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A dynamic programming approach for L^0 optimal control design

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Abstract

The present work investigates the optimal control problems with L^0-control cost. The value function is characterized as the unique viscosity solution of a Hamilton-Jacobi-Bellman (HJB) equation. The sparsity properties of optimal controllers induced by L^0-penalty is analyzed under different cases of control constraints. The existence of optimal controllers is discussed for the time-discretized problem. The value function and the optimal control are computed by solving the corresponding HJB equation. Numerical examples are presented under different types of control constraints and different penalization parameters with special attention to the sparsity. Comparisons between L^0-controller and other types of controllers are also illustrated.

1 Introduction

In this paper, we study the finite horizon optimal control problems of dynamical system with L^0-control cost and closed loop control design. This type of cost functional induces sparsity properties of the optimal controls, that is the optimal controls can be identically 0 on subsets of positive measure. However, the cost functional is non-smooth and non-convex, which is the main difficulty for the analysis of the problem. L^p-type control cost with $1 \leq p < \infty$ for this type of problem has been studied in the literature. In Falcone et al. (2014), L^p-penalized minimal time problems with $1 \leq p < \infty$ have been analyzed and the closed loop controls were computed via a semi-Lagrangian scheme. L^1-control cost, which is non-smooth but useful for the

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sparse control design, was considered in Vossen et al. (2006) and Alt et al. (2015) for open-loop optimal control of dynamical systems. More recently, L^p-type control cost with $p \in [0, 1[$ has been recognized as an useful tool for generating sparsity. In Ito et al. (2014b), sparsity optimization problems have been studied in infinite dimensional sequence space ℓ^p with $p \in [0, 1]$. In a recent work Kalise et al. (2016), infinite horizon optimal control problems concerning L^p-cost functionals with $p \in]0, 1]$ have been considered and sparsity properties of optimal controllers have been discussed. Ito et al. (2014a) has investigated optimal control problems with L^p, $p \in [0, 1[$, control cost, and existence results and necessary optimality conditions have been obtained for the problems with regularized cost functionals.

The cost functional involving L^0-control cost investigated in this paper is non-convex, therefore the existence of optimal controls is not guaranteed due to the absence of weak lower semi-continuity properties. In the literature, the available existence result has been obtained in the case of discrete optimization problem in infinite dimensional sequence space in Ito et al. (2014b), and for some restricted class of problems under specific conditions in Ito et al. (2014a). In this work, we at first assume the existence of optimal controls and derive the necessary optimality conditions. Our attention is focused on the sparsity properties of optimal controls deduced from the optimality conditions. The problem is under control constraints, and two different types of control constraints are considered: the ball constraints and the box constraints. Sparsity results are obtained in both cases. In the next step, we discretize the original optimal control problem in time, and we discuss the existence result for the time-discretized problem as in Kalise et al. (2016). The control space is then restricted to the space of piecewise constant functions, and then the existence result can be derived. The motivation of investigating the time-discretized version of problem comes from the numerical experiments since the numerical optimal solutions are piecewise constant for each time step. Finally, numerical examples with Eikonal dynamics under ball control constraints and box control constraints are computed via HJB approach. The numerical results confirm the analysis on the sparsity properties of optimal controls.

The paper is organized as follows. The framework of the optimal control problem is introduced in Section 2. The value function is defined and the HJB equation is given. Section 3 is devoted to the analysis on the sparsity properties of optimal controls derived from necessary optimality conditions. The time-discretized problem is discussed in Section 4 and the existence result is proved. In Section 5, numerical simulations on L^0-control are presented with comparisons to L^2- and L^1-controls and special attention is paid to the sparsity with respect to the penalization parameter.

2 Setting of the problem

Let $T > 0$, $x_d \in \mathbb{R}^d$, $\gamma > 0$ and A be a compact and convex subset of $\mathbb{R}^m$. For any $a = (a_1, \dots, a_m) \in A$, L^0-norm is defined by

$$\|a\|_0 = \sum_{i=1}^{m} |a_i|^0,$$

with the convention $0^0 = 0$. An important property of L^0-norm is the lower semi-continuity: for any $a_n \to a$ in $\mathbb{R}^m$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|a_n\|_0 \geq \|a\|_0.$$

