ 1, & k \neq l \text{ and } \mathcal{N}^k, \mathcal{N}^l \text{ are connected according to the coupling} \\ & \text{procedure of Definition 1 via the coupling pair } (\mathrm{Re}_c^k, \mathrm{De}_c^l), \\ 0, & \text{else.} \end{cases}$$

The graph $\mathcal{G}_{\mathrm{coup}}$ associated with B is called the *coupling graph*.

In the following, we assume that two networks $\mathcal{N}^k$, $\mathcal{N}^l$ are coupled at most by one pair of boundary conditions. Coupling two networks via several boundary conditions, corresponds to coupling a network with itself, which corresponds to changing its internal structure. Hence, in the following, we assume that the coupling graph $\mathcal{G}_{\mathrm{coup}}$ is simple.

Coupling two networks $\mathcal{N}^k, \mathcal{N}^l$ by a coupling reservoir from $\mathcal{N}^k$ and a demand from $\mathcal{N}^l$, we can thus number the elements such that the coupling is performed by the pair $(\mathrm{Re}_c^k, \mathrm{De}_c^l)$. Furthermore, every coupling boundary condition indeed is coupled.

Given the adjacency matrix, we specify the structure of the coupled network $\mathcal{N}$.

Lemma 2 *Consider networks $\mathcal{N}^1, ..., \mathcal{N}^K$ as in (1). Let $B \in \mathbb{R}^{K\times K}$ the adjacency matrix of a simple coupling graph $\mathcal{G}_{coup}$. The coupling of $\mathcal{N}^1, ..., \mathcal{N}^K$ according to the adjacency matrix B is the network $\mathcal{N}$*

$$\mathcal{N} = \bigcup_{\substack{k,l\in\{1,...,K\}\\ s.t.\ B_{kl}=1}} \left(\mathcal{N}^k \setminus \{Re_c^k, P_c^k(Jc^k, Re_c^k)\}\right) \cup \left(\mathcal{N}^l \setminus \{De_c^l\}\right) \cup \left\{P_c^k(Jc^k, Jc_c^l)\right\}.$$

The incidence matrix of $\mathcal{N}$ is given by

$$A_{\mathcal{N}} = \begin{bmatrix} A_{Jc,P} & A_{Jc,De} \\ A_{Re,P} & 0 \end{bmatrix},$$

where

$$A_{Jc,P} = \begin{bmatrix} A_{Jc^1,P^1} & & A_{coup,kl} \\ & \ddots & \\ A_{coup,kl} & & A_{Jc^K,P^K} \end{bmatrix},$$

$$A_{Jc,De} = \begin{bmatrix} A_{Jc^1,De_{\bullet}^1} & & \\ & \ddots & \\ & & A_{Jc^K,De_{\bullet}^K} \end{bmatrix}, \quad A_{Re,P} = \begin{bmatrix} A_{Re_c^1,P^1} & & \\ & \ddots & \\ & & A_{Re_c^K,P^K} \end{bmatrix}.$$

In $A_{Jc,P}$, the diagonal blocks are partitioned according to

$$A_{Jc,P} = \begin{bmatrix} A_{Jc_{\bullet}^k,P_{\bullet}^k} & A_{Jc_{\bullet}^k,P_c^k} \\ A_{Jc_c^k,P_{\bullet}^k} & A_{Jc_c^k,P_c^k} \end{bmatrix},$$

and the off-diagonal blocks $A_{coup,kl}$ are given (up to permutation) by

$$A_{coup,kl} = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & A_{Re_{c,l}^k,P_{c,l}^k} \end{bmatrix}, & \text{if } B_{kl} = 1, \\ 0, & \text{if } B_{kl} = 0. \end{cases}$$

Proof The assertion follows from Definition 1 and the structure of the incidence matrix (9).

□

If the coupling graph $\mathcal{G}_{\mathrm{coup}}$ is simple, then $\mathcal{N}$ satisfies Assumption 1 if the subnetworks $\mathcal{N}^1, ..., \mathcal{N}^K$ do.

