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Lattice rules with random n achieve nearly the optimal $\mathcal{O}(n^{-\alpha-1/2})$ error independently of the dimension

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Lattice rules with random n achieve nearly the optimal $\mathcal{O}(n^{-\alpha-1/2})$ error independently of the dimension

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Abstract

We analyze a random algorithm for numerical integration of d-variate functions from weighted Sobolev spaces with dominating mixed smoothness $\alpha \ge 0$ and product weights $1 \ge \gamma_1 \ge \gamma_2 \ge \cdots > 0$. The algorithm is based on rank-1 lattice rules with a random number of points n. A variant of this algorithm was first introduced by Bakhvalov in 1961. For the case $\alpha > 1/2$, we prove that the algorithm achieves almost the optimal order of convergence of $\mathcal{O}(n^{-\alpha-1/2})$, where the implied constant is independent of d if the weights satisfy $\sum_{j=1}^{\infty} \gamma_j^{1/\alpha} < \infty$. The same rate of convergence holds for the more general case $\alpha > 0$ by adding a random shift to the lattice rule with random n. This shows, in particular, that the exponent of strong tractability in the randomized setting equals $1/(\alpha+1/2)$, if the weights decay fast enough. We obtain a lower bound to indicate that our results are essentially optimal.

1 Introduction

We study the problem of numerical integration of d-variate functions, i.e., the approximation of

$$I_d(f) := \int_{[0,1]^d} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$$

for f in a weighted Sobolev space $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$ with smoothness parameter $\alpha \ge 0$ and product weights $\boldsymbol{\gamma} = (\gamma_j)_{j\ge1}$ (details are provided below). To this end we use a randomized algorithm M_n that uses at most $n \in \mathbb{N}$ function evaluations. The basic building blocks of the algorithm M_n are so-called rank-1 lattice rules of the form

$$Q_{d,p,\boldsymbol{z}}(f) := \frac{1}{p} \sum_{k=0}^{p-1} f\left(\left\{\frac{k\boldsymbol{z}}{p}\right\}\right), \tag{1}$$

where p is a prime number, $\boldsymbol{z} \in \mathbb{Z}^d$ is known as the generating vector, and $\{x\}$ denotes the fractional part of a real number x and is applied componentwise to a vector. Lattice rules as in (1) are very well studied in the field of quasi-Monte Carlo methods (see [4] and [24] for overviews on lattice rules, and [6, 16] for further introductions to the field of quasi-Monte Carlo methods). Here, however, we do *not* study one fixed quasi-Monte Carlo rule, but an algorithm based on randomly choosing one of a certain set of lattice rules. Indeed, the random algorithm M_n, for given $n \in \mathbb{N}$, is defined by $Q_{d,p,\boldsymbol{z}}$ with randomly chosen prime $p \in \{\lceil n/2 \rceil + 1, \ldots, n\}$ and $\boldsymbol{z} \in \{1, \ldots, p-1\}^d$. To be precise, we choose a random prime number p in the given range and then a random generating vector $\boldsymbol{z}$ from a certain set $\mathcal{Z}_p$ of "good" generating vectors (see (12) below). We call such an algorithm M_n a *randomized lattice algorithm.*

An algorithm of such a form was first analyzed by Bakhvalov in [1], see also [2], for generating vectors of a special form (commonly known as the "Korobov type"), i.e., $\boldsymbol{z} = (1, z, \ldots, z^{d-1})$ for some $z \in \mathbb{N}$. Bakhvalov proved that this algorithm has almost the optimal order of convergence in a Sobolev space with dominating mixed smoothness. However, with that algorithm we were not able to prove dimension-independent upper bounds. Here we continue this study and prove that there exist good generating vectors (not restricted to the Korobov type) such that our randomized lattice algorithm M_n achieves almost the optimal order of convergence in weighted Sobolev spaces, with the error bound independent of d under a summability condition on the weights $\boldsymbol{\gamma}$ that is common in this field of research.

More precisely, we analyze the *randomized* (*worst case*) *error* for our randomized lattice algorithm M_n in the unit ball of $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$, defined by

$$e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \;:=\; \sup_{\substack{f\in\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}\\ \|f\|_{\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}}\le 1}} \mathbb{E}\Big[\big|M_n(f) - I_d(f)\big|\Big]. \tag{2}$$

The details of the expectation will be made clear when we formally specify our randomized lattice algorithm in Section 3. We have three main theorems in this paper. In Theorem 1 below, we prove for $\alpha > 1/2$ and n sufficiently large that

$$e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \;=\; \mathcal{O}(n^{-a-1/2}) \qquad \text{for } a<\alpha \text{ arbitrarily close to } \alpha,$$

where the implied constant is independent of d if

$$\sum_{j=1}^{\infty} \gamma_j^{1/\alpha} \;<\; \infty. \tag{3}$$

(In the literature, the exponent in (3) sometimes differs by a factor of two, depending on how α and $\boldsymbol{\gamma}$ enter the definition of the norm, see Section 2.1.)

Additionally we will analyze the *randomized lattice algorithm with shift*, which is defined by

$$\widetilde{M}_n(f) \;:=\; M_n\big(f(\cdot + U \bmod 1)\big),$$

where U is a random variable that is uniformly distributed on $[0,1]^d$. Again, the algorithm uses at most n function evaluations. The advantage of the algorithm $\widetilde{M}_n$ is that it can treat the case $\alpha \in (0,1/2]$, i.e., when the integrands are not necessarily continuous. In Theorem 2 below, we prove for $\alpha > 0$ that

$$e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(\widetilde{M}_n) \;=\; \mathcal{O}(n^{-a-1/2}) \qquad \text{for } a<\alpha \text{ arbitrarily close to } \alpha,$$

where again the implied constant is independent of d if (3) holds. Moreover, the result also holds with the randomized error replaced by the root-mean-squared error of $\widetilde{M}_n$, which is larger.

Then, we show that the upper bounds for the algorithms M_n and $\widetilde{M}_n$ are essentially best possible, by proving in Theorem 3 the lower bound

$$\frac{\gamma_1}{2}\,\frac{\sqrt{\log n}}{n^{\alpha+1/2}}$$

for the randomized errors of the above algorithms.

The presented upper bounds are almost optimal as the optimal order of convergence that can be achieved by any randomized algorithm that uses only function values is $\Theta(n^{-\alpha-1/2})$, even in the unweighted situation. This has been proven only recently by one of the authors [30] using the algorithm of [12]. However, the algorithm in [30] is quite impractical in high dimensions, in contrast to the algorithm that will be analyzed in the following. This mainly comes from the fact that the involved point sets, i.e., certain subsets of irrational lattices in $\mathbb{R}^d$, seem to be very hard to implement, see, e.g. [10, 31]. Moreover, the presented upper bound in [30] is at least exponential in d and probably not improvable, see [30, Remark 3.2].

The optimal order in the randomized setting should be compared to the optimal order in the deterministic setting, where it is $\Theta(n^{-\alpha}(\log n)^{(d-1)/2})$, see [3, 7] or [28, 31] and the references therein. A tutorial on the proof of the upper bound can be found in [29]. By now there are many known constructions for *optimal* algorithms if $\alpha \leq 1$, see, e.g., [6, 16, 19], while for $\alpha > 1$ they are still rare, see [7, 8]. By introducing weights in these function spaces, one can achieve the rate $\mathcal{O}(n^{-a})$ for $a < \alpha$, with the implied constant independent of d, see [27, Theorem 3]. For a component-by-component construction that achieves this bound, see, e.g., [4, 13].

The rest of the paper is structured as follows. In Section 2, we define the function space, summarize some known results from the deterministic error setting and show a few auxiliary results that will be needed in this paper. In Section 3, we state and prove the main results, Theorems 1 and 2. The lower bound in Theorem 3 will be shown in Section 4. The independence of the error bounds on the dimension is the essence of *strong tractability*, and we will provide a brief discussion about this in Section 5. We end the paper with a short conclusion.

2 Previous and auxiliary results

In this section we define the function space and review some known results for the error of lattice rules in the deterministic setting (this includes the definitions necessary for the analysis), and we also prove a few helpful results.

