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Recent development of robust monolithic fluid-structure interaction solvers

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Abstract. In the last few years, from the modeling point of view, the monolithic approach for fluid-structure interaction problems in many different application fields has been adopted by more and more researchers. Meanwhile, the development of monolithic solvers in the solution procedure for solving such coupled fluid-structure interaction problems all at once is in general a very hard task and has received a lot of attention. Due to the coupling conditions on the interface, it is challenging to design efficient preconditioners for the linearized coupled system of equations, that are robust with respect to the mesh size, time step size and material parameters. Further, it is nontrivial to realize scalable parallel implementations for solving such large scale coupled systems, which requires special care for handling the interface conditions. In this survey, we present an overview of some recent results on robust monolithic fluid-structure interaction solvers, that are mainly based on the block factorization, geometric and algebraic multigrid, and domain decomposition methods.

Keywords. Fluid-structure interaction, monolithic solvers, block factorization, geometric multigrid methods, algebraic multigrid methods, domain decomposition methods, preconditioners.

AMS classification. 74S05, 74F10, 65M55, 65M60.

1 Introduction

Partitioned fluid-structure interaction (FSI) methods are traditionally adopted in engineering applications since it is easy to combine available solvers for fluid and solid mechanical problems; see, e.g., [23, 27, 32, 4, 53, 31, 37, 18, 9, 10, 19, 20, 78, 59] for recent development of partitioned methods. However, the separation of the solid from the fluid part in the solution process usually yield a loss in efficiency and robustness mainly due to the so-called added-mass effect; see, e.g., [60, 88, 23, 74, 34]. Another critical point is the control of the systematic errors caused by the accuracy of the fluid, solid (structure) and mesh sub-problem solvers in each iteration step for solving the non-linear "Schur-complement-system" (discrete Steklov-Poicaré problem). That is the main reason why the monolithic methods for solving the FSI problems have received more and more interests during the last few years; see, e.g., [13] for the aspects of algorithms and computations in the monolithic methods for FSI problems. The methods may be mainly classified into two groups (A and B),

A: the first one relies on the fluid, structure and mesh sub-problem solvers, where efficient sub-problem solvers on the underlying basis have to be called in some

way, see, e.g., [43, 44, 25, 26, 7, 61, 6, 31, 5, 55, 66, 36, 58, 54],

B: the second one relies on solving many small local problems, which are characterized by the same form as the global problem, i.e., a restriction of the global problem onto the local sub-domains or patches, and featured as black-box type solvers, see, e.g., [47, 64, 11, 92, 12].

The methods mentioned in the above two groups are mainly developed from the classical methods for solving the individual field problems, e.g., block factorization [68, 85, 51], geometric multigrid [41], algebraic multigrid [73, 67] and domain decomposition [63, 77, 72] methods. Therefore, we discuss the monolithic methods in the following corresponding sections:

(1) Section 2, block factorization; see [43, 44, 25, 26, 7, 61, 6, 5, 55],

(2) Section 3, geometric multigrid; see [47, 64, 66],

(3) Section 4, algebraic multigrid; see [36, 58, 54],

(4) Section 5, domain decomposition; see [92, 11, 12].

We mention that some of the methods are combination of two classical ones, e.g., [25, 26, 7] based on the block factorization and domain decomposition methods, and [55, 36, 54] based on block factorization and algebraic multigrid methods. We characterize the methods by their main features. Recently, apart from the above mentioned work, that we will discuss in detail from the practical point of view, some results on the analysis of optimal FSI preconditioners have been published in [93]. There optimal FSI preconditioners have been developed under the framework give in [57, 96], where the uniform well-posedness of the saddle point system and corresponding Riesz operator are exploited.