Lemma 3 *Consider networks $\mathcal{N}^1, ..., \mathcal{N}^K$ as in (1). Let $B \in \mathbb{R}^{K\times K}$ be the adjacency matrix of a simple coupling graph $\mathcal{G}_{coup}$ and let $\mathcal{N}$ be the coupling of $\mathcal{N}^1, ..., \mathcal{N}^K$ according to B. If $\mathcal{N}^1, ..., \mathcal{N}^K$ satisfy Assumption 1, then the coupled network $\mathcal{N}$ satisfies Assumption 1.*

Proof As the coupling graph $\mathcal{G}_{\text{coup}}$ is simple, two networks $\mathcal{N}^k, \mathcal{N}^l$ are connected at most by one coupling edge. As $\mathcal{N}^1, ..., \mathcal{N}^K$ are simple by assumption, this implies that also $\mathcal{N}$ satisfies Assumption 1 (N1). As the coupling edges keep their orientation and the networks $\mathcal{N}^1, ..., \mathcal{N}^K$ are oriented, also $\mathcal{N}$ is oriented, hence (N2) is satisfied. As $\mathcal{N}^1, ..., \mathcal{N}^K$ are connected, connecting them to a new graph $\mathcal{N}$, also $\mathcal{N}$ is connected, hence (N3) is satisfied. The coupling procedure of Definition 1 only removes boundary conditions, it does not add any reservoirs or demands. As $\mathcal{N}^1, ..., \mathcal{N}^K$ satisfy (N4), also $\mathcal{N}$ satisfies (N4).

□

To assemble the network of the coupled network $\mathcal{N}$, we partition the network function $F_{\mathcal{N}^k}$ of a network $\mathcal{N}^k$ according to the coupling elements into, cp. (5),

$$F_{\mathcal{N}^k,1_e}(q_{\mathrm{Pi}_e^k}, p_{\mathrm{Jc}^k}, p_{\mathrm{Re}_e^k}) = \dot{q}_{\mathrm{Pi}_e^k} - f_{\mathrm{Pi}_e^k}(q_{\mathrm{Pi}_e^k}, p_{\mathrm{Jc}^k}, p_{\mathrm{Re}_e^k}) \tag{10a}$$

$$F_{\mathcal{N}^k,1_c}(q_{\mathrm{Pi}_c^k}, p_{\mathrm{Jc}^k}, p_{\mathrm{Re}_c^k}) = \dot{q}_{\mathrm{Pi}_c^k} - f_{\mathrm{Pi}_c^k}(q_{\mathrm{Pi}_c^k}, p_{\mathrm{Jc}^k}, p_{\mathrm{Re}_c^k}) \tag{10b}$$

$$F_{\mathcal{N}^k,2_e}(q_{\mathrm{Pu}_e^k}, p_{\mathrm{Jc}^k}, p_{\mathrm{Re}_e^k}) = A^T_{\mathrm{Jc}^k,\mathrm{Pu}_e^k} p_{\mathrm{Jc}^k} + A^T_{\mathrm{Re}_e^k,\mathrm{Pu}_e} p_{\mathrm{Re}_e^k} - f_{\mathrm{Pu}_e^k}(q_{\mathrm{Pu}_e^k}) \tag{10c}$$

$$F_{\mathcal{N}^k,2_c}(q_{\mathrm{Pu}_c^k}, p_{\mathrm{Jc}^k}, p_{\mathrm{Re}_c^k}) = A^T_{\mathrm{Jc}^k,\mathrm{Pu}_c^k} p_{\mathrm{Jc}^k} + A^T_{\mathrm{Re}_c^k,\mathrm{Pu}_c} p_{\mathrm{Re}_c^k} - f_{\mathrm{Pu}_c^k}(q_{\mathrm{Pu}_c^k}) \tag{10d}$$

$$F_{\mathcal{N}^k,3_e}(q_{\mathrm{Pi}^k}, q_{\mathrm{Pu}^k}, q_{\mathrm{De}_e^k}) = A_{\mathrm{Jc}_e^k,\mathrm{Pi}^k} q_{\mathrm{Pi}^k} + A_{\mathrm{Jc}_e^k,\mathrm{Pu}^k} q_{\mathrm{Pu}^k} + A_{\mathrm{Jc}_e^k,\mathrm{De}^k} q_{\mathrm{De}_e^k} \tag{10e}$$

$$F_{\mathcal{N}^k,3_c}(q_{\mathrm{Pi}^k}, q_{\mathrm{Pu}^k}, q_{\mathrm{De}_c^k}) = A_{\mathrm{Jc}_c^k,\mathrm{Pi}^k} q_{\mathrm{Pi}^k} + A_{\mathrm{Jc}_c^k,\mathrm{Pu}^k} q_{\mathrm{Pu}^k} + A_{\mathrm{Jc}_c^k,\mathrm{De}_c^k} q_{\mathrm{De}_e^k} \tag{10f}$$

$$F_{\mathcal{N}^k,4_e}(q_{\mathrm{De}_c^k}) = q_{\mathrm{De}_c^k} - \bar{q}_{\mathrm{De}_c^k} \tag{10g}$$

$$F_{\mathcal{N}^k,4_c}(q_{\mathrm{De}_e^k}) = q_{\mathrm{De}_e^k} - \bar{q}_{\mathrm{De}_e^k} \tag{10h}$$