2.1 Function spaces

Let us now define the function spaces under consideration in this paper. These spaces are so-called *Sobolev spaces with dominating mixed smoothness*, where we assume (product) weights as introduced by Sloan and Woźniakowski in [26]. The basic idea of weighted spaces is that the weights, occurring in the norm of the space, allow us to model different influence of different coordinates on the integration problem, where larger weights mean more influence and smaller weights mean less influence.

Let $\alpha \geq 0$, let $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 1}$ be a non-increasing sequence of positive weights bounded by 1, and let $d \in \mathbb{N}$. We denote by $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$ the Sobolev space of functions defined on $[0,1]^d$, with finite norm

$$\|f\|_{d,\alpha,\boldsymbol{\gamma}} := \left(\sum_{\boldsymbol{h}\in\mathbb{Z}^d} \left| r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})\, \hat{f}(\boldsymbol{h}) \right|^2 \right)^{1/2},$$

where $\hat{f}(\boldsymbol{h}) := \int_{[0,1]^d} f(\boldsymbol{x})\, e^{-2\pi i \boldsymbol{h}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}$ and

$$r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) = \prod_{j=1}^{d} r_{\alpha,\gamma_j}(h_j),$$

with

$$r_{\alpha,\gamma_j}(h_j) := \max\left\{1, \frac{|h_j|^\alpha}{\gamma_j}\right\}$$

In the recent literature on lattice rules, these spaces are often called *weighted Korobov spaces.*

It is straightforward to check that, for $\alpha \in \mathbb{N}$,

$$\|f\|^2_{d,\alpha,\boldsymbol{\gamma}} = \sum_{\mathfrak{u}\subseteq\{1:d\}} (2\pi)^{-2\alpha|\mathfrak{u}|} \left(\prod_{j\in\mathfrak{u}} \gamma_j^{-2} \right) \left(\int_{[0,1]^{|\mathfrak{u}|}} \left| \int_{[0,1]^{d-|\mathfrak{u}|}} \left(\prod_{j\in\mathfrak{u}} \frac{\partial}{\partial x_j} \right)^{\alpha} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}_{\{1:d\}\setminus\mathfrak{u}} \right|^2 \mathrm{d}\boldsymbol{x}_{\mathfrak{u}} \right),$$

where $\{1:d\} := \{1,\ldots,d\}$ and $\boldsymbol{x} = (x_1,\ldots,x_d)$ with $\boldsymbol{x}_{\mathfrak{u}}$ and $\boldsymbol{x}_{\{1:d\}\setminus\mathfrak{u}}$ denoting the projection of $\boldsymbol{x}$ onto the coordinates in $\mathfrak{u}$ and $\{1:d\}\setminus\mathfrak{u}$, respectively. Moreover, it is well known that $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$ consists only of continuous functions if $\alpha > 1/2$.

It is known that $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}} \subseteq L_2([0,1]^d)$ (with equality for $\alpha = 0$), and that its elements can be expressed in terms of their Fourier series, i.e.,

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^d} \hat{f}(\boldsymbol{h})\, e^{2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{x}} \quad \text{for} \quad f \in \mathcal{H}_{d,\alpha,\boldsymbol{\gamma}},$$

where by "$\cdot$" we denote the usual Euclidean inner product. Note that the convergence of the Fourier series holds pointwise if $\alpha > 1/2$, and that $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$ contains only 1-periodic functions in this case. If $\alpha \le 1/2$, then the Fourier series of a function in $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$ does not necessarily converge pointwise, and the above equation is to be understood almost everywhere, which is enough for our purposes.

Remark 1. *If we replace the definition of $r_{\alpha,\gamma_j}(h_j)$ by*

$$r^*_{\alpha,\gamma_j}(h_j) := 1 + \frac{|2\pi h_j|^\alpha}{\gamma_j},$$

then, for $\alpha \in \mathbb{N}$, we obtain the norm

$$\|f\|_*^2 = \sum_{\mathfrak{u}\subseteq\{1:d\}} \left(\prod_{j\in\mathfrak{u}} \gamma_j^{-2}\right) \left\| \left(\prod_{j\in\mathfrak{u}} \frac{\partial}{\partial x_j}\right)^{\alpha} f \right\|^2_{L_2([0,1]^d)} = \sum_{\boldsymbol{b}\in\{0,1\}^d} \boldsymbol{\gamma}^{-2\boldsymbol{b}} \|D^{\alpha\boldsymbol{b}} f\|^2_{L_2([0,1]^d)},$$

with $\boldsymbol{\gamma}^{-2\boldsymbol{b}} = \prod_{j=1}^d \gamma_j^{-2b_j}$, which recovers the standard norm for the Sobolev space when $\boldsymbol{\gamma} \equiv \mathbf{1}$. We could obtain the later theorems for this norm in the same way, since $r_{\alpha,\gamma_j}(h_j) \le r^*_{\alpha,\gamma_j}(h_j)$, which implies $\|f\|_{d,\alpha,\boldsymbol{\gamma}} \le \|f\|_*$.

2.2 Lattice rule error

For a prime p and $\boldsymbol{z} = (z_1,\ldots,z_d) \in \{1:p-1\}^d$, where $\{1:p-1\} := \{1,2,\ldots,p-1\}$, it easily follows that the error of a (rank-1) lattice rule $Q_{d,p,\boldsymbol{z}}$ is given by

$$Q_{d,p,\boldsymbol{z}}(f) - I_d(f) = \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0}} \hat{f}(\boldsymbol{h}), \tag{4}$$

where $\boldsymbol{h}\cdot\boldsymbol{z} \equiv_p 0$ means that $\boldsymbol{h}\cdot\boldsymbol{z}$ is congruent to zero modulo p. Hence, the performance of a lattice rule $Q_{d,p,\boldsymbol{z}}$ in the worst case setting depends solely on the structure of the *dual lattice*, i.e., the set of all $\boldsymbol{h} \in \mathbb{Z}^d$ such that $\boldsymbol{h}\cdot\boldsymbol{z} \equiv_p 0$. There are several measures, or *figures of merit*, for the quality of a lattice rule, see, e.g. [15, 16].

One figure of merit is

$$P_{\alpha,\boldsymbol{\gamma}}(p,\boldsymbol{z}) := \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0}} \frac{1}{r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})},$$

for which the unweighted version was defined independently by Hlawka in [9] and Korobov in [11]. This quantity is the worst case error in the class of functions $E_{\alpha,\boldsymbol{\gamma}}$, which consists of all functions f with $|\hat{f}(\boldsymbol{h})| \le r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})^{-1}$ for all $\boldsymbol{h} \in \mathbb{Z}^d$. For the weighted Sobolev space that is considered in this paper, we clearly obtain by Cauchy–Schwarz inequality that

$$\begin{aligned} |Q_{d,p,\boldsymbol{z}}(f) - I_d(f)| \;&\le\; \|f\|_{d,\alpha,\boldsymbol{\gamma}} \left(\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0}} \frac{1}{[r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})]^2} \right)^{1/2} \\ &=\; \|f\|_{d,\alpha,\boldsymbol{\gamma}} \sqrt{P_{2\alpha,\boldsymbol{\gamma}^2}(p,\boldsymbol{z})}, \end{aligned} \tag{5}$$

where we use that $[r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})]^2 = r_{2\alpha,\boldsymbol{\gamma}^2}(\boldsymbol{h})$. It is easily seen that equality holds in (5) for some (worst case) function f, and hence we conclude that the (deterministic) worst case error of a single lattice rule $Q_{d,p,\boldsymbol{z}}$ is precisely

$$e^{\mathrm{wor}}_{d,\alpha,\boldsymbol{\gamma}}(Q_{d,p,\boldsymbol{z}}) \;:=\; \sup_{\substack{f\in\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}} \\ \|f\|_{\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}}\le 1}} |Q_{d,p,\boldsymbol{z}}(f) - I_d(f)| \;=\; \sqrt{P_{2\alpha,\boldsymbol{\gamma}^2}(p,\boldsymbol{z})}. \tag{6}$$

Another relevant figure of merit is

$$\rho_{\alpha,\boldsymbol{\gamma}}(p,\boldsymbol{z}) \;:=\; \min_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0}} r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}),$$

which is a weighted version of the *Zaremba index* for higher smoothness. Clearly,

$$\frac{1}{\rho_{\alpha,\boldsymbol{\gamma}}(p,\boldsymbol{z})} \;<\; P_{\alpha,\boldsymbol{\gamma}}(p,\boldsymbol{z}). \tag{7}$$

2.3 The existence of good generating vectors

Here, we show by a standard averaging argument that there exist generating vectors which make the worst case error of the corresponding lattice rule small. In addition, we show that many such vectors exist, which will be essential for the proof of our main result.