Given $t \in [0, T]$ and $x \in \mathbb{R}^d$, consider the following dynamical system:

$$\begin{cases} \dot{y}(s) = f_0(y(s)) + \sum_{i=1}^{m} f_i(y(s))\alpha_i(s) & \text{for } s \in]t, T[, \\ y(t) = x, \end{cases} \tag{1}$$

where $\alpha = (\alpha_1, \dots, \alpha_m) \in L^\infty(t, T; A)$ is the input control. Throughout the paper, we assume that for $i = 0, \dots, m$, $f_i : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous. The cost functional is defined by

$$J(t, x, \alpha) = \int_t^T \left[\frac{1}{2} \|y(s) - x_d\|_2^2 + \gamma \|\alpha(s)\|_0 \right] ds,$$

where $\|\cdot\|_2$ is the Euclidean norm and (y, α) satisfies (1). The optimal control problem is the following:

$$v(t, x) = \inf\{J(t, x, \alpha) \, : \, \alpha \in L^\infty(0, T; A)\}, \tag{2}$$

where v is called the value function. v satisfies the following dynamic programming principle: for any $h \geq 0$,

$$v(t, x) = \inf_{y,\alpha} \left\{ v(t + h, y(t + h)) + \int_t^{t+h} \ell(y(s), \alpha(s)) ds \right\},$$

where (y, α) satisfies (1) and

$$\ell(x, a) := \frac{1}{2} \|x - x_d\|_2^2 + \gamma \|a\|_0, \ \forall\, x \in \mathbb{R}^d, \ a \in A.$$

By standard arguments in Bardi et al. (1997), v is the unique viscosity solution of the HJB equation

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x)) = 0 & \text{in }]0,T[\times\mathbb{R}^d, \\ u(T,x) = 0 & \text{in } \mathbb{R}^d. \end{cases} \tag{3}$$

Here the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is defined by

$$H(x,p) = \sup_{a\in A}\{-p\cdot f(x,a) - \ell(x,a)\},$$

where $f(x,a) := f_0(x) + \sum_{i=1}^m f_i(x)a_i$ for $(x,a) \in \mathbb{R}^d \times \mathbb{R}^m$.

3 Sparsity properties

The standard existence results do not apply for problem (2) due to the non-convexity of L^0-norm. Here we proceed by deriving the maximum principle and focus on the sparsity properties arising from L^0-norm, assuming the existence of an optimal control. The maximum principle, which does not require the convexity of the cost functional with respect to the control variable, is taken from Theorem 7.4.17 in Cannarsa et al. (2004) and recalled as follows.

Lemma 3.1. *Assume that f_i are continuously differentiable for $i = 0,\dots,m$. Given $(t,x) \in [0,T]\times\mathbb{R}^d$, let $\bar{\alpha} : [t,T] \to A$ be an optimal control for problem (2) with the initial data (t,x) and let $\bar{y} : [t,T] \to \mathbb{R}^d$ be the corresponding optimal trajectory. Let $p : [t,T] \to \mathbb{R}^d$ be the solution of the adjoint equation*

$$\begin{cases} \dot{p}(s) = -f_x(\bar{y}(s), \bar{\alpha}(s))p(s) - \bar{y}(s) + x_d & \text{for } s \in]t,T[, \\ p(T) = 0. \end{cases}$$

Then p satisfies, for $s \in [t,T]$ a.e.,

$$\bar{\alpha}(s) \in \underset{a\in A}{arg\,max}\{-f(\bar{y}(s),a)\cdot p(s) - \ell(\bar{y}(s),a)\}. \tag{4}$$

According to the definitions of f and ℓ, for $s \in [t,T]$ a.e., (4) is equivalent to

$$\bar{\alpha}(s) \in \underset{a\in A}{arg\,max}\left\{-\sum_{i=1}^m \langle f_i(\bar{y}(s)), p(s)\rangle a_i - \gamma\|a\|_0\right\}. \tag{5}$$

In the following, the properties of optimal control are investigated under precise control constraints: the ball constraints and the box constraints.

3.1 Ball control constraints

Given $\rho > 0$, consider the ball control constraints

$$A = \{a \in \mathbb{R}^m \, : \, \sum_{i=1}^{m} a_i^2 \leq \rho^2\}.$$

For $s \in [t, T]$ a.e., we set

$$c_i(s) = -\langle f_i(\bar{y}(s)), p(s) \rangle, \; i = 1, \ldots, m, \tag{6}$$

and

$$\begin{aligned} I^-(s) &= \{i \, : \, |c_i(s)|\rho < \gamma, \; i = 1, \ldots, m\}, \\ I^0(s) &= \{i \, : \, |c_i(s)|\rho = \gamma, \; i = 1, \ldots, m\}, \\ I^+(s) &= \{i \, : \, |c_i(s)|\rho > \gamma, \; i = 1, \ldots, m\}. \end{aligned}$$

The sparsity properties of α can be deduced from (5).