Concerning temporal, spatial discretization, linearization for the coupled model FSI problems, we refer to [13] for details. In short, the computational FSI domain $\overline{\Omega^t} = \overline{\Omega_f} \cup \overline{\Omega_s}$ is decomposed into two deformable sub-domains, the fluid Ω_f and structure Ω_s. The intersection of boundaries of two sub-domains yields the moving interface $\Gamma_{fs} = \partial\Omega_f \cap \partial\Omega_s$. By notations Γ_f^D, Γ_f^N, and Γ_s^D, Γ_s^N, we denote the Dirichlet and Neumann boundaries for the fluid and structure sub-problem, respectively. We refer to Fig. 1 for a schematic illustration of the sub-domains, interface and boundaries.

Depending on the particular applications, the incompressible Navier-Stokes equations under the arbitrary Lagrangian-Eulerian (ALE) framework [28, 49, 33, 52] are considered in the fluid sub-domain Ω_f; the elasticity and hyperelasticity models [45, 91, 17] under the Lagrangian framework are usually used in the structure reference sub-domain $\hat{\Omega}_s$. On the interface Γ_{fs}, the no-slip condition and balance of surface tractions from the fluid and structure sides are fulfilled, i.e., the velocities match and the surface tractions balance. Further, a geometric adherence condition is imposed, i.e., the fluid domain displacement and structure displacement on the interface equal to each other.

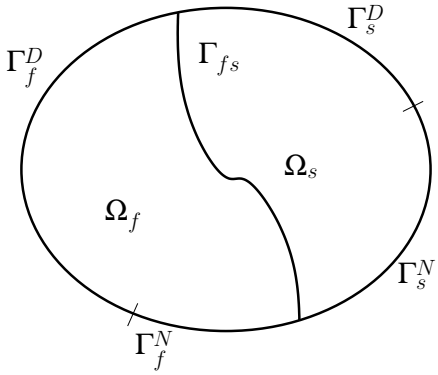

Figure 1. A schematic illustration of the computational FSI domain: $\overline{\Omega^t} = \overline{\Omega_f} \cup \overline{\Omega_s}$.

In this paper, we focus on a short overview of recent development of the monolithic methods for solving the FSI linear system of algebraic equations:

$$Kx = b. \tag{1.1}$$

In order to ease the demonstration of each method, K denotes the Jacobian system matrix at the current state, which will be detailed for each case in the respective section, x the unknown corrections of possible combination of fluid velocity, fluid pressure, structure displacement, structure velocity, structure pressure, mesh displacement, and b the corresponding residual. Further, we use the notation P with the corresponding subscription to indicate the respective preconditioner. In addition, Δt indicates the time step size.

The remainder of the paper is organized as follows: Section 2 deals with FSI preconditioners based on block factorizations. In Section 3, we discuss geometric multigrid methods for solving FSI problems. Section 4 concerns algebraic multigrid methods applied to FSI problems. Section 5 is devoted to FSI preconditioners based on domain decomposition techniques. Finally, some conclusions are drawn in Section 7.

2 Block factorization based preconditioners

2.1 Direct LU decomposition

In [48], with a skyline profile technique, a direct LU decomposition of the system matrix is used for preconditioning in solving the discretized FSI problems (in small scale) using the space-time finite element methods [76, 42, 15]. The BiCGStab solver (see, e.g., [81]) combined with this preconditioner is quite efficient for smaller examples, i.e., it only needs a few iterations. However, this method is hardly applicable to large 3D examples.

2.2 Block-triangular approximations

In [43], three block-triangular approximations for the Jacobian matrix

$$K = \begin{bmatrix} S & C_{su} & C_{sp} \\ C_{us} & F & G \\ C_{ps} & D & \end{bmatrix} \tag{2.1}$$

are used as FSI (fluid in 2D coupled with a 1D thin-walled elastic beam) preconditioners by neglecting certain fluid-structure interaction blocks, i.e.,

$$\tilde{P}_{sup} = \begin{bmatrix} S & C_{su} & C_{sp} \\ & F & G \\ & D & \end{bmatrix}, \; \tilde{P}_{sub} = \begin{bmatrix} S & & \\ C_{us} & F & G \\ C_{ps} & D & \end{bmatrix}, \; \tilde{P}_{diag} = \begin{bmatrix} S & & \\ & F & G \\ & D & \end{bmatrix}. \tag{2.2}$$

Here, S represents the Jacobian matrix from the contributions of the tangent stiffness matrix of the wall equations and the fluid-structure interaction, F the fluid momentum equation, G the discrete gradient operator , D the discrete divergence operator, and the remaining non-zero off-diagonal blocks the coupling between different fields.

