

Truncation dimension for linear problems on multivariate function spaces

**A. Hinrichs, P. Kritzer, F. Pillichshammer,
G. Wasilkowski**

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Truncation Dimension for Linear Problems on Multivariate Function Spaces

Aicke Hinrichs, Peter Kritzer*, Friedrich Pillichshammer†, G.W. Wasilkowski

Abstract

The paper considers linear problems on weighted spaces of high-dimensional functions. The main questions addressed are: When is it possible to approximate the original function of very many variables by the same function; however with all but the first k variables set to zero, so that the corresponding error is small? What is the *truncation dimension*, i.e., the smallest number $k = k(\varepsilon)$ such that the corresponding error is bounded by a given error demand ε ? Surprisingly, $k(\varepsilon)$ could be very small.

1 Introduction

This paper is a continuation of our study initiated in [5, 7] on the truncation dimension for functions with a huge (or even infinite) number of variables. Since our notion of the *truncation dimension* is very different from the one used in the statistical literature, we next provide its definition and then compare it to the statistical one.

Let \mathcal{F} be a normed linear space of s -variate functions defined on D^s . Here s is very large or even infinite. For simplicity, we assume that $0 \in D$ where D is a subinterval of \mathbb{R} (we also allow D to be unbounded). Let

$$\mathcal{S} : \mathcal{F} \rightarrow \mathcal{G}$$

be a continuous linear operator into another normed linear space \mathcal{G} . Consider the problem of approximating $\mathcal{S}(f)$. Since the number s of variables is huge (or even $s = \infty$), it is natural to ask whether approximations of functions $f_k(\mathbf{x}) = f(x_1, \dots, x_k, 0, 0, \dots)$, that depend only on $k \ll s$ variables (here \ll indicates in an informal way that k is “much” smaller than s), are good enough to approximate $\mathcal{S}(f)$. This leads to the following notion of *truncation dimension*.

Suppose that for any $f \in \mathcal{F}$ and any $k \in \mathbb{N} := \{1, 2, 3, \dots\}$ the function f_k obtained from f by setting all the variables x_j with $j > k$ to be zero,

$$f_k(\mathbf{x}) := f(x_1, \dots, x_k, 0, 0, \dots) \quad \text{for } \mathbf{x} = (x_1, x_2, \dots, x_s), \quad (1) \quad \boxed{\text{starstar}}$$

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also belongs to \mathcal{F} . For given $k \in \mathbb{N}$, by the k^{th} truncation error we mean

$$\text{err}^{\text{trnc}}(k) := \sup_{f \in \mathcal{F}} \frac{\|\mathcal{S}(f - f_k)\|_{\mathcal{G}}}{\|f - f_k\|_{\mathcal{F}}}.$$

:truncdim

Definition 1 For a given error demand $\varepsilon > 0$, the ε -truncation dimension for approximating \mathcal{S} (or truncation dimension for short) is defined as

$$\text{dim}^{\text{trnc}}(\varepsilon) := \min \{k : \text{err}^{\text{trnc}}(k) \leq \varepsilon\}.$$

We stress that the truncation dimension is a property of the problem. In particular, it depends on the spaces \mathcal{F}, \mathcal{G} , the operator \mathcal{S} , and the error demand ε . This is in a contrast with the truncation dimension concept in statistical literature (see e.g. [1, 10, 11, 13]), which depends on the particular function under consideration. Moreover, it is defined via ANOVA decomposition which is hard to approximate directly using only function evaluations.

In the rest of the paper we estimate the truncation dimension for a special, yet important, class of γ -weighted spaces \mathcal{F} with *anchored decomposition* and \mathcal{S} having a tensor product form.

Roughly speaking, functions from \mathcal{F} have a decomposition

$$f(\mathbf{x}) = \sum_{\mathbf{u}} f_{\mathbf{u}}(\mathbf{x}),$$

where the sum is with respect to finite subsets $\mathbf{u} \subseteq \{1, \dots, s\}$ (or $\mathbf{u} \subset \mathbb{N}$ if $s = \infty$), and each function $f_{\mathbf{u}}$ depends only on the variables listed in \mathbf{u} . Moreover, each $f_{\mathbf{u}}$ belongs to a normed space $F_{\mathbf{u}}$, and we define the norm in \mathcal{F} by

$$\|f\|_{\mathcal{F}} = \left(\sum_{\mathbf{u}} \gamma_{\mathbf{u}}^{-p} \|f_{\mathbf{u}}\|_{F_{\mathbf{u}}}^p \right)^{1/p} < \infty$$

for some $1 \leq p \leq \infty$ and positive numbers $\gamma_{\mathbf{u}}$ called weights. Of course $\|f\|_{\mathcal{F}} = \sup_{\mathbf{u}} \gamma_{\mathbf{u}}^{-1} \|f_{\mathbf{u}}\|_{F_{\mathbf{u}}}$ when $p = \infty$.

For $s = \infty$, these spaces include spaces of special functions of the form

$$f(\mathbf{x}) = g(\mathbf{X}(t)),$$

where $\mathbf{X}(t)$ is the value of the stochastic process $\mathbf{X}(t) = \sum_{i=1}^{\infty} x_i \phi_i(t)$ at time t . Here the x_i 's are i.i.d. random variables and the base functions $\phi_i(t)$ converge to zero sufficiently fast. Clearly

$$f(\mathbf{x}) = g\left(\sum_{i=1}^{\infty} x_i \phi_i(t)\right) \quad \text{whereas} \quad f_k(\mathbf{x}) = g\left(\sum_{i=1}^k x_i \phi_i(t)\right).$$

Moreover, the weights $\gamma_{\mathbf{u}}$ can be written as product weights of the form

$$\gamma_{\mathbf{u}} = \prod_{j \in \mathbf{u}} \phi_j^{\beta}(t)$$

for some $\beta \leq 1$. We prove that then (cf. Theorem 2) the truncation dimension can be bounded from above by the smallest integer k such that

$$\left(\prod_{j=1}^{\infty} (1 + (C_1 \phi_j^\beta(t))^{p^*}) \left(1 - \exp \left(-C_1^{p^*} \sum_{j=k+1}^{\infty} (\phi_j^\beta(t))^{p^*} \right) \right) \right)^{1/p^*} \leq \varepsilon.$$

Here and elsewhere, p^* is the conjugate of p (i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$), $C_1 > 0$ is a number such that $\|S_1\| \leq C_1$ and $\|S_1\|$ is the norm of the operator \mathcal{S} restricted to the space $F_{\{1\}}$ of functions depending only on one variable. (One has to apply the usual adaptations if $p = 1$, cf. Theorem 2 again.) From this result it can be seen that faster decay of the $\phi_j^\beta(t)$ leads to smaller truncation dimension.

We illustrate this for different values of ε , p , and γ_u . We use product weights of the form $\gamma_u = \prod_{j \in u} j^{-\alpha}$ for $\alpha \in \{2, 3, 4, 5\}$. For simplicity, we assume that $C_1 = 1$. For $p = 2$ we have:

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	
$\dim^{\text{trnc}}(\varepsilon)$	4	19	90	416	1933	8973	$\alpha = 2$
$\dim^{\text{trnc}}(\varepsilon)$	2	5	13	33	84	210	$\alpha = 3$
$\dim^{\text{trnc}}(\varepsilon)$	2	3	6	12	22	43	$\alpha = 4$
$\dim^{\text{trnc}}(\varepsilon)$	1	2	4	7	11	18	$\alpha = 5$

In particular, for the error demand $\varepsilon = 10^{-3}$ it is enough to work with only 90 variables when $\alpha = 2$, only 13 variables when $\alpha = 3$, and with 6 or 4 when $\alpha = 4$ or 5, respectively.

For $p = 1$ we have $\dim^{\text{trnc}}(\varepsilon) = \lceil \varepsilon^{-1/\alpha} - 1 \rceil$, which leads to even better results. The following table already appeared in [7]:

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	
$\dim^{\text{trnc}}(\varepsilon)$	3	9	31	99	316	999	$\alpha = 2$
$\dim^{\text{trnc}}(\varepsilon)$	2	4	9	21	46	99	$\alpha = 3$
$\dim^{\text{trnc}}(\varepsilon)$	1	3	5	9	17	31	$\alpha = 4$
$\dim^{\text{trnc}}(\varepsilon)$	1	2	3	6	9	15	$\alpha = 5$

The content of the paper is as follows. In Section 2, we provide basic definitions and the main result. In Section 3, we propose spaces that are generalizations of anchored Sobolev spaces with bounded mixed derivatives of order one, that have been considered extensively in the literature. We next apply the general results to these special spaces. In Section 4, we study some unanchored spaces and show when they are equivalent to their anchored counterparts. Note that the equivalence implies that algorithms with small errors for anchored spaces also have small errors for the corresponding unanchored spaces. The results in Section 4 are simple extensions of results in [2, 3, 4, 6].

2 Weighted Anchored Spaces of Multivariate Functions

We begin by introducing the notation used throughout the paper. For $s \in \mathbb{N}$ and

$$[s] := \{1, 2, \dots, s\},$$

we will use \mathbf{u}, \mathbf{v} to denote subsets of $[s]$, i.e., $\mathbf{u}, \mathbf{v} \subseteq [s]$. Some of the results hold for functions with infinitely many variables; then $s = \infty$, $[s] = \mathbb{N}$, and \mathbf{u}, \mathbf{v} denote finite subsets of \mathbb{N} .

We assume that the functions $f \in \mathcal{F}$ have the following decomposition

$$f = \sum_{\mathbf{u} \subseteq [s]} f_{\mathbf{u}},$$

where each $f_{\mathbf{u}}$ belongs to a normed linear space $F_{\mathbf{u}}$ such that $F_{\mathbf{u}} \cap F_{\mathbf{v}} = \{0\}$ if $\mathbf{u} \neq \mathbf{v}$, $f_{\mathbf{u}}$ depends only on $\mathbf{x}_{\mathbf{u}} = (x_j)_{j \in \mathbf{u}}$, and

$$f_{\mathbf{u}}(\mathbf{x}) = 0 \quad \text{if } x_j = 0 \text{ for } j \in \mathbf{u}. \quad (2) \quad \boxed{\text{star}}$$

For $\mathbf{u} = \emptyset$, F_{\emptyset} is the space of constant functions with the absolute value as its norm. Clearly, the property (2) yields that for any $f \in \mathcal{F}$ and $k \in [s]$,

$$f_k(\mathbf{x}) = \sum_{\mathbf{u} \subseteq [k]} f_{\mathbf{u}}(\mathbf{x}), \quad (3) \quad \boxed{\text{dec-tr}}$$

where f_k is defined in (1).