Theorem 3.2. *For $s \in [t, T]$ a.e., the following holds.*

- *For any $i \in I^-(s)$, $\bar{\alpha}_i(s) = 0$.*
- *If $I^+(s) = \emptyset$, then*

$$\begin{aligned} &\forall \, i \in I^0(s), \; \bar{\alpha}_i(s) \in \{0, \rho \, sgn(c_i(s))\}, \\ &\forall \, i, j \in I^0(s), \; i \neq j, \; \bar{\alpha}_i(s)\bar{\alpha}_j(s) = 0. \end{aligned}$$

- *If $I^+(s) \neq \emptyset$, then $\bar{\alpha}_i(s) = 0$ for any $i \in I^0(s)$.*

Proof. The arguments are carried out for $s \in [t, T]$ a.e.. To simplify the notation, the dependence of c_i on s is not indicated. For any $a \in A$, we define

$$G(a) := \sum_{i=1}^{m} \left(c_i a_i - \gamma |a_i|^0 \right).$$

By (5), $\bar{\alpha}(s)$ is a maximizer of G in A. We note that for any $i = 1, \ldots, m$ and $a_i \neq 0$,

$$c_i a_i - \gamma |a_i|^0 = c_i a_i - \gamma \leq |c_i|\rho - \gamma,$$

where the equality holds when $a_i = \rho \, sgn(c_i)$. Therefore,

$$c_i a_i - \gamma |a_i|^0 \leq 0, \; \forall \, i \in I^-(s),$$

and the equality holds if and only if $a_i = 0$. Thus,

$$\bar{\alpha}_i(s) = 0, \ \forall\, i \in I^-(s).$$

We proceed to discuss the coordinates in $I^0(s)$. At first, we consider the case when $I^+(s) = \emptyset$. For any $i \in I^0(s)$, $c_i a_i - \gamma |a_i|^0 \leq 0$ and the equality holds when $a_i \in \{0, \rho\, sgn(c_i)\}$. Therefore,

$$\bar{\alpha}_i(s) \in \{0, \rho\, sgn(c_i)\}.$$

Besides, due to the ball control constraints A, at most one coordinate of $\bar{\alpha}(s)$ can take the value $\rho\, sgn(c_i)$. This is equivalent to say that

$$\forall\, i, j \in I^0(s), \ i \neq j, \ \bar{\alpha}_i(s)\bar{\alpha}_j(s) = 0.$$

If $I^+(s) \neq \emptyset$, by contradiction we assume that there exists some $i \in I^0(s)$ such that $\bar{\alpha}_i(s) \neq 0$. Therefore,

$$\bar{\alpha}_i(s) = \rho\, sgn(c_i).$$

Due to the definition of the ball constraints A, we deduce that $\bar{\alpha}_j(s) = 0$ for any $j \in I^+(s)$. Consequently,

$$G(\alpha(s)) = 0.$$

Let $k \in I^+(s)$ and we define $\tilde{a} \in A$ by

$$\tilde{a}_k = \rho\, sgn(c_k), \text{ and } \tilde{a}_i = 0 \ \forall\, i \neq k, \ i = 1, \ldots, m.$$

Thus,

$$G(\tilde{a}) = |c_k|\rho - \gamma > 0 = G(\bar{\alpha}(s)),$$

which contradicts the fact that $\bar{\alpha}(s)$ is the maximizer of G. Therefore, $\bar{\alpha}_i(s) = 0$ for any $i \in I^0(s)$. $\square$

Remark 3.3. *The above theorem presents the sparsity properties for the coordinates of control in $I^-(s)$ and the switching properties for the coordinates of control in $I^0(s)$. However, the coordinates of control in $I^+(s)$ have not been discussed because the sparsity for these coordinates of control does not hold in general. Here we provide partial results without proofs. If $I^+(s)$ is a singleton, then*

$$\bar{\alpha}_i(s) = \rho\, sgn(c_i(s)).$$

If there are two elements i, j in $I^+(s)$ with $|c_i| \geq |c_j|$, some comparisons should be made to determine the maximizer. If $a_i a_j = 0$, then

$$c_i a_i + c_j a_j - \gamma |a_i|^0 - \gamma |a_j|^0 \leq |c_i| \rho - \gamma.$$

If $a_i a_j \neq 0$, then by Cauchy-Schwarz inequality

$$c_i a_i + c_j a_j - \gamma |a_i|^0 - \gamma |a_j|^0 \leq \rho \sqrt{c_i^2 + c_j^2} - 2\gamma.$$

Therefore, if $\rho \left(\sqrt{c_i^2 + c_j^2} - c_i \right) < \gamma$, then

$$\bar{\alpha}_i(s) = \rho\, sgn(c_i(s)), \;\; \bar{\alpha}_j(s) = 0.$$

If $\rho \left(\sqrt{c_i^2 + c_j^2} - c_i \right) > \gamma$, then

$$\bar{\alpha}_i(s) = \frac{\rho c_i}{\sqrt{c_i^2 + c_j^2}}, \;\; \bar{\alpha}_j(s) = \frac{\rho c_j}{\sqrt{c_i^2 + c_j^2}}.$$

Finally, if there are more than two elements in $I^+(s)$, similar comparisons should be made to determine the maximizer.