The most expensive fluid sub-block in the approximations (2.2) are replaced by a cheap global pressure Schur complement preconditioner [79], which leads to three FSI preconditioners

$$P_{sup} = \begin{bmatrix} S & C_{su} & C_{sp} \\ & F & G \\ & & X \end{bmatrix}, \; P_{sub} = \begin{bmatrix} S & & \\ C_{us} & F & G \\ C_{ps} & & X \end{bmatrix}, \; P_{diag} = \begin{bmatrix} S & & \\ & F & G \\ & & X \end{bmatrix}, \tag{2.3}$$

where $X = -(DG)(DFG)^{-1}(DG)$ is Elman's BFBt approximation [29] to the pressure Schur complement $-DF^{-1}G$. These preconditioners show relatively good performance with respect to the Reynolds number and the mesh size.

Further, in [44], two block triangular FSI preconditioners P_1 and P_2 are proposed by neglecting the fluid-structure and structure-fluid interaction blocks in the reordered Jacobian matrix compared to (2.1)

$$K = \begin{bmatrix} F & G & C_{us} \\ D & & C_{ps} \\ C_{su} & C_{sp} & S \end{bmatrix}, \tag{2.4}$$

i.e.,

$$P_1 = \begin{bmatrix} F & G & \\ D & & \\ C_{su} & C_{sp} & S \end{bmatrix}, \; P_2 = \begin{bmatrix} F & G & C_{us} \\ D & & C_{ps} \\ & & S \end{bmatrix}. \tag{2.5}$$

Each preconditioning step requires solving the fluid and structure sub-problems approximately. Compared to the earlier work [43], for the fluid sub-problem, the least squares commutator (LSC) preconditioner [30] is employed with inclusion of a scaling factor $\hat{Q}$ in the pressure Schur complement, that is the diagonal of the velocity mass matrix. To be more precise, the LSC preconditioner for the fluid sub-problem has the following form:

$$P_{LSC} = \begin{bmatrix} F & G \\ & \tilde{X} \end{bmatrix}, \tag{2.6}$$

where $\tilde{X} = (D\hat{Q}G)^{-1}(D\hat{Q}^{-1}F\hat{Q}^{-1}G)(D\hat{Q}^{-1}G)^{-1}$ represents a special choice of proper approximations to the pressure Schur complement. The mesh-independent convergence rates of these two preconditioners are observed using uniform and adaptive mesh refinement.

2.3 Combination with parallel preconditioners

In [25, 26], the two serial block Gauss-Seidel preconditioners $P_{GS}^{(1)}$ and $P_{GS}^{(2)}$ are further developed in parallel context by performing block factorization of the augmented Jacobian matrix

$$K = \begin{bmatrix} C_{ff} & C_{f\Gamma} & & & \\ C_{\Gamma f} & C_{\Gamma\Gamma} & & & I \\ & & N_{ss} & N_{s\Gamma} & \\ & & N_{s\Gamma} & N_{\Gamma\Gamma} & -I \\ & I & & -I/\Delta t & \end{bmatrix} \tag{2.7}$$

using the geometry-convective explicit time discretization, neglecting the term $-I$,

$$\begin{aligned} P_{GS}^{(1)} &= \begin{bmatrix} C_{ff} & C_{f\Gamma} & & & \\ C_{\Gamma f} & C_{\Gamma\Gamma} & & & I \\ & & N_{ss} & N_{s\Gamma} & \\ & & N_{s\Gamma} & N_{\Gamma\Gamma} & \\ & I & & -I/\Delta t & \end{bmatrix} \\ &= \begin{bmatrix} I & & & & \\ & I & & & \\ & & N_{ss} & N_{s\Gamma} & \\ & & N_{s\Gamma} & N_{\Gamma\Gamma} & \\ & & & & I \end{bmatrix} \begin{bmatrix} C_{ff} & C_{f\Gamma} & & & \\ C_{\Gamma f} & C_{\Gamma\Gamma} & & & I \\ & & I & & \\ & & & I & \\ & I & & -I/\Delta t & \end{bmatrix} \\ &= P_{GS,1}^{(1)} P_{GS,2}^{(1)}, \end{aligned} \tag{2.8}$$