We assume that for given positive weights $\gamma = (\gamma_{\mathbf{u}})_{\mathbf{u} \subseteq [s]}$, the norm in \mathcal{F} is given by

$$\|f\|_{\mathcal{F}} = \left(\sum_{\mathbf{u} \subseteq [s]} \gamma_{\mathbf{u}}^{-p} \|f_{\mathbf{u}}\|_{F_{\mathbf{u}}}^p \right)^{1/p}.$$

Let $S_{\mathbf{u}}$ be \mathcal{S} restricted to $F_{\mathbf{u}}$, and let $\|S_{\mathbf{u}}\|$ be its operator norm,

$$\|S_{\mathbf{u}}\| := \sup_{f_{\mathbf{u}} \in F_{\mathbf{u}}} \frac{\|S_{\mathbf{u}}(f_{\mathbf{u}})\|_{\mathcal{G}}}{\|f_{\mathbf{u}}\|_{F_{\mathbf{u}}}}.$$

We have the following simple proposition.

op:simple

Proposition 1 *For every $k \leq s$ we have*

$$\text{err}^{\text{trnc}}(k) \leq \left(\sum_{\mathbf{u} \not\subseteq [k]} \|S_{\mathbf{u}}\|^{p^*} \gamma_{\mathbf{u}}^{p^*} \right)^{1/p^*},$$

where here and throughout this paper $\sum_{\mathbf{u} \not\subseteq [k]}$ means summation over all $\mathbf{u} \subseteq [s]$ with $\mathbf{u} \not\subseteq [k]$. In particular,

$$\dim^{\text{trnc}}(\varepsilon) \leq \min \left\{ k : \left(\sum_{\mathbf{u} \not\subseteq [k]} \|S_{\mathbf{u}}\|^{p^*} \gamma_{\mathbf{u}}^{p^*} \right)^{1/p^*} \leq \varepsilon \right\}.$$

Of course, for $p^* = \infty$, we have

$$\text{err}^{\text{trnc}}(k) \leq \sup_{\mathbf{u} \not\subseteq [k]} \|S_{\mathbf{u}}\| \gamma_{\mathbf{u}} \quad \text{and} \quad \dim^{\text{trnc}}(\varepsilon) \leq \min \left\{ k : \sup_{\mathbf{u} \not\subseteq [k]} \|S_{\mathbf{u}}\| \gamma_{\mathbf{u}} \leq \varepsilon \right\}.$$

Proof. We have

$$\begin{aligned} \|\mathcal{S}(f) - \mathcal{S}(f_k)\|_{\mathcal{G}} &= \left\| \sum_{\mathbf{u} \subseteq [k]} S_{\mathbf{u}}(f_{\mathbf{u}}) \right\|_{\mathcal{G}} \leq \sum_{\mathbf{u} \subseteq [k]} \gamma_{\mathbf{u}} \|S_{\mathbf{u}}\| \gamma_{\mathbf{u}}^{-1} \|f_{\mathbf{u}}\|_{F_{\mathbf{u}}} \\ &\leq \left(\sum_{\mathbf{u} \subseteq [k]} \|S_{\mathbf{u}}\|^{p^*} \gamma_{\mathbf{u}}^{p^*} \right)^{1/p^*} \|f - f_k\|_{\mathcal{F}}, \end{aligned}$$

where, in the last step, we used $\|f - f_k\|_{\mathcal{F}} = (\sum_{\mathbf{u} \subseteq [k]} \gamma_{\mathbf{u}}^{-p} \|f_{\mathbf{u}}\|_{F_{\mathbf{u}}}^p)^{1/p}$ together with Hölder's inequality. From this the result follows. \square

In this paper we mainly concentrate on *product weights*, introduced in [12], that have the form

$$\gamma_{\mathbf{u}} = \prod_{j \in \mathbf{u}} \gamma_j$$

for a decreasing sequence of positive numbers γ_j for $j \in \mathbb{N}$.

hm:simple

Theorem 2 *Suppose that the weights γ have product and that there exists a constant C_1 such that*

$$\|S_{\mathbf{u}}\| \leq C_1^{|\mathbf{u}|} \quad \text{for all } \mathbf{u}. \quad (4) \quad \text{ass:tnspr}$$

For $p > 1$ and every $k \leq s$ we have

$$\text{err}^{\text{trnc}}(k) \leq \left(\prod_{j=1}^s (1 + (C_1 \gamma_j)^{p^*}) \left(1 - e^{-C_1^{p^*} \sum_{j=k+1}^s \gamma_j^{p^*}} \right) \right)^{1/p^*}.$$

In particular, $\dim^{\text{trnc}}(\varepsilon)$ is bounded from above by

$$\min \left\{ k : \prod_{j=1}^s (1 + (C_1 \gamma_j)^{p^*})^{1/p^*} \left(1 - e^{-C_1^{p^*} \sum_{j=k+1}^s \gamma_j^{p^*}} \right)^{1/p^*} \leq \varepsilon \right\}.$$

For $p = 1$ and every $k \leq s$ we have

$$\text{err}^{\text{trnc}}(k) \leq \max_{\mathbf{u} \subseteq [k]} C_1^{|\mathbf{u}|} \gamma_{\mathbf{u}}$$

and if additionally $C_1 \leq 1$ then $\text{err}^{\text{trnc}}(k) \leq C_1 \gamma_{k+1}$.

Proof. We have

$$\begin{aligned} \sum_{\mathbf{u} \subseteq [k]} \|S_{\mathbf{u}}\|^{p^*} \gamma_{\mathbf{u}}^{p^*} &\leq \sum_{\mathbf{u} \subseteq [k]} C_1^{|\mathbf{u}| p^*} \\ &= \sum_{\mathbf{u} \subseteq [s]} \prod_{j \in \mathbf{u}} (C_1 \gamma_j)^{p^*} - \sum_{\mathbf{u} \subseteq [k]} \prod_{j \in \mathbf{u}} (C_1 \gamma_j)^{p^*} \\ &= \prod_{j=1}^s (1 + (C_1 \gamma_j)^{p^*}) - \prod_{j=1}^k (1 + (C_1 \gamma_j)^{p^*}) \end{aligned}$$

$$\begin{aligned}
&= \prod_{j=1}^s (1 + (C_1 \gamma_j)^{p^*}) \left(1 - e^{-\sum_{j=k+1}^s \log(1 + (C_1 \gamma_j)^{p^*})} \right) \\
&\leq \prod_{j=1}^s (1 + (C_1 \gamma_j)^{p^*}) \left(1 - e^{-C_1^{p^*} \sum_{j=k+1}^s \gamma_j^{p^*}} \right),
\end{aligned}$$

since $\log(1+x) \leq x$ for all $x > -1$. From this the result follows. \square

Remark 1 It is well known that (4) holds if F_1 is a Hilbert space and, for every $\mathbf{u} \neq \emptyset$, the spaces $F_{\mathbf{u}}$ are $|\mathbf{u}|$ -fold tensor products of F_1 and also $S_{\mathbf{u}}$ are $|\mathbf{u}|$ -fold tensor products of S_1 , i.e.,

$$S_{\mathbf{u}} \left(\bigotimes_{j \in \mathbf{u}} f_j \right) = \bigotimes_{j \in \mathbf{u}} S_1(f_j).$$

Actually then we have

$$\|S_{\mathbf{u}}\| = C_1^{|\mathbf{u}|} \quad \text{and} \quad \|S_1\| = C_1.$$

However, (4) also holds with inequality for Banach spaces $F_{\mathbf{u}}$ and operators $S_{\mathbf{u}}$ that we consider in the next sections.

We introduce some further notation. For $\mathbf{x} = (x_1, x_2, \dots, x_s) \in D^s$ and $\mathbf{u} \subseteq [s]$, $[\mathbf{x}_{\mathbf{u}}; \mathbf{0}_{-\mathbf{u}}]$ denotes the s -dimensional vector with all x_j for $j \notin \mathbf{u}$ replaced by zero, i.e.,

$$[\mathbf{x}_{\mathbf{u}}; \mathbf{0}_{-\mathbf{u}}] = (y_1, y_2, \dots, y_s) \quad \text{with} \quad y_j = \begin{cases} x_j & \text{if } j \in \mathbf{u}, \\ 0 & \text{if } j \notin \mathbf{u}. \end{cases}$$

As shown in [5] for the integration problem, the importance of the ε -truncation dimension lies in the fact that when approximating $\mathcal{S}(f)$ for functions $f \in \mathcal{F}$ it is sufficient to approximate \mathcal{S} only for k -variate functions

$$f_k(\mathbf{x}) = f(x_1, \dots, x_k, 0, 0, \dots) = f([\mathbf{x}_{[k]}; \mathbf{0}_{-[k]}])$$

with $k \geq \dim^{\text{trnc}}(\varepsilon)$ since $f - f_k = \sum_{\mathbf{u} \not\subseteq [k]} f_{\mathbf{u}}$ and, therefore,

$$\|\mathcal{S}(f) - \mathcal{S}(f_k)\|_{\mathcal{G}} \leq \varepsilon \left\| \sum_{\mathbf{u} \not\subseteq [k]} f_{\mathbf{u}} \right\|_{\mathcal{F}}.$$