Recall that, for $\boldsymbol{h} \in \mathbb{Z}$, we have

$$r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) \;=\; \prod_{j=1}^{d} \max\Big\{1,\, |h_j|^\alpha/\gamma_j\Big\}.$$

We start with some auxiliary results.

Lemma 1. *Let $d \in \mathbb{N}$, $\beta > 1$, and $\boldsymbol{\gamma} \in (0,1]^{\mathbb{N}}$. Then we have*

$$\sum_{\boldsymbol{h}\in\mathbb{Z}^d} \frac{1}{r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h})} \;=\; \prod_{j=1}^{d} \left(1 + 2\gamma_j\, \zeta(\beta)\right), \tag{8}$$

where ζ denotes the Riemann zeta function.

Proof. The result follows easily by the definition of the function $r_{\beta,\boldsymbol{\gamma}}$. □

The quantity on the right-hand side of (8) will be crucial in the following computations. To simplify notation, we define, for $d \in \mathbb{N}$, $\beta > 1$, and $\boldsymbol{\gamma} \in (0,1]^{\mathbb{N}}$,

$$V_d(\beta,\boldsymbol{\gamma}) \;:=\; 3\prod_{j=1}^{d} \left(1 + 2\gamma_j\, \zeta(\beta)\right) \;\le\; 3\exp\left(2\,\zeta(\beta)\sum_{j=1}^{d}\gamma_j\right), \tag{9}$$

where we use $1 + x \le e^x$ for $x \in \mathbb{R}$. Note that $V_d(\beta,\boldsymbol{\gamma})$ is bounded independently of d for all $\beta > 1$ if $\sum_{j=1}^{\infty} \gamma_j < \infty$.

As a direct consequence of Lemma 1 we obtain a bound on the number of $\boldsymbol{h} \in \mathbb{Z}^d$ such that $r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h})$ is small.

Corollary 2. *Let $d \in \mathbb{N}$, $\beta > 1$, $T > 0$, and $\boldsymbol{\gamma} \in (0,1]^{\mathbb{N}}$, and define*

$$\mathcal{A}_{\beta,\boldsymbol{\gamma}}(T) := \{\boldsymbol{h} \in \mathbb{Z}^d \colon r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h}) \le T\}.$$

Then we have

$$|\mathcal{A}_{\beta,\boldsymbol{\gamma}}(T)| \le T\, V_d(\beta,\boldsymbol{\gamma}).$$

Proof. From Lemma 1 we obtain

$$V_d(\beta,\boldsymbol{\gamma}) \ge \sum_{\boldsymbol{h}\in\mathcal{A}_{\beta,\gamma}(T)} \frac{1}{r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h})} \ge \frac{|\mathcal{A}_{\beta,\boldsymbol{\gamma}}(T)|}{T}.$$

This proves the result. □

The next lemma is useful in bounding the number of points in the dual lattice for a given generating vector.

Lemma 3. *For every prime number p, every $d \in \mathbb{N}$, and $\boldsymbol{h} \in \mathbb{Z}^d$ we have*

$$\#\big\{\boldsymbol{z} \in \{1:p-1\}^d \colon \boldsymbol{h}\cdot\boldsymbol{z} \equiv_p 0\big\} \le \begin{cases} (p-1)^d & \text{if } \boldsymbol{h} \equiv_p \boldsymbol{0}, \\ (p-1)^{d-1} & \text{otherwise,} \end{cases}$$

with equality in the first case, where by $\boldsymbol{h} \equiv_p \boldsymbol{0}$ we mean that each component of $\boldsymbol{h}$ is congruent to zero modulo p.

Proof. Recall that $\boldsymbol{h}\cdot\boldsymbol{z} = \sum_{j=1}^d h_j z_j$. If $\boldsymbol{h} \equiv_p \boldsymbol{0}$, then every $\boldsymbol{z} \in \{1:p-1\}^d$ is a solution to $\boldsymbol{h}\cdot\boldsymbol{z} \equiv_p 0$. Hence, there are $(p-1)^d$ solutions in this case. Otherwise, there is an $\ell \in \{1:d\}$ such that $h_\ell \not\equiv_p 0$. Now, for any fixed z_j, $j \ne \ell$, we have $\boldsymbol{h}\cdot\boldsymbol{z} \equiv_p 0$ if and only if $h_\ell z_\ell \equiv_p \sum_{j\ne\ell} h_j z_j$. Since $h_\ell \not\equiv_p 0$ there is at most one such $z_\ell \in \{1:p-1\}$. This shows that there are no more than $(p-1)^{d-1}$ solutions to $\boldsymbol{h}\cdot\boldsymbol{z} \equiv_p 0$ in this case. □

A similar result to following proposition can be found in many papers.

Proposition 4. *Let p be prime, $d \in \mathbb{N}$, and $\boldsymbol{\gamma} \in (0,1]^{\mathbb{N}}$. For any $\beta > 1$, we have*

$$\frac{1}{(p-1)^d} \sum_{\boldsymbol{z}\in\{1:p-1\}^d} P_{\beta,\boldsymbol{\gamma}}(p,\boldsymbol{z}) \le \frac{V_d(\beta,\boldsymbol{\gamma})}{p}.$$

Proof. By definition we have

$$\begin{aligned} \frac{1}{(p-1)^d} \sum_{\boldsymbol{z}\in\{1:p-1\}^d} P_{\beta,\boldsymbol{\gamma}}(p,\boldsymbol{z}) &= \frac{1}{(p-1)^d} \sum_{\boldsymbol{z}\in\{1:p-1\}^d} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0}} \frac{1}{r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h})} \\ &= \frac{1}{(p-1)^d} \sum_{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}} \frac{\#\big\{\boldsymbol{z} \in \{1:p-1\}^d \colon \boldsymbol{h}\cdot\boldsymbol{z} \equiv_p 0\big\}}{r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h})}. \end{aligned}$$

We split this sum into a sum over all $\boldsymbol{h}$ with $\boldsymbol{h} \equiv_p \boldsymbol{0}$ and the remaining $\boldsymbol{h}$, and bound the sums separately. By Lemma 3 we have for the sum over $\boldsymbol{h} \equiv_p \boldsymbol{0}$,

$$\frac{1}{(p-1)^d} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\equiv_p\boldsymbol{0}}} \frac{\#\big\{\boldsymbol{z} \in \{1:p-1\}^d \colon \boldsymbol{h}\cdot\boldsymbol{z} \equiv_p 0\big\}}{r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h})} \le \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\equiv_p\boldsymbol{0}}} \frac{1}{r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h})}$$

$$
\begin{aligned}
&= \sum_{\boldsymbol{l}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}} \frac{1}{r_{\beta,\boldsymbol{\gamma}}(p\,\boldsymbol{l})} \\
&\le \frac{1}{p^\beta} \sum_{\boldsymbol{l}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}} \frac{1}{r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{l})} \\
&\le \frac{1}{3p}\, V_d(\beta,\boldsymbol{\gamma}).
\end{aligned}
$$

The penultimate inequality easily follows from the definition of $r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h})$ and $\boldsymbol{h}\neq\boldsymbol{0}$, and the last inequality follows from Lemma 1.

