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A. S. Fokas*† and K. Kalimeris‡

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Abstract

The unified transform, also known as the Fokas method, provides a powerful methodology for studying boundary value problems. Employing this methodology, we analyze inviscid, irrotational, two dimensional water waves in a bounded domain, and in particular we study the generation of waves by a moving piecewise horizontal bottom, as it occurs in tsunamis. We show that this problem is characterized by two equations which involve only first order derivatives. It is argued that under the assumptions of "small amplitude waves" but not of "long waves", the above two equations can be treated numerically via a recently introduced numerical technique for elliptic PDEs in a polygonal domain. In the particular case that the moving bottom is horizontal and under the assumption of "small amplitude waves", but not of "long waves", these equations yield a non-local generalization of the Boussinesq system. Furthermore, under the additional assumption of "long waves" the above system yields a Boussinesq-type system, which however includes the effect of the moving boundary.

1 Introduction

The study of surface water waves has a long and illustrious history dating back to the classical works of Stokes and his contemporaries in the 19th century. A new formulation of this problem, based on the so called *unified transform* or *Fokas method* [1, 2], was presented in [3]. This unexpected development has led to important advances in water waves [4]-[11] and in related areas [12]-[14]. However, for water waves, the application of this new approach has been limited to unbounded domains, or to the simple case of periodic boundary conditions.

In many physical applications, such as tsunami generation in enclosed seas, or reservoir sloshing, the problem is formulated in a bounded domain and involves moving boundaries [15, 16].

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The problem of water waves can be formulated in terms of the Laplace equation, for which the unified transform has given rise to novel numerical techniques. In particular, in [17], following earlier works by Fornberg and others [18]-[22], an efficient numerical algorithm has been presented for the case of the interior of a polygonal domain.

In what follows, by employing the unified method, we study two–dimensional surface water waves in a bounded domain with moving vertical boundaries and a moving piecewise horizontal bottom. Our analysis yields two equations which contain only first order derivatives. We argue that under the assumption of the "small amplitude" approximation, these two equations can be treated numerically via the technique of [17].

We then concentrate in the case of a moving horizontal bottom, and show that now the above two equations yield a nonlocal generalisation of the Boussinesq system. Under the additional assumption of the "long waves" approximation, the above nonlocal system yields a Boussinesq–type system. To our knowledge, this is the first time that a well posed problem for a Boussinesq–type system has been obtained for a moving seafloor.

The derivation of the "small amplitude" and "long waves" approximations relies on the introduction of suitable dimensionless parameters. In contrast to the usual ad hoc non-dimensionalization of the water wave equations, our approach relies only on physical parameters, known a priori from the forcing.

In the general case, we consider a piecewise horizontal moving bottom, depicted in Figure 1, where the two lower parts are located at $y = -h_2(t)$, while the higher part is located at $y = -h_1(t)$. The relevant vertical fluid speeds are given by $\{b_j(x,t)\}_1^2$ for the lower parts and by $b(x,t)$ for the higher part. We assume that the left vertical boundary at $x = 0$ is fixed, whereas the vertical boundary at $L_1(t)$ is moving with speed $l_1(y,t)$ and the vertical boundaries at $L_2(t)$, and $L(t)$ are moving with speeds $r_1(y,t)$ and $r(y,t)$, respectively.

In the case of a horizontal bottom, only two boundaries are moving, namely the boundary $x = L(t)$ with speed $r(y,t)$, and the boundary $y = -h(t)$ with speed $b(x,t)$. Clearly, the physically interesting situation corresponds to

$$b_1(x,t) = b_2(x,t) = -\dot{h}_2(t), \qquad b(x,t) = -\dot{h}(t),$$

$$l_1(y,t) = \dot{L}_1(t), \qquad r_1(y,t) = \dot{L}_2(t), \qquad r(y,t) = \dot{L}(t),$$

where dot denotes the differentiation with respect to time.

In what follows, we denote the free surface position with respect to its position at rest by $\eta(x,t)$, and the velocity potential by $\phi(x,y,t)$.

2 Derivation of the Global Relation

The function $\phi(x,y,t)$ satisfies Laplace's equation in the domain $\Omega(t)$, depicted by the grey area in Figure 1, for $t > 0$, i.e.,

$$\phi_{xx} + \phi_{yy} = 0, \qquad (x,y) \in \Omega(t), \quad t > 0, \tag{1}$$

along with the boundary conditions given below.

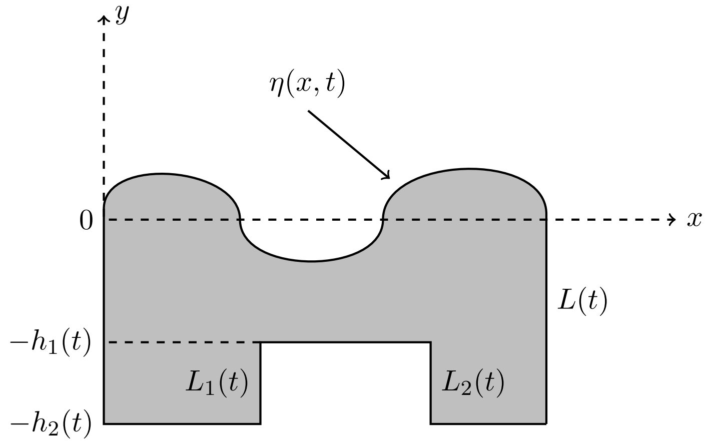

Figure 1: Two dimensional water waves with piecewise horizontal moving bottom.

- On the vertical boundaries:

$$\phi_x(0,y,t)=0, \qquad -h_2(t)<y<\eta(0,t), \quad t>0, \tag{2}$$
$$\phi_x(L_1(t),y,t)=l_1(y,t), \qquad -h_2(t)<y<-h_1(t), \quad t>0, \tag{3}$$
$$\phi_x(L_2(t),y,t)=r_1(y,t), \qquad -h_2(t)<y<-h_1(t), \quad t>0, \tag{4}$$
$$\phi_x(L(t),y,t)=r(y,t), \qquad -h_2(t)<y<\eta(L(t),t), \quad t>0. \tag{5}$$

- On the piecewise horizontal bottom:

$$\phi_y(x,-h_2(t),t)=b_1(x,t), \qquad 0<x<L_1(t), \quad t>0, \tag{6}$$
$$\phi_y(x,-h_1(t),t)=b(x,t), \qquad L_1(t)<x<L_2(t), \quad t>0, \tag{7}$$
$$\phi_y(x,-h_2(t),t)=b_2(x,t), \qquad L_2(t)<x<L(t), \quad t>0. \tag{8}$$

- On the free boundary:

$$\eta_t+\eta_x\phi_x=\phi_y, \qquad \text{on} \quad y=\eta(x,t). \tag{9}$$

Equations (1)-(9) define a Neumann boundary value problem for Laplace's equation, involving the unknown boundary $\eta(x,t)$. The latter function can be determined, in principle, by supplementing the above equations with the additional condition

$$\phi_t+\frac{1}{2}(\phi_x^2+\phi_y^2)+g\eta=0, \text{ on } y=\eta(x,t). \tag{10}$$

We denote by $q(x,t)$ the value of $\phi(x,y,t)$, on the free surface, i.e.,

$$q(x,t)=\phi(x,\eta(x,t),t). \tag{11}$$

Our aim is to find the equations satisfied by $\eta(x,t)$ and $q(x,t)$. In this respect, differentiating (11) with respect to x, we find

$$q_x = \phi_x + \phi_y \eta_x, \qquad \text{on} \;\; y = \eta(x,t).$$

This equation together with equation (9) can be used to express the values of ϕ_x and ϕ_y at $y = \eta$, in terms of the following derivatives of q and η:

$$\phi_x = \frac{q_x - \eta_x \eta_t}{1+\eta_x^2}, \quad \phi_y = \frac{\eta_t + \eta_x q_x}{1+\eta_x^2}, \qquad \text{on} \;\; y = \eta(x,t). \tag{12}$$

Differentiating (11) with respect to t, we obtain

$$q_t = \phi_t + \phi_y \eta_t, \qquad \text{on} \;\; y = \eta(x,t).$$

Using in this equation the second of equations (12), we can express ϕ_t in terms of derivatives of q and η:

$$\phi_t = q_t - \frac{\eta_t + \eta_x q_x}{1+\eta_x^2}\eta_t, \qquad \text{on} \;\; y = \eta(x,t).$$

Using the above expression together with equations (12) in condition (10), we obtain the first equation coupling q and η:

$$q_t + g\eta + \frac{1}{2}q_x^2 - \frac{1}{2}\frac{(\eta_t + \eta_x q_x)^2}{1+\eta_x^2} = 0. \tag{13}$$

In order to obtain a second equation relating q and η, we introduce the complex variable $z = x + iy$. Then, Laplace's equation becomes

$$\phi_{z\bar{z}} = 0,$$

where the over–bar denotes complex conjugation. Thus, ϕ also satisfies the equation

$$(e^{i\lambda z}\phi_z)_{\bar{z}} = 0,$$

where λ is an arbitrary complex function of t. This implies that $e^{i\lambda z}\phi_z$ is an analytic function, hence Cauchy's theorem yields the so called "global relation" (GR):

$$\int_{\partial\Omega} e^{i\lambda z}\phi_z dz = 0, \qquad \lambda \in \mathbb{C}, \tag{14}$$

where

$$\phi_z = \frac{1}{2}(\phi_x - i\phi_y),$$

and $\partial\Omega$ denotes the boundary of the domain.

Using the boundary conditions (2)-(8) together with the identities (12), it is possible to compute the LHS of the global relation. We denote by F_1 the contribution of the free surface $y = \eta(x,t)$ and by $\{F_j\}_2^8$ the contributions of the intervals which form the rest of the boundary from the corner $\big(L(t), \eta(L(t),t)\big)$ to the corner $\big(0, \eta(0,t)\big)$, in the clockwise direction.