For $k \leq s$, let

$$\mathcal{F}_k = \bigoplus_{\mathbf{u} \subseteq [k]} F_{\mathbf{u}}$$

be the subspace of \mathcal{F} consisting of k -variate functions $f([\cdot]_{[k]}; \mathbf{0}_{-[k]})$, and let $A_{k,n}$ be an algorithm for approximating $\mathcal{S}_{[k]}(f)$ for functions from \mathcal{F}_k that uses n function values. The worst case error of $A_{k,n}$ with respect to the space \mathcal{F}_k is

$$e(A_{k,n}; \mathcal{F}_k) := \sup_{f \in \mathcal{F}_k} \frac{\|\mathcal{S}_{[k]}(f) - A_{k,n}(f)\|_{\mathcal{G}}}{\|f\|_{\mathcal{F}_k}}.$$

Now let

$$\mathcal{A}_{s,k,n}^{\text{trnc}}(f) = A_{k,n}(f([\cdot]_{[k]}; \mathbf{0}_{-[k]}))$$

be an algorithm for approximating functions from the whole space \mathcal{F} . The worst case error of $\mathcal{A}_{s,k,n}^{\text{trnc}}$ is defined as

$$e(\mathcal{A}_{s,k,n}^{\text{trnc}}; \mathcal{F}) := \sup_{f \in \mathcal{F}} \frac{\|\mathcal{S}(f) - \mathcal{A}_{s,k,n}^{\text{trnc}}(f)\|_{\mathcal{G}}}{\|f\|_{\mathcal{F}}}.$$

This yields the following theorem.

thm:main **Theorem 3** For given $\varepsilon > 0$ and $k \geq \dim^{\text{trnc}}(\varepsilon)$ we have

$$e(\mathcal{A}_{s,k,n}^{\text{trnc}}; \mathcal{F}) \leq \left(\varepsilon^{p^*} + e(A_{k,n}; \mathcal{F}_k)^{p^*} \right)^{1/p^*}.$$

Proof. The spaces \mathcal{F}_k are subspaces of \mathcal{F} . Moreover any $f_k = f([\cdot]_{[k]}; \mathbf{0}_{-[k]})$ belongs to \mathcal{F}_k and

$$\|f_k\|_{\mathcal{F}_k} = \|f_k\|_{\mathcal{F}}.$$

Therefore, for any $f \in \mathcal{F}$ we have

$$\begin{aligned} \|\mathcal{S}(f) - \mathcal{A}_{s,k,n}^{\text{trnc}}(f)\|_{\mathcal{G}} &\leq \|\mathcal{S}(f_k) - A_{k,n}(f_k)\|_{\mathcal{G}} + \|\mathcal{S}(f) - \mathcal{S}(f_k)\|_{\mathcal{G}} \\ &\leq e(A_{k,n}; \mathcal{F}_k) \left\| \sum_{\mathbf{u} \subseteq [k]} f_{\mathbf{u}} \right\|_{\mathcal{F}} + \varepsilon \left\| \sum_{\mathbf{u} \not\subseteq [k]} f_{\mathbf{u}} \right\|_{\mathcal{F}} \\ &\leq \left(\varepsilon^{p^*} + e(A_{k,n}; \mathcal{F}_k)^{p^*} \right)^{1/p^*} \|f\|_{\mathcal{F}}, \end{aligned}$$

with the last inequality due to Hölder's inequality. \square

In the following section we consider anchored Sobolev spaces of multivariate functions and show that the assumptions above are justified. As examples for the linear approximation problem we consider function approximation and integration.

3 Anchored Spaces of Multivariate Functions

defspace In this section, we begin by recalling the definitions and basic properties of weighted anchored Sobolev spaces of s -variate functions with mixed partial derivatives of order one bounded in L_p -norm. More detailed information can be found in [3, 4, 14]. Such spaces have often been assumed in the context of *quasi-Monte Carlo methods*. However, for us they serve as a motivation to consider more general classes of anchored spaces.

3.1 Anchored Sobolev Spaces

sec1.1 Here we follow [3]. We use the notations $[s]$, and $\mathbf{u}, \mathbf{v} \subseteq [s]$ as above. We also write $\mathbf{x}_{\mathbf{u}}$ to denote the $|\mathbf{u}|$ -dimensional vector $(x_j)_{j \in \mathbf{u}}$ and

$$f^{(\mathbf{u})} = \frac{\partial^{|\mathbf{u}|} f}{\partial \mathbf{x}_{\mathbf{u}}} = \prod_{j \in \mathbf{u}} \frac{\partial}{\partial x_j} f \quad \text{with} \quad f^{(\emptyset)} = f.$$

For a family of weights $\gamma = (\gamma_{\mathbf{u}})_{\mathbf{u} \subseteq [s]}$, which are non-negative numbers, and for $p \in [1, \infty]$ the corresponding γ -weighted *anchored* space $\mathcal{F}_{s,p,\gamma}$ is the Banach space of functions defined on $D^s = [0, 1]^s$ with the norm

$$\|f\|_{\mathcal{F}_{s,p,\gamma}} = \left(\sum_{\mathbf{u} \subseteq [s]} \gamma_{\mathbf{u}}^{-p} \|f^{(\mathbf{u})}([\cdot; \mathbf{0}_{-\mathbf{u}}])\|_{L_p([0,1]^{|\mathbf{u}|})}^p \right)^{1/p}.$$

For $p = \infty$, the norm reduces to

$$\|f\|_{\mathcal{F}_{s,p,\gamma}} = \sup_{\mathbf{u} \subseteq [s]} \gamma_{\mathbf{u}}^{-1} \|f^{(\mathbf{u})}([\cdot; \mathbf{0}_{-\mathbf{u}}])\|_{L_{\infty}([0,1]^{|\mathbf{u}|})}.$$

As shown in [3] the functions from $\mathcal{F}_{s,p,\gamma}$ have the unique decomposition

$$f = \sum_{\mathbf{u} \subseteq [s]} f_{\mathbf{u}},$$

where each $f_{\mathbf{u}}$, although formally a function of \mathbf{x} , depends only on the variables $\mathbf{x}_{\mathbf{u}}$, and is an element of a space $F_{\mathbf{u}}$ given by

$$F_{\mathbf{u}} = K_{\mathbf{u}}(L_p([0, 1]^{|\mathbf{u}|})).$$

Here, for $\mathbf{u} \neq \emptyset$,

$$f_{\mathbf{u}}(\mathbf{x}) = K_{\mathbf{u}}(g_{\mathbf{u}})(\mathbf{x}) = \int_{[0,1]^{|\mathbf{u}|}} g_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) \prod_{j \in \mathbf{u}} (x_j - t_j)_+^0 d\mathbf{t}_{\mathbf{u}},$$

where $(x - t)_+^0 = 1$ if $t < x$ and $(x - t)_+^0 = 0$ otherwise, and

$$\|f_{\mathbf{u}}\|_{F_{\mathbf{u}}} = \|g_{\mathbf{u}}\|_{L_p([0,1]^{|\mathbf{u}|})}.$$

For $\mathbf{u} = \emptyset$, $F_{\mathbf{u}}$ is the space of constant functions with the absolute value as its norm.

An important property of these spaces is that they are anchored at 0, i.e., for any $\mathbf{u} \neq \emptyset$ and any $f_{\mathbf{u}} \in F_{\mathbf{u}}$,

$$f_{\mathbf{u}}(\mathbf{x}) = 0 \quad \text{if } x_j = 0 \text{ for some } j \in \mathbf{u}.$$

This implies that

$$f^{(\mathbf{u})}([\mathbf{x}_{\mathbf{u}}; \mathbf{0}_{-\mathbf{u}}]) = f_{\mathbf{u}}^{(\mathbf{u})} \quad \text{and} \quad \|f\|_{\mathcal{F}_{s,p,\gamma}} = \left(\sum_{\mathbf{u} \subseteq [s]} \gamma_{\mathbf{u}}^{-p} \|f_{\mathbf{u}}\|_{F_{\mathbf{u}}}^p \right)^{1/p}.$$

3.2 More General Anchored Spaces

In this section we extend the definition of $\mathcal{F}_{s,p,\gamma}$ from the previous section to spaces of functions

$$f = \sum_{\mathbf{u} \subseteq [s]} f_{\mathbf{u}}$$

with the components $f_{\mathbf{u}}$ given by

$$f_{\mathbf{u}} = K_{\mathbf{u}}(g_{\mathbf{u}}) = \int_{D^{|\mathbf{u}|}} g_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) \kappa_{\mathbf{u}}(\cdot, \mathbf{t}_{\mathbf{u}}) d\mathbf{t}_{\mathbf{u}} \quad \text{with} \quad \kappa_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) = \prod_{j \in \mathbf{u}} \kappa(x_j, t_j),$$

where $\kappa(x, t)$ could be more general than $(x - t)_+^0$, and $g_{\mathbf{u}}$ could be from a more general ψ -weighted L_p space.

More specifically, let D be an interval in \mathbb{R} that, without any loss of generality, contains 0. This includes both bounded intervals like $D = [0, 1]$ from the previous subsection, as well as unbounded ones, e.g., $D = [0, \infty)$ or $D = \mathbb{R}$.