For the sum over $\boldsymbol{h}\not\equiv_p 0$ we use again Lemma 1 and Lemma 3, and obtain

$$
\begin{aligned}
\frac{1}{(p-1)^d} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\not\equiv_p\boldsymbol{0}}} \frac{\#\big\{\boldsymbol{z}\in\{1:p-1\}^d\colon\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0\big\}}{r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h})} &\le \frac{1}{p-1}\sum_{\boldsymbol{h}\in\mathbb{Z}^d}\frac{1}{r_{\beta,\boldsymbol{\gamma}}(\boldsymbol{h})} \\
&\le \frac{1}{3(p-1)}\, V_d(\beta,\boldsymbol{\gamma}).
\end{aligned}
$$

Combining the last two estimates with $1/(p-1)\le 2/p$ leads to the desired result. □

Clearly, there must be one choice of $\boldsymbol{z}$ such that $P_{\beta,\boldsymbol{\gamma}}(p,\boldsymbol{z})$ is as good as the average. By this argument, and using (5), it is typically shown that there exists a generating vector that makes the worst case error small. Although this is clearly not a constructive argument, there are methods to generate such vectors in an efficient way, in particular component-by-component algorithms. These constructions, dating back to Korobov, were re-invented in 2002 in [25], and proven to yield optimal results in [5] and [13]. Moreover, there exists a fast variant due to [22] which is heavily used nowadays.

For the upcoming analysis, it is not enough to have a single "good" generating vector. However, as the next corollary shows, there are actually many of them.

Corollary 5. *Let p be prime, $d\in\mathbb{N}$, and $\boldsymbol{\gamma}\in(0,1]^{\mathbb{N}}$. For any $\beta>1$ and $\tau\in(0,1)$, there exist at least $\lceil\tau(p-1)^d\rceil$ generating vectors $\boldsymbol{z}\in\{1:p-1\}^d$ such that*

$$
P_{\beta,\boldsymbol{\gamma}}(p,\boldsymbol{z}) \;\le\; \frac{1}{1-\tau}\cdot\frac{V_d(\beta,\boldsymbol{\gamma})}{p}\,.
$$

Proof. Suppose to the contrary that $P_{\beta,\boldsymbol{\gamma}}(p,\boldsymbol{z})$ is smaller than the right-hand side above for χ choices of $\boldsymbol{z}$, where $\chi\le\lceil\tau(p-1)^d\rceil-1\le\tau(p-1)^d$. The number of $\boldsymbol{z}$ such that $P_{\beta,\boldsymbol{\gamma}}(p,\boldsymbol{z})$ is larger, i.e., $(p-1)^d-\chi$, therefore satisfies $(p-1)^d-\chi\ge(1-\tau)(p-1)^d$. Then the average over all $\boldsymbol{z}$ is bigger than $V_d(\beta,\boldsymbol{\gamma})/p$, which contradicts Proposition 4. □

Most of the results discussed so far in this section hold for arbitrary $\beta>1$. However, for the further analysis we need a bound on $\rho_{\alpha,\boldsymbol{\gamma}}(p,\boldsymbol{z})$ for many $\boldsymbol{z}$ for all $\alpha>0$. It is known that the classical Zaremba index, corresponding to $\alpha=1$ and $\boldsymbol{\gamma}=\boldsymbol{1}$, has a growth behavior like p (with additional logarithmic terms), see e.g. [16], so we expect a growth behavior of $\rho_{\alpha,\boldsymbol{\gamma}}(p,\boldsymbol{z})$ comparable to p^α. This is made more precise in the following lemma.

Lemma 6. *Let p be prime, $d\in\mathbb{N}$, $\alpha>0$, and $\boldsymbol{\gamma}\in(0,1]^{\mathbb{N}}$. For any $\tau\in(0,1)$, there exist at least $\lceil\tau(p-1)^d\rceil$ generating vectors $\boldsymbol{z}\in\{1:p-1\}^d$ such that*

$$
\rho_{\alpha,\boldsymbol{\gamma}}(p,\boldsymbol{z}) \;\ge\; \left(\frac{(1-\tau)\,p}{V_d\big(\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda}\big)}\right)^{\lambda} \quad \text{for all } \lambda\in(0,\alpha)\,. \tag{10}
$$

Proof. Using (7), the monotonicity of the ℓ_p-norm, and $r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) = (r_{\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda}}(\boldsymbol{h}))^\lambda$ for all $\lambda \in (0,\alpha)$, we obtain

$$\frac{1}{\rho_{\alpha,\boldsymbol{\gamma}}(p,\boldsymbol{z})} = \frac{1}{(\rho_{\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda}}(p,\boldsymbol{z}))^\lambda} < (P_{\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda}}(p,\boldsymbol{z}))^\lambda \le \left(\frac{V_d(\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda})}{(1-\tau)p}\right)^\lambda \tag{11}$$

for all $\lambda \in (0,\alpha)$. Now for each $\lambda \in (0,\alpha)$, we know from Corollary 5 that the last inequality in (11) holds for at least $\lceil \tau(p-1)^d \rceil$ different generating vectors $\boldsymbol{z}$. To obtain the set of generating vectors that satisfies (11) for all $\lambda \in (0,\alpha)$ we consider the infimum of the final expression of (11) over λ. If this infimum is attained for some $\lambda^* \in (0,\alpha)$, then those generating vectors $\boldsymbol{z}$ which correspond to λ^* will satisfy the desired bound (10) for all other values of λ. If it is not attained, then the infimum is the limit as $\lambda \to 0$ of this final expression. (The limit $\lambda \to \alpha$ is clearly infinity.) However, this limit equals 1, which makes the statement of the theorem trivial. □

Remark 2. *For a given prime $p \ge 2$, let us choose a generating vector $\boldsymbol{z} \in \{1 : p-1\}^d$ uniformly at random. By Lemma 6 we have that the probability of obtaining, by this procedure, a "good" generating vector, i.e., a vector $\boldsymbol{z}$ that satisfies the bound* (10)*, is at least τ. Besides possible implementation issues, we did not find any advantage of adjusting τ. Hence, to ease the notation, we set $\tau = 1/2$ in the following.*

3 New results in the randomized setting

3.1 The randomized lattice algorithm without shift

Let $n \in \mathbb{N}$, $n \ge 2$, and

$$\mathcal{P}_n := \{p\colon\ p \text{ is prime and } n/2+1 \le p \le n\}.$$

Let $d \in \mathbb{N}$, $\alpha > 0$, and $\boldsymbol{\gamma} \in (0,1]^{\mathbb{N}}$. For each $p \in \mathcal{P}_n$, let $\mathcal{Z}_p$ denote the set of good generating vectors $\boldsymbol{z}$ in the sense of Lemma 6, with $\tau = 1/2$, that is,

$$\mathcal{Z}_p := \mathcal{Z}_{p,\alpha,\boldsymbol{\gamma}} := \left\{\boldsymbol{z} \in \{1:p-1\}^d\colon\ \rho_{\alpha,\boldsymbol{\gamma}}(p,\boldsymbol{z}) \ge \left(\frac{p}{2\,V_d(\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda})}\right)^\lambda \text{ for all } \lambda \in (0,\alpha)\right\}. \tag{12}$$

We know from Lemma 6 that $|\mathcal{Z}_p| \ge \lceil (p-1)^d/2 \rceil$.

Our random algorithm M_n is defined by first randomly and uniformly selecting a prime $p \in \mathcal{P}_n$ and then randomly and uniformly selecting a generating vector $\boldsymbol{z} \in \mathcal{Z}_p$. The randomized error (2) for the lattice algorithm M_n is then given precisely by

$$e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) = \sup_{\substack{f \in \mathcal{H}_{d,\alpha,\boldsymbol{\gamma}} \\ \|f\|_{\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}} \le 1}} \left(\frac{1}{|\mathcal{P}_n|}\sum_{p\in\mathcal{P}_n}\frac{1}{|\mathcal{Z}_p|}\sum_{\boldsymbol{z}\in\mathcal{Z}_p}|Q_{d,p,\boldsymbol{z}}(f) - I_d(f)|\right).$$

We stress that the randomized error is *not* the same as the average of the worst case errors (6) of a set of deterministic lattice rules. (Here the averaging occurs inside the supremum rather than outside.)