The GR is given by

$$\sum_{j=1}^{8} F_j = 0, \tag{15}$$

where the functions $\{F_j\}_1^8$ will be computed below.

On the free surface

$$z = x + i\eta(x,t),$$

hence, using $dz = (1 + i\eta_x)dx$, and replacing ϕ_x, ϕ_y by equations (12), we find

$$F_1 = \int_0^{L(t)} e^{i\lambda\big(x+i\eta(x,t)\big)} \left[\frac{q_x - \eta_x\eta_t}{1+\eta_x^2} - i\frac{\eta_t + \eta_x q_x}{1+\eta_x^2}\right](1+i\eta_x)dx,$$

which remarkably simplifies to

$$F_1 = \int_0^{L(t)} e^{i\lambda x - \lambda\eta}\left(q_x - i\eta_t\right)dx. \tag{16}$$

On the vertical boundary $x = L(t)$, we have

$$z = L(t) + iy, \quad \phi_x = r(y,t),$$

thus,

$$F_2 = \int_{\eta(L(t),t)}^{-h_2(t)} e^{i\lambda(L(t)+iy)}\left[\phi_y(L(t),y,t) + ir(y,t)\right]dy. \tag{17}$$

On the right part of the flat bottom, $y = -h_2(t), \quad x \in (L_2(t), L(t))$, we have

$$z = x - ih_2(t), \quad \phi_y = b_2(x,t),$$

thus,

$$F_3 = \int_{L(t)}^{L_2(t)} e^{i\lambda(x-ih_2(t))}\left[\phi_x(x,-h_2(t),t) - ib_2(x,t)\right]dx. \tag{18}$$

On the vertical boundary $x = L_2(t)$, we have

$$z = L_2(t) + iy, \quad \phi_x = r_1(y,t),$$

thus,

$$F_4 = \int_{-h_2(t)}^{-h_1(t)} e^{i\lambda(L_2(t)+iy)}\left[\phi_y(L_2(t),y,t) + ir_1(y,t)\right]dy. \tag{19}$$

On the middle part of the flat bottom, $y = -h_1(t), \quad x \in (L_1(t), L_2(t))$, we have

$$z = x - ih_1(t), \quad \phi_y = b(x,t),$$

thus,

$$F_5 = \int_{L_2(t)}^{L_1(t)} e^{i\lambda(x - ih_1(t))} \left[\phi_x(x, -h_1(t), t) - ib(x,t)\right] dx. \tag{20}$$

On the vertical boundary $x = L_1(t)$, we have

$$z = L_1(t) + iy, \quad \phi_x = l_1(y,t),$$

thus,

$$F_6 = \int_{-h_1(t)}^{-h_2(t)} e^{i\lambda(L_1(t) + iy)} \left[\phi_y(L_1(t), y, t) + il_1(y,t)\right] dy. \tag{21}$$

On the left part of the flat bottom, $y = -h_2(t), \quad x \in (0, L_1(t))$, we have

$$z = x - ih_2(t), \quad \phi_y = b_1(x,t),$$

thus,

$$F_7 = \int_{L_1(t)}^{0} e^{i\lambda(x - ih_2(t))} \left[\phi_x(x, -h_2(t), t) - ib_1(x,t)\right] dx. \tag{22}$$

On the vertical boundary $x = 0$, we have

$$z = iy, \quad \phi_x = 0,$$

thus,

$$F_8 = \int_{-h_2(t)}^{\eta(0,t)} e^{-\lambda y} \phi_y(0, y, t) dy. \tag{23}$$

Hence, the global relation becomes

$$\begin{aligned}
&\int_0^{L(t)} e^{i\lambda x - \lambda\eta}(q_x - i\eta_t)dx - e^{\lambda h_1(t)} \int_{L_1(t)}^{L_2(t)} e^{i\lambda x} \phi_x(x, -h_1(t), t)dx \\
&- e^{\lambda h_2(t)} \int_{L_2(t)}^{L(t)} e^{i\lambda x} \phi_x(x, -h_2(t), t)dx - e^{\lambda h_2(t)} \int_0^{L_1(t)} e^{i\lambda x} \phi_x(x, -h_2(t), t)dx \\
&+ e^{i\lambda L_2(t)} \int_{-h_2(t)}^{-h_1(t)} e^{-\lambda y} \phi_y(L_2(t), y, t)dy - e^{i\lambda L_1(t)} \int_{-h_2(t)}^{-h_1(t)} e^{-\lambda y} \phi_y(L_1(t), y, t)dy \\
&+ \int_{-h_2(t)}^{\eta(0,t)} e^{-\lambda y} \phi_y(0, y, t)dy - e^{i\lambda L(t)} \int_{-h_2(t)}^{\eta(L(t),t)} e^{-\lambda y} \phi_y(L(t), y, t)dy \; = \; iF(\lambda, t), \qquad \lambda \in \mathbb{C},
\end{aligned} \tag{24}$$

where the known function $F(\lambda, t)$ is given by

$$\begin{aligned}
F(\lambda, t) = e^{i\lambda L(t)} &\int_{-h_2(t)}^{\eta(L(t),t)} e^{-\lambda y} r(y,t)dy - e^{\lambda h_1(t)} \int_{L_1(t)}^{L_2(t)} e^{i\lambda x} b(x,t)dx \\
- e^{\lambda h_2(t)} &\int_0^{L_1(t)} e^{i\lambda x} b_1(x,t)dx - e^{\lambda h_2(t)} \int_{L_2(t)}^{L(t)} e^{i\lambda x} b_2(x,t)dx \\
+ e^{i\lambda L_1(t)} &\int_{-h_2(t)}^{-h_1(t)} e^{-\lambda y} l_1(y,t)dy - e^{i\lambda L_2(t)} \int_{-h_2(t)}^{-h_1(t)} e^{-\lambda y} r_1(y,t)dy.
\end{aligned} \tag{25}$$

In summary, inviscid, irrotational, two dimensional water waves in the domain $\Omega(t)$ *depicted in Figure 1 are characterised by the equation* (13)*, as well as by the global relation* (24)*. The latter equation contains a single unknown function on its segment of the boundary of* $\Omega(t)$*, except of the free surface where it contains two unknown functions, namely* $q(x,t)$ *and* $\eta(x,t)$*. Both equations* (13) *and* (24) *involve only first order derivatives.*

3 The Non-Dimensional Form of the Basic Equations

In order to study two interesting limits of equations (13) and (24) we replace all variables with primed variables, and then we introduce dimensionless variables:

$$x' = L_0 x, \quad y' = h_0 y, \quad \eta' = \alpha\eta, \quad \lambda' = \frac{\lambda}{L_0}, \quad t' = Tt, \quad q' = Qq, \tag{26}$$

where α is a typical wave amplitude, and T and Q are given by the expressions

$$T = \frac{L_0}{c_0}, \qquad Q = \epsilon c_0 L_0, \tag{27}$$

with

$$\epsilon = \frac{\alpha}{h_0}, \quad \delta = \frac{h_0}{L_0}, \quad c_0^2 = gh_0. \tag{28}$$

Remark 1. *Small* ϵ *is indicative of small amplitude and small* δ *is indicative of long waves.*

Remark 2. *A rigorous derivation of* (27) *is presented in Appendix A.*

Using the above dimensionless variables, equation (13) becomes,

$$q_t + \eta + \frac{1}{2}\epsilon q_x^2 - \frac{1}{2}\epsilon\delta^2 \frac{(\eta_t + \epsilon\eta_x q_x)^2}{1 + \epsilon^2\delta^2\eta_x^2} = 0. \tag{29}$$

We choose

$$r'(y', t') = \frac{A}{T} r(y,t), \qquad r_1'(y', t') = \frac{A}{T} r_1(y,t), \qquad l_1'(y', t') = \frac{A}{T} l_1(y,t)$$

and

$$b'(x', t') = \frac{B}{T} b(x,t), \qquad b_1'(x', t') = \frac{B}{T} b_1(x,t), \qquad b_2'(x', t') = \frac{B}{T} b_2(x,t),$$

where A and B are typical amplitudes of displacements of the horizontal and vertical boundaries respectively, and T is given by (27).

We next find the dimensionless form of the global relationship (24). In this connection, we scale the ϕ_x on the horizontal boundaries with $c_0\epsilon\hat{\epsilon}$ and ϕ_y on the vertical boundaries with $c_0\epsilon\hat{\delta}$, respectively, namely, let

$$
\begin{aligned}
&\phi'_{x'}(x', -h_1(t'), t') = c_0\epsilon\hat{\epsilon}q_b(x,t), && \phi'_{x'}(x', -h_2(t'), t') = c_0\epsilon\hat{\epsilon}q_d(x,t) \\
&\phi'_{y'}(0, y', t') = c_0\epsilon\hat{\delta}q_l(y,t), && \phi'_{y'}(L(t'), y', t') = c_0\epsilon\hat{\delta}q_r(y,t), \\
&\phi'_{y'}(L_1(t'), y', t') = c_0\epsilon\hat{\delta}q_{l_1}(y,t), && \phi'_{y'}(L_2(t'), y', t') = c_0\epsilon\hat{\delta}q_{r_1}(y,t),
\end{aligned}
\tag{30}
$$

where $\hat{\epsilon}$ and $\hat{\delta}$ will be determined in the following section.