3.2 Box control constraints

For $i = 1, \ldots, m$, let $L_i < 0$ and $R_i > 0$. Consider the box control constraints:

$$A = \{a \in \mathbb{R}^m \,:\, L_i \leq a_i \leq R_i, \; i = 1, \ldots, m\}.$$

Theorem 3.4. *For $i = 1, \ldots, m$ and $s \in [t, T]$ a.e., the following holds.*

$$\begin{cases} \bar{\alpha}_i(s) = 0 & \frac{\gamma}{L_i} < c_i(s) < \frac{\gamma}{R_i}. \\ \bar{\alpha}_i(s) = L_i & c_i(s) < \frac{\gamma}{L_i}. \\ \bar{\alpha}_i(s) = R_i & c_i(s) > \frac{\gamma}{R_i}. \\ \bar{\alpha}_i(s) \in \{0, L_i\} & c_i(s) = \frac{\gamma}{L_i}. \\ \bar{\alpha}_i(s) \in \{0, R_i\} & c_i(s) = \frac{\gamma}{R_i}, \end{cases}$$

where $c_i(s)$ are defined as in (6).

Proof. For $i = 1, \ldots, m$ and $s \in [t, T]$ a.e., we consider three different cases according to the sign of $c_i(s)$. If $c_i(s) = 0$, then for any $a_i \in [L_i, R_i]$,

$$c_i(s) a_i - \gamma |a_i|^0 = -\gamma |a_i|^0 \leq 0,$$

where the equality holds if and only if $a_i = 0$. Therefore,

$$\bar{\alpha}(s) = 0.$$

If $c_i(s) > 0$, for any $a_i \in [L_i, R_i]\backslash\{0\}$,

$$c_i(s)a_i - \gamma |a_i|^0 = c_i(s)a_i - \gamma \leq c_i(s)R_i - \gamma.$$

Therefore,

$$\begin{cases} \bar{\alpha}_i(s) = 0 & \text{if } c_i(s)R_i < \gamma, \\ \bar{\alpha}_i(s) = R_i & \text{if } c_i(s)R_i > \gamma, \\ \bar{\alpha}_i(s) \in \{0, R_i\} & \text{if } c_i(s)R_i = \gamma. \end{cases}$$

If $c_i(s) < 0$, for any $a_i \in [L_i, R_i]\backslash\{0\}$,

$$c_i(s)a_i - \gamma |a_i|^0 = c_i(s)a_i - \gamma \leq c_i(s)L_i - \gamma.$$

Therefore,

$$\begin{cases} \bar{\alpha}_i(s) = 0 & \text{if } c_i(s)L_i < \gamma, \\ \bar{\alpha}_i(s) = L_i & \text{if } c_i(s)L_i > \gamma, \\ \bar{\alpha}_i(s) \in \{0, L_i\} & \text{if } c_i(s)L_i = \gamma. \end{cases}$$

The desired result is then concluded. □

4 Existence result

We have mentioned that the existence of optimal control for problem (2) does not hold in general because the cost function J is not convex in α. However, the existence result can be derived for the time-discretized problem in the space of piecewise constant control functions. The time-discretized version of problem (2) is described as follows. Given $K \in \mathbb{N}$, consider the time sequence $(t_k)_{k=0,\dots,K}$ with

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots < t_K = T,$$

and denote by $I_k = [t_k, t_{k+1}[$ for $k = 0, \dots, K-1$. The control α lives in the following set of piecewise constant functions:

$$\mathcal{A} = \{\alpha \in L^\infty(0, T; A) : \alpha|_{I_k} \equiv a_k \in A,\ k = 0, \dots, K-1\},$$

where the vector $a_k = (a_{k,1}, \dots, a_{k,m})$. The time-discretized model is the following: given $k \in \{0, \dots, K-1\}$ and $x \in \mathbb{R}^d$, consider

$$\inf_{\alpha \in \mathcal{A}} \int_{t_k}^{T} \left[\frac{1}{2} \|y(s) - x_d\|_2^2 + \gamma \sum_{j=k}^{K-1} \|a_j\|_0 \mathbb{1}_{I_j}(s) \right] ds, \tag{7}$$

where (y, α) satisfies for $j = k, \dots, K-1$,

$$\begin{cases} \dot{y}(s) = f_0(y(s)) + \sum_{i=1}^m f_i(y(s)) a_{j,i} & \text{for } s \in I_j, \\ y(t_k) = x. \end{cases} \tag{8}$$

Theorem 4.1. *There exists a solution $\bar{\alpha} \in \mathcal{A}$ for problem* (7).