Theorem 1. *For $\alpha > 1/2$, $\lambda \in (1/2,\alpha)$, $\delta \in (0,\lambda-1/2)$, and*

$$n \ge 4\,V_d(\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda}), \tag{13}$$

the randomized error of the randomized lattice algorithm M_n satisfies

$$e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \;\le\; C_{\lambda,\delta}\left[V_d\big(\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda}\big)\right]^{\lambda} n^{-\lambda-1/2+\delta},$$

where V_d is defined as in (9) and $C_{\lambda,\delta}$ is a constant depending only on λ and δ. The upper bound is independent of d if $\sum_{j=1}^{\infty}\gamma_j^{1/\lambda}<\infty$.

Proof. We first define

$$B_n \;:=\; B_{n,\alpha,\boldsymbol{\gamma},\lambda} \;:=\; \left(\frac{n}{4\,V_d\big(\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda}\big)}\right)^{\lambda}. \tag{14}$$

Then the condition (13) ensures that $B_n \ge 1$.

It follows from the definition of $\mathcal{Z}_p$ in (12) that for all $p\in\mathcal{P}_n$ and $\boldsymbol{z}\in\mathcal{Z}_p$ we have $p>n/2$ and $\rho_{\alpha,\boldsymbol{\gamma}}(p,\boldsymbol{z})>B_n$; and consequently for every $\boldsymbol{h}\in\mathbb{Z}^d$ with $\boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0$ we have $r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})>B_n$. From (4) we obtain

$$\begin{aligned}
\frac{1}{|\mathcal{P}_n|}\sum_{p\in\mathcal{P}_n}\frac{1}{|\mathcal{Z}_p|}\sum_{\boldsymbol{z}\in\mathcal{Z}_p}|Q_{d,p,\boldsymbol{z}}(f)-I_d(f)| &= \frac{1}{|\mathcal{P}_n|}\sum_{p\in\mathcal{P}_n}\frac{1}{|\mathcal{Z}_p|}\sum_{\boldsymbol{z}\in\mathcal{Z}_p}\Bigg|\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0}}\hat{f}(\boldsymbol{h})\Bigg| \\
&\le \frac{1}{|\mathcal{P}_n|}\sum_{p\in\mathcal{P}_n}\frac{1}{|\mathcal{Z}_p|}\sum_{\boldsymbol{z}\in\mathcal{Z}_p}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0}}|\hat{f}(\boldsymbol{h})| \\
&= \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})>B_n}}\omega_n(\boldsymbol{h})\,|\hat{f}(\boldsymbol{h})| \\
&\le \Bigg(\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})>B_n}}\left(\frac{\omega_n(\boldsymbol{h})}{r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})}\right)^2\Bigg)^{1/2}\|f\|_{\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}},
\end{aligned}$$

where

$$\omega_n(\boldsymbol{h}) \;:=\; \omega_{n,\alpha,\boldsymbol{\gamma},\lambda}(\boldsymbol{h}) \;:=\; \frac{1}{|\mathcal{P}_n|}\sum_{p\in\mathcal{P}_n}\frac{1}{|\mathcal{Z}_p|}\sum_{\boldsymbol{z}\in\mathcal{Z}_p}\mathbb{I}(\boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0)\,, \tag{15}$$

with $\mathbb{I}(\cdot)$ denoting the indicator function. The last inequality was obtained by multiplying and dividing the penultimate expression by $r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})$ and then applying the Cauchy–Schwarz inequality. Hence we conclude that

$$e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \le \Bigg(\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})>B_n}}\left(\frac{\omega_n(\boldsymbol{h})}{r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})}\right)^2\Bigg)^{1/2}.$$

We now proceed to obtain a bound on $\omega_n(\boldsymbol{h})$. For fixed p, if $\boldsymbol{h}\equiv_p\boldsymbol{0}$ then $\boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0$ holds for all $\boldsymbol{z}\in\mathcal{Z}_p$. On the other hand, if $\boldsymbol{h}\not\equiv_p\boldsymbol{0}$ then we may bound the last sum by the number of all $\boldsymbol{z}\in\{1:p-1\}^d$ with $\boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0$. We already computed this number in Lemma 3. Thus we have

$$\begin{cases}
\displaystyle\frac{1}{|\mathcal{Z}_p|}\sum_{\boldsymbol{z}\in\mathcal{Z}_p}\mathbb{I}(\boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0) \;=\; 1 & \text{if } \boldsymbol{h}\equiv_p\boldsymbol{0}, \\
\displaystyle\frac{1}{|\mathcal{Z}_p|}\sum_{\boldsymbol{z}\in\mathcal{Z}_p}\mathbb{I}(\boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0) \;\le\; \frac{(p-1)^{d-1}}{|\mathcal{Z}_p|} \;\le\; \frac{(p-1)^{d-1}}{\lceil (p-1)^d/2\rceil} \;\le\; \frac{4}{n} & \text{if } \boldsymbol{h}\not\equiv_p\boldsymbol{0},
\end{cases}$$

and therefore

$$\omega_n(\boldsymbol{h}) \;\le\; \frac{1}{|\mathcal{P}_n|}\sum_{p\in\mathcal{P}_n}\left(\mathbb{I}(\boldsymbol{h}\equiv_p \boldsymbol{0}) + \frac{4\,\mathbb{I}(\boldsymbol{h}\not\equiv_p \boldsymbol{0})}{n}\right) \;\le\; \frac{1}{|\mathcal{P}_n|}\sum_{p\in\mathcal{P}_n}\mathbb{I}(\boldsymbol{h}\equiv_p \boldsymbol{0}) + \frac{4}{n}\,.$$

Note that any number $h\in\mathbb{N}$ has at most $\log_M(h)$ prime divisors greater than $M\in\mathbb{N}$. So for $\boldsymbol{h}\neq\boldsymbol{0}$ the number of primes $p\ge\lceil n/2\rceil+1$ for which $\boldsymbol{h}\equiv_p\boldsymbol{0}$ holds is at most $\log_{\lceil n/2\rceil+1}(|\boldsymbol{h}|_\infty)$, that is,

$$\sum_{p\in\mathcal{P}_n}\mathbb{I}(\boldsymbol{h}\equiv_p 0) \;\le\; \log_{\lceil n/2\rceil+1}(|\boldsymbol{h}|_\infty) \;=\; \frac{\ln(|\boldsymbol{h}|_\infty)}{\ln(\lceil n/2\rceil+1)} \;\le\; \frac{2\ln(|\boldsymbol{h}|_\infty)}{\ln(n)}$$

for all $n\ge 2$. Combining this with the estimate $|\mathcal{P}_n| > c'n/\ln(n)$ for some $c'>0$, see, e.g., [23], we conclude that

$$\omega_n(\boldsymbol{h}) \;\le\; \frac{2}{c'\,n}\ln(|\boldsymbol{h}|_\infty) + \frac{4}{n} \;\le\; \frac{c}{n}\ln(|\boldsymbol{h}|_\infty) \tag{16}$$

for some $c<\infty$, which yields

$$e^{\rm ran}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \;\le\; \frac{c}{n}\Bigg(\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})>B_n}}\frac{\ln^2(|\boldsymbol{h}|_\infty)}{(r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}))^2}\Bigg)^{1/2} \;=\; \frac{c}{n}\Bigg(\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h})>B_n^{1/\lambda}}}\frac{\ln^2(|\boldsymbol{h}|_\infty)}{(r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h}))^{2\lambda}}\Bigg)^{1/2}$$

with $\beta:=\alpha/\lambda>1$ and $\boldsymbol{\gamma}'=\boldsymbol{\gamma}^{1/\lambda}$. Here we used $r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) = (r_{\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda}}(\boldsymbol{h}))^\lambda$. Now we apply $|\boldsymbol{h}|_\infty\le r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h})$ and $\log(x)\le\log(1+x)\le x^\delta/\delta$, which holds for all $\delta>0$ and $x>0$, and obtain

$$e^{\rm ran}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \;\le\; \frac{c}{\delta n}\Bigg(\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h})>B_n^{1/\lambda}}}\frac{1}{(r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h}))^{2(\lambda-\delta)}}\Bigg)^{1/2}. \tag{17}$$