Then, the global relation (24) becomes

$$
\begin{aligned}
&\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x-\epsilon\delta\lambda\eta}q_x dx - i\delta\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x-\epsilon\delta\lambda\eta}\eta_t dx - \hat{\epsilon}e^{\lambda\delta\frac{h_1(t)}{h_0}}\int_{\frac{L_1(t)}{L_0}}^{\frac{L_2(t)}{L_0}} e^{i\lambda x}q_b(x,t)\,dx \\
&- \hat{\epsilon}e^{\lambda\delta\frac{h_2(t)}{h_0}}\int_{\frac{L_2(t)}{L_0}}^{\frac{L(t)}{L_0}} e^{i\lambda x}q_d(x,t)\,dx - \hat{\epsilon}e^{\lambda\delta\frac{h_2(t)}{h_0}}\int_0^{\frac{L_1(t)}{L_0}} e^{i\lambda x}q_d(x,t)\,dx \\
&+ \delta\hat{\delta}e^{i\lambda\frac{L_1(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y}q_{r_1}(y,t)dy - \delta\hat{\delta}e^{i\lambda\frac{L_2(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y}q_{l_1}(y,t)\,dy \\
&+ \delta\hat{\delta}\int_{-\frac{h_2(t)}{h_0}}^{\epsilon\eta(0,t)} e^{-\delta\lambda y}q_l(y,t)dy - \delta\hat{\delta}e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{\epsilon\eta\left(\frac{L(t)}{L_0},t\right)} e^{-\delta\lambda y}q_r(y,t)\,dy \\
=& -i\frac{B}{\epsilon L_0}\left\{e^{\lambda\delta\frac{h_1(t)}{h_0}}\int_{\frac{L_1(t)}{L_0}}^{\frac{L_2(t)}{L_0}} e^{i\lambda x}b(x,t)dx + e^{\lambda\delta\frac{h_2(t)}{h_0}}\left[\int_0^{\frac{L_1(t)}{L_0}} e^{i\lambda x}b_1(x,t)dx + \int_{\frac{L_2(t)}{L_0}}^{\frac{L(t)}{L_0}} e^{i\lambda x}b_2(x,t)dx\right]\right\} \\
&- i\frac{A\delta}{\epsilon L_0}\left[-e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{\epsilon\eta\left(\frac{L(t)}{L_0},t\right)} e^{-\delta\lambda y}r(y,t)dy\right. \\
&\qquad\left. - e^{i\lambda\frac{L_1(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y}l_1(y,t)dy + e^{i\lambda\frac{L_2(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y}r_1(y,t)dy\right].
\end{aligned}
\tag{31}
$$

The balance on the scaling of the RHS and the second term of LHS, for the $\mathcal{O}(\delta)$ coefficients as $\delta \to 0$, implies

$$\delta = \frac{B}{\epsilon L_0} = \frac{A\delta}{\epsilon L_0}.$$

The first equality reads

$$B = \epsilon\delta L_0 = \frac{\alpha}{h_0}\frac{h_0}{L_0}L_0 = \alpha$$

and the second equality reads

$$B = A\delta.$$

Hence, we obtain the following conditions:

$$B = \alpha, \quad A = \frac{\alpha}{\delta}. \tag{32}$$

The equation $\alpha = B$ is consistent with the fact that it has been finally established after a long lasting controversy that, at least in long wave theory when applied to tsunami generation from seafloor displacement, the surface amplitude scales with the seafloor displacement [23].

Using (26), we observe that the first order term of the RHS of the second of equations (12), as $\epsilon \to 0$, scales with $\dfrac{\alpha}{T} = c_0\epsilon\delta$. Thus the second of equations (12) implies that ϕ_y on the unknown boundary $y = \eta(x,t)$ scales with $c_0\epsilon\delta$. Similarly, the first of equations (12) implies that ϕ_x on the unknown boundary $y = \eta(x,t)$ scales with $\dfrac{Q}{L_0} = c_0\epsilon$.

Thus, we observe that for small ϵ:

- On the unknown boundary $y = \eta(x,t)$, ϕ_y scales with $c_0\epsilon\delta$, whereas ϕ_x scales with $c_0\epsilon$.
- On the piecewise horizontal bottom $\left\{y = b(x,t),\ y = b_1(x,t),\ y = b_2(x,t)\right\}$, the quantity ϕ_y scales with $\dfrac{B}{T} = c_0\epsilon\delta$.
- On the vertical boundaries $\left\{x = 0,\ x = L_1(t),\ x = L_2(t),\ x = L(t)\right\}$, the quantity ϕ_x scales with $\dfrac{A}{T} = c_0\epsilon$.

The above observations suggest that on the entire boundary, ϕ_y scales with $c_0\epsilon\delta$ and ϕ_x scales with $c_0\epsilon$, which implies that in equation (30) we have

$$\hat{\epsilon} = 1, \qquad \hat{\delta} = \delta. \tag{33}$$

Remark 3. *A rigorous derivation of the scaling* (33) *through the global relation, is given at the Appendix B.*

Applying equations (28), (32) and (33) in (31), we obtain the following form of the global relation:

$$
\begin{aligned}
&\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x-\epsilon\delta\lambda\eta}\left(q_x-i\delta\eta_t\right)dx - e^{\lambda\delta\frac{h_1(t)}{h_0}}\int_{\frac{L_1(t)}{L_0}}^{\frac{L_2(t)}{L_0}} e^{i\lambda x}q_b\left(x,t\right)dx\\
&- e^{\lambda\delta\frac{h_2(t)}{h_0}}\int_{\frac{L_2(t)}{L_0}}^{\frac{L(t)}{L_0}} e^{i\lambda x}q_d\left(x,t\right)dx - e^{\lambda\delta\frac{h_2(t)}{h_0}}\int_0^{\frac{L_1(t)}{L_0}} e^{i\lambda x}q_d\left(x,t\right)dx\\
&+\delta^2 e^{i\lambda\frac{L_1(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y}q_{r_1}(y,t)dy - \delta^2 e^{i\lambda\frac{L_2(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y}q_{l_1}\left(y,t\right)dy\\
&+\delta^2\int_{-\frac{h_2(t)}{h_0}}^{\epsilon\eta(0,t)} e^{-\delta\lambda y}q_l(y,t)dy - \delta^2 e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{\epsilon\eta\left(\frac{L(t)}{L_0},t\right)} e^{-\delta\lambda y}q_r\left(y,t\right)dy\\
=&-i\delta\left\{e^{\lambda\delta\frac{h_1(t)}{h_0}}\int_{\frac{L_1(t)}{L_0}}^{\frac{L_2(t)}{L_0}} e^{i\lambda x}b(x,t)dx + e^{\lambda\delta\frac{h_2(t)}{h_0}}\left[\int_0^{\frac{L_1(t)}{L_0}} e^{i\lambda x}b_1(x,t)dx + \int_{\frac{L_2(t)}{L_0}}^{\frac{L(t)}{L_0}} e^{i\lambda x}b_2(x,t)dx\right]\right\}\\
&+i\delta\left[e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{\epsilon\eta\left(\frac{L(t)}{L_0},t\right)} e^{-\delta\lambda y}r(y,t)dy\right.\\
&\qquad\left.+e^{i\lambda\frac{L_1(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y}l_1(y,t)dy - e^{i\lambda\frac{L_2(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y}r_1(y,t)dy\right]. \qquad (34)
\end{aligned}
$$

In summary, the non-dimensional form of the basic equations (13) *and* (24) *is given by the equations* (29) *and* (34) *respectively, with ϵ and δ are defined in equation* (28).

4 The Small Amplitude Approximation.

We evaluate the non-dimensional form of the first of equations (12) at the endpoints of the unknown boundary, i.e. at $x=0$ and $x=\frac{L(t)}{L_0}$. Applying the conditions (2) and (5) in the resulting equalities, and neglecting terms of $\mathcal{O}\left(\epsilon^2\right)$, we obtain the following conditions:

$$q_x(0,t)=0, \qquad q_x\left(\frac{L(t)}{L_0},t\right)=r(0,t). \qquad (35)$$

Similarly, evaluating the second of equations (12) at the endpoints of the unknown boundary, i.e. at $x=0$ and $x=\frac{L(t)}{L_0}$, and applying the conditions (30) we obtain the following conditions:

$$q_l(0,t)=\eta_t(0,t), \qquad q_r\left(0,t\right)=\eta_t\left(\frac{L(t)}{L_0},t\right). \qquad (36)$$

Using the non-dimensional formulation of section 3, and neglecting terms of $\mathcal{O}\left(\epsilon^2\right)$, equations (29) and (34), become

$$q_t+\eta+\frac{1}{2}\epsilon q_x^2-\frac{1}{2}\epsilon\delta^2\eta_t^2=0, \qquad (37)$$

and

$$\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}(q_x - i\delta\eta_t)dx - \epsilon\delta\lambda \int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}(\eta q_x - i\delta\eta\eta_t)dx - e^{\lambda\delta\frac{h_1(t)}{h_0}} \int_{\frac{L_1(t)}{L_0}}^{\frac{L_2(t)}{L_0}} e^{i\lambda x} q_b(x,t)\,dx$$
$$- e^{\lambda\delta\frac{h_2(t)}{h_0}} \int_{\frac{L_2(t)}{L_0}}^{\frac{L(t)}{L_0}} e^{i\lambda x} q_d(x,t)\,dx - e^{\lambda\delta\frac{h_2(t)}{h_0}} \int_0^{\frac{L_1(t)}{L_0}} e^{i\lambda x} q_d(x,t)\,dx$$
$$+ \delta^2 e^{i\lambda\frac{L_1(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} q_{r_1}(y,t)dy - \delta^2 e^{i\lambda\frac{L_2(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} q_{l_1}(y,t)\,dy$$
$$+ \delta^2 \int_{-\frac{h_2(t)}{h_0}}^{0} e^{-\delta\lambda y} q_l(y,t)dy - \delta^2 e^{i\lambda\frac{L(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{0} e^{-\delta\lambda y} q_r(y,t)\,dy$$
$$+ \epsilon\delta^2\left[\eta(0,t)q_l(0,t) - e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)q_r(0,t)\right]$$
$$= -i\delta\left\{e^{\lambda\delta\frac{h_1(t)}{h_0}} \int_{\frac{L_1(t)}{L_0}}^{\frac{L_2(t)}{L_0}} e^{i\lambda x} b(x,t)dx + e^{\lambda\delta\frac{h_2(t)}{h_0}}\left[\int_0^{\frac{L_1(t)}{L_0}} e^{i\lambda x} b_1(x,t)dx + \int_{\frac{L_2(t)}{L_0}}^{\frac{L(t)}{L_0}} e^{i\lambda x} b_2(x,t)dx\right]\right\}$$
$$+ i\delta\left[e^{i\lambda\frac{L(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{0} e^{-\delta\lambda y} r(y,t)dy\right.$$
$$\left. + e^{i\lambda\frac{L_1(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} l_1(y,t)dy - e^{i\lambda\frac{L_2(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} r_1(y,t)dy\right].$$
$$+ i\epsilon\delta e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)r(0,t). \tag{38}$$