$$F_{\mathcal{N}^k,5_e}(p_{\mathrm{Re}_e^k}) = p_{\mathrm{Re}_e^k} - \bar{p}_{\mathrm{Re}_e^k} \tag{10i}$$

$$F_{\mathcal{N}^k,5_c}(p_{\mathrm{Re}_c^k}) = p_{\mathrm{Re}_c^k} - \bar{p}_{\mathrm{Re}_c^k}, \tag{10j}$$

where $f_{\mathrm{Pi}_e^k}(q_{\mathrm{Pi}_e^k}, p_{\mathrm{Jc}^k}, p_{\mathrm{Re}_e^k}) = [f_{\mathrm{Pi}_{e,l}^k}(q_{\mathrm{Pi}_{e,l}^k}, p_{\mathrm{Jc}^k}, p_{\mathrm{Re}_e^k})]_{l=1,...,|\mathcal{PI}_e|}$ and $f_{\mathrm{Pi}_c^k}(q_{\mathrm{Pi}_c^k}, p_{\mathrm{Jc}^k}, p_{\mathrm{Re}_c^k}) = [f_{\mathrm{Pi}_{c,l}^k}(q_{\mathrm{Pi}_{c,j}^k}, p_{\mathrm{Jc}^k}, p_{\mathrm{Re}_c^k})]_{l=1,...,|\mathcal{PI}_e|}$ as well as $f_{\mathrm{Pu}_e^k}(q_{\mathrm{Pu}_e^k}) = [f_{\mathrm{Pu}_{e,l}^k}(q_{\mathrm{Pu}_{e,l}^k})]_{l=1,...,|\mathcal{PU}_e|}]$ and $f_{\mathrm{Pu}_c^k}(q_{\mathrm{Pu}_c^k}) = [f_{\mathrm{Pu}_{c,l}^k}(q_{\mathrm{Pu}_{c,l}^k})]_{l=1,...,|\mathcal{PU}_c|}]$.

Then, the DAE of the coupled system is given as follows.

Lemma 4 *Consider networks $\mathcal{N}^1, ..., \mathcal{N}^K$ as in (1) that satisfy Assumption 1. Let $B \in \mathbb{R}^{K\times K}$ be the adjacency matrix of a simple coupling graph $\mathcal{G}_{coup}$ and let $\mathcal{N}$ be the coupling of $\mathcal{N}^1, ..., \mathcal{N}^K$ according to B. The network function $F_{\mathcal{N}}$ of $\mathcal{N}$ is given by*

$$F_{\mathcal{N},1}(q_{Pi}, p_{Jc}, p_{Re_e}) = \begin{bmatrix} F_{\mathcal{N}^k,1_e}(q_{Pi_e^k}, p_{Jc^k}, p_{Re_e^k}) \\ F_{\mathcal{N}^k,1_c}(q_{Pi_c^k}, p_{Jc^k}, p_{Jc_c^l}) \end{bmatrix}_{k=1,...,K,\, l\neq k\in\{1,...,K\}} \tag{11a}$$

$$F_{\mathcal{N},2}(q_{Pu_e}, p_{Jc}, p_{Re_e}) = \begin{bmatrix} F_{\mathcal{N}^k,2_e}(q_{Pu_e^k}, p_{Jc^k}, p_{Re_e^k}) \\ F_{\mathcal{N}^k,2_c}(q_{Pu_c^k}, p_{Jc^k}, p_{Jc_c^l}) \end{bmatrix}_{k=1,...,K,\, l\neq k\in\{1,...,K\}} \tag{11b}$$

$$F_{\mathcal{N},3}(q_{Pi}, q_{Pu}, q_{De_e}) = \begin{bmatrix} F_{\mathcal{N}^k,3_e}(q_{Pi^k}, q_{Pu^k}, q_{De_e^k}) \\ F_{\mathcal{N}^k,3_c}(q_{Pi^k}, q_{Pu^k}, q_{P_c^l}) \end{bmatrix}_{k=1,...,K,\, l\neq k\in\{1,...,K\}} \tag{11c}$$

$$F_{\mathcal{N},4}(q_{De_e}) = \left[F_{\mathcal{N}^k,4_e}(q_{De_e^k})\right]_{k=1,...,K} \tag{11d}$$

$$F_{\mathcal{N},5}(p_{Re}) = \left[F_{\mathcal{N}^k,5_e}(p_{Re_e^k})\right]_{k=1,...,K}, \tag{11e}$$

If $F_{\mathcal{N}^k} \in C^2(\mathbb{D}^k, \mathbb{R}^n)$ for $k = 1, 2$, then $F_{\mathcal{N}} \in C^2(\mathbb{D}, \mathbb{R}^n)$.