To complete the proof we will show a suitable upper bound on

$$\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h})>B_n^{1/\lambda}}}\frac{1}{(r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h}))^{2(\lambda-\delta)}}$$

for $\lambda<\alpha$ and $\delta>0$ with $\lambda-\delta>1/2$. For $T>0$, we define the set

$$\mathcal{A}(T) \;:=\; \mathcal{A}_{\beta,\boldsymbol{\gamma}'}(T) \;=\; \{\boldsymbol{h}\in\mathbb{Z}^d\colon r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h})\le T\}.$$

Recall from Corollary 2 that $|\mathcal{A}(T)|\le T\,V_d(\alpha/\lambda,\boldsymbol{\gamma}^{1/\lambda})$. Moreover, we have for $c>1$ and $T\ge 1$,

$$T^{-c}-(T+1)^{-c} \;=\; c\int_T^{T+1} x^{-c-1}\,{\rm d}x \;\le\; c\,(T+1)^{-c-1}.$$

Since $\boldsymbol{h}\notin\mathcal{A}(T)$ implies $1/r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h})<1/T$, we obtain for $1/2<1/2+\delta<\lambda$,

$$\begin{aligned}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h})>B_n^{1/\lambda}}}\frac{1}{(r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h}))^{2(\lambda-\delta)}} \;&\le\; \sum_{T=\lfloor B_n^{1/\lambda}\rfloor}^{\infty}\;\sum_{\boldsymbol{h}\in\mathcal{A}(T+1)\setminus\mathcal{A}(T)}\frac{1}{(r_{\beta,\boldsymbol{\gamma}'}(\boldsymbol{h}))^{2(\lambda-\delta)}}\\ &\le\; \sum_{T=\lfloor B_n^{1/\lambda}\rfloor}^{\infty} T^{-2(\lambda-\delta)}\Big(|\mathcal{A}(T+1)|-|\mathcal{A}(T)|\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{T=\lfloor B_n^{1/\lambda}\rfloor}^{\infty} \left(T^{-2(\lambda-\delta)} - (T+1)^{-2(\lambda-\delta)}\right) |\mathcal{A}(T+1)| \\
&\leq 2(\lambda-\delta) \sum_{T=\lfloor B_n^{1/\lambda}\rfloor}^{\infty} (T+1)^{-2(\lambda-\delta)-1} |\mathcal{A}(T+1)| \\
&\leq 2(\lambda-\delta)\, V_d(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda}) \sum_{T=\lfloor B_n^{1/\lambda}\rfloor}^{\infty} (T+1)^{-2(\lambda-\delta)} \\
&\leq 2(\lambda-\delta)\, V_d(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda}) \int_{\lfloor B_n^{1/\lambda}\rfloor}^{\infty} x^{-2(\lambda-\delta)}\, \mathrm{d}x \\
&\leq \frac{2(\lambda-\delta)\, V_d(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda})}{2(\lambda-\delta)+1} \left(\lfloor B_n^{1/\lambda}\rfloor\right)^{-2(\lambda-\delta)+1} \\
&\leq \frac{2(\lambda-\delta)\, V_d(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda})}{2(\lambda-\delta)+1}\, 2^{2(\lambda-\delta)-1} \left(B_n^{1/\lambda}\right)^{-2(\lambda-\delta)+1},
\end{aligned}
$$

where we used $\lfloor x\rfloor \geq x/2$ for $x \geq 1$ in the last inequality.

Together with (14) and (17), and using $1/\lambda > 1/\alpha$, this shows

$$
\begin{aligned}
e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) &\leq \frac{c\, 2^{\lambda}}{\delta n} \left(\frac{(\lambda-\delta)\, V_d(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda})}{\lambda-\delta+1/2} \left(\frac{4\, V_d\big(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda}\big)}{n} \right)^{2(\lambda-\delta)-1} \right)^{1/2} \\
&\leq \frac{c\, 2^{3\lambda-2\delta-1}}{\delta} \sqrt{\frac{\lambda-\delta}{\lambda-\delta+1/2}}\, [V_d\big(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda}\big)]^{\lambda}\, n^{-\lambda-1/2+\delta}.
\end{aligned}
$$

Note that the coefficient of $n^{-\lambda-1/2+\delta}$ in the latter term is bounded under the assumptions we made, i.e., $\tau \in (0,1)$, $\alpha > \lambda > 1/2$ and $0 < \delta < \lambda - 1/2$. This completes the proof. □

3.2 The randomized lattice algorithm with shift

We are now going to prove in Theorem 2 below that $\widetilde{M}_n$, i.e., the random lattice rule with an additional shift, satisfies a similar upper bound on its randomized error as M_n, but also for functions from $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$ with $0 < \alpha \leq 1/2$. Note that, by allowing general $\alpha > 0$ in Theorem 2, we are considering a larger function class than in Theorem 1.

The randomized error (2) for the algorithm $\widetilde{M}_n$ is given by

$$
e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}\big(\widetilde{M}_n\big) = \sup_{\substack{f\in\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}} \\ \|f\|_{\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}}\leq 1}} \left(\frac{1}{|\mathcal{P}_M|} \sum_{p\in\mathcal{P}_M} \frac{1}{|\mathcal{Z}_p|} \sum_{\boldsymbol{z}\in\mathcal{Z}_p} \mathbb{E}_U \Big[\big| Q_{d,p,\boldsymbol{z}}\big(f(\cdot + U \bmod 1)\big) - I_d(f) \big| \Big] \right),
$$

where $\mathbb{E}_U$ denotes expectation with respect to the random shift U. For this algorithm we can even bound the *root-mean-square error* (or standard deviation)

$$
\begin{aligned}
e^{\mathrm{rmse}}_{d,\alpha,\boldsymbol{\gamma}}\big(\widetilde{M}_n\big) &:= \sup_{\substack{f\in\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}} \\ \|f\|_{\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}}\leq 1}} \sqrt{\mathbb{E}\left[\left| \widetilde{M}_n(f) - I_d(f) \right|^2 \right]} \\
&= \sup_{\substack{f\in\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}} \\ \|f\|_{\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}}\leq 1}} \left(\frac{1}{|\mathcal{P}_M|} \sum_{p\in\mathcal{P}_M} \frac{1}{|\mathcal{Z}_p|} \sum_{\boldsymbol{z}\in\mathcal{Z}_p} \mathbb{E}_U \left[\big| Q_{d,p,\boldsymbol{z}}\big(f(\cdot + U \bmod 1)\big) - I_d(f) \big|^2 \right] \right)^{1/2}.
\end{aligned}
$$

Clearly, $e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}\big(\widetilde{M}_n\big) \le e^{\mathrm{rmse}}_{d,\alpha,\boldsymbol{\gamma}}\big(\widetilde{M}_n\big)$.

Theorem 2. *For $\alpha > 0$, $\lambda \in (0, \alpha)$, and $\delta > 0$, the randomized worst case error of the randomized lattice algorithm with shift $\widetilde{M}_n$ satisfies*

$$e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}\big(\widetilde{M}_n\big) \;\le\; e^{\mathrm{rmse}}_{d,\alpha,\boldsymbol{\gamma}}\big(\widetilde{M}_n\big) \;\le\; \frac{c}{\alpha\delta}\Big(4\,V_d\big(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda}\big)\Big)^{\lambda}\; n^{-\lambda-1/2+\delta\lambda/2},$$

where again V_d is defined as in (9) *and c is an absolute constant. The upper bound is independent of d if $\sum_{j=1}^{\infty}\gamma_j^{1/\lambda} < \infty$.*