Note that all the integral terms in (38) vanish as $\lambda \to -\infty$, except for the second term in the LHS of (38) which is[1] of $\mathcal{O}(1)$ as $\lambda \to -\infty$. Integrating this term by parts, we find

$$\epsilon\delta\lambda \int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}(\eta q_x - i\delta\eta\eta_t)dx = i\epsilon\delta\left[\eta(0,t)q_x(0,t) - e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)q_x\left(\frac{L(t)}{L_0},t\right)\right]$$
$$+ \int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}\left[i\epsilon\delta(\eta q_x)_x + \epsilon\delta^2(\eta\eta_t)_x\right]dx,$$

[1] Here, we keep track of the λ dependence only.

where the first term of the RHS is of order $\mathcal{O}(1)$ and the second is of order $\mathcal{O}\left(\frac{1}{|\lambda|}\right)$, as $\lambda \to -\infty$. Thus, equation (38) can be rewritten in the following form:

$$
\begin{aligned}
&\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}\left[q_x - i\delta\eta_t - i\epsilon\delta(\eta q_x)_x - \epsilon\delta^2(\eta\eta_t)_x\right]dx - e^{\lambda\delta\frac{h_1(t)}{h_0}}\int_{\frac{L_1(t)}{L_0}}^{\frac{L_2(t)}{L_0}} e^{i\lambda x} q_b\left(x,t\right)dx \\
&- e^{\lambda\delta\frac{h_2(t)}{h_0}}\int_{\frac{L_2(t)}{L_0}}^{\frac{L(t)}{L_0}} e^{i\lambda x} q_d\left(x,t\right)dx - e^{\lambda\delta\frac{h_2(t)}{h_0}}\int_0^{\frac{L_1(t)}{L_0}} e^{i\lambda x} q_d\left(x,t\right)dx \\
&+ \delta^2 e^{i\lambda\frac{L_1(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} q_{r_1}(y,t)dy - \delta^2 e^{i\lambda\frac{L_2(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} q_{l_1}\left(y,t\right)dy \\
&+ \delta^2\int_{-\frac{h_2(t)}{h_0}}^{0} e^{-\delta\lambda y} q_l(y,t)dy - \delta^2 e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{0} e^{-\delta\lambda y} q_r\left(y,t\right)dy \\
&+ i\delta\left\{e^{\lambda\delta\frac{h_1(t)}{h_0}}\int_{\frac{L_1(t)}{L_0}}^{\frac{L_2(t)}{L_0}} e^{i\lambda x} b(x,t)dx + e^{\lambda\delta\frac{h_2(t)}{h_0}}\left[\int_0^{\frac{L_1(t)}{L_0}} e^{i\lambda x} b_1(x,t)dx + \int_{\frac{L_2(t)}{L_0}}^{\frac{L(t)}{L_0}} e^{i\lambda x} b_2(x,t)dx\right]\right\} \\
&- i\delta\left[e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{0} e^{-\delta\lambda y} r(y,t)dy\right. \\
&\qquad\left. + e^{i\lambda\frac{L_1(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} l_1(y,t)dy - e^{i\lambda\frac{L_2(t)}{L_0}}\int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} r_1(y,t)dy\right]. \\
&= i\epsilon\delta\left[\eta(0,t)q_x(0,t) - e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)q_x\left(\frac{L(t)}{L_0},t\right)\right] \\
&+ \epsilon\delta^2\left[\eta(0,t)\eta_t(0,t) - e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)\eta_t\left(\frac{L(t)}{L_0},t\right)\right] \\
&- \epsilon\delta^2\left[\eta(0,t)q_l(0,t) - e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)q_r(0,t)\right] \\
&+ i\epsilon\delta e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)r(0,t).
\end{aligned}
\tag{39}
$$

We observe that the LHS of the above expression is of order $\mathcal{O}\left(\frac{1}{|\lambda|}\right)$, while the RHS is of order $\mathcal{O}(1)$, as $\lambda \to -\infty$. Using consistent asymptotics, both should vanish. We emphasise that the conditions (35) and (36) imply that the RHS of (39) vanishes.

Hence, the global relation takes the form

$$\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}\left[q_x - i\delta\eta_t - i\epsilon\delta(\eta q_x)_x - \epsilon\delta^2(\eta\eta_t)_x\right]dx - e^{\lambda\delta\frac{h_1(t)}{h_0}} \int_{\frac{L_1(t)}{L_0}}^{\frac{L_2(t)}{L_0}} e^{i\lambda x} q_b\left(x,t\right)dx$$

$$- e^{\lambda\delta\frac{h_2(t)}{h_0}} \int_{\frac{L_2(t)}{L_0}}^{\frac{L(t)}{L_0}} e^{i\lambda x} q_d\left(x,t\right)dx - e^{\lambda\delta\frac{h_2(t)}{h_0}} \int_0^{\frac{L_1(t)}{L_0}} e^{i\lambda x} q_d\left(x,t\right)dx$$

$$+ \delta^2 e^{i\lambda\frac{L_1(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} q_{r_1}(y,t)dy - \delta^2 e^{i\lambda\frac{L_2(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} q_{l_1}\left(y,t\right)dy$$

$$+ \delta^2 \int_{-\frac{h_2(t)}{h_0}}^{0} e^{-\delta\lambda y} q_l(y,t)dy - \delta^2 e^{i\lambda\frac{L(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{0} e^{-\delta\lambda y} q_r\left(y,t\right)dy$$

$$= -i\delta\left\{ e^{\lambda\delta\frac{h_1(t)}{h_0}} \int_{\frac{L_1(t)}{L_0}}^{\frac{L_2(t)}{L_0}} e^{i\lambda x} b(x,t)dx + e^{\lambda\delta\frac{h_2(t)}{h_0}} \left[\int_0^{\frac{L_1(t)}{L_0}} e^{i\lambda x} b_1(x,t)dx + \int_{\frac{L_2(t)}{L_0}}^{\frac{L(t)}{L_0}} e^{i\lambda x} b_2(x,t)dx\right]\right\}$$

$$+ i\delta\left[e^{i\lambda\frac{L(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{0} e^{-\delta\lambda y} r(y,t)dy\right.$$

$$\left. + e^{i\lambda\frac{L_1(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} l_1(y,t)dy - e^{i\lambda\frac{L_2(t)}{L_0}} \int_{-\frac{h_2(t)}{h_0}}^{-\frac{h_1(t)}{h_0}} e^{-\delta\lambda y} r_1(y,t)dy\right], \qquad (40)$$

where we have neglected terms of $\mathcal{O}\left(\epsilon^2\right)$.

In summary, the small amplitude limit of surface waves is characterised by equations (37) *and* (40)*, together with the conditions* (35) *and* (36)*, i.e.,*

$$q_x(0,t) = 0, \qquad q_x\left(\frac{L(t)}{L_0}, t\right) = r(0,t),$$

$$q_l(0,t) = \eta_t(0,t), \qquad q_r(0,t) = \eta_t\left(\frac{L(t)}{L_0}, t\right). \qquad (41)$$

4.1 Small Amplitude and Long Waves Approximations.

We recall that small δ is indicative of long waves. Thus, under this additional assumption, so that terms of $\mathcal{O}(\epsilon\delta^2)$ can be neglected, equation (37) becomes

$$q_t + \eta + \frac{\epsilon}{2} q_x^2 = 0. \qquad (42)$$

Furthermore, the global relationship (40) has the same form, but now the term $(\eta q_x)_x$ is omitted from the first integral.

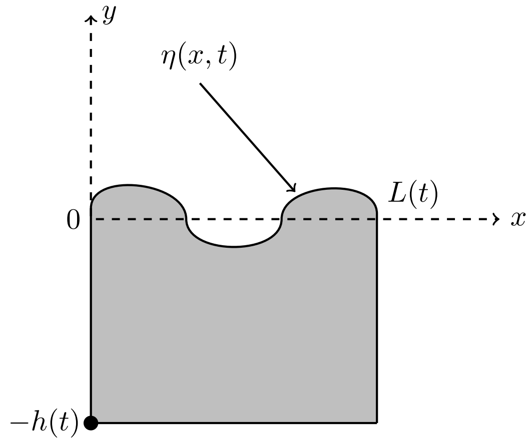

Figure 2: Two dimensional water waves with a horizontal (moving) bottom.