Proof The coupling procedure of Definition 1 does not change the internal structure of the networks $\mathcal{N}^1, ..., \mathcal{N}^K$, so the solution q, p of the coupled network $\mathcal{N}$ must naturally satisfy the DAEs

$$F_{\mathcal{N}^k}(q^k, p^k) = 0, \quad k = 1, ..., K \tag{12}$$

with $F_{\mathcal{N}^k}$ given by (10). By the coupling, we obtain the additional condition

$$q_{\mathrm{De}^k_{c,l}} = q_{\mathrm{P}^l_{c,k}} \qquad\qquad p_{\mathrm{Re}^k_{c,m}} = p_{\mathrm{Jc}^m_{c,k}} \tag{13}$$

for $l = 1, ..., |\mathcal{DE}_c|$, $m = 1, ..., |\mathcal{RE}_c|$ and $k = 1, ..., K$. With (13), we can eliminate the coupling boundary conditions in (12) and obtain (11). The smoothness of $F_{\mathcal{N}}$ then follows directly from the smoothness of the subnetwork functions.

Alternatively, we can construct (11) in the same manner as (5) using the incidence matrix $A_{\mathcal{N}}$.

□

Hence, the dynamics of the coupled network $\mathcal{N}$ are described by the DAE

$$F_{\mathcal{N}}(q_{\mathrm{Pi}}, q_{\mathrm{Pu}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}_\diamond}, q_{\mathrm{De}_\diamond}) = 0 \tag{14}$$

with $F_{\mathcal{N}}$ given by (11). The set of initial values is defined as

$$\mathcal{C}_{IV} := \{(t_0, q_0, p_0) \in \mathcal{I} \times \mathbb{R}^{n_\mathcal{E}} \times \mathbb{R}^{n_\mathcal{V}} \mid \exists\, \dot{q}_0, \dot{p}_0 \colon F_{\mathcal{N}}(t_0, q_0, p_0, \dot{q}_0, \dot{p}_0) = 0\}.$$

We characterize the solvability of the DAE (14).

Theorem 3 *Consider networks $\mathcal{N}^1, ..., \mathcal{N}^K$ as in (1). Let $B \in \mathbb{R}^{K\times K}$ be the adjacency matrix of a simple coupling graph $\mathcal{G}_{coup}$ and let $\mathcal{N}$ be the coupling of $\mathcal{N}^1, ..., \mathcal{N}^K$ according to B. Let $n_{Re^k_\diamond} > 0$ for at least one $k \in \{1, ..., K\}$ and let $V_2^T\, \mathrm{D}f_{Pu} V_2$ be pointwise nonsingular for $\operatorname{span}(V_2) = \ker(A_{Jc,Pu})$. Then, the following assertions hold.*

1. *The DAE (14) has regular s-index $\mu = 1$ (d-index 2).*
2. *The DAE (14) is uniquely solvable for every $(t_0, q_0, p_0) \in \mathcal{C}_{IV}$ and the solution is $(q, p) \in C^1(\mathcal{I}, \mathbb{R}^n)$.*

Proof Considering the reservoir part (11e) of the coupled network function, we find that $n_{\mathrm{Re}} = \sum_{k=1}^K n_{\mathrm{Re}^k_\diamond}$. Hence, if $n_{\mathrm{Re}^k_\diamond} > 0$ for at least one $k \in \{1, ..., K\}$, then $n_{\mathrm{Re}} > 0$. If, in addition, $V_2^T\, \mathrm{D}f_{\mathrm{Pu}} V_2$ is pointwise nonsingular for $\operatorname{span}(V_2) = \ker(A_{\mathrm{Jc,Pu}})$, then the network function (11) satisfies the assumptions of Theorem 1 and the assertions 1. and 2. follow. In particular, the set of initial values corresponds to the consistent initial values.

□

Coupling the networks $\mathcal{N}^1, ..., \mathcal{N}^K$ with a simple coupling graph, the solvability and index result of Theorem 1 straightforward extend to the coupled network $\mathcal{N}$.

The structural part of the solvability condition, i.e., $n_{\mathrm{Re}} > 0$, can be easily verified. If there is at least one non-coupling reservoir in one of the subnetworks, there is at least one reservoir in the coupled network. Under certain conditions, the element part of the solvability condition, i.e., $V_2^T\, \mathrm{D}f_{\mathrm{Pu}} V_2$ is pointwise nonsingular for $\operatorname{span}(V_2) = \ker(A_{\mathrm{Jc,Pu}})$, can be also deduced from the subnetworks, cp. Lemma 5.

Regarding the simulation of the coupled network $\mathcal{N}$, we consider the s-free surrogate model of $\mathcal{N}$.