and $I/\Delta t$

$$
\begin{aligned}
P_{GS}^{(2)} &= \begin{bmatrix} C_{ff} & C_{f\Gamma} & & & \\ C_{\Gamma f} & C_{\Gamma\Gamma} & & & I \\ & & N_{ss} & N_{s\Gamma} & \\ & & N_{s\Gamma} & N_{\Gamma\Gamma} & -I \\ & I & & & \end{bmatrix} \\
&= \begin{bmatrix} C_{ff} & C_{f\Gamma} & & & \\ C_{\Gamma f} & C_{\Gamma\Gamma} & & & I \\ & & I & & \\ & & & I & \\ & I & & & \end{bmatrix} \begin{bmatrix} I & & & & \\ & I & & & \\ & & N_{ss} & N_{s\Gamma} & \\ & & N_{s\Gamma} & N_{\Gamma\Gamma} & -I \\ & & & & I \end{bmatrix} \\
&= P_{GS,1}^{(2)} P_{GS,2}^{(2)},
\end{aligned}
\tag{2.9}
$$

on the off-diagonal part, respectively, and replacing the fluid and structure sub-blocks with proper parallel preconditioners. Here C with subscripts f and Γ denotes the fluid stiffness matrix associated with the degrees of freedom in the fluid domain and on the interface. The same applies to the structure stiffness matrix N. We mention that, in the augmented system, the two interface conditions are imposed explicitly. In [25], the same technique is used to construct the FSI preconditioners for the geometry-convective explicit time discretized FSI system.

Particularly, the three parallel preconditioners $P_{AS}(K)$, $P_{GS-AS}^{(1)}$ and $P_{GS-AS}^{(2)}$ are then built by applying the one-level algebraic additive Schwarz preconditioner [63, 77, 72] to each factor of the full Jacobian matrix, the approximated one $P_{GS}^{(1)}$ and $P_{GS}^{(2)}$, respectively. The preconditioners $P_{GS-AS}^{(1)}$ and $P_{GS-AS}^{(2)}$ preserve modularity and have demonstrated similar or better performance than $P_{AS}(K)$ with increasing number of processors.

A similar preconditioning technique is employed in [7] for the FSI simulation using anisotropic polyconvex hyperelastic and viscoelastic arterial models at finite strains. Such an approximated monolithic Dirichlet-Neumann preconditioner P_{DN} is obtained by performing a block factorization to the Jacobian matrix

$$
K = \begin{bmatrix} F & & C_1^T & D_{d_f} F \\ & D_{d_s} S & C_3^T & \\ C_1 & C_2 & & \\ & C_4 & & H \end{bmatrix}
\tag{2.10}
$$

neglecting the coupling block C_3^T, i.e.,

$$
\begin{aligned}
P_{DN} &= \begin{bmatrix} F & & C_1^T & D_{d_f}F \\ & D_{d_s}S & & \\ C_1 & C_2 & & \\ & C_4 & & H \end{bmatrix} \\
&= \begin{bmatrix} I & & & \\ & D_{d_s}S & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & C_4 & & H \end{bmatrix} \begin{bmatrix} F & & C_1^T & D_{d_f}F \\ & I & & \\ C_1 & C_2 & & \\ & & & I \end{bmatrix},
\end{aligned} \tag{2.11}
$$

where F denotes the fluid stiffness matrix, S the structure stiffness matrix, H the geometry stiffness matrix, the remaining stemming from the continuity of two interface conditions and geometric adherence, $D_{d_f}F$ the linearization of the fluid sub-problem with respect to the fluid domain displacement, $D_{d_s}S$ the linearization of the structure sub-problem with respect to the structure displacement. The one level algebraic additive Schwarz preconditioners from [72] are used to approximate the inverse of each factor in P_{DN}.