5 The case of a horizontal bottom

In this section we consider the case of a horizontal bottom, see Figure 2. The non-local formulation is now obtained as a special case of the formulation (24)-(25), by making the substitutions

$$L_1(t) \equiv 0, \qquad L_2(t) \equiv L(t), \qquad h_2(t) \equiv h_1(t) =: h(t).$$

Thus, the boundary conditions (2)-(8) take the form

$$\phi_x(0, y, t) = 0, \qquad -h(t) < y < \eta(0, t), \quad t > 0, \tag{43}$$

$$\phi_x(L(t), y, t) = r(y, t), \qquad -h(t) < y < \eta(L(t), t), \quad t > 0, \tag{44}$$

$$\phi_y(x, -h(t), t) = b(x, t), \qquad 0 < x < L(t), \quad t > 0. \tag{45}$$

Moreover, F_1 remains invariant, whereas

$$F_3 = F_7 = F_4 = F_6 = 0.$$

Also, F_2, F_5, F_8 take the following form:

$$F_2 = \hat{F}_2 := \int_{\eta(L(t),t)}^{-h(t)} e^{i\lambda(L(t)+iy)} \left[\phi_y(L(t), y, t) + ir(y, t)\right] dy,$$

$$F_5 = \hat{F}_5 := \int_{L(t)}^{0} e^{i\lambda(x-ih(t))} \left[\phi_x(x, -h(t), t) - ib(x, t)\right] dx,$$

$$F_8 = \hat{F}_8 := \int_{-h(t)}^{\eta(0,t)} e^{-\lambda y} \phi_y(0, y, t) dy.$$

Thus, equation (15) now becomes

$$\sum_{j=1}^{4} G_j = 0,$$

where

$$G_1 = F_1, \quad G_2 = \hat{F}_2, \quad G_3 = \hat{F}_5, \text{ and } G_4 = \hat{F}_8.$$

Hence, the global relation becomes

$$\int_0^{L(t)} e^{i\lambda x - \lambda\eta}(q_x - i\eta_t)dx - e^{\lambda h(t)} \int_0^{L(t)} e^{i\lambda x}\phi_x(x, -h(t), t)dx + \int_{-h(t)}^{\eta(0,t)} e^{-\lambda y}\phi_y(0, y, t)dy$$
$$- e^{i\lambda L(t)} \int_{-h(t)}^{\eta(L(t),t)} e^{-\lambda y}\phi_y(L(t), y, t)dy = iG(\lambda, t), \qquad \lambda \in \mathbb{C}, \tag{46}$$

where the known function $G(\lambda, t)$ is given by

$$G(\lambda, t) = e^{i\lambda L(t)} \int_{-h(t)}^{\eta(L(t),t)} e^{-\lambda y} r(y, t)dy - e^{\lambda h(t)} \int_0^{L(t)} e^{i\lambda x} b(x, t)dx. \tag{47}$$

5.1 The non-dimensional form of the main equations

In the case of a horizontal bottom, the main equation (34) takes the following form:

$$\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x - \epsilon\delta\lambda\eta}(q_x - i\delta\eta_t)dx - e^{\lambda\delta\frac{h(t)}{h_0}} \int_0^{\frac{L(t)}{L_0}} e^{i\lambda x} q_b(x, t)\, dx$$
$$+ \delta^2 \int_{-\frac{h(t)}{h_0}}^{\epsilon\eta(0,t)} e^{-\delta\lambda y} q_l(y, t)dy - \delta^2 e^{i\lambda\frac{L(t)}{L_0}} \int_{-\frac{h(t)}{h_0}}^{\epsilon\eta\left(\frac{L(t)}{L_0}, t\right)} e^{-\delta\lambda y} q_r(y, t)\, dy$$
$$= i\delta \left[-e^{\lambda\delta\frac{h(t)}{h_0}} \int_0^{\frac{L(t)}{L_0}} e^{i\lambda x} b(x, t)dx + e^{i\lambda\frac{L(t)}{L_0}} \int_{-\frac{h(t)}{h_0}}^{\epsilon\eta\left(\frac{L(t)}{L_0}, t\right)} e^{-\delta\lambda y} r(y, t)dy \right]. \tag{48}$$

Furthermore, under the "small amplitude" approximation we obtain the analogue of equation (40) for the case of a horizontal bottom, which takes the following form:

$$\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x} \left[q_x - i\delta\eta_t - i\epsilon\delta(\eta q_x)_x - \epsilon\delta^2(\eta\eta_t)_x\right] dx - e^{\lambda\delta\frac{h(t)}{h_0}} \int_0^{\frac{L(t)}{L_0}} e^{i\lambda x} q_b(x, t)\, dx$$
$$+ \delta^2 \int_{-\frac{h(t)}{h_0}}^{0} e^{-\delta\lambda y} q_l(y, t)dy - \delta^2 e^{i\lambda\frac{L(t)}{L_0}} \int_{-\frac{h(t)}{h_0}}^{0} e^{-\delta\lambda y} q_r(y, t)\, dy \tag{49}$$
$$= i\delta \left[-e^{\lambda\delta\frac{h(t)}{h_0}} \int_0^{\frac{L(t)}{L_0}} e^{i\lambda x} b(x, t)dx + e^{i\lambda\frac{L(t)}{L_0}} \int_{-\frac{h(t)}{h_0}}^{0} e^{-\delta\lambda y} r(y, t)dy \right],$$

where we have neglected terms of $\mathcal{O}\left(\epsilon^2\right)$.

Under the additional "long wave" approximation, we obtain the equation

$$\begin{aligned}
&\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}\left[q_x - i\delta\eta_t - i\epsilon\delta(\eta q_x)_x\right]dx - e^{\lambda\delta\frac{h(t)}{h_0}}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x} q_b\left(x,t\right)dx \\
&+\delta^2\int_{-\frac{h(t)}{h_0}}^{0} e^{-\delta\lambda y} q_l(y,t)dy - \delta^2 e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h(t)}{h_0}}^{0} e^{-\delta\lambda y} q_r\left(y,t\right)dy \\
&= i\delta\left[-e^{\lambda\delta\frac{h(t)}{h_0}}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x} b(x,t)dx + e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h(t)}{h_0}}^{0} e^{-\delta\lambda y} r(y,t)dy\right],
\end{aligned} \tag{50}$$

where we have neglected terms of $\mathcal{O}\left(\epsilon\delta^2\right)$.

It turns out that only in the case of a horizontal bottom we can eliminate the unknown boundary values of the GR (46)*, so that we can obtain a single equation coupling q and η, see equation* (51) *below. Furthermore, equation* (49) *yields a nonlocal Boussinesq equation, see* (61) *below, whereas equation* (50) *yields a Boussinesq-type equation, see* (63) *below.*

The derivation of (51), (61), (63) are provided in sections 5.2, 5.3, 5.4 respectively.

5.2 The Non-Local Formulation coupling q and η

Starting with equation (46), we can obtain a nonlocal equation coupling q and η. Indeed, we will show that equation (46) implies the equation

$$\begin{aligned}
&\int_0^{L(t)}\Big\{q_x\sin(\lambda x)\sinh\left[\lambda(\eta+h(t))\right] \\
&\qquad + \eta_t\cos(\lambda x)\cosh\left[\lambda(\eta+h(t))\right]\Big\}dx = S(\lambda,t), \quad \lambda = \frac{n\pi}{L(t)},
\end{aligned} \tag{51}$$

where $S(\lambda,t)$ is defined by

$$\begin{aligned}
S(\lambda,t) = &-\cos(\lambda L(t))\int_{-h(t)}^{\eta(L(t),t)}\cosh\left[\lambda(y+h(t))\right]r(y,t)dy \\
&+\int_0^{L(t)}\cos(\lambda x)b(x,t)dx, \qquad \lambda\in\mathbb{C},\ t>0.
\end{aligned} \tag{52}$$

In order to derive (51) we take the complex conjugate of (46), and then replace $\bar{\lambda}$ with $-\lambda$:

$$\begin{aligned}
&\int_0^{L(t)} e^{i\lambda x+\lambda\eta}(q_x+i\eta_t)dx - e^{-\lambda h(t)}\int_0^{L(t)} e^{i\lambda x}\phi_x(x,-h(t),t)dx + \int_{-h(t)}^{\eta(0,t)} e^{\lambda y}\phi_y(0,y,t)dy \\
&- e^{i\lambda L(t)}\int_{-h(t)}^{\eta(L(t),t)} e^{\lambda y}\phi_y(L(t),y,t)dy = -i\bar{G}(-\lambda,t), \qquad \lambda\in\mathbb{C}.
\end{aligned} \tag{53}$$

Multiplying equations (46) and (53) by $e^{-\lambda h(t)}$ and $e^{\lambda h(t)}$ respectively, and then subtracting the resulting equations, we obtain an equation which does not contain $\phi_x(x,-h(t),t)$:

$$\int_0^{L(t)} e^{i\lambda x}\Big\{q_x \sinh\big[\lambda(\eta+h(t))\big] + i\eta_t \cosh\big[\lambda(\eta+h(t))\big]\Big\}dx$$
$$+\int_{-h(t)}^{\eta(0,t)} \sinh[\lambda(y+h(t))]\phi_y(0,y,t)dy - e^{i\lambda L(t)}\int_{-h(t)}^{\eta(L(t),t)} \sinh[\lambda(y+h(t))]\phi_y(L(t),y,t)dy$$
$$= -\frac{i}{2}\left[e^{-\lambda h(t)}G(\lambda,t) + e^{\lambda h(t)}\bar{G}(-\lambda,t)\right], \qquad \lambda\in\mathbb{C}. \tag{54}$$

Replacing in (54) λ with $-\lambda$ we find

$$\int_0^{L(t)} e^{-i\lambda x}\Big\{-q_x \sinh\big[\lambda(\eta+h(t))\big] + i\eta_t \cosh\big[\lambda(\eta+h(t))\big]\Big\}dx$$
$$-\int_{-h(t)}^{\eta(0,t)} \sinh[\lambda(y+h(t))]\phi_y(0,y,t)dy + e^{-i\lambda L(t)}\int_{-h(t)}^{\eta(L(t),t)} \sinh[\lambda(y+h(t))]\phi_y(L(t),y,t)dy$$
$$= -\frac{i}{2}\left[e^{\lambda h(t)}G(-\lambda,t) + e^{-\lambda h(t)}\bar{G}(\lambda,t)\right], \qquad \lambda\in\mathbb{C}. \tag{55}$$

Adding (54) and (55) we obtain

$$\int_0^{L(t)} q_x \sin(\lambda x)\sinh\big[\lambda(\eta+h(t))\big] + \eta_t\cos(\lambda x)\cosh\big[\lambda(\eta+h(t))\big]dx$$
$$-\sin(\lambda L(t))\int_{-h(t)}^{\eta(L(t),t)} \sinh[\lambda(y+h(t))]\phi_y(L(t),y,t)dy = S(\lambda,t), \qquad \lambda\in\mathbb{C}, \tag{56}$$

where $S(\lambda,t)$ is defined by

$$S(\lambda,t) = -\frac{1}{4}e^{-\lambda h(t)}\Big(G(\lambda,t)+\bar{G}(\lambda,t)\Big) - \frac{1}{4}e^{\lambda h(t)}\Big(G(-\lambda,t)+\bar{G}(-\lambda,t)\Big).$$

In order to eliminate the term involving $\phi_y(L(t),y,t)$, we evaluate equation (56) at $\lambda = \frac{n\pi}{L(t)}$, $n\in\mathbb{Z}$ and then (51) follows.