Theorem 4 *Consider networks $\mathcal{N}^1, ..., \mathcal{N}^K$ as in (1). Let $B \in \mathbb{R}^{K\times K}$ be the adjacency matrix of a simple coupling graph $\mathcal{G}_{coup}$ and let $\mathcal{N}$ be the coupling of $\mathcal{N}^1, ..., \mathcal{N}^K$ according to B. Let $n_{Re_e^k} > 0$ for at least one $k \in \{1, ..., K\}$ and let $V_2^T \,\mathrm{D}f_{Pu} V_2$ be pointwise nonsingular for $\mathrm{span}(V_2) = \ker(A_{Jc,Pu})$. The* s-free model *of $\mathcal{N}$ is given by*

$$0 = \Pi_2^T F_{\mathcal{N},1}(q_{Pi}, p_{Jc}, p_{Re_e}) \tag{15a}$$

$$\begin{aligned} 0 = {} & U_{2,e}^T A_{Jc_e,Pi} f_{Pi}(q_{Pi}, p_{Jc}, p_{Re_e}) + U_{2,e}^T A_{Jc_e,De_e} \dot{q}_{De_e^k} \\ & + U_{2,c}^T A_{Jc_c,Pi} f_{Pi}(q_{Pi}, p_{Jc}, p_{Re_c}) + U_{2,c}^T A_{Jc_c,De_c} f_{Pi_c}(q_{Pi_c}, p_{Jc}, p_{Jc_c}) \end{aligned} \tag{15b}$$

$$0 = F_{\mathcal{N},2}(q_{Pu_e}, p_{Jc}, p_{Re_e}) \tag{15c}$$

$$0 = F_{\mathcal{N},3}(q_{Pi}, q_{Pu}, q_{De_e}) \tag{15d}$$

$$0 = F_{\mathcal{N},4}(q_{De_e}) \tag{15e}$$

$$0 = F_{\mathcal{N},5}(p_{Re_e}), \tag{15f}$$

where $f_{Pi} = [f_{Pi_e}^T, f_{Pi_c}^T]^T$ and $U_2 = [U_{2,e}^T, U_{2,c}^T]^T$ and Π_1 are such that $\mathrm{span}(U_2) = \mathrm{coker}(A_{Jc,Pu})$ and $\mathrm{corange}(U_2^T A_{Jc,Pu}) = \mathrm{span}(\Pi_1)$.

1. *The* s-free model *has regular s-index $\mu = 0$ (d-index 1).*
2. *A function $(q, p) \in C^1(\mathcal{I}, \mathbb{R}^{n_\mathcal{E}} \times \mathbb{R}^{n_\mathcal{V}})$ solves (14) if and only if it solves (15).*

Proof The surrogate model (15) follows straightforward from Theorem 2. Note that in the hidden constraints (15b), the coupling edges P_c play the role of the demands De_c as the coupling implies that $q_{\mathrm{P}_c} = q_{\mathrm{De}_c}$. With $\dot{q}_{\mathrm{P}_c} = f_{\mathrm{Pi}_c}(q_{\mathrm{Pi}_c}, p_{\mathrm{Jc}}, p_{\mathrm{Jc}_c})$, equation (15b) follows.

The assertions 1. and 2. also follow from Theorem 2.

□

Having specified the solvability conditions as well as the surrogate model of the coupled network, we ask how the knowledge about the Substructures 1 to 3 can be exploited to assemble the corresponding substructures of the coupled network $\mathcal{N}$.

Lemma 5 *Consider networks $\mathcal{N}^1, ..., \mathcal{N}^K$ as in (1). Let $B \in \mathbb{R}^{K\times K}$ be the adjacency matrix of a simple coupling graph $\mathcal{G}_{coup}$ and let $\mathcal{N}$ be the coupling of $\mathcal{N}^1, ..., \mathcal{N}^K$ according to B. For $k = 1, ..., K$, let V_2^k, U_2^k be such that $\mathrm{span}(V_2^k) = \ker(A_{Jc^k,Pu^k})$, $\mathrm{span}(U_2^k) = \mathrm{coker}(A_{Jc^k,Pu^k})$ and let $[\Pi_1^k, \Pi_2^k]$ be a permutation such that $\mathrm{span}(\Pi_1^k) = \mathrm{corange}(U_2^{k,T} A_{Jc^k,Pu^k})$. Let V_2, U_2 be such that $\mathrm{span}(V_2) = \ker(A_{Jc,Pu})$, $\mathrm{span}(U_2) = \mathrm{coker}(A_{Jc,Pu})$ and let $[\Pi_1, \Pi_2]$ be a permutation with $\mathrm{span}(\Pi_1) = \mathrm{corange}(U_2^T A_{Jc,Pu})$.*