Proof. Let $\{\alpha^n\}_{n\in\mathbb{N}}$ be a minimizing sequence with

$$\alpha^n|_{I_j} \equiv a_j^n = (a_{j,1}^n, \dots, a_{j,m}^n) \in A, \ j = k, \dots, K-1.$$

Since A is compact, there exists $\bar{a}_j \in A$ such that up to a subsequence

$$\lim_{n\to\infty} a_j^n = \bar{a}_j, \ j = k, \dots, K-1.$$

We set $\bar{\alpha} \in \mathcal{A}$ with

$$\bar{\alpha}|_{I_j} \equiv \bar{a}_j, \ j = k, \dots, K = 1.$$

Let y^n be the corresponding trajectories to α^n. By the compactness of A and Lipschitz continuity of f_i, $i = 0, \dots, m$, we deduce that $\{y^n\}_{n\in\mathbb{N}}$ is uniformly bounded on $[t_k, T]$ and equicontinuous. Therefore, up to a subsequence, there exists $\bar{y} : [t_k, T] \to \mathbb{R}^d$ such that

$$y^n \to \bar{y} \text{ pointwise in } [t_k, T], \text{ as } n \to \infty,$$

by Arzelà-Ascoli theorem. From the uniform boundedness of y^n, we can also deduce that $\dot{y}^n$ is uniformly bounded in L^1. Thus, there exists $q \in L^1(t_k, T; \mathbb{R}^d)$ such that

$$\dot{y}^n \to q \text{ weakly in } L^1(t_k, T; \mathbb{R}^d).$$

By the compactness and convexity of A and (Aubin et al., 1984, Theorem 1, pp. 60),

$$\dot{\bar{y}} = q \text{ a.e. in } [t_k, T].$$

Therefore, $(\bar{y}, \bar{\alpha})$ satisfies (8). By Fatou's lemma and the lower semi-continuity of L^0-norm, we deduce that

$$\begin{aligned} &\lim_{n\to\infty} \int_{t_k}^T \left[\frac{1}{2} \|y^n(s) - x_d\|_2^2 + \gamma \sum_{j=k}^{K-1} \|a_j^n\|_0 \mathbb{1}_{I_j}(s) \right] ds, \\ \geq \ & \int_{t_k}^T \left[\frac{1}{2} \|\bar{y}(s) - x_d\|_2^2 + \gamma \sum_{j=k}^{K-1} \|\bar{a}_j\|_0 \mathbb{1}_{I_j}(s) \right] ds, \end{aligned}$$

which implies that $\bar{\alpha}$ is optimal for the problem (7). $\square$

5 Numerical examples

We provide numerical examples to illustrate L^0 optimal controllers, in particular the sparsity properties compared to other types of control costs. The optimal control problem is the following: given $\gamma > 0$, $T > 0$ and $x \in \mathbb{R}^2$, consider

$$\inf_{y,\alpha} \int_0^T \left(\frac{1}{2}\|y(s)\|_2^2 + \gamma\psi(\alpha(s)) \right) ds,$$

where $y(0) = x$ and (y, α) satisfies the following Eikonal dynamical system

$$\begin{cases} \dot{y}(s) = \alpha(s) & \text{in } (0, T), \\ y(0) = x. \end{cases}$$

Here the function $\psi : A \to \mathbb{R}$ is the control cost of our choice. The HJB equation is the same as (3), and we are interested in the value function at time $t = 0$. For each test, the computation domain is $[-5, 5]^2$ with 101^2 grid points. The feedback optimal controls are obtained by solving the corresponding HJB equations and the numerical tests are carried out by first-order finite difference schemes for the space and Euler scheme for the time. For each test, we display the value function at $t = 0$, the first coordinate of optimal control at $t = 0$ with respect to the state variable x and the optimal trajectory starting from the initial point $(-2.4, -2.2)$ with the corresponding optimal control. Additionally, we display the Euclidean norm of optimal control with respect to the space at $t = 0$ in the tests of ball constraints, and the ℓ^∞-norm in the test of box constraints.

5.1 Ball constraints

The first case is under ball control constraints:

$$A = \{(a_1, a_2) \in \mathbb{R}^2 \, : \, a_1^2 + a_2^2 \leq 1\}.$$

Test 1: $T = 2$, $\gamma = 2$ and

$$\psi(a) = \|a\|_2^2 = a_1^2 + a_2^2 \text{ for } a \in A,$$

which is a quadratic control penalization. Results in Figure 1 illustrate that the L^2-control has no sparsity and the trajectory is smooth.