Let

$$\psi : D \rightarrow \mathbb{R}_+$$

be a measurable and (a.e.) positive *weight* function. For $p_1 \in [1, \infty]$, by $L_{p_1, \psi} = L_{p_1, \psi}(D)$ we denote the space of scalar functions with the norm

$$\|g\|_{L_{p_1, \psi}} = \left(\int_D |g(t)|^{p_1} \psi(t) dt \right)^{1/p_1}.$$

For non-empty \mathbf{u} , $L_{\mathbf{u}, p_1, \psi} = L_{\mathbf{u}, p_1, \psi}(D^{|\mathbf{u}|})$ is the space of $|\mathbf{u}|$ -variate functions with the norm given by

$$\|g_{\mathbf{u}}\|_{L_{\mathbf{u}, p_1, \psi}} = \left(\int_{D^{|\mathbf{u}|}} |g_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}})|^{p_1} \psi_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) d\mathbf{t}_{\mathbf{u}} \right)^{1/p_1} \quad \text{with} \quad \psi_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) = \prod_{j \in \mathbf{u}} \psi(t_j).$$

Let

$$\kappa : D \times D \rightarrow \mathbb{R}$$

be a given measurable function. For non-empty \mathbf{u} , define

$$\kappa_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) := \prod_{j \in \mathbf{u}} \kappa(x_j, t_j)$$

and

$$K_{\mathbf{u}}(g_{\mathbf{u}})(\mathbf{x}_{\mathbf{u}}) := \int_{D^{|\mathbf{u}|}} g_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) \kappa_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) d\mathbf{t}_{\mathbf{u}} \quad \text{for} \quad g_{\mathbf{u}} \in L_{\mathbf{u}, p_1, \psi}.$$

We assume that

$$\widehat{\kappa}_{p_1}(x) := \left(\int_D \left(\frac{|\kappa(x, t)|}{\psi^{1/p_1}(t)} \right)^{p_1^*} dt \right)^{1/p_1^*} < \infty \quad \text{for all } x \in D. \quad (5) \quad \boxed{\text{ass: def}}$$

Of course, for $p_1 = 1$, $\widehat{\kappa}_1(x) = \text{ess sup}_{t \in D} |\kappa(x, t)|/\psi(t) < \infty$ for all $x \in D$. Then

$$f_{\mathbf{u}}(\mathbf{x}) = K_{\mathbf{u}}(g_{\mathbf{u}})(\mathbf{x}) \quad (g \in L_{\mathbf{u}, p_1, \psi})$$

are well defined functions since

$$|f_{\mathbf{u}}(\mathbf{x})| \leq \|g_{\mathbf{u}}\|_{L_{\mathbf{u}, p_1, \psi}} \widehat{\kappa}_{\mathbf{u}, p_1}(\mathbf{x}), \quad \text{where} \quad \widehat{\kappa}_{\mathbf{u}, p_1}(\mathbf{x}) := \prod_{j \in \mathbf{u}} \widehat{\kappa}_{p_1}(x_j).$$

We also assume that $K_{\mathbf{u}}$ is an injective operator, i.e.,

$$K_{\mathbf{u}}(g_{\mathbf{u}}) \equiv 0 \quad \text{implies} \quad g_{\mathbf{u}} = 0 \quad \text{a.e.} \quad (6) \quad \boxed{\text{ass: 1-1}}$$

We define the following Banach spaces

$$F_{\mathbf{u}} = K_{\mathbf{u}}(L_{\mathbf{u},p_1,\psi}(D^{|\mathbf{u}|})) \quad \text{with} \quad \|K_{\mathbf{u}}(g_{\mathbf{u}})\|_{F_{\mathbf{u}}} := \|g_{\mathbf{u}}\|_{L_{\mathbf{u},p_1,\psi}}.$$

We assume also that

$$\kappa(0, \cdot) \equiv 0. \quad (7) \quad \boxed{\text{ass:anchor}}$$

Then the spaces $F_{\mathbf{u}}$ are anchored at zero since for every $f_{\mathbf{u}} \in F_{\mathbf{u}}$ we have

$$f_{\mathbf{u}}(\mathbf{x}) = 0 \quad \text{if } x_j = 0 \text{ for some } j \in \mathbf{u}.$$

As in the previous section, F_{\emptyset} is the space of constant functions.

Finally, for $p_2 \in [1, \infty]$, consider the Banach space $\mathcal{F}_{s,p_1,p_2,\gamma}$

$$\mathcal{F}_{s,p_1,p_2,\gamma} = \bigoplus_{\mathbf{u} \subseteq [s]} F_{\mathbf{u}} \quad \text{of functions} \quad f = \sum_{\mathbf{u} \subseteq [s]} f_{\mathbf{u}}, \quad \text{where } f_{\mathbf{u}} \in F_{\mathbf{u}},$$

with the norm given by

$$\|f\|_{\mathcal{F}_{s,p_1,p_2,\gamma}} := \left(\sum_{\mathbf{u} \subseteq [s]} \gamma_{\mathbf{u}}^{-p_2} \|f_{\mathbf{u}}\|_{F_{\mathbf{u}}}^{p_2} \right)^{1/p_2}. \quad (8) \quad \boxed{\text{def:norm}}$$

rem1

Remark 2 For functions with infinitely many variables, $\mathcal{F}_{\infty,p_1,p_2,\gamma}$ is the completion of $\bigcup_{\mathbf{u}} F_{\mathbf{u}}$ with respect to the norm (8). In general, it is a space of sequences $f = (f_{\mathbf{u}})_{\mathbf{u}}$ since $\sum_{\mathbf{u}} f_{\mathbf{u}}(\mathbf{x})$ may not exist when \mathbf{x} has infinitely many non-zero x_j 's. Of course, it exists for $\mathbf{x} = [\mathbf{x}_{\mathbf{u}}; \mathbf{0}_{-\mathbf{u}}]$. However, $\mathcal{F}_{\infty,p_1,p_2,\gamma}$ is a function space if

$$\left(\sum_{\mathbf{u}} (\gamma_{\mathbf{u}} \widehat{\kappa}_{\mathbf{u},p_1}(\mathbf{x}))^{p_2^*} \right)^{1/p_2^*} < \infty \quad \text{for all } \mathbf{x} \in D^{\mathbb{N}} \quad (9) \quad \boxed{\text{ass:inft}}$$

since then

$$\left| \sum_{\mathbf{u}} f_{\mathbf{u}}(\mathbf{x}) \right| \leq \|f\|_{\mathcal{F}_{\infty,p_1,p_2,\gamma}} \left(\sum_{\mathbf{u}} (\gamma_{\mathbf{u}} \widehat{\kappa}_{\mathbf{u},p_1}(\mathbf{x}))^{p_2^*} \right)^{1/p_2^*} < \infty$$

are well defined.

We end this section with the following examples.

exmp1

Example 1 As in Section 3.1, $D = [0, 1]$, $\psi \equiv 1$ and $\kappa(x, t) = (x - t)_{+}^0$. Then the assumptions (5)–(7) are satisfied and

$$\widehat{\kappa}_{p_1}(x) = x^{1/p_1^*}.$$

Moreover, for product weights and $s = \infty$, (9) holds iff $\sum_{j=1}^{\infty} \gamma_j^{p_2^*} < \infty$ (or $\sup_{\mathbf{u}} \gamma_{\mathbf{u}} < \infty$ if $p_2^* = \infty$) since

$$\left(\sum_{\mathbf{u}} (\gamma_{\mathbf{u}} \widehat{\kappa}_{\mathbf{u},p_1}(\mathbf{x}))^{p_2^*} \right)^{1/p_2^*} = \prod_{j=1}^{\infty} \left(1 + \gamma_j^{p_2^*} x_j^{p_2^*/p_1^*} \right)^{1/p_2^*}.$$

exmp3

Example 2 Let $D = [0, \infty)$, $\kappa(x, t) = (x - t)_+^{r-1}/(r-1)!$ for $r \geq 1$, and $\psi(t) = e^{\lambda t}$ for given $\lambda \in \mathbb{R}$. Recall that $(x - t)_+^{r-1} = (\max(0, x - t))^{r-1}$. For $s = \infty$, the space $\mathcal{F}_{\infty, p_1, p_2, \gamma}$ is a sequence space. Hence we consider here only finite s .

For $p_1 = 1$, $\widehat{\kappa}_1(x) (r-1)! = \max_{t \in [0, x]} (x-t)^{r-1} e^{-\lambda t}$. For $\lambda \geq 0$ or $x \leq (r-1)/|\lambda|$, the maximum above is attained at $t = 0$. Otherwise, it is attained at $t = x - (r-1)/|\lambda|$. Hence

$$\widehat{\kappa}_1(x) = \begin{cases} \frac{x^{r-1}}{(r-1)!} & \text{if } \lambda \geq 0 \text{ or } x \leq \frac{r-1}{|\lambda|}, \\ \frac{(r-1)^{r-1} e^{|\lambda|x-(r-1)}}{|\lambda|^{r-1} (r-1)!} & \text{if } \lambda < 0 \text{ and } x > \frac{r-1}{|\lambda|}. \end{cases}$$

For $p_1 > 1$, $p_1^* < \infty$ and $p_1^*/p_1 = p_1^* - 1$. Hence

$$\widehat{\kappa}_{p_1}(x) = \frac{1}{(r-1)!} \left(\int_0^x (x-t)^{(r-1)p_1^*} e^{-\lambda t(p_1^*-1)} dt \right)^{1/p_1^*}.$$

If $\lambda \geq 0$, then

$$\widehat{\kappa}_{p_1}(x) \leq \frac{x^{r-1/p_1}}{(r-1)! ((r-1)p_1^* + 1)^{1/p_1^*}}.$$

If $\lambda < 0$, then

$$\widehat{\kappa}_{p_1}(x) (r-1)! \leq \left(x^{(r-1)p_1^*} \frac{e^{|\lambda|x(p_1^*-1)}}{|\lambda|(p_1^*-1)} \right)^{1/p_1^*}$$

and

$$\widehat{\kappa}_{p_1}(x) (r-1)! \leq \left(e^{|\lambda|x(p_1^*-1)} \frac{x^{(r-1)p_1^*+1}}{(r-1)p_1^*+1} \right)^{1/p_1^*}.$$

Hence, for $\lambda < 0$,

$$\widehat{\kappa}_{p_1}(x) \leq e^{|\lambda|x/p_1} \frac{x^{r-1}}{(r-1)!} \left(\min \left(\frac{x}{(r-1)p_1^*+1}, \frac{1}{|\lambda|(p_1^*-1)} \right) \right)^{1/p_1^*}.$$

For this example, $f(0) = f'(0) = \dots = f^{(r-1)}(0) = 0$. Our result also holds for functions of the form $f(x) = \sum_{j=1}^{r-1} a_j x^j/j! + K_1(g)$ with the norm changed to $\|f\| = (\sum_{j=1}^{r-1} |f^{(j)}(0)|^{p_2} + \|g\|_{L_{p_1, \psi}}^{p_2})^{1/p_2}$.