1. *If the coupling of $\mathcal{N}^1, ..., \mathcal{N}^K$ is performed by pipes only, i.e., $\mathcal{PU}_c^k = \emptyset$ for $k = 1, ..., K$, then $V_2 = diag(V_2^k)_k$, $U_2 = diag(U_2^k)_k$ and $\Pi_1 = diag(\Pi_1^k)_k$.*
2. *If the coupling graph $\mathcal{G}_{coup}$ is a tree and the network $\mathcal{N}^k$ does not contain cycles of pumps or paths of pumps connecting two reservoirs for $k = 1, ..., K$, i.e., $\ker(A_{Jc^k,Pu^k}) = \{0\}$, then $\ker(A_{Jc,Pu}) = \{0\}$.*
3. *For $k = 1, ..., K$, if the network $\mathcal{N}^k$ does not contain paths of pumps connecting elements of $\mathcal{RE}_e^k$ and $\mathcal{RE}_c^k$, then $V_2 = diag(V_2^k)_k$.*

Proof The connection matrix $A_{\mathrm{Jc,Pu}}$ of the junction and pump set $\mathcal{G}_{\mathrm{Jc,Pu}}$ in $\mathcal{N}$ is given by, cp. Lemma 2,

$$A_{\mathrm{Jc,Pu}} = \begin{bmatrix} A_{\mathrm{Jc}^1,\mathrm{Pu}^1} & & A_{\mathrm{coup},kl} \\ & \ddots & \\ A_{\mathrm{coup},kl} & & A_{\mathrm{Jc}^K,\mathrm{Pu}^K} \end{bmatrix},$$

with

$$A_{\mathrm{Jc}^k,\mathrm{Pu}^k} = \begin{bmatrix} A_{\mathrm{Jc}^k_e,\mathrm{Pu}^k_e} & A_{\mathrm{Jc}^k_e,\mathrm{Pu}^k_c} \\ A_{\mathrm{Jc}^k_c,\mathrm{Pu}^k_e} & A_{\mathrm{Jc}^k_c,\mathrm{Pu}^k_c} \end{bmatrix}, \quad A_{\mathrm{coup},kl} = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & A_{\mathrm{Re}^l_{c,k},\mathrm{Pu}^l_{c,k}} \end{bmatrix}, & \text{if } B_{kl} = 1, \\ 0, & \text{if } B_{kl} = 0. \end{cases}$$

1. If the coupling is performed by pipes only, then $A_{\mathrm{coup},kl} = 0$ for $k, l = 1, ..., K$, and $A_{\mathrm{Jc,Pu}}$ is block diagonal. It follows that $\ker(A_{\mathrm{Jc,Pu}}) = \mathrm{span}(\mathrm{diag}(V_2^k)_k)$ with $\mathrm{span}(V_2^k) = \ker(A_{\mathrm{Jc}^k,\mathrm{Pu}^k})$ and $U_2 = \mathrm{diag}(U_2^k)_k$ with $\mathrm{span}(U_2^k) = \mathrm{coker}(A_{\mathrm{Jc}^k,\mathrm{Pu}^k})$. From the latter, in particular, it follows that $U_2^T A_{\mathrm{Jc,Pu}} = \mathrm{diag}(U_2^{k,T} A_{\mathrm{Jc}^k,\mathrm{Pu}^k})$. Hence, $\mathrm{corange}(U_2^T A_{\mathrm{Jc,Pu}}) = \mathrm{span}(\Pi_1)$ for $\Pi_1 = \mathrm{diag}(\Pi_1^k)_k$ with $\mathrm{span}(\Pi_1^k) = \mathrm{corange}(U_2^{k,T} A_{\mathrm{Jc}^k,\mathrm{Pu}^k})$ for $k = 1, ..., K$.

2. If the coupling graph $\mathcal{G}_{\mathrm{coup}}$ is a tree, then $\mathcal{G}_{\mathrm{coup}}$ is not strongly connected and its adjacency matrix B is reducible cp. [17]. Hence, there exists a permutation, such that $A_{\mathrm{Jc,Pu}}$ is block triangular. If $\ker(A_{\mathrm{Jc}^k,\mathrm{Pu}^k}) = \{0\}$ for $k = 1, ..., K$, then the triangular block structure of $A_{\mathrm{Jc,Pu}}$ implies that $\ker(A_{\mathrm{Jc,Pu}}) = \{0\}$. In the same manner, we find that $\mathrm{coker}(A_{\mathrm{Jc,Pu}}) = \{0\}$ if $\mathrm{coker}(A_{\mathrm{Jc}^k,\mathrm{Pu}^k}) = \{0\}$ for $k = 1, ..., K$. From the latter, in particular, we get that $U_2^T A_{\mathrm{Jc,Pu}} = \mathrm{diag}(U_2^{k,T} A_{\mathrm{Jc}^k,\mathrm{Pu}^k})$. Hence, $\mathrm{corange}(U_2^T A_{\mathrm{Jc,Pu}}) = \mathrm{span}(\Pi_1)$ for $\Pi_1 = \mathrm{diag}(\Pi_1^k)_k$ with $\mathrm{span}(\Pi_1^k) = \mathrm{corange}(U_2^{k,T} A_{\mathrm{Jc}^k,\mathrm{Pu}^k})$ for $k = 1, ..., K$.