Test 2: $T = 2$, $\gamma = 2$ and

$$\psi(a) = \|a\|_1 = |a_1| + |a_2| \text{ for } a \in A,$$

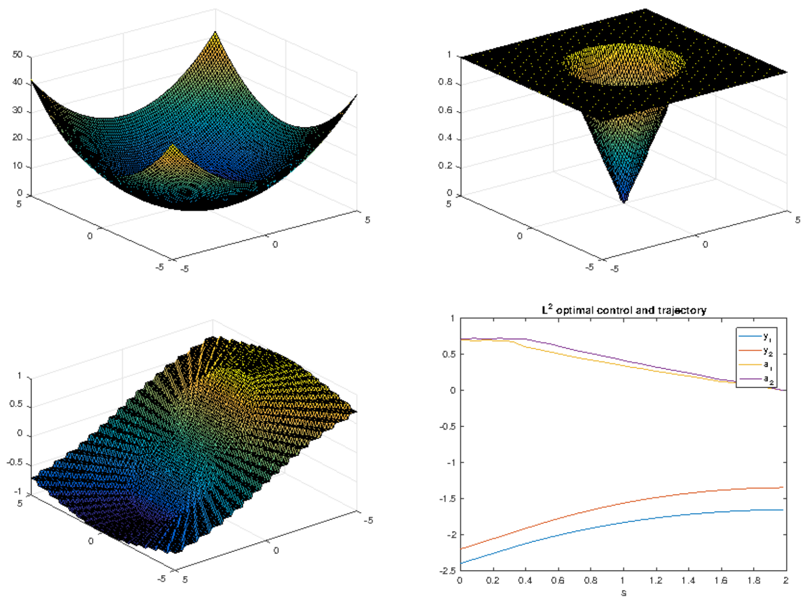

Figure 1: L^2-control with $\gamma = 2$: the value function, the ℓ^2-norm of control, the first coordinate of control, and the optimal trajectory starting from $(-2.4, -2.2)$.

which is a nonsmooth control penalization. Results in Figure 2 illustrate that a sparsity region is created around the origin. The trajectory is not smooth and the singularity appears when the control is switched off.

Test 3: $T = 2$, $\gamma = 2$ and

$$\psi(a) = \|a\|_0 = |a_1|^0 + |a_2|^0 \text{ for } a \in A,$$

which is a non-smooth and non-convex control penalization. Results in Figure 3 illustrate that the sparsity is created around the origin as the L^1-control case and the Euclidean norm constraint is active outside the sparsity region. But the sparsity is enhanced for each coordinate of control compared to L^1-control. Concerning the optimal trajectory with the corresponding optimal control, one coordinate of control is always sparse and the other one is of bang-zero type. The control has the feather of switching control since at most one coordinate of control is active at each moment, and this feather is not illustrated by L^1-control.

Test 4: $T = 2$, $\gamma = 0.2$ and

$$\psi(a) = \|a\|_0 = |a_1|^0 + |a_2|^0 \text{ for } a \in A.$$

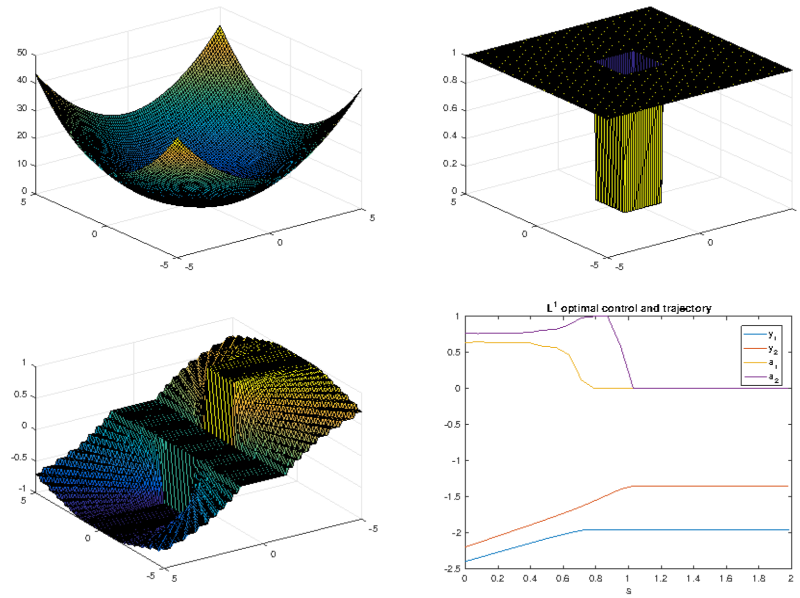

Figure 2: L^1-control with $\gamma = 2$: the value function, the ℓ^2-norm of control, the first coordinate of control, and the optimal trajectory starting from $(-2.4, -2.2)$.

This test is made with a smaller L^0-penalization parameter compared to *Test 3*. Results in Figure 4 illustrate that the sparsity region is reduced with small γ. This phenomenon corresponds to Theorem 3.2.

5.2 Box constraints

The second case is under box constraints:

$$A := \{(a_1, a_2) \in \mathbb{R}^2 \,:\, -1 \leq a_i \leq 1,\ i = 1, 2\}.$$

Test 5: $T = 2$, $\gamma = 2$, and

$$\psi(a) = \|a\|_2^2 = a_1^2 + a_2^2 \text{ for } a \in A.$$

Results in Figure 5 illustrate that the L^2-control has no sparsity and the trajectory is smooth.