2.4 The augmentation preconditioner

In [61], for the linearized saddle point system

$$
K_{PS} = \begin{bmatrix} E & C_{xl} \\ C_{lx} & \end{bmatrix} \tag{2.12}
$$

arising from the pseudo-solid mesh problem (large-displacement elasticity), the so-called augmentation preconditioner [38, 39, 65]

$$
P_{saug} = \begin{bmatrix} E_{aug} & \\ & W \end{bmatrix} = \begin{bmatrix} E + C_{xl}W^{-1}C_{lx} & \\ & W \end{bmatrix} \tag{2.13}
$$

is analyzed, which is used to update the moving fluid mesh. Here, E denotes the tangent stiffness matrix of the unconstrained pseudo-solid problem, C_{xl} and C_{lx} results from the imposition of the displacement constraint by the Lagrange multipliers [14], W is an matrix such that P_{aug} is effective and E_{aug} sparse. In [61], a new optimal preconditioner is developed by choosing a special block diagonal form of W.

For the fully coupled FSI system, the argumentation preconditioner P_{fsiaug} is obtained by replacing the bottom-right 2×2 block (corresponding to the pseudo-solid

mesh problem) in the FSI Jacobian matrix

$$K = \begin{bmatrix} F & & C_{fx} & \\ C_{sf} & S & C_{sx} & \\ & & E & C_{xl} \\ & C_{ls} & C_{lx} & \end{bmatrix} \tag{2.14}$$

with the above pseudo-solid augmentation preconditioner P_{saug}, i.e.,

$$P_{fsiaug} = \begin{bmatrix} F & & C_{fx} & \\ C_{sf} & S & C_{sx} & \\ & & E_{aug} & \\ & C_{ls} & & W \end{bmatrix}. \tag{2.15}$$

Here, F denotes the fluid Jacobian, S the tangent stiffness matrix of the structure, and the non-zero off-diagonal blocks the coupling between the various fields. Subject to the constraint that the tangent stiffness matrix of the unconstrained pseudo-solid problem is symmetric positive definite, the non-unit eigenvalues of the preconditioned FSI system are bounded by the ones of the preconditioned pseudo-solid problem. This efficient FSI preconditioner leads to a nearly optimal solver with respect to computational cost, i.e., the time for solving the linearized systems scales linearly with the number of unknowns.

2.5 Block LU type factorization

In [6], an inexact block LU factorization is employed to construct a FSI preconditioner for the FSI system using 1-dimensional structures

$$K = \begin{bmatrix} C_{ff} & G_f & C_{f\sigma} \\ D_f & & D_\sigma \\ C_{\sigma f} & G_\sigma & C_{\sigma\sigma} + N \end{bmatrix}, \tag{2.16}$$

where $C_{\alpha\beta}$, $\alpha, \beta \in \{\sigma, f\}$ represents the fluid matrix associated with the viscous, convective and $1/\Delta t$ scaled mass terms, with σ being nodes on the interface and f in the fluid domain, G_α the divergence term, $D_\alpha = G_\alpha^T$, and N the structure matrix corresponding to the no-slip interface condition. For the d-dimensional structures, the arising linear system has the same block structure as (2.16). An exact factorization leads to the following splitting:

$$K = \begin{bmatrix} C_{ff} & & \\ D_f & S_{pp} & S_{p\sigma} \\ C_{\sigma f} & S_{\sigma p} & S_{\sigma\sigma} \end{bmatrix} \begin{bmatrix} I & C_{ff}^{-1} G_f & C_{ff}^{-1} C_{f\sigma} \\ & I & \\ & & I \end{bmatrix} = LU, \tag{2.17}$$

where the Schur complements matrices $S_{pp} = -D_f C_{ff}^{-1} G_f$, $S_{p\sigma} = D_\sigma - D_f C_{ff}^{-1} C_{f\sigma}$, $S_{\sigma p} = G_\sigma - C_{\sigma f} C_{ff}^{-1} G_f$ and $S_{\sigma\sigma} = C_{\sigma\sigma} + N - C_{\sigma f} C_{ff}^{-1} C_{f\sigma}$.