3. WLOG, we assume that the matrix V_2^k with $\mathrm{span}(V_2^k) = \ker(A_{\mathrm{Jc}^k,\mathrm{Pu}^k})$ is sorted such that $V_2^k = [V_{2,1}^{k,T}, V_{2,2}^{k,T}]^T$, where $V_{2,1}^k$ selects paths of pumps connecting elements of $\mathcal{RE}_e^k$ and $\mathcal{RE}_e^k$ and cycles of pumps and $V_{2,2}^k$ selects paths of pumps connecting elements of $\mathcal{RE}_e^k$ and $\mathcal{RE}_c^k$. If $\mathcal{N}^k$ is free of paths of pumps connecting elements of $\mathcal{RE}_e^k$ and $\mathcal{RE}_c^k$, then $V_{2,2}^k = 0$, implying that $A_{\mathrm{coup},kl} V_2^k = 0$. As the coupling graph is simple, there is only one non-zero off-diagonal entry in each columng associated with a coupling pump. Hence, $\ker(A_{\mathrm{Jc,Pu}}) = \mathrm{span}(\mathrm{diag}(V_2^k)_k)$ with $\mathrm{span}(V_2^k) = \ker(A_{\mathrm{Jc}^k,\mathrm{Pu}^k})$.

□

As the Jacobian of the pump function is given by $\mathrm{D}f_{\mathrm{Pu}} = \mathrm{diag}(\mathrm{D}f_{\mathrm{Pu}^k})_k$, it follows that $V_2^T \mathrm{D}f_{\mathrm{Pu}} V_2 = \mathrm{diag}(V_2^{k,T} \mathrm{D}f_{\mathrm{Pu}^k} V_2^k)_k$ if one of the assertions of Lemma 5 is satisfied. Then, the matrix $V_2^T \mathrm{D}f_{\mathrm{Pu}} V_2$ is pointwise nonsingular if $V_2^{k,T} \mathrm{D}f_{\mathrm{Pu}^k} V_2^k$ are pointwise nonsingular.

Hence, under one of the assertions of Lemma 5, the solvability of the coupled network $\mathcal{N}$ is characterized by the subnetworks $\mathcal{N}^1, ..., \mathcal{N}^K$. There is no need for an extra solvability analysis of the coupled system.

If the coupling is performed by pipes only, then not only the solvability of the coupled network $\mathcal{N}$ is characterized by the subnetworks $\mathcal{N}^1, ..., \mathcal{N}^K$, but also the surrogate model of the coupled network can be assembled directly from the

subnetworks. The hidden constraints of the coupled network correspond to the hidden constraints of the subnetworks and the set of differential equations of the coupled system corresponds to the differential equations of the subnetworks. In particular, this implies that the set of consistent initial values $\mathcal{C}_{IV}$ of $\mathcal{N}$ corresponds to the union of the sets of consistent initial values $\mathcal{C}_{IV}^k$ of $\mathcal{N}^k$.