Proof. By a variation of Poisson's summation formula, we have

$$\mathbb{E}_U\Big[\big|Q_{d,p,\boldsymbol{z}}\big(f(\cdot + U \bmod 1)\big) - I_d(f)\big|^2\Big] \;=\; \mathbb{E}_U\Bigg[\Big|\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0}} \hat{f}(\boldsymbol{h})e^{2\pi i\,\boldsymbol{h}U}\Big|^2\Bigg] \;=\; \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0}} \big|\hat{f}(\boldsymbol{h})\big|^2,$$

and therefore

$$\begin{aligned}
\big(e^{\mathrm{rmse}}_{d,\alpha,\boldsymbol{\gamma}}\big(\widetilde{M}_n\big)\big)^2 \;&=\; \sup_{\substack{f\in\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}\\ \|f\|_{\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}}\le 1}} \frac{1}{|\mathcal{P}_n|}\sum_{p\in\mathcal{P}_n}\frac{1}{|\mathcal{Z}_p|}\sum_{\boldsymbol{z}\in\mathcal{Z}_p}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv_p 0}} \big|\hat{f}(\boldsymbol{h})\big|^2\\
&=\; \sup_{\substack{f\in\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}\\ \|f\|_{\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}}\le 1}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})>B_n}} \omega_n(\boldsymbol{h})\,\big|\hat{f}(\boldsymbol{h})\big|^2\\
&\le\; \sup_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})>B_n}} \frac{\omega_n(\boldsymbol{h})}{(r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}))^2}
\end{aligned}$$

with $\omega_n(\boldsymbol{h})$ from (15). From (16) we know that

$$\omega_n(\boldsymbol{h}) \le c\ln(|\boldsymbol{h}|_\infty)/n = c\ln((|\boldsymbol{h}|_\infty)^\alpha)/(\alpha n).$$

We apply $(|\boldsymbol{h}|_\infty)^\alpha \le r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})$ and $\log(x) \le \log(1+x) \le x^\delta/\delta$ and obtain

$$\big(e^{\mathrm{rmse}}_{d,\alpha,\boldsymbol{\gamma}}\big(\widetilde{M}_n\big)\big)^2 \;\le\; \frac{c}{\alpha\delta n}\sup_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})>B_n}} \frac{1}{(r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}))^{2-\delta}} \;\le\; \frac{c}{\alpha\delta n}\,B_n^{\delta-2}.$$

Together with (14) this yields

$$\begin{aligned}
e^{\mathrm{rmse}}_{d,\alpha,\boldsymbol{\gamma}}\big(\widetilde{M}_n\big) \;&\le\; \frac{c}{\alpha\delta}\Big(4\,V_d\big(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda}\big)\Big)^{\lambda(1-\delta/2)} n^{\lambda(\delta-2)/2-1/2}\\
&\le\; \frac{c}{\alpha\delta}\Big(4\,V_d\big(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda}\big)\Big)^{\lambda}\; n^{-\lambda-1/2+\delta\lambda/2},
\end{aligned}$$

which proves the result. □

4 Lower bound

In the last section we proved that the algorithms M_n and $\widetilde{M}_n$ can achieve an order of convergence that is arbitrarily close to the optimal $n^{-\alpha-1/2}$, see [30]. However, we will show in Theorem 3 that the exact optimal order cannot be achieved by these algorithms, no matter how we choose the weights $\boldsymbol{\gamma}$. This shows that the main order in our results in Theorems 1 and 2 is essentially best possible. Note that the lower bound also holds for the case $\alpha = 0$. However, it would be interesting to find an upper bound of the form $C(\log n)^q/n^{\alpha+1/2}$ with C, q independent of n and d. It is not clear if such a bound exists.

Theorem 3. *Let $\alpha \geq 0$. Then the randomized errors of the random algorithms M_n and $\widetilde{M}_n$ are bounded from below by*

$$e^{\mathrm{rmse}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \;\geq\; e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \;\geq\; \frac{\gamma_1}{2}\,\frac{\sqrt{\log n}}{n^{\alpha+1/2}}$$

and

$$e^{\mathrm{rmse}}_{d,\alpha,\boldsymbol{\gamma}}(\widetilde{M}_n) \;\geq\; \frac{\gamma_1}{2}\,\frac{\sqrt{\log n}}{n^{\alpha+1/2}}.$$

Proof. The bound $e^{\mathrm{rmse}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \geq e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n)$ is obvious. To prove the lower bound on $e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n)$ it is enough to construct, for each $n \in \mathbb{N}$, $n \geq 2$, a function, say f_n, that satisfies the lower bound. To this end, we define f_n such that

$$\hat{f}_n(\boldsymbol{h}) \;=\; \begin{cases} \left(r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})\,\sqrt{|\mathcal{P}_n|}\right)^{-1} & \text{if } h_1 \in \mathcal{P}_n, h_2 = \cdots = h_d = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\|f_n\|_{d,\alpha,\boldsymbol{\gamma}} = 1$ and $I_d(f_n) = 0$. Moreover, for $p \in \mathcal{P}_n$, we have

$$\begin{aligned} |Q_{d,p,\boldsymbol{z}}(f_n) - I_d(f_n)| \;&=\; \left(r_{\alpha,\boldsymbol{\gamma}}(p,0,\ldots,0)\,\sqrt{|\mathcal{P}_n|}\right)^{-1} \;=\; \frac{\gamma_1}{p^\alpha\,\sqrt{|\mathcal{P}_n|}} \\ &\geq\; \frac{\gamma_1\sqrt{\log n}}{2\,n^{\alpha+1/2}}, \end{aligned}$$

where we used $|\mathcal{P}_n| \leq \frac{2n}{\log n}$ from [23]. This proves

$$e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \;\geq\; \left(\frac{1}{|\mathcal{P}_M|}\sum_{p\in\mathcal{P}_M}\frac{1}{|\mathcal{Z}_p|}\sum_{\boldsymbol{z}\in\mathcal{Z}_p}|I_d(f_n) - Q_{d,p,\boldsymbol{z}}(f_n)|\right) \;\geq\; \frac{\gamma_1\sqrt{\log n}}{2\,n^{\alpha+1/2}}.$$

Using

$$\mathbb{E}_U\left[\left|Q_{d,p,\boldsymbol{z}}\big(f_n(\cdot + U \bmod 1)\big) - I_d(f_n)\right|^2\right] \;=\; \left(r_{\alpha,\boldsymbol{\gamma}}(p,0,\ldots,0)\,\sqrt{|\mathcal{P}_n|}\right)^{-2},$$

we can can proceed exactly as above to prove the lower bound on $e^{\mathrm{rmse}}_{d,\alpha,\boldsymbol{\gamma}}(\widetilde{M}_n)$. □

5 Results on tractability

Let us now briefly comment on tractability results.

Suppose now that for fixed $d \in \mathbb{N}$ and $\varepsilon \in (0,1)$, we would like to achieve $e^{\mathrm{ran}}_{d,\alpha,\boldsymbol{\gamma}}(M_n) \leq \varepsilon$. Then from Theorem 1 we conclude that for $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$ with $\alpha > 1/2$, provided n satisfies (13), it is sufficient to choose n such that

$$n \;\geq\; \left(C_{\lambda,\delta}V_d(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda})\right)^{\frac{1}{\lambda+1/2-\delta}}\,\varepsilon^{-\frac{1}{\lambda+1/2-\delta}}.$$

Hence, the *information complexity* $n(\varepsilon, d)$, i.e., the minimal number of function evaluations that is required by any kind of random algorithm to achieve a randomized error within the threshold ε, satisfies

$$n(\varepsilon,d) \le \max\left\{ \left\lceil 4\,V_d(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda}) \right\rceil, \left\lceil \left(C_{\lambda,\delta} V_d(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda}) \right)^{\frac{1}{\lambda+1/2-\delta}} \varepsilon^{-\frac{1}{\lambda+1/2-\delta}} \right\rceil \right\}.$$

More generally, for $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$ with $\alpha > 0$, we obtain from Theorem 2 that

$$n(\varepsilon,d) \le \left\lceil \left(\frac{c}{\alpha\delta} (4\,V_d(\alpha/\lambda, \boldsymbol{\gamma}^{1/\lambda}))^{\lambda} \right)^{\frac{1}{\lambda+1/2-\delta\lambda/2}} \varepsilon^{-\frac{1}{\lambda+1/2-\delta\lambda/2}} \right\rceil.$$

In both cases, we have $n(\varepsilon, d) = \mathcal{O}(\varepsilon^{-p})$, for p arbitrarily close to $1/(\lambda + 1/2)$, where the implied constant is independent of d and ε if $\sum_{j=1}^{\infty} \gamma_j^{1/\lambda} < \infty$. The integration problem is then said to be *strongly tractable in the randomized setting*, with the *exponent of strong tractability* being the infimum of those p.