Test 6: $T = 2$, $\gamma = 2$ and

$$\psi(a) = \|a\|_0 = |a_1|^0 + |a_2|^0 \text{ for } a \in A.$$

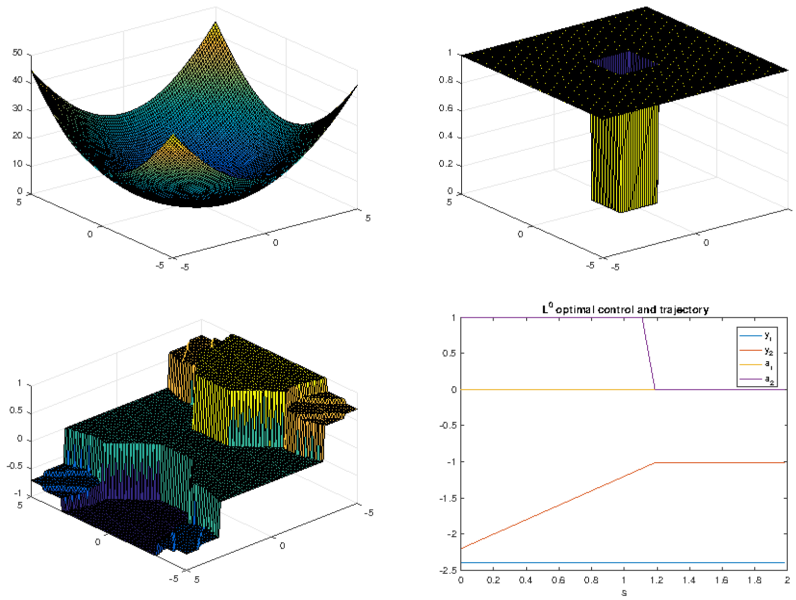

Figure 3: L^0-control with $\gamma = 2$: the value function, the ℓ^2-norm of control and the first coordinate of control, and the optimal trajectory starting from $(-2.4, -2.2)$.

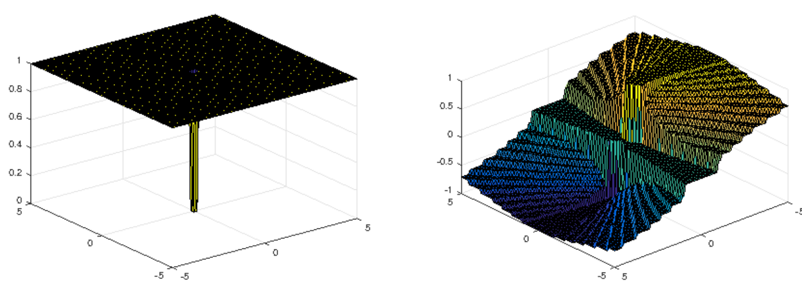

Figure 4: L^0-control with $\gamma = 0.2$: the ℓ^2-norm of control and the first coordinate of control.

Results in Figure 6 illustrate that a sparsity region is created around the origin and each coordinate of control is -1, 0 or 1, which corresponds to Theorem 3.4.

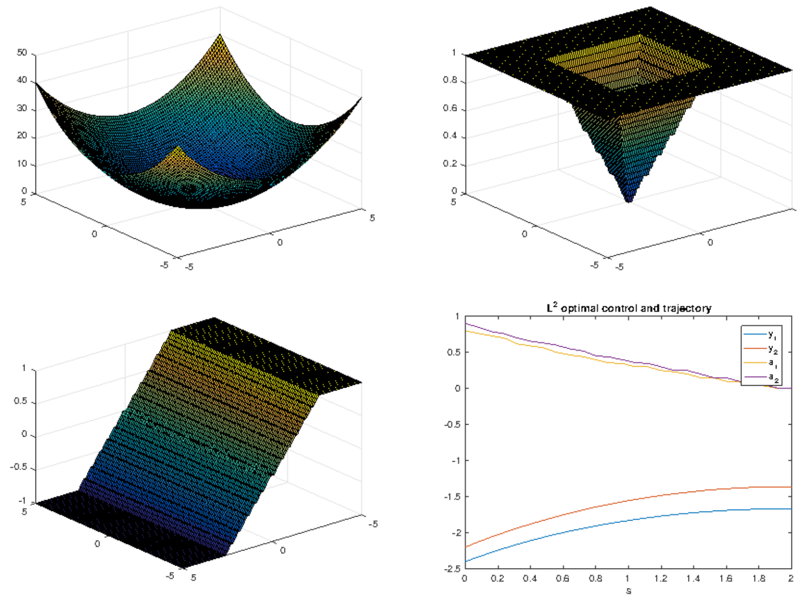

Figure 5: L^2-control with $\gamma = 2$: the value function, the ℓ^∞-norm of control, the first coordinate of control, and the optimal trajectory starting from $(-2.4, -2.2)$.

Test 7: $T = 2$, $\gamma = 2$ and

$$\psi(a) = \|a\|_0 = |a_1|^0 + |a_2|^0 \text{ for } a \in A.$$

Results concerning the optimal control and the optimal trajectory are exactly the same as in Figure 6, which illustrate that the optimal control has the same structure as L^1-control. This is different from the case of ball constraints (*Test 3*), under the reason that the constraints for each control coordinate are independent in this test.