For $r = 1$ and $p_1 > 1$, one can get exact values

$$\widehat{\kappa}_{p_1}(x) = \begin{cases} x^{1/p^*} & \text{if } \lambda = 0, \\ \left(\frac{p_1 - 1}{\lambda} \right)^{1/p_1^*} \left(1 - e^{-\lambda x/(p_1-1)} \right)^{1/p_1^*} & \text{if } \lambda > 0, \\ \left(\frac{p_1 - 1}{\lambda} \right)^{1/p_1^*} \left(e^{|\lambda|x/(p_1-1)} - 1 \right)^{1/p_1^*} & \text{if } \lambda < 0. \end{cases}$$

exam5

Example 3 Consider $D = [0, \infty)$ and

$$\kappa(x, t) = G(xt)$$

for a smooth function G with $G(0) = 0$. Then the functions

$$f(x) = \int_0^\infty \kappa(x, t) g(t) dt = \int_0^\infty G(xt) g(t) dt$$

with $g \in L_{p_1, \psi}$ have all derivatives continuous given by

$$f^{(n)}(x) = \int_0^\infty G^{(n)}(x t) t^n g(t) dt$$

provided that

$$\left(\int_0^\infty \left| \frac{t^n}{\psi^{1/p_1}(t)} \right|^{p_1^*} dt \right)^{1/p_1^*} < \infty \quad \text{for all } n, \quad (10) \quad \boxed{\text{condxy}}$$

and $\|G^{(n)}\|_{L_\infty} < \infty$, e.g. for $G(y) = 1 - e^{-y}$ or $G(y) = 1 - \cos(y)$.

Indeed, consider first $n = 1$. Then

$$\begin{aligned} f'(x) &= \lim_{n \rightarrow \infty} \int_0^\infty g(t) \frac{G((x + 1/n)t) - G(xt)}{1/n} dt \\ &= \lim_{n \rightarrow \infty} \int_{0+}^\infty g(t) t \frac{G(xt + t/n) - G(xt)}{t/n} dt \\ &= \int_0^\infty g(t) t G'(xt) dt. \end{aligned}$$

The last equality holds due to the dominated convergence theorem because

$$\lim_{n \rightarrow \infty} g(t) t \frac{G(xt + t/n) - G(xt)}{t/n} = g(t) t G'(xt),$$

$$\left| g(t) t \frac{G(xt + t/n) - G(xt)}{t/n} \right| \leq |g(t) t| \|G'\|_{L_\infty} \quad \text{due to mean value theorem,}$$

and $|g(t) t|$ is integrable, since by Hölder's inequality and (10)

$$\int_0^\infty |g(t) t| dt \leq \|g\|_{L_{p_1, \psi}} \left(\int_0^\infty \left| \frac{t}{\psi^{1/p_1}(t)} \right|^{p_1^*} dt \right)^{1/p_1^*} < \infty.$$

The proof for an arbitrary n is by induction. Since the inductive step is very similar to the basic one for $n = 1$, we omit it.

Assume additionally that $G^{(n)}(0) \neq 0$ for all $n = 1, 2, \dots$ and that there exists x such that

$$\int_0^\infty G(xt) dt \neq 0. \quad (11) \quad \boxed{\text{ass:hateit}}$$

Then (6) holds. Indeed, consider g such that

$$f = \int_0^\infty G(\cdot t) g(t) dt \equiv 0.$$

Then

$$0 = f^{(n)}(0) = G^{(n)}(0) \int_0^\infty t^n g(t) dt.$$

i.e., g is orthogonal to all polynomials $p(t) = t^n$ for $n = 1, 2, \dots$, i.e., g is constant. However then, due to (11), $g \equiv 0$.

The assumptions above are quite restrictive. For some special functions G , (6) holds under weaker conditions like e.g.

$$\frac{1}{\psi^{1/p_1}} \in L_{p_1^*}([0, \infty)). \quad (12) \quad \boxed{\text{ass:def_uniform}}$$

We illustrate this for $G(y) = 1 - e^{-y}$ and $G(y) = 1 - \cos(y)$. It is enough to consider $\mathbf{u} = \{1\}$ in (6), i.e. to show that the operator K given by

$$(Kg)(x) := \int_D g(t) \kappa(x, t) dt$$

satisfies $g = 0$ almost everywhere whenever $Kg = 0$ and $g \in L_{p_1, \psi}$.

Indeed, using Hölder's inequality and (12), we get that $L_{p_1, \psi} \subseteq L_1([0, \infty))$. Hence it is enough to show that $Kg = 0$ for some $g \in L_1([0, \infty))$ implies $g = 0$ in $L_1([0, \infty))$.

For $G(y) = 1 - e^{-y}$, i.e.

$$(Kg)(x) := \int_0^\infty g(t) (1 - e^{-xt}) dt,$$

this follows from the properties of the Laplace transform \mathcal{L} . Indeed, observe that with $c = \int_0^\infty g(t) dt$, we have

$$0 = (Kg)(x) = c - (\mathcal{L}g)(x) \quad \text{for all } x \in [0, \infty).$$

Hence $\mathcal{L}g = c$ is a constant, which is only possible if this constant is $c = 0$. But then $\mathcal{L}g = 0$, and $g = 0$ almost everywhere follows from the injectivity property of the Laplace transform.

For $G(y) = 1 - \cos(y)$, i.e.

$$(Kg)(x) := \int_0^\infty g(t) (1 - \cos(xt)) dt,$$

we can argue similarly with the Fourier transform instead of the Laplace transform by extending g to an even function on $(-\infty, \infty)$.

3.3 The Function Approximation Problem

We follow [14]. Let ω be a probability density on D and let $q \in [1, \infty]$. For non-empty \mathbf{u} , let $L_{\mathbf{u}, q, \omega} = L_{\mathbf{u}, q, \omega}(D^{|\mathbf{u}|})$ be the space of functions with finite semi-norm

$$\|f\|_{L_{\mathbf{u}, q, \omega}} = \left(\int_{D^{|\mathbf{u}|}} |f(\mathbf{t}_{\mathbf{u}})|^q \omega_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) d\mathbf{t}_{\mathbf{u}} \right)^{1/q} \quad \text{with} \quad \omega_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) = \prod_{j \in \mathbf{u}} \omega(t_j).$$

For $\mathbf{u} = \emptyset$, the corresponding space is $L_{\emptyset, q, \omega}$, the space of constant functions.

Consider next the embedding operators

$$S_{\mathbf{u}}(f_{\mathbf{u}}) = f_{\mathbf{u}} \in L_{\mathbf{u}, q, \omega}.$$

For them to be well defined, we assume that

$$\tilde{\kappa}_{q, p_1, \omega} := \|\widehat{\kappa}_{p_1}\|_{L_{\{1\}, q, \omega}} = \left(\int_D |\widehat{\kappa}_{p_1}(x)|^q \omega(x) dx \right)^{1/q} < \infty.$$

Then for any $f_{\mathbf{u}} \in F_{\mathbf{u}}$ we have

$$\|f_{\mathbf{u}}\|_{L_{\mathbf{u}, q, \omega}} \leq \|f_{\mathbf{u}}\|_{F_{\mathbf{u}}} \tilde{\kappa}_{q, p_1, \omega}^{|\mathbf{u}|}, \quad \text{i.e.,} \quad \|S_{\mathbf{u}}\| \leq \tilde{\kappa}_{q, p_1, \omega}^{|\mathbf{u}|}. \quad (13) \quad \text{help}$$

This means that (4) holds with

$$C_1 = \tilde{\kappa}_{q,p_1,\omega}.$$

Of course, $C_1 = \tilde{\kappa}_{q,p_1,\omega}$ depends also on ψ .

Let $\mathcal{L}_{s,q,\omega}$ be a space containing $\mathcal{F}_{s,p_1,p_2,\gamma}$ and endowed with a semi-norm such that for every \mathbf{u} and $f_{\mathbf{u}} \in F_{\mathbf{u}}$

$$\|f_{\mathbf{u}}\|_{\mathcal{L}_{s,q,\omega}} \leq \|f_{\mathbf{u}}\|_{L_{\mathbf{u},q,\omega}}.$$

Finally, let \mathcal{S}_s be the embedding operator

$$\mathcal{S}_s : \mathcal{F}_{s,p_1,p_2,\gamma} \rightarrow \mathcal{L}_{s,q,\omega}.$$

Of course, it depends on all the parameters, $p_1, p_2, q, \psi, \omega$, and the weights γ . We assume that these parameters satisfy the following condition

$$\left(\sum_{\mathbf{u} \subseteq [s]} \left(\gamma_{\mathbf{u}} \tilde{\kappa}_{q,p_1,\omega}^{|\mathbf{u}|} \right)^{p_2^*} \right)^{1/p_2^*} < \infty$$

since then

$$\|\mathcal{S}_s\| \leq \left(\sum_{\mathbf{u} \subseteq [s]} \left(\gamma_{\mathbf{u}} \tilde{\kappa}_{q,p_1,\omega}^{|\mathbf{u}|} \right)^{p_2^*} \right)^{1/p_2^*}.$$

Note that for product weights the embedding operator is of tensor product form.

We illustrate the assumptions above for the examples from the previous section.

Example 4 We continue Example 1 here. Consider $\omega \equiv 1$. This case was studied in [7]. We have

$$\|S_1\| \leq \tilde{\kappa}_{q,p_1,\omega} = \begin{cases} 1 & \text{if } q = \infty \text{ or } p_1 = 1, \\ (1 + q/p_1^*)^{-1/q} & \text{otherwise.} \end{cases}$$

Example 5 We return to Example 2 and assume that $\omega(x) = \mu e^{-\mu x}$ for some $\mu > 0$. In what follows and some other places we use the well known fact that

$$\int_0^{\infty} x^a e^{-bx} dx = \frac{\Gamma(a+1)}{b^{a+1}} \quad \text{for } a, b > 0.$$

We begin with the case of $p_1 = 1$. It is easy to see that

$$\tilde{\kappa}_{q,p_1,\omega} = \begin{cases} \frac{1}{(r-1)! \mu^{r-1}} (\Gamma((r-1)q+1))^{1/q} & \text{if } q < \infty \\ \infty & \text{if } q = \infty, \end{cases}$$

for $\lambda \geq 0$. For $\lambda < 0$, $\tilde{\kappa}_{\infty,p_1,\omega} = \infty$ if $q = \infty$ or $\mu + \lambda q/p_1 \leq 0$. Otherwise, for $\lambda < 0$,

$$\tilde{\kappa}_{q,p_1,\omega} \leq \left(\frac{r-1}{|\lambda|} \right)^{r-1} \left(e^{-\mu(r-1)/|\lambda|} + e^{-(r-1)/|\lambda|} \right)^{1/q}.$$

We now consider the case of $p_1 > 1$. For $q = \infty$ and any λ we have

$$(r-1)! \tilde{\kappa}_{\infty,p_1,\omega} = \sup_{x \geq 0} \left(\int_0^x (x-t)^{(r-1)p_1^*} e^{-\lambda t(p_1^*-1)} dt \right)^{1/p_1^*}$$

$$\geq \sup_{x \geq 1} \left(\int_0^1 (x-t)^{(r-1)p_1^*} e^{-\lambda t(p_1^*-1)} dt \right)^{1/p_1^*} = \infty.$$

Therefore, for the rest of this example, we consider $q < \infty$.