Equation (13) and either of the two equations (46) or (51), are the basic equations characterising the boundary value problem defined in (1) with the boundary conditions (43)-(45) and (9).

Using (26) and (32), equation (13) takes the form (29), and equation (51) becomes

$$\int_0^{\frac{L(t)}{L_0}} \left\{q_x \sin(\lambda x)\frac{1}{\delta}\sinh\left[\lambda\delta\left(\frac{h(t)}{h_0}+\epsilon\eta\right)\right] + \eta_t\cos(\lambda x)\cosh\left[\lambda\delta\left(\frac{h(t)}{h_0}+\epsilon\eta\right)\right]\right\}dx$$
$$= \int_0^{\frac{L(t)}{L_0}} \cos(\lambda x)b(x,t)dx - (-1)^n\int_{-\frac{h(t)}{h_0}}^{\epsilon\eta\left(\frac{L(t)}{L_0},t\right)} \cosh\left[\lambda\delta\left(y+\frac{h(t)}{h_0}\right)\right] r(y,t)dy, \qquad \lambda = \frac{n\pi L_0}{L(t)}. \tag{57}$$

5.3 A generalisation of Boussinesq equations for the small amplitude approximation

Letting $\epsilon \to 0$ in equation (57) and neglecting terms of $\mathcal{O}\left(\epsilon^2\right)$, we find

$$\begin{aligned}
&\int_0^{\frac{L(t)}{L_0}} \left\{ q_x \sin(\lambda x) \left[\frac{1}{\delta} \sinh\left(\lambda\delta \frac{h(t)}{h_0}\right) + \epsilon\lambda\eta \cosh\left(\lambda\delta\frac{h(t)}{h_0}\right)\right] \right.\\
&\qquad \left. + \eta_t \cos(\lambda x) \left[\cosh\left(\lambda\delta\frac{h(t)}{h_0}\right) + \epsilon\delta\lambda\eta \sinh\left(\lambda\delta\frac{h(t)}{h_0}\right)\right]\right\} dx \\
&= \int_0^{\frac{L(t)}{L_0}} \cos(\lambda x) b(x,t) dx - (-1)^n \int_{-\frac{h(t)}{h_0}}^0 \cosh\left[\lambda\delta\left(y + \frac{h(t)}{h_0}\right)\right] r(y,t) dy \\
&-\epsilon(-1)^n \cosh\left(\lambda\delta\frac{h(t)}{h_0}\right) \eta\left(\frac{L(t)}{L_0}, t\right) r(0,t), \qquad \lambda = \frac{n\pi L_0}{L(t)}. \qquad (58)
\end{aligned}$$

Conditions (41) imply that the term of the last line of (58) cancels with the expression which comes from integrating by parts the term that involves $(q_x\eta)$ of the LHS, thus equation (58) takes the form

$$\begin{aligned}
&\int_0^{\frac{L(t)}{L_0}} \left[q_x \frac{\sin(\lambda x)}{\delta} + \epsilon\delta\lambda\eta\eta_t \cos(\lambda x)\right] \sinh\left(\lambda\delta\frac{h(t)}{h_0}\right) dx \\
&+ \int_0^{\frac{L(t)}{L_0}} \left[\eta_t + \epsilon\left(\eta q_x\right)_x\right] \cos(\lambda x) \cosh\left(\lambda\delta\frac{h(t)}{h_0}\right) dx \qquad (59)\\
&= \int_0^{\frac{L(t)}{L_0}} \cos(\lambda x) b(x,t) dx - (-1)^n \int_{-\frac{h(t)}{h_0}}^0 \cosh\left[\lambda\delta\left(y + \frac{h(t)}{h_0}\right)\right] r(y,t) dy, \qquad \lambda = \frac{n\pi L_0}{L(t)}.
\end{aligned}$$

By fixing the vertical boundary, namely

$$L(t) = L_0, \qquad \text{and} \qquad r(y,t) = 0,$$

the last two terms of the RHS of (58) vanish. Thus, dividing this simplified equation by $\cosh\left(\lambda\delta\frac{h(t)}{h_0}\right)$, and then using the inverse cosine series formula, we find that the following equations provide a previously underived generalization of the Boussinesq equations, for a moving seafloor boundary, and, importantly, without the common "long wave" approximation:

$$\begin{aligned}
&\eta_t + \sum_{n=0}^{\infty} \cos(\lambda x) \left\{ \tanh\left(\lambda\delta\frac{h}{h_0}\right) \int_0^1 \left[q_\xi \frac{\sin(\lambda\xi)}{\delta} + \epsilon\delta\lambda\eta\eta_t\cos(\lambda\xi)\right] d\xi + \epsilon\lambda\int_0^1 \eta q_\xi \sin(\lambda\xi) d\xi \right\} \\
&= \sum_{n=0}^{\infty} \frac{\cos(\lambda x)}{\cosh(\lambda\delta\frac{h}{h_0})} \int_0^1 b(\xi,t)\cos(\lambda\xi) d\xi, \qquad \text{for} \quad \lambda = n\pi. \qquad (60)
\end{aligned}$$

Equation (60), can be further simplified. Following exactly the same procedure as above, but now starting from (59), we obtain the following form of the generalization of the Boussinesq equations:

$$\eta_t + \epsilon\left(\eta q_x\right)_x + \sum_{n=0}^{\infty}\cos(\lambda x)\left\{\tanh\left(\lambda\delta\frac{h}{h_0}\right)\int_0^1\left[q_\xi\frac{\sin(\lambda\xi)}{\delta} + \epsilon\delta\lambda\eta\eta_t\cos(\lambda\xi)\right]d\xi\right\}$$
$$= \sum_{n=0}^{\infty}\frac{\cos(\lambda x)}{\cosh(\lambda\delta\frac{h}{h_0})}\int_0^1 b(\xi,t)\cos(\lambda\xi)d\xi, \qquad \text{for } \lambda = n\pi. \tag{61}$$

Remark 4. *Equation* (61) *can also be obtained directly from* (60) *by integrating by parts the term* $\epsilon\lambda\int_0^1 \eta q_\xi\sin(\lambda\xi)d\xi$, *and observing that* (41), *now, give the conditions*

$$q_x(0,t) = q_x(1,t) = 0. \tag{62}$$

5.4 A Boussinesq-type equation for the Small Amplitude and Long Waves approximation

We recall that small δ is indicative of long waves. In this section, using the dimensionless variables, we analyse the interesting physical situation:

$$r(y,t) = \frac{\dot{L}(t)}{L_0}, \qquad b(x,t) = -\frac{\dot{h}(t)}{h_0},$$

where dot denotes time derivative.

Under the additional assumption that δ is small, so that terms of $\mathcal{O}(\epsilon\delta^2)$ can be neglected, equation (37) becomes (42). Furthermore, the global relationship (38) can be simplified further, thus its numerical integration is simpler than the integration of (40). Also, equation (58) yield the following well posed boundary value problem for the newly derived Boussinesq–type equation:

$$\eta_t + \frac{h(t)}{h_0}q_{xx} + \epsilon(\eta q_x)_x - \frac{\delta^2}{2}\left(\frac{h(t)}{h_0}\right)^2\eta_{txx} - \frac{\delta^2}{6}\left(\frac{h(t)}{h_0}\right)^3 q_{xxxx} = -\frac{\dot{h}(t)}{h_0}, \tag{63}$$

along with the boundary conditions

$$q_x(0,t) = 0, \qquad q_x\left(\frac{L(t)}{L_0},t\right) = \frac{\dot{L}(t)}{L_0}, \tag{64}$$

$$q_{xxx}(0,t) = 0, \qquad q_{xxx}\left(\frac{L(t)}{L_0},t\right) = 3\frac{h_0}{h(t)}\frac{\dddot{L}(t)}{L_0}. \tag{65}$$

In order to obtain (63)-(65), we apply in equation (58), the approximations

$$\cosh x \approx 1 + \frac{x^2}{2}, \qquad \sinh x \approx x + \frac{x^3}{6}, \qquad \text{for small } x.$$

Therefore, equation (58) becomes

$$\int_0^{\frac{L(t)}{L_0}} \cos(\lambda x)\eta_t dx + \int_0^{\frac{L(t)}{L_0}} \lambda \sin(\lambda x) \left[\frac{h(t)}{h_0} q_x + \epsilon \eta q_x\right] dx$$
$$+ \frac{\delta^2}{2}\left(\frac{h(t)}{h_0}\right)^2 \int_0^{\frac{L(t)}{L_0}} \lambda^2 \cos(\lambda x)\eta_t dx + \frac{\delta^2}{6}\left(\frac{h(t)}{h_0}\right)^3 \int_0^{\frac{L(t)}{L_0}} \lambda^3 \sin(\lambda x) q_x dx$$
$$= -\frac{\dot{h}(t)}{h_0}\int_0^{\frac{L(t)}{L_0}} \cos(\lambda x) dx - (-1)^n \frac{h(t)}{h_0}\frac{\dot{L}(t)}{L_0}$$
$$- (-1)^n \frac{1}{6}\delta^2\lambda^2 \left(\frac{h(t)}{h_0}\right)^3 \frac{\dot{L}(t)}{L_0} - (-1)^n \epsilon\eta\left(\frac{L(t)}{L_0}, t\right)\frac{\dot{L}(t)}{L_0}, \qquad \lambda = \frac{n\pi L_0}{L(t)}, \tag{66}$$

where we have neglected terms of $\mathcal{O}(\epsilon\delta^2)$. We use integration by parts in the LHS of (66) in order to replace the terms $\{\lambda\sin(\lambda x),\ \lambda^2\cos(\lambda x),\ \lambda^3\sin(\lambda x)\}$ with $\cos(\lambda x)$.