In the inexact factorization, the inverse of the fluid sub-block C_f^{-1} is replaced by the zero-order term of its Neumann expansion:

$$C_{ff}^{-1} = \left(\frac{1}{\Delta t} M_{ff} + K_{ff} \right)^{-1} = \Delta t M_{ff}^{-1} + \mathcal{O}\left(\Delta t^2\right) \simeq \Delta t M_{ff}^{-1}, \tag{2.18}$$

where K_{ff} represents the fluid stiffness matrix associated with the viscous and convective terms, and M_{ff} the fluid velocity mass matrix. The approximations of the Schur complements follow $S_{pp} \simeq T_{pp} = -\Delta t D_f M_{ff}^{-1} G_f$, $S_{p\sigma} \simeq T_{p\sigma} = D_\sigma - \Delta t D_f M_{ff}^{-1} C_{f\sigma}$, $S_{\sigma p} \simeq T_{\sigma p} = G_\sigma - \Delta t C_{\sigma f} M_{ff}^{-1} G_f$ and $S_{\sigma\sigma} \simeq T_{\sigma\sigma} = C_{\sigma\sigma} + N - \Delta t C_{\sigma f} M_{ff}^{-1} C_{f\sigma}$. Now the approximated factors $\hat{L}$ and $\hat{U}$ are given by

$$\hat{L} = \begin{bmatrix} C_{ff} & & \\ D_f & T_{pp} & T_{p\sigma} \\ C_{\sigma f} & T_{\sigma p} & T_{\sigma\sigma} \end{bmatrix}, \; \hat{U} = \begin{bmatrix} I & \Delta t M_{ff}^{-1} G_f & \Delta t M_{ff}^{-1} C_{f\sigma} \\ & I & \\ & & I \end{bmatrix}. \tag{2.19}$$

If the fluid sub-block approximation (2.18) is used in both the lower and upper block-triangular factors, a pressure-correction like method (Chorin-Temam projection scheme) is obtained,

$$P_{PIC} = \hat{L}\hat{U}, \tag{2.20}$$

that is original designed for the numerical solution of the Navier-Stokes equations [24, 75], and is recently employed and analyzed in the explicit-implicit splitting algorithm for the FSI problems [31]. If the approximation is only used in the lower block triangular factor, a Yosida like method arrives

$$P_{FSY} = \hat{L}U, \tag{2.21}$$

that was originally proposed as an inexact factorization of the matrix arising from the numerical solution of the Navier-Stokes equations [62]. These two preconditioners are rather stable for the simulation of a pressure pulse propagation in a blood flow vessel.

In [5], an incomplete LU factorization P_{ILU}, the so-called $ILUT$ preconditioner [68] is applied to the diagonally scaled system:

$$P_{ILU}^{-1} D^{-1} K x = P_{ILU}^{-1} D^{-1} b, \tag{2.22}$$

where D denotes the diagonal coefficients of K. In addition, an inexact block LU factorization is used to construct the FSI preconditioner and the same zeroth order term as in (2.18) is utilized in both the factors. To reduce the computational cost, a lumped mass matrix in (2.18) is used. These preconditioners show good behavior in the large range of the added-mass effect [60].