If $\lambda \geq 0$, then

$$\tilde{\kappa}_{q,p_1,\omega} \leq \frac{(\Gamma((r-1/p_1)q+1))^{1/q}}{(r-1)!((r-1)p_1^*+1)^{1/p_1^*} \mu^{r-1/p_1}}$$

Consider next $\lambda < 0$. Since

$$(r-1)! \widehat{\kappa}_{p_1}(x) \geq \left(\int_0^{x-1} e^{\lambda t(p_1^*-1)} dt \right)^{1/p_1^*},$$

we conclude that

$$\tilde{\kappa}_{q,p_1,\omega} = \infty \quad \text{if} \quad \mu + \lambda \frac{q}{p_1} \leq 0.$$

If $\mu + \lambda q/p_1 > 0$, then $\tilde{\kappa}_{q,p_1,\omega}$ is bounded from above by

$$\frac{(\Gamma((r-1)q+1))^{1/q}}{(r-1)! (|\lambda| (p_1^*-1))^{1/p_1^*} \mu^{1/q} (\mu + \lambda q/p_1)^{r-1+1/q}}$$

and

$$\frac{(\Gamma((r-1)q+2))^{1/q}}{(r-1)! ((r-1)p_1^*+1)^{1/p_1^*} \mu^{1/q} (\mu + \lambda q/p_1)^{r-1+2/q}}.$$

Let the assumptions from the previous section be satisfied.

Remark 3 In this setting the ε -truncation dimension from Definition 1 is the smallest natural number k such that

$$\left\| \sum_{u \in \mathcal{Z}[k]} f_u \right\|_{\mathcal{L}_{s,q,\omega}} \leq \varepsilon \left\| \sum_{u \in \mathcal{Z}[k]} f_u \right\|_{\mathcal{F}_{s,p_1,p_2,\gamma}} \quad \text{for all } f = \sum_{u \subseteq [s]} f_u \in \mathcal{F}_{s,p_1,p_2,\gamma}.$$

We obtain the following corollary of Proposition 1 and Theorem 2.

Corollary 4 *We have*

$$\dim^{\text{trnc}}(\varepsilon) \leq \min \left\{ k : \left(\sum_{u \in \mathcal{Z}[k]} (\gamma_u \tilde{\kappa}_{q,p_1,\omega}^{|u|})^{p_2^*} \right)^{1/p_2^*} \leq \varepsilon \right\} \quad (14) \quad \boxed{\mathbf{k-e}}$$

which reduces to $\dim^{\text{trnc}}(\varepsilon) \leq \min \left\{ k : \sup_{u \in \mathcal{Z}[k]} \gamma_u \tilde{\kappa}_{q,p_1,\omega}^{|u|} \leq \varepsilon \right\}$ for $p_2 = 1$.

For product weights, $\dim^{\text{trnc}}(\varepsilon)$ is bounded from above by

$$\min \left\{ k : \prod_{j=1}^s (1 + (\gamma_j \tilde{\kappa}_{q,p_1,\omega})^{p_2^*})^{1/p_2^*} \left(1 - e^{-\sum_{j=k+1}^s (\gamma_j \tilde{\kappa}_{q,p_1,\omega})^{p_2^*}} \right)^{1/p_2^*} \leq \varepsilon \right\}.$$

For $k \leq s$ let

$$\mathcal{F}_{k,p_1,p_2,\gamma} = \bigoplus_{\mathbf{u} \subseteq [k]} F_{\mathbf{u}}$$

be the subspace of $\mathcal{F}_{s,p_1,p_2,\gamma}$ consisting of k -variate functions $f([\cdot]_{[k]}; \mathbf{0}_{-[k]})$, and let $A_{k,n}$ be an algorithm for approximating functions from $\mathcal{F}_{k,p_1,p_2,\gamma}$ that uses n function values. The worst case error of $A_{k,n}$ with respect to the space $\mathcal{F}_{k,p_1,p_2,\gamma}$ is

$$e(A_{k,n}; \mathcal{F}_{k,p_1,p_2,\gamma}) := \sup_{f \in \mathcal{F}_{k,p_1,p_2,\gamma}} \frac{\|f - A_{k,n}(f)\|_{\mathcal{L}_{k,q,\omega}}}{\|f\|_{\mathcal{F}_{k,p_1,p_2,\gamma}}}.$$

Now let

$$\mathcal{A}_{s,k,n}^{\text{trnc}}(f) = A_{k,n}(f([\cdot]_{[k]}; \mathbf{0}_{-[k]}))$$

be an algorithm for approximating functions from the whole space $\mathcal{F}_{s,p_1,p_2,\gamma}$. The worst case error of $\mathcal{A}_{s,k,n}^{\text{trnc}}$ is defined as

$$e(\mathcal{A}_{s,k,n}^{\text{trnc}}; \mathcal{F}_{s,p_1,p_2,\gamma}) := \sup_{f \in \mathcal{F}_{s,p_1,p_2,\gamma}} \frac{\|f - \mathcal{A}_{s,k,n}^{\text{trnc}}(f)\|_{\mathcal{L}_{s,q,\omega}}}{\|f\|_{\mathcal{F}_{s,p_1,p_2,\gamma}}}.$$

This yields the following corollary of Theorem 3.

Corollary 5 *For given $\varepsilon > 0$ and $k \geq \dim^{\text{trnc}}(\varepsilon)$ we have*

$$e(\mathcal{A}_{s,k,n}^{\text{trnc}}; \mathcal{F}_{s,p_1,p_2,\gamma}) \leq \left(\varepsilon^{p_2^*} + e(A_{k,n}; \mathcal{F}_{k,p_1,p_2,\gamma})^{p_2^*} \right)^{1/p_2^*}$$

which reduces to $e(\mathcal{A}_{s,k,n}^{\text{trnc}}; \mathcal{F}_{s,p_1,1,\gamma}) \leq \max(\varepsilon, e(A_{k,n}; \mathcal{F}_{k,p_1,1,\gamma}))$ for $p_2 = 1$.

3.4 The Integration Problem

secint

In this subsection we assume that (9) is satisfied. We consider the problem of numerically approximating the integral

$$\mathcal{I}_s(f) = \int_{D^s} f(\mathbf{x}) \omega_{[s]}(\mathbf{x}) \, d\mathbf{x},$$

where $f \in \mathcal{F}_{s,p_1,p_2,\gamma}$, where ω is a probability density function on D , and $\omega_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) = \prod_{j \in \mathbf{u}} \omega(x_j)$ for $\mathbf{u} \subseteq [s]$.

We require in this section that $\bar{\kappa}_{p_1,\omega}$, defined by

$$\bar{\kappa}_{p_1,\omega}(t) := \int_D \frac{|\kappa(x,t)|}{\psi^{1/p_1}(t)} \omega(x) \, dx,$$

is such that $\|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}} < \infty$.

Let now $f \in \mathcal{F}_{s,p_1,p_2,\gamma}$. For non-empty \mathbf{u} , let $g_{\mathbf{u}} \in L_{\mathbf{u},p_1,\psi}$ be such that $\|f_{\mathbf{u}}\|_{F_{\mathbf{u}}} = \|g_{\mathbf{u}}\|_{L_{\mathbf{u},p_1,\psi}}$ (as outlined in Section 3.2). We then have

$$|\mathcal{I}_s(f_{\mathbf{u}})| = \left| \int_{D^{|\mathbf{u}|}} f_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) \omega_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) \, d\mathbf{x}_{\mathbf{u}} \right| = \left| \int_{D^{|\mathbf{u}|}} g_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) \psi_{\mathbf{u}}^{1/p_1}(\mathbf{t}_{\mathbf{u}}) \prod_{j \in \mathbf{u}} \bar{\kappa}_{p_1,\omega}(t_j) \, d\mathbf{t}_{\mathbf{u}} \right|$$

$$\leq \|g_u\|_{L_{u,p_1,\psi}} \|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}^{|\mathbf{u}|}} = \|f_u\|_{F_u} \|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}^{|\mathbf{u}|}}.$$

Since Hölder's inequality is sharp we conclude that

$$\|I_u\| = \|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}^{|\mathbf{u}|}},$$

where I_u is the restriction of \mathcal{I}_s to F_u . This means that (4) hold with equality for

$$C_1 = \|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}}.$$

Example 6 Let us once more return to Example 2 with $\omega(x) = \mu e^{-\mu x}$. Then the $L_{p_1^*}$ -norm of $\bar{\kappa}_{p_1,\omega}$ is given by

$$\frac{1}{(r-1)!} \left(\int_0^\infty e^{-\lambda t p_1^*/p_1} \left(\mu \int_t^\infty (x-t)^{r-1} e^{-\mu x} dx \right)^{p_1^*} dt \right)^{1/p_1^*}.$$

The inner integral, after the change of variables $z = x - t$, is equal to

$$\mu \int_0^\infty z^{r-1} e^{-\mu(z+t)} dz = \frac{1}{\mu^{r-1}} e^{-\mu t} \Gamma(r)$$

and, therefore,

$$\|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}} = \begin{cases} \frac{\Gamma(r)}{(r-1)! \mu^{r-1}} \left(\frac{1}{p_1^* (\lambda/p_1 + \mu)} \right)^{1/p_1^*} & \text{if } \mu + \lambda/p_1 > 0, \\ \infty & \text{if } \mu + \lambda/p_1 \leq 0 \end{cases}$$

with the convention that $(1/(p_1^*(\lambda + \mu)))^{1/p_1^*} = 1$ for $p_1^* = \infty$.