In particular, we use the following identities:

- $\displaystyle\int_0^p \lambda \sin(\lambda x) F(x) dx = -\cos(\lambda p) F(p) + F(0) + \int_0^p \cos(\lambda x) F_x(x) dx,$

  for $F = \dfrac{h(t)}{h_0} q_x + \epsilon \eta q_x,$

- $\displaystyle\int_0^p \lambda^2 \cos(\lambda x) F(x) dx = \cos(\lambda p) F_x(p) - F_x(0) - \int_0^p \cos(\lambda x) F_{xx}(x) dx,$

  for $F = \eta_t,$

- $\displaystyle\int_0^p \lambda^3 \sin(\lambda x) F(x) dx = -\lambda^2 \cos(\lambda p) F(p) + \lambda^2 F(0)$

  $\displaystyle + \cos(\lambda p) F_{xx}(p) - F_{xx}(0) - \int_0^p \cos(\lambda x) F_{xxx}(x) dx, \qquad \text{for } F = q_x,$

where $\lambda = \frac{n\pi}{p}$ and $p = \frac{L(t)}{L_0}$.

Thus, equation (66) takes the form

$$\int_0^{\frac{L(t)}{L_0}} \cos(\lambda x)\left[\eta_t + \frac{h(t)}{h_0} q_{xx} + \epsilon(\eta q_x)_x - \frac{\delta^2}{2}\left(\frac{h(t)}{h_0}\right)^2 \eta_{txx} - \frac{\delta^2}{6}\left(\frac{h(t)}{h_0}\right)^3 q_{xxxx} + \frac{\dot{h}(t)}{h_0}\right] dx$$
$$+ \frac{h(t)}{h_0} q_x(0,t) + \epsilon\eta(0,t) q_x(0,t) - (-1)^n \left[\frac{h(t)}{h_0} q_x\left(\frac{L(t)}{L_0}, t\right) + \epsilon\eta\left(\frac{L(t)}{L_0}, t\right) q_x\left(\frac{L(t)}{L_0}, t\right)\right]$$
$$- \frac{\delta^2}{2}\left(\frac{h(t)}{h_0}\right)^2 \left[\eta_{tx}(0,t) - (-1)^n \eta_{tx}\left(\frac{L(t)}{L_0}, t\right)\right]$$
$$+ \frac{\delta^2\lambda^2}{6}\left(\frac{h(t)}{h_0}\right)^3 \left[q_x(0,t) - (-1)^n q_x\left(\frac{L(t)}{L_0}, t\right)\right] - \frac{\delta^2}{6}\left(\frac{h(t)}{h_0}\right)^3 \left[q_{xxx}(0,t) - (-1)^n q_{xxx}\left(\frac{L(t)}{L_0}, t\right)\right]$$
$$= -(-1)^n \left[\frac{h(t)}{h_0}\frac{\dot{L}(t)}{L_0} + \frac{1}{6}\delta^2\lambda^2\left(\frac{h(t)}{h_0}\right)^3 \frac{\dot{L}(t)}{L_0} + \epsilon\eta\left(\frac{L(t)}{L_0}, t\right)\frac{\dot{L}(t)}{L_0}\right], \qquad \lambda = \frac{n\pi L_0}{L(t)}. \tag{67}$$

Employing the completeness of $\left\{\cos(\lambda x),\ \lambda = \dfrac{n\pi L_0}{L(t)}, n \in \mathbb{N}\right\}$, equation (67) yields (63), (64), as well as the following equations:

$$\eta_{tx}(0,t) + \frac{1}{3}\frac{h(t)}{h_0} q_{xxx}(0,t) = 0, \tag{68}$$

$$\eta_{tx}\left(\frac{L(t)}{L_0}, t\right) + \frac{1}{3}\frac{h(t)}{h_0} q_{xxx}\left(\frac{L(t)}{L_0}, t\right) = 0. \tag{69}$$

Indeed, balancing the $O(\epsilon)$ terms of (67) yields (64). Applying (64) in (67), and balancing the $O(\delta^2)$ terms of the resulting expression, yields the conditions (68) and (69).

Using equation (42) and conditions (64), we can further simplify conditions (68) and (69). In particular, differentiating (42) with respect to t and to x gives

$$\eta_{tx} = -q_{xtt} - \epsilon q_{xt} q_{xx} - \epsilon q_x q_{xxt}. \tag{70}$$

Equation (64) yields

$$q_x(0,t) = q_{xt}(0,t) = q_{xtt}(0,t) = 0,$$

thus, evaluating (70) at $x = 0$, yields $\eta_{tx}(0,t) = 0$. Therefore, the condition (68) simplifies to the first of the conditions (65). Similar considerations for $x = L(t)$, yield

$$\eta_{tx}\left(\frac{L(t)}{L_0}, t\right) = -\frac{\dddot{L}(t)}{L_0} - \epsilon\frac{\ddot{L}(t)}{L_0} q_{xx}\left(\frac{L(t)}{L_0}, t\right) - \epsilon\frac{\dot{L}(t)}{L_0} q_{xxt}\left(\frac{L(t)}{L_0}, t\right).$$

Thus, recalling that we have neglected in (67) terms of $O\left(\epsilon\delta^2\right)$, condition (69) simplifies to the second of the conditions (65).

For the special case of fixed vertical boundaries, namely

$$L(t) = L_0, \qquad \text{and} \qquad r(y,t) = 0,$$

the boundary value problem given in equation (63) remains invariant, but the boundary conditions (64)-(65) simplify to

$$q_x(0,t) = q_x(1,t) = q_{xxx}(0,t) = q_{xxx}(1,t) = 0, \tag{71}$$

since $L(t)$ is constant.

Furthermore, $\eta_{tx}(0,t) = \eta_{tx}(1,t) = 0$.

6 Discussion

Two dimensional, inviscid, irrotational water waves formulated in the domain $\Omega(t)$ depicted in Figure 1, under the “small amplitude”, but without the “long wave” approximation, are characterised by (37) and the global relation (38). These equations can be treated numerically using the newly developed technique of [17]. Furthermore, in the case of a horizontal bottom, these equations yield a nonlocal generalization of the Boussinesq system defined by (37) and (61), supplemented with the boundary conditions (62). Under the further approximation of "long waves", the above problem is characterised by the Boussinesq system (42) and (63), supplemented with the boundary conditions (71).

In the case of horizontal bottom, the global relation (38) takes the form (49), which is similar with the equation obtained for the analysis of the Laplace equation formulated in the interior of a rectangle with corners at

$$\left\{0,\ \ \frac{L(t)}{L_0},\ \ \frac{L(t)}{L_0}-i\frac{h(t)}{h_0},\ \ -i\frac{h(t)}{h_0}\right\},$$

but with two important differences: First, the boundary values on the side $\left(0, \frac{L(t)}{L_0}\right)$ are quadratically nonlinear, and second all boundary values depend on t. This implies that equation (40) can be integrated numerically via the method presented in [17], with the following modifications:

(i) $q_x(x,t),\ \eta(x,t),\ q_b(x,t),\ q_l(y,t)$, and $q_r(y,t)$ should be expanded in terms of Chebysev instead of Legendre polynomials, since Chebysev polynomials have the important property that

$$T_n(x)T_m(x) = \frac{1}{2}T_{n-m}(x) + \frac{1}{2}T_{n+m}(x).$$

(ii) The coefficients of the expansions are now functions of time.

Appendix A (Derivation of the scaling (27) through the global relation)

We introduce dimensionless variables:

$$x' = L_0 x,\ y' = h_0 y,\ \eta' = \alpha\eta,\ \lambda' = \frac{\lambda}{L_0},\ t' = tT,\ q' = Qq,$$

where α is a typical wave amplitude, whereas Q, T will be determined below.

Equation (13) becomes

$$\frac{Q}{T}q_t + \epsilon c_0^2\eta + \frac{1}{2}\frac{Q^2}{L_0^2}q_x^2 - \frac{1}{2}\frac{\alpha^2}{T^2}\frac{(\eta_t + \frac{TQ}{L_0^2}\eta_x q_x)^2}{1+\epsilon^2\delta^2\eta_x^2} = 0, \tag{72}$$

with

$$\epsilon = \frac{\alpha}{h_0},\ \ \delta = \frac{h_0}{L_0},\ \ c_0^2 = gh_0. \tag{73}$$

We choose

$$r'(y',t') = \frac{A}{T}r(y,t), \qquad b'(x',t') = \frac{B}{T}b(x,t),$$

where A and B are typical amplitudes of the boundary displacements at $x = L(t)$ and $y = -h(t)$, respectively.

Equation (51) becomes

$$L_0 \int_0^{\frac{L(t)}{L_0}} \left\{ \frac{Q}{L_0} q_x \sin(\lambda x) \sinh\left[\delta\lambda\left(\epsilon\eta + \frac{h(t)}{h_0}\right)\right] + \frac{\alpha}{T}\eta_t \cos(\lambda x)\cosh\left[\delta\lambda\left(\epsilon\eta + \frac{h(t)}{h_0}\right)\right]\right\} dx$$
$$= \frac{BL_0}{T}\int_0^{\frac{L(t)}{L_0}} \cos(\lambda x) b(x,t)dx - \frac{Ah_0}{T}\cos\left(\lambda\frac{L(t)}{L_0}\right)\int_{-\frac{h(t)}{h_0}}^{\epsilon\eta\left(\frac{L(t)}{L_0},t\right)} \cosh\left[\lambda\delta\left(y + \frac{h(t)}{h_0}\right)\right] r(y,t)dy. \tag{74}$$

The balance on the scaling of the first order terms, as $\delta \to 0$, of the above equation implies

$$\delta Q = \frac{\alpha L_0}{T} = \frac{BL_0}{T} = \frac{Ah_0}{T}.$$

We note that in the first term of the above equalities, δ multiplying Q comes from the first order term of $\sinh\left[\delta\lambda\left(\epsilon\eta + \frac{h(t)}{h_0}\right)\right]$ as $\delta \to 0$.