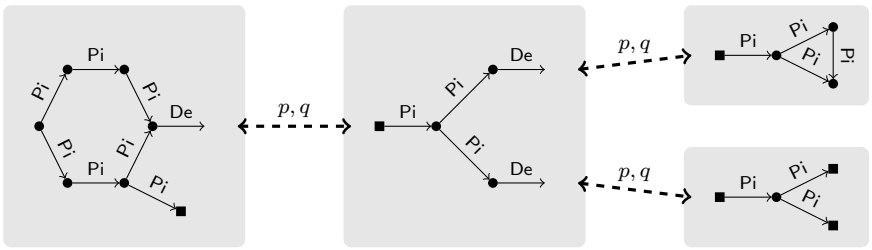

Fig. 13 Network of liquid flow systems

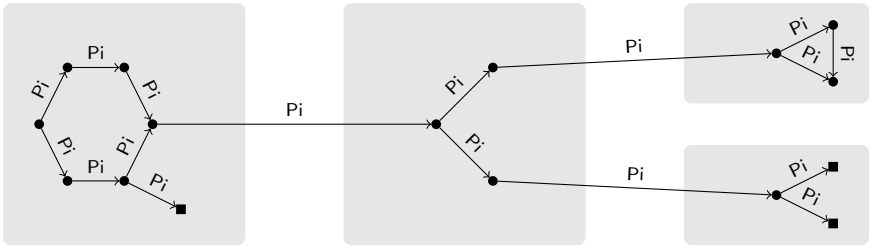

Fig. 14 Surrogate network for the analysis of liquid flow systems

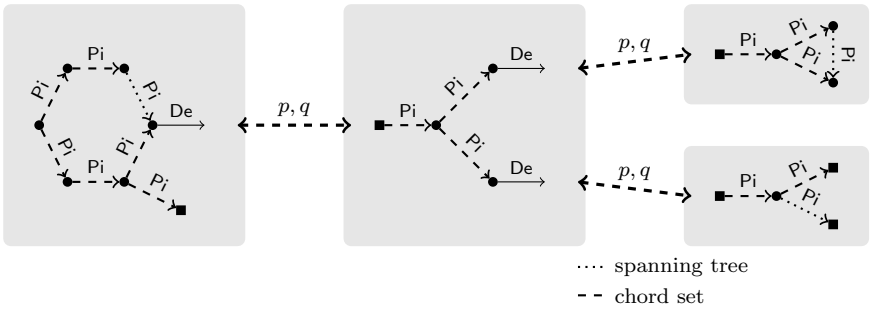

Fig. 15 Spanning tree network of liquid flow systems

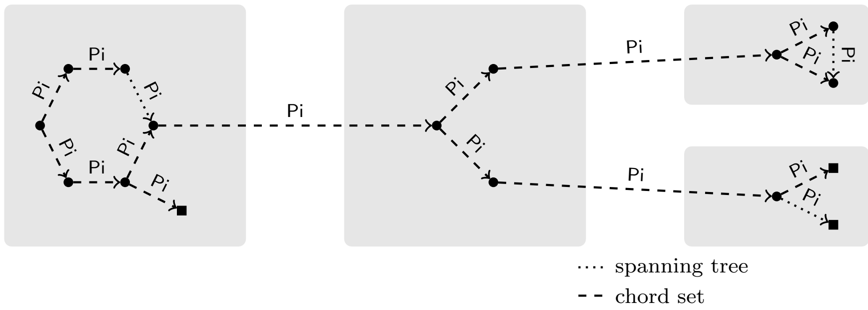

Fig. 16 Spanning tree for surrogate network for the analysis of liquid flow systems

6 Conclusion and discussion

So far physical networks have been considered mainly as isolated and stand alone, but in many application they are not. The derivation of physical based topological conditions is also required for coupled systems of physical DAEs. We have shown, that it is very promising to derive additional topology based rules for coupled system. Nevertheless they seem to be not sufficient for all type of constellations. It is quite remarkable, that already the analysis of the uniform network has to be done in an appropriate way to provide the basic framework for a constructive analysis of the coupled networks. As a specific graph theoretical problem we have identified the *unique* and *good* choice of the spanning trees of the graphs of the underlying network. Consequently, the existing tools for single networks (Modified nodal analysis, Topological based index analysis) have to be re-evaluated against the possibility to be used for the topological analysis in multi-network structures.

The main driving feature for the presented approach is the possibility to combine assembled models without changing the models itself (or changing initial conditions). Indeed this feature can not be guaranteed by using purely graph theoretical approaches like Pantelides or the Σ-Method. Anyhow, those algorithms have their right to exists in all applications, where a tight connection to the underlying physics is not relevant or not available. For modular system simulation software the strong connection to the physics increases the applicability in engineering approaches and therefore has to preferred.

Furthermore, the analysis of multi-network structures provides the basic tools for the treatment of black-box elements (which physical coupling conditions) within physical networks. At that point it is required to marriage purely graph theoretical approaches with physical based topological methods in order to extract the advantages of both worlds. One very recent example in automotive applications is the incorporation of FMUs in physical networks. Therein FMUs with appropriate coupling conditions provide internal dependency graphs, that can be re-interpreted as additional class of components. E.g. in liquid flow networks those components can form an additional class next to pipes, pumps, demands, junctions and reservoirs. In this case the topological criteria may be extended to this new classes.

In this work we have considered liquid flow network as a representative example. Indeed, those topics are also relevant for other physical domains like gas–dynamics and electric networks. To the authors best knowledge, the coupling of

electrical network via defined interface and coupling conditions has not been considered so far. For the *Modified nodal analysis* applied to networks of electric networks, the challenge definitely is hidden in the identification and correct treatment of CV-loops and IL-cutsets, that arise through the coupling procedure. At least the case of CV-loops in electrical networks may be equivalent to the case of pump circles in liquid flow networks.

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