Let the assumptions from the previous section be satisfied.

Remark 4 In this setting the ε -truncation dimension from Definition 1 is the smallest natural number k such that

$$\left| \sum_{\mathbf{u} \notin [k]} \mathcal{I}_s(f_u) \right| \leq \varepsilon \left\| \sum_{\mathbf{u} \notin [k]} f_u \right\|_{\mathcal{F}_{s,p_1,p_2,\gamma}} \quad \text{for all } f = \sum_{\mathbf{u} \subseteq [s]} f_u \in \mathcal{F}_{s,p_1,p_2,\gamma}.$$

We obtain the following corollary of Proposition 1 and Theorem 2.

Corollary 6 *We have*

$$\dim^{\text{trnc}}(\varepsilon) \leq \min \left\{ k : \left(\sum_{\mathbf{u} \notin [k]} (\gamma_u \|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}^{|\mathbf{u}|}})^{p_2^*} \right)^{1/p_2^*} \leq \varepsilon \right\}. \quad (15) \quad \boxed{\text{k-eint}}$$

which reduces to $\dim^{\text{trnc}}(\varepsilon) \leq \min \left\{ k : \sup_{\mathbf{u} \notin [k]} \gamma_u \|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}^{|\mathbf{u}|}} \leq \varepsilon \right\}$ for $p_2 = 1$.

For product weights $\dim^{\text{trnc}}(\varepsilon)$ is bounded from above by the smallest k for which

$$\prod_{j=1}^s (1 + (\gamma_j \|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}})^{p_2^*})^{1/p_2^*} \left(1 - e^{-\sum_{j=k+1}^s (\gamma_j \|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}})^{p_2^*}} \right)^{1/p_2^*} \leq \varepsilon.$$

For $k \leq s$ let $A_{k,n}$ be an algorithm for integrating functions from $\mathcal{F}_{k,p_1,p_2,\gamma}$ that uses n function values. The worst case error of $A_{k,n}$ with respect to the space $\mathcal{F}_{k,p_1,p_2,\gamma}$ is

$$e(A_{k,n}; \mathcal{F}_{k,p_1,p_2,\gamma}) := \sup_{f \in \mathcal{F}_{k,p_1,p_2,\gamma}} \frac{|\mathcal{I}_k(f) - A_{k,n}(f)|}{\|f\|_{\mathcal{F}_{k,p_1,p_2,\gamma}}}.$$

Now let

$$\mathcal{A}_{s,k,n}^{\text{trnc}}(f) = A_{k,n}(f([\cdot]_k; \mathbf{0}_{-[k]}))$$

be an algorithm for integrating functions from the whole space $\mathcal{F}_{s,p_1,p_2,\gamma}$. The worst case error of $\mathcal{A}_{s,k,n}^{\text{trnc}}$ is defined as

$$e(\mathcal{A}_{s,k,n}^{\text{trnc}}; \mathcal{F}_{s,p_1,p_2,\gamma}) := \sup_{f \in \mathcal{F}_{s,p_1,p_2,\gamma}} \frac{|\mathcal{I}_s(f) - \mathcal{A}_{s,k,n}^{\text{trnc}}(f)|}{\|f\|_{\mathcal{F}_{s,p_1,p_2,\gamma}}}.$$

This yields the following corollary of Theorem 3.

Corollary 7 *For given $\varepsilon > 0$ and $k \geq \dim^{\text{trnc}}(\varepsilon)$ we have*

$$e(\mathcal{A}_{s,k,n}^{\text{trnc}}; \mathcal{F}_{s,p_1,p_2,\gamma}) \leq \left(\varepsilon^{p_2^*} + e(A_{k,n}; \mathcal{F}_{k,p_1,p_2,\gamma})^{p_2^*} \right)^{1/p_2^*}$$

which reduces to $e(\mathcal{A}_{s,k,n}^{\text{trnc}}; \mathcal{F}_{s,p_1,1,\gamma}) \leq \max(\varepsilon, e(A_{k,n}; \mathcal{F}_{k,p_1,1,\gamma}))$ for $p_2 = 1$.

4 Unanchored Spaces of Multivariate Functions

Let κ , ω , and \mathcal{I}_s be as in the previous section. Also here we assume that $\|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}} < \infty$. In what follows we use I to denote the ω -weighted integral operator for univariate functions,

$$I(f) = \int_D f(x) \omega(x) dx.$$

Of course $\|I\| = \|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}}$.

Consider

$$\kappa_{\mathbf{u},\omega}(\mathbf{x}_{\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) = \prod_{j \in \mathbf{u}} (\kappa(x_j, t_j) - I(\kappa(\cdot, t_j)))$$

and

$$K_{\mathbf{u},\omega}(g_{\mathbf{u}})(\mathbf{x}_{\mathbf{u}}) = \int_{D^{|\mathbf{u}|}} g_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) \kappa_{\mathbf{u},\omega}(\mathbf{x}_{\mathbf{u}}, \mathbf{t}_{\mathbf{u}}) d\mathbf{t}_{\mathbf{u}}$$

and the corresponding space $\mathcal{F}_{s,p_1,p_2,\gamma,\omega}$ of functions

$$f(\mathbf{x}) = \sum_{\mathbf{u}} f_{\mathbf{u},\omega}(\mathbf{x}_{\mathbf{u}}) \quad \text{with} \quad f_{\mathbf{u},\omega}(\mathbf{x}_{\mathbf{u}}) = K_{\mathbf{u},\omega}(g_{\mathbf{u}})(\mathbf{x}_{\mathbf{u}})$$

such that

$$\|f\|_{\mathcal{F}_{s,p_1,p_2,\gamma,\omega}} := \left(\sum_{\mathbf{u}} \gamma_{\mathbf{u}}^{-p_2} \|g_{\mathbf{u}}\|_{L_{\mathbf{u},p_1,\psi}}^{p_2} \right)^{1/p_2} < \infty.$$

Instead of being anchored, the functions $f_{u,\omega}$ satisfy the following property

$$\int_D f_{u,\omega}(\mathbf{x}_u) \prod_{k \in u} \omega(x_k) dx_j = 0 \quad \text{if } j \in u.$$

As in [3], one can show that the spaces $\mathcal{F}_{s,p_1,p_2,\gamma}$ and $\mathcal{F}_{s,p_1,p_2,\gamma,\omega}$ as sets of functions are equal if and only if

$$\gamma_u > 0 \quad \text{implies that} \quad \gamma_v > 0 \quad \text{for all } v \subseteq u. \quad (16) \quad \boxed{\text{weights}}$$

From now on, we assume that (16) is satisfied. Of course (16) always holds true for product weights.

Let ι_{p_1,p_2} be the embedding

$$\iota_{p_1,p_2} : \mathcal{F}_{s,p_1,p_2,\gamma} \rightarrow \mathcal{F}_{s,p_1,p_2,\gamma,\omega} \quad \text{and} \quad \iota_{p_1,p_2}(f) = f,$$

and let ι_{p_1,p_2}^{-1} be its inverse. As in [6], see also [2], one can check that

$$\|\iota_{p_1,p_2}\| = \|\iota_{p_1,p_2}^{-1}\|.$$

Moreover, following the approach in [2], one can provide exact formulas for the norms of the embeddings for $p_1, p_2 \in \{1, \infty\}$ and next, using interpolation theory (as in [4], see also [2]), derive upper bounds for arbitrary values of p_1 and p_2 .

More precisely, we have the following proposition.

Proposition 8 *Suppose that $\|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}} < \infty$ for $p_1 \in \{1, \infty\}$. Then*

$$\|\iota_{p_1,p_2}\| = \begin{cases} \max_u \sum_{v \subseteq u} \frac{\gamma_u}{\gamma_v} \|\bar{\kappa}_{1,\omega}\|_{L_\infty}^{|\mathbf{u}|-|\mathbf{v}|} & \text{if } p_1 = 1 \text{ and } p_2 = 1, \\ \max_u \sum_{v \subseteq [s] \setminus u} \frac{\gamma_{u \cup v}}{\gamma_u} \|\bar{\kappa}_{1,\omega}\|_{L_\infty}^{|\mathbf{v}|} & \text{if } p_1 = 1 \text{ and } p_2 = \infty, \\ \max_u \sum_{v \subseteq u} \frac{\gamma_u}{\gamma_v} \|\bar{\kappa}_{\infty,\omega}\|_{L_1}^{|\mathbf{u}|-|\mathbf{v}|} & \text{if } p_1 = \infty \text{ and } p_2 = 1, \\ \max_u \sum_{v \subseteq [s] \setminus u} \frac{\gamma_{u \cup v}}{\gamma_u} \|\bar{\kappa}_{\infty,\omega}\|_{L_1}^{|\mathbf{v}|} & \text{if } p_1 = \infty \text{ and } p_2 = \infty. \end{cases}$$

To give a flavor of the proof, we prove the proposition for $p_1 = p_2 = 1$.