The above equalities yield

$$\alpha = B, \quad A = \frac{BL_0}{h_0}, \quad \delta Q = \frac{BL_0}{T}.$$

The last equation above together with the equation $\epsilon c_0^2 = \frac{Q}{T}$ which follows from balancing the first two terms of equation (72), imply

$$\delta Q = \frac{BL_0\epsilon c_0^2}{Q},$$

$$Q^2 = \epsilon c_0^2 \frac{BL_0}{\delta} = \epsilon c_0^2 \frac{\epsilon}{h_0} L_0 \delta = \epsilon^2 c_0^2 L_0^2.$$

Hence,

$$\alpha = B, \quad A = \frac{B}{\delta}, \quad Q = \epsilon c_0 L_0, \quad T = \frac{L_0}{c_0}. \tag{75}$$

Appendix B (Derivation of the scaling (33) through the global relation)

In the case of a horizontal bottom, by applying (28) and (32) in equation (31), we obtain the following form of the global relation:

$$\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x-\epsilon\delta\lambda\eta}(q_x - i\delta\eta_t)dx - \hat{\epsilon}e^{\lambda\delta\frac{h(t)}{h_0}}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}q_b\left(x,t\right)dx$$
$$+\delta\hat{\delta}\int_{-\frac{h(t)}{h_0}}^{\epsilon\eta(0,t)} e^{-\delta\lambda y}q_l(y,t)dy - \delta\hat{\delta}e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h(t)}{h_0}}^{\epsilon\eta\left(\frac{L(t)}{L_0},t\right)} e^{-\delta\lambda y}q_r\left(y,t\right)dy$$
$$= i\delta\left[-e^{\lambda\delta\frac{h(t)}{h_0}}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}b(x,t)dx + e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h(t)}{h_0}}^{\epsilon\eta\left(\frac{L(t)}{L_0},t\right)} e^{-\delta\lambda y}r(y,t)dy\right], \tag{76}$$

which is a slight variant of (48).

By making the small amplitude approximation in (76), and neglecting terms of $\mathcal{O}\left(\epsilon^2\right)$, we obtain the equation

$$\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}\left[q_x - i\delta\eta_t - i\epsilon\delta(\eta q_x)_x - \epsilon\delta^2(\eta\eta_t)_x\right]dx - \hat{\epsilon}e^{\lambda\delta\frac{h(t)}{h_0}}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}q_b\left(x,t\right)dx$$
$$+\delta\hat{\delta}\int_{-\frac{h(t)}{h_0}}^{0} e^{-\delta\lambda y}q_l(y,t)dy - \delta\hat{\delta}e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h(t)}{h_0}}^{0} e^{-\delta\lambda y}q_r\left(y,t\right)dy$$
$$-i\delta\left[-e^{\lambda\delta\frac{h(t)}{h_0}}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}b(x,t)dx + e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h(t)}{h_0}}^{0} e^{-\delta\lambda y}r(y,t)dy\right]$$
$$= i\epsilon\delta\left[\eta(0,t)q_x(0,t) - e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)q_x\left(\frac{L(t)}{L_0},t\right)\right]$$
$$+\epsilon\delta^2\eta(0,t)\eta_t(0,t) - \epsilon\delta^2 e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)\eta_t\left(\frac{L(t)}{L_0},t\right)$$
$$-\epsilon\delta\hat{\delta}\eta(0,t)q_l(0,t) + \epsilon\delta\hat{\delta}e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)q_r(0,t)$$
$$+i\epsilon\delta e^{i\lambda\frac{L(t)}{L_0}}\eta\left(\frac{L(t)}{L_0},t\right)r(0,t). \tag{77}$$

Following the discussion below equation (39), consistent asymptotics imply that the RHS of (77) vanishes, hence we obtain the following conditions:

$$iq_x(0,t) + \delta\eta_t(0,t) - \hat{\delta}q_l(0,t) = 0,$$
$$i\left[q_x\left(\frac{L(t)}{L_0},t\right) - r(0,t)\right] + \delta\eta_t\left(\frac{L(t)}{L_0},t\right) - \hat{\delta}q_r(0,t) = 0,$$

which yield conditions (41) and

$$\hat{\delta} = \delta.$$

Consequently, (77) becomes

$$\begin{aligned}
&\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}\left[q_x - i\delta\eta_t - i\epsilon\delta(\eta q_x)_x - \epsilon\delta^2(\eta\eta_t)_x\right]dx - \hat{\epsilon}e^{\lambda\delta\frac{h(t)}{h_0}}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x} q_b\left(x,t\right)dx \\
&+\delta^2\int_{-\frac{h(t)}{h_0}}^0 e^{-\delta\lambda y} q_l(y,t)dy - \delta^2 e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h(t)}{h_0}}^0 e^{-\delta\lambda y} q_r\left(y,t\right)dy \\
&= i\delta\left[-e^{\lambda\delta\frac{h(t)}{h_0}}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x} b(x,t)dx + e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h(t)}{h_0}}^0 e^{-\delta\lambda y} r(y,t)dy\right].
\end{aligned} \tag{78}$$

The derivation that $\hat{\epsilon} = 1$, is more involved: We let $\epsilon \to 0$, so the last two terms of the first integral of the LHS of (78) vanish. Then, by integrating all the terms of (78) by parts, we obtain the equation

$$\begin{aligned}
&-\frac{i}{\lambda}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}\left(q_x - i\delta\eta_t\right)_x dx + \hat{\epsilon}\frac{i}{\lambda}e^{\lambda\delta\frac{h(t)}{h_0}}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x}\left(q_b\right)_x dx \\
&-\frac{\delta}{\lambda}\int_{-\frac{h(t)}{h_0}}^0 e^{-\delta\lambda y}\left(q_l\right)_y dy + \frac{\delta}{\lambda}e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h(t)}{h_0}}^0 e^{-\delta\lambda y}\left(q_r\right)_y dy \\
&+\frac{\delta}{\lambda}e^{\lambda\delta\frac{h(t)}{h_0}}\int_0^{\frac{L(t)}{L_0}} e^{i\lambda x} b_x dx + \frac{i}{\lambda}e^{i\lambda\frac{L(t)}{L_0}}\int_{-\frac{h(t)}{h_0}}^0 e^{-\delta\lambda y} r_y dy \\
=&-\frac{i}{\lambda}e^{i\lambda\frac{L(t)}{L_0}} q_x\left(\frac{L(t)}{L_0},t\right) - \frac{\delta}{\lambda}e^{i\lambda\frac{L(t)}{L_0}}\eta_t\left(\frac{L(t)}{L_0},t\right) + \frac{i}{\lambda}q_x\left(0,t\right) + \frac{\delta}{\lambda}\eta_t\left(0,t\right) \\
&+\hat{\epsilon}\frac{i}{\lambda}e^{\lambda\delta\frac{h(t)}{h_0}}e^{i\lambda\frac{L(t)}{L_0}} q_b\left(\frac{L(t)}{L_0},t\right) - \hat{\epsilon}\frac{i}{\lambda}e^{\lambda\delta\frac{h(t)}{h_0}} q_b\left(0,t\right) \\
&-\frac{\delta}{\lambda}\left[q_l\left(0,t\right) - e^{\delta\lambda\frac{h(t)}{h_0}} q_l\left(-\frac{h(t)}{h_0},t\right)\right] + \frac{\delta}{\lambda}e^{i\lambda\frac{L(t)}{L_0}}\left[q_r\left(0,t\right) - e^{\delta\lambda\frac{h(t)}{h_0}} q_r\left(-\frac{h(t)}{h_0},t\right)\right] \\
&+\frac{\delta}{\lambda}e^{\lambda\delta\frac{h(t)}{h_0}}\left[e^{i\lambda\frac{L(t)}{L_0}} b\left(\frac{L(t)}{L_0},t\right) - b(0,t)\right] \\
&+\frac{i}{\lambda}e^{i\lambda\frac{L(t)}{L_0}} r(0,t) - \frac{i}{\lambda}e^{\lambda\delta\frac{h(t)}{h_0}}e^{i\lambda\frac{L(t)}{L_0}} r\left(-\frac{h(t)}{h_0},t\right).
\end{aligned} \tag{79}$$

Using equations (41) and observing that the boundary conditions (43)-(45) imply the conditions

$$q_b\left(0,t\right) = 0, \qquad q_b\left(\frac{L(t)}{L_0},t\right) = r\left(-\frac{h(t)}{h_0},t\right),$$

$$q_l\left(-\frac{h(t)}{h_0},t\right) = b\left(0,t\right), \qquad q_r\left(-\frac{h(t)}{h_0},t\right) = b\left(\frac{L(t)}{L_0},t\right),$$

the RHS of (79) simplifies to

$$\left(\hat{\epsilon} - 1\right)\frac{i}{\lambda}e^{\lambda\delta\frac{h(t)}{h_0}}e^{i\lambda\frac{L(t)}{L_0}} r\left(-\frac{h(t)}{h_0},t\right).$$

Hence, demanding that the RHS of (79) vanishes, yields $\hat{\epsilon} = 1$.

Remark 5. *Letting in* (78) $\lambda \to -\infty$ *we find that all terms do* *<u>not</u>* *vanish at the same rate. Indeed, the first term is of order* $\mathcal{O}\left(\frac{1}{|\lambda|}\right)$, *whereas all the other terms are of order* $\mathcal{O}\left(e^{\lambda\delta}\right)$. *Integrating by parts the latter terms, give terms that have exponential behaviour, but also terms that do not. The procedure used above shows that balancing the terms which vanish exponentially yields* $\hat{\epsilon} = 1$. *Moreover, in order to balance the terms that vanish at* $\mathcal{O}\left(\frac{1}{|\lambda|}\right)$ *we need the contribution from the term coming from the integration by parts of the first term of* (78).

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