We summarize these fact results in the following theorem.

Theorem 4. *Let $\alpha \ge 0$, let $\boldsymbol{\gamma} = (\gamma_j)_{j\ge 1}$ be a non-increasing sequence of positive weights bounded by 1, and let $d \in \mathbb{N}$. Additionally, let $\lambda_0 \ge 0$ be the supremum of the numbers $\lambda > 0$ such that*

$$\sum_{j=1}^{\infty} \gamma_j^{1/\lambda} < \infty$$

with the convention that $\lambda_0 = 0$ if no such λ exists.
Then the integration problem in the class $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$ is strongly tractable in the randomized setting, with the exponent of strong tractability lying in the interval

$$\left[\frac{1}{\alpha + 1/2}, \max\left\{ \frac{1}{\lambda_0 + 1/2}, \frac{1}{\alpha + 1/2} \right\} \right].$$

In particular, if $\lambda_0 \ge \alpha$, then the exponent of strong tractability is $1/(\alpha + 1/2)$.

Proof. The upper bound on the exponent of strong tractability for $\alpha > 0$ follows from Theorem 2 and the discussion above. For $\alpha = 0$ it is well-known that the exponent of strong tractability in the randomized setting is 1/2, which can be achieved, i.e., by the simple Monte Carlo method. For the lower bound note that $n(\varepsilon, d) \ge n(\varepsilon, 1) \ge c \cdot \varepsilon^{-1/(\alpha+1/2)}$ for some $c > 0$, see e.g. [17]. □

Remark 3. *In the same way as it was done in [27], we could also comment on other forms of tractability, like polynomial tractability, under modified summability assumptions on the weights. We omit the details.*

For further information on tractability in various settings and an overview of various kinds of error criteria in the context of tractability, see the trilogy [18]–[20].

6 Conclusion

We showed in this paper that the randomized lattice algorithm M_n achieves a randomized integration error with a convergence order arbitrarily close to $\mathcal{O}(n^{-\alpha-1/2})$ in $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$ with $\alpha > 1/2$, where the implied constant is independent of d as long as $\sum_{j=1}^{\infty} \gamma_j^{1/\alpha} < \infty$. By additionally making use of a random shift, this result can be extended to all $\alpha > 0$.

If the weights only satisfy a weaker summability condition $\sum_{j=1}^{\infty} \gamma_j^{1/\lambda} < \infty$ for some $\lambda < \alpha$, then we can still obtain error bounds that are independent of d but the convergence rate is reduced correspondingly to be close to $\mathcal{O}(n^{-\lambda-1/2})$.

The question whether component-by-component constructions can be used in the context of this setting remains open for future research.

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References

[1] N.S. Bakhvalov. An estimate of the mean remainder term in quadrature formulae. Zh. Vych. Mat. **1**, 64–77, 1961.

[2] N.S. Bakhvalov, On the approximate calculation of multiple integrals. J. Complexity **31**, 502–516, 2015 [English translation; the original appeared in: Vestnik MGU, Ser. Math. Mech. Astron. Phys. Chem., **4**, 3–18, 1959.]

[3] V. A. Bykovskii, On the correct order of the error of optimal cubature formulas in spaces with dominant derivative, and on quadratic deviations of grids info. Preprint, Computing Center Far-Eastern Scientific Center, Akad. Sci. USSR, Vladivostok, 1985.

[4] J. Dick, F.Y. Kuo, I.H. Sloan. High-dimensional integration—the quasi-Monte Carlo way. Acta Numer. **22**, 133–288, 2013.

[5] J. Dick, F.Y. Kuo, F. Pillichshammer, I.H. Sloan. Construction algorithms for polynomial lattice rules for multivariate integration. Math. Comp. **74**, 1895–1921, 2005.

[6] J. Dick, F. Pillichshammer. *Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration.* Cambridge University Press, Cambridge, 2010.

[7] K.K. Frolov. Upper error bounds for quadrature formulas on function classes. Dokl. Akad. Nauk SSSR **231**, 818–821, 1976.

[8] T. Goda, K. Suzuki, T. Yoshiki. An explicit construction of optimal order quasi–Monte Carlo rules for smooth integrands. SIAM J. Numer. Anal. **54**, 2664–2683, 2016.

[9] E. Hlawka. Zur angenäherten Berechnung mehrfacher Integrale. Monatsh. Math. **66**, 140-151, 1962 (in German).

[10] C. Kacwin, J. Oettershagen, T. Ullrich. On the orthogonality of the Chebyshev-Frolov lattice and applications. Preprint, 2016. arXiv.org: 1606.00492.

[11] N.M. Korobov. Approximate evaluation of repeated integrals. Dokl. Akad. Nauk. SSSR **124**, 1207-1210, 1959 (in Russian).

[12] D. Krieg, E. Novak. A universal algorithm for multivariate integration. To appear in: Found. Comput. Math., 2017.

[13] F.Y. Kuo. Component-by-component constructions achieve the optimal rate of convergence for multivariate integration in weighted Korobov and Sobolev spaces. J. Complexity **19**, 301–320, 2003.

[14] F.Y. Kuo, I.H. Sloan, H. Woźniakowski. Lattice rules for multivariate approximation in the worst case setting. In: H. Niederreiter, D. Talay (eds.). *Monte Carlo and Quasi-Monte Carlo Methods 2004.* Springer, Berlin, pp. 289–330, 2006.

[15] H. Niederreiter. Quasi-Monte Carlo methods and pseudo-random numbers. Bull. Amer. Math. Soc. **84**, 957–1041, 1978.

[16] H. Niederreiter. *Random Number Generation and Quasi-Monte Carlo Methods*. No. 63 in CBMS-NSF Series in Applied Mathematics. SIAM, Philadelphia, 1992.

[17] E. Novak, *Deterministic and Stochastic Error Bounds in Numerical Analysis*, LNiM **1349**, Springer Verlag, Berlin, 1988.

[18] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems, Volume I: Linear Information*. EMS, Zurich, 2008.

[19] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems, Volume II: Standard Information for Functionals*. EMS, Zurich, 2010.

[20] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems, Volume III: Standard Information for Operators*. EMS, Zurich, 2012.

[21] V. K. Nguyen, M. Ullrich, T. Ullrich. Change of variable in spaces of mixed smoothness and numerical integration of multivariate functions on the unit cube. To appear in: Constr. Approx., 2017.

[22] D. Nuyens, R. Cools. Fast algorithms for component-by-component constructions of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces. Math. Comp. **75**, 903–920, 2006.

[23] J. B. Rosser, L. Schoenfeld. Approximate formulas for some functions of prime numbers. Illinois J. Math. **6**, 64–94, 1962.

[24] I.H. Sloan, S. Joe. *Lattice Methods for Multiple Integration*. Oxford University Press, New York and Oxford, 1994.

[25] I.H. Sloan, A.V. Reztsov. Component-by-component construction of good lattice rules. Math. Comp. **71**, 263–273, 2002.

[26] I.H. Sloan, H. Woźniakowski. When are quasi-Monte Carlo algorithms efficient for high-dimensional integrals? J. Complexity **14**, 1–33, 1998.

[27] I.H. Sloan, H. Woźniakowski. Tractability of Multivariate Integration for Weighted Korobov Classes. J. Complexity **17**(4), 697–721, 2001.

[28] V. N. Temlyakov. *Approximation of periodic functions*, Computational Mathematics and Analysis Series, Nova Science Publishers, Inc.. Commack, NY, 1993.

[29] M. Ullrich. On "Upper error bounds for quadrature formulas on function classes" by K.K. Frolov. In: Ronald Cools and Dirk Nuyens (eds.): *Monte Carlo and Quasi-Monte Carlo Methods*, Springer, Cham, pp. 571–582, 2016.

[30] M. Ullrich. A Monte Carlo method for integration of multivariate smooth functions. To appear in: SIAM J. Numer. Anal., 2017.

[31] M. Ullrich, T. Ullrich, The role of Frolov's cubature formula for functions with bounded mixed derivative. SIAM J. Numer. Anal., **54**, 969–993, 2016.

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