Proof. For $f = \sum_u f_{u,\omega}$ we have

$$\begin{aligned} f_{u,\omega} &= K_{u,\omega}(g_u) = \int_{D^{|\mathbf{u}|}} g_u(\mathbf{t}_u) \prod_{j \in u} (\kappa(\cdot, t_j) - I(\kappa(\cdot, t_j))) d\mathbf{t}_u \\ &= \sum_{v \subseteq u} \int_{D^{|\mathbf{u}|}} g_u(\mathbf{t}_u) \kappa_v(\cdot, \mathbf{t}_v) \prod_{j \in u \setminus v} (-1) I(\kappa(\cdot, t_j)) d\mathbf{t}_u. \end{aligned}$$

Therefore

$$f = \sum_u \sum_{v \subseteq u} \int_{D^{|\mathbf{u}|}} g_u(\mathbf{t}_u) \kappa_v(\cdot, \mathbf{t}_v) \prod_{j \in u \setminus v} (-1) I(\kappa(\cdot, t_j)) d\mathbf{t}_u$$

$$= \sum_{\mathfrak{v}} \int_{D^{|\mathfrak{v}|}} \kappa_{\mathfrak{v}}(\cdot, \mathbf{t}_{\mathfrak{v}}) \sum_{\mathfrak{w}, \mathfrak{w} \cap \mathfrak{v} = \emptyset} \int_{D^{|\mathfrak{w}|}} g_{\mathfrak{v} \cup \mathfrak{w}}(\mathbf{t}_{\mathfrak{v}}, \mathbf{t}_{\mathfrak{w}}) \prod_{j \in \mathfrak{w}} (-1) I(\kappa(\cdot, t_j)) d\mathbf{t}_{\mathfrak{w}} d\mathbf{t}_{\mathfrak{v}},$$

where $(\mathbf{t}_{\mathfrak{v}}, \mathbf{t}_{\mathfrak{w}}) = \mathbf{t}_{\mathfrak{v} \cup \mathfrak{w}}$, which implies that $f = \sum_{\mathfrak{v}} K_{\mathfrak{v}}(h_{\mathfrak{v}})$ with

$$h_{\mathfrak{v}}(\mathbf{t}_{\mathfrak{v}}) = \sum_{\mathfrak{w}, \mathfrak{w} \cap \mathfrak{v} = \emptyset} \int_{D^{|\mathfrak{w}|}} g_{\mathfrak{v} \cup \mathfrak{w}}(\mathbf{t}_{\mathfrak{v}}, \mathbf{t}_{\mathfrak{w}}) \prod_{j \in \mathfrak{w}} (-1) I(\kappa(\cdot, t_j)) d\mathbf{t}_{\mathfrak{w}}.$$

Clearly

$$\|h_{\mathfrak{v}}\|_{L_{\mathfrak{v},1,\psi}} \leq \sum_{\mathfrak{w}, \mathfrak{w} \cap \mathfrak{v} = \emptyset} \|g_{\mathfrak{v} \cup \mathfrak{w}}\|_{L_{\mathfrak{v} \cup \mathfrak{w},1,\psi}} \|\bar{\kappa}_{1,\omega}\|_{L_{\infty}}^{|\mathfrak{w}|}$$

and using $\mathfrak{u} = \mathfrak{v} \cup \mathfrak{w}$ we get

$$\begin{aligned} \sum_{\mathfrak{v}} \gamma_{\mathfrak{v}}^{-1} \|h_{\mathfrak{v}}\|_{L_{\mathfrak{v},1,\psi}} &\leq \sum_{\mathfrak{u}} \gamma_{\mathfrak{u}}^{-1} \|g_{\mathfrak{u}}\|_{L_{\mathfrak{u},1,\psi}} \gamma_{\mathfrak{u}} \sum_{\mathfrak{v} \subseteq \mathfrak{u}} \gamma_{\mathfrak{v}}^{-1} \|\bar{\kappa}_{1,\omega}\|_{L_{\infty}}^{|\mathfrak{u}|-|\mathfrak{v}|} \\ &\leq \|f\|_{\mathcal{F}_{s,1,1,\gamma,\omega}} \max_{\mathfrak{u}} \sum_{\mathfrak{v} \subseteq \mathfrak{u}} \frac{\gamma_{\mathfrak{u}}}{\gamma_{\mathfrak{v}}} \|\bar{\kappa}_{1,\omega}\|_{L_{\infty}}^{|\mathfrak{u}|-|\mathfrak{v}|}. \end{aligned}$$

This proves the bound on $\|\iota^{-1}\|$. Since the Hölder inequality is sharp, we actually have equality. The proof for ι is identical. \square

For product weights $\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$ the expressions in the proposition above reduce to

$$\prod_{j=1}^s (1 + \gamma_j \|\bar{\kappa}_{1,\omega}\|_{L_{\infty}}) \quad \text{if } p_1 = 1$$

and to

$$\prod_{j=1}^s (1 + \gamma_j \|\bar{\kappa}_{\infty,\omega}\|_{L_1}) \quad \text{if } p_1 = \infty.$$

Applying interpolation theory we get, as in [2]:

Corollary 9 *Suppose that $\|\bar{\kappa}_{p_1,\omega}\|_{L_{p_1^*}} < \infty$ for any $p_1 \in [1, \infty]$. If $p_1 \leq p_2$ then*

$$\|\iota_{p_1,p_2}\| \leq \|\iota_{1,\infty}\|^{1/p_1-1/p_2} \|\iota_{1,1}\|^{1/p_2} \|\iota_{\infty,\infty}\|^{1-1/p_1},$$

and if $p_2 < p_1$ then

$$\|\iota_{p_1,p_2}\| \leq \|\iota_{\infty,1}\|^{1/p_2-1/p_1} \|\iota_{1,1}\|^{1/p_1} \|\iota_{\infty,\infty}\|^{1-1/p_2}.$$

For product weights we have

$$\|\iota_{p_1,p_2}\| \leq \prod_{j=1}^s (1 + \gamma_j \|\bar{\kappa}_{1,\omega}\|_{L_{\infty}})^{1/p_1} (1 + \gamma_j \|\bar{\kappa}_{\infty,\omega}\|_{L_1})^{1-1/p_1}.$$

It was shown in [4] for product weights and in [6] for a number of different types of weights that the upper bounds in Corollary 9 are sharp.

Suppose now that $\sum_{j=1}^{\infty} \gamma_j < \infty$. Then the norms of the embeddings are uniformly bounded,

$$\|\iota_{p_1,p_2}\| \leq \prod_{j=1}^{\infty} (1 + \gamma_j \|\bar{\kappa}_{1,\omega}\|_{L_{\infty}})^{1/p_1} (1 + \gamma_j \|\bar{\kappa}_{\infty,\omega}\|_{L_1})^{1-1/p_1},$$

for any s including $s = \infty$. Hence the results of previous sections are applicable for unanchored spaces considered in this section.

Remark 5 It is possible to consider even more general unanchored spaces. Indeed, consider a linear functional ℓ that is continuous for the space of univariate functions, i.e., with

$$\|\bar{\ell}_{p_1}\|_{L_{p_1}^*} < \infty, \quad \text{where} \quad \bar{\ell}_{p_1}(t) := \frac{\ell(\kappa(\cdot, t))}{\psi^{1/p_1}}.$$

Suppose also that

$$\ell\left(\int_D g(t) \kappa(\cdot, t) dt\right) = \int_D g(t) \ell(\kappa(\cdot, t)) dt \quad \text{for all } g \in L_{p_1, \psi}(D).$$

For nonempty \mathbf{u} , define

$$K_{\mathbf{u}, \ell}(g_{\mathbf{u}})(\mathbf{x}_{\mathbf{u}}) := \int_{D^{|\mathbf{u}|}} g_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) \prod_{j \in \mathbf{u}} (\kappa(x_j, t_j) - \ell(\kappa(\cdot; t_j))) d\mathbf{t}_{\mathbf{u}}.$$

Then the corresponding functions $f_{\mathbf{u}, \ell} = K_{\mathbf{u}, \ell}(g_{\mathbf{u}})$ satisfy

$$\ell_j(f_{\mathbf{u}, \ell}) = 0 \quad \text{if } j \in \mathbf{u}.$$

Here ℓ_j denotes the functional ℓ acting on functions with respect to the j^{th} variable. More formally,

$$\ell_j = \bigotimes_{n=1}^s T_n \quad \text{with} \quad T_n = \begin{cases} \text{id} & \text{if } n \neq j, \\ \ell & \text{if } n = j, \end{cases}$$

where id is an identity operator. For instance for $\ell(g) = g(0) + \int_D g(t) dt$,

$$\ell_j(f)(\mathbf{x}) = f([\mathbf{x}_{\mathbf{u}}; \mathbf{0}_{-\mathbf{u}}]) + \int_D f(\mathbf{x}) dx_j \quad \text{with } \mathbf{u} = \{j\}.$$

Let $\mathcal{F}_{s, p_1, p_2, \gamma, \ell}$ be the Banach space of functions $f = \sum_{\mathbf{u}, \gamma_{\mathbf{u}} > 0} K_{\mathbf{u}, \ell}(g_{\mathbf{u}})$ with the norm

$$\|f\|_{\mathcal{F}_{s, p_1, p_2, \gamma, \ell}} = \left(\sum_{\mathbf{u}, \gamma_{\mathbf{u}} > 0} \gamma_{\mathbf{u}}^{-1/p_2} \|g_{\mathbf{u}}\|_{L_{\mathbf{u}, p_1, \psi}}^{p_2} \right)^{1/p_2}.$$

It is easy to extend all the results of this section provided that $\|\bar{\ell}_{p_1}\|_{L_{p_1}^*}$ is finite for all p_1 . In particular, Proposition 8 and Corollary 9 hold with $\|\bar{\kappa}_{p_1, \omega}\|_{L_{p_1}^*}$ replaced by $\|\bar{\ell}_{p_1}\|_{L_{p_1}^*}$.

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Authors' addresses:

Aicke Hinrichs
 Institute for Analysis
 Johannes Kepler University Linz
 Altenbergerstr. 69, 4040 Linz, Austria.
Aicke.Hinrichs@jku.at

Peter Kritzer
 Johann Radon Institute for Computational and Applied Mathematics (RICAM)
 Austrian Academy of Sciences
 Altenbergerstr. 69, 4040 Linz, Austria.
peter.kritzer@oeaw.ac.at

Friedrich Pillichshammer
Department of Financial Mathematics and Applied Number Theory
Johannes Kepler University Linz
Altenbergerstr. 69, 4040 Linz, Austria
`friedrich.pillichshammer@jku.at`

G.W. Wasilkowski
Computer Science Department
University of Kentucky
301 David Marksbury Building, Lexington, KY 40506, USA
`greg@cs.uky.edu`