Test 8: $T = 2$, $\gamma = 0.2$ and

$$\psi(a) = \|a\|_0 = |a_1|^0 + |a_2|^0 \text{ for } a \in A.$$

This test is made with a smaller L^0-penalization parameter compared to *Test 7*. Results in Figure 8 illustrate that the sparsity region is reduced with small γ. This phenomenon corresponds to Theorem 3.4.

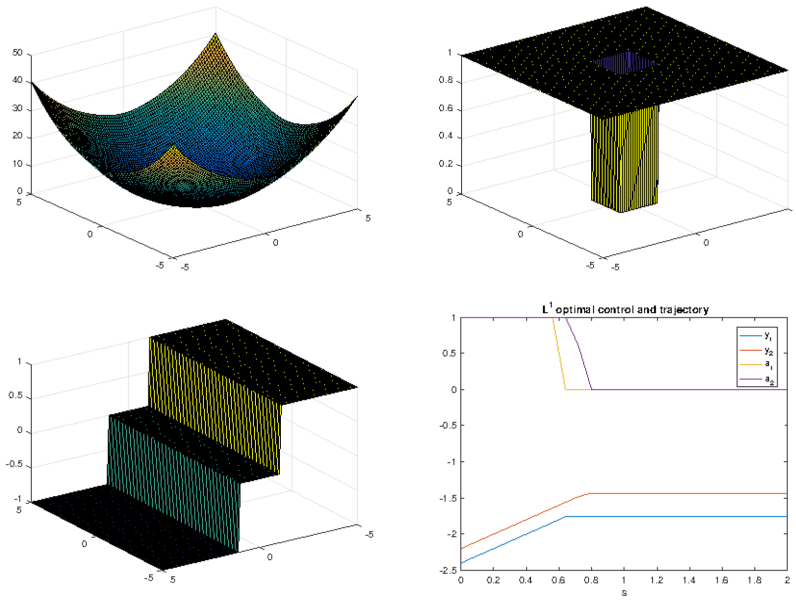

Figure 6: L^1-control with $\gamma = 2$: the value function, the first coordinate of control, and the optimal trajectory starting from $(-2.4, -2.2)$.

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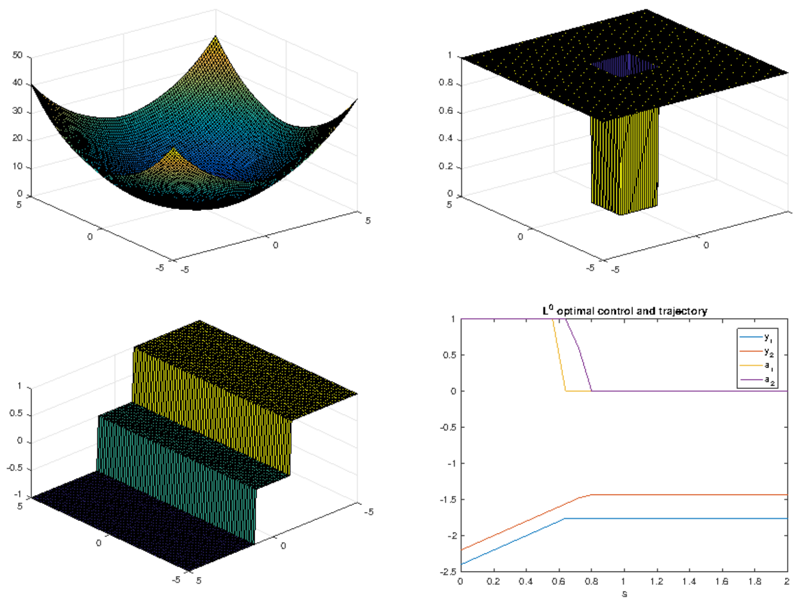

Figure 7: L^0-control with $\gamma = 2$: the value function, the ℓ^∞-norm of control, the first coordinate of control, and the optimal trajectory starting from $(-2.4, -2.2)$.

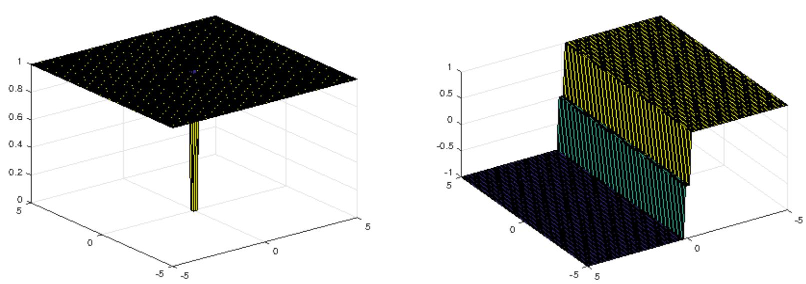

Figure 8: L^0-control with $\gamma = 0.2$: the ℓ^∞-norm of control and the first coordinate of control.

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