2.6 A LDU block factorization

In [55], a new FSI preconditioner is constructed in form of $\hat{L}\hat{D}\hat{U}$ with $\hat{L}$, $\hat{D}$ and $\hat{U}$ being proper approximations to the matrices L, D and U in the LDU block factorization of the fully coupled and reordered Jacobian matrix

$$K = \begin{bmatrix} A_m & A_{ms} & \\ & A_s & A_{sf} \\ A_{fm} & A_{fs} & A_f \end{bmatrix}, \tag{2.23}$$

where the stiffness matrices A_m, A_s and A_f for the mesh movement, structure and fluid sub-problem are respectively lying on the main diagonal, and the coupling matrices $A_{\beta\alpha}$, $\beta, \alpha \in \{f, m, s\}$ and $\beta \neq \alpha$ among different fields on the off-diagonal. An exact LDU factorization of K reads:

$$K = \begin{bmatrix} I & & \\ & I & \\ A_{fm}A_m^{-1} & \tilde{A}_{fs}A_s^{-1} & I \end{bmatrix} \begin{bmatrix} A_m & & \\ & A_s & \\ & & S \end{bmatrix} \begin{bmatrix} I & A_m^{-1}A_{ms} & \\ & I & A_s^{-1}A_{sf} \\ & & I \end{bmatrix}, \tag{2.24}$$

with the fluid Schur complement

$$S = A_f - \tilde{A}_{fs}A_s^{-1}A_{sf} = A_f - (A_{fs} - A_{fm}A_m^{-1}A_{ms})A_s^{-1}A_{sf}.$$

The new FSI preconditioner $P_{LDU} = \hat{L}\hat{D}\hat{U}$ is formed as

$$P_{LDU} = \begin{bmatrix} I & & \\ & I & \\ A_{fm}\hat{A}_m^{-1} & \hat{\tilde{A}}_{fs}\hat{A}_s^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{A}_m & & \\ & \hat{A}_S & \\ & & \hat{S} \end{bmatrix} \begin{bmatrix} I & \hat{A}_m A_{ms} & \\ & I & \hat{A}_s^{-1}A_{sf} \\ & & I \end{bmatrix}, \tag{2.25}$$

where the Schur complement approximation $\hat{S}$ corresponds to the perturbed fluid sub-problem, that is obtained by modifying corresponding entries in the original fluid matrix with an explicitly constructed approximation to the exact perturbation stemming from the sparse matrix-matrix multiplications:

$$\hat{S} = A_f - \hat{\tilde{A}}_{fs}\hat{\tilde{A}}_s^{-1}A_{sf} = A_f - (A_{fs} - A_{fm}\hat{\tilde{A}}_m^{-1}A_{ms})\hat{\tilde{A}}_s^{-1}A_{sf}, \tag{2.26}$$

where $\hat{\tilde{A}}_m = \text{diag}[A_m]$ and $\hat{\tilde{A}}_s = \text{diag}[A_s]$ denote the point-wise block diagonal of the corresponding mesh movement and structure matrices.

The inverse operations of $\hat{A}_m^{-1}$, $\hat{A}_s^{-1}$ and $\hat{A}_f^{-1}$ in each factor of the Jacobian matrix are realized by applying a few W-cycles of a special class of algebraic multigrid methods [50, 86, 87, 40, 94] to the corresponding mesh, fluid and structure sub-problems with 0 as initial guess.

Further, dropping $\hat{L}$ leads to a reduced FSI preconditioner $\hat{P}_{LDU}$ in form of $\hat{P}_{LDU} = \hat{D}\hat{U}$:

$$\hat{P}_{LDU} = \begin{bmatrix} \hat{A}_m & & \\ & \hat{A}_S & \\ & & \hat{S} \end{bmatrix} \begin{bmatrix} I & \hat{A}_m A_{ms} & \\ & I & \hat{A}_s^{-1} A_{sf} \\ & & I \end{bmatrix}, \tag{2.27}$$

that is based on an inexact Uzawa method (see, e.g., [16]) and originally analyzed in [71] for the fluid problem. The preconditioners demonstrate relatively good robustness with respect to temporal and spatial discretization parameters, and certain material parameter.

3 Geometric multigrid methods

3.1 A Vanka-like smoother

In [47, 64], a special monolithic ALE FSI system with respect to displacement, velocity and pressure unknowns on the whole computational FSI domain is formulated in a unified manner; see also [89]. For such a formulation, the conforming biquadratic (for displacement and velocity), discontinuous linear (for pressure) Q_2P_1 finite element pair is chosen. This leads to a natural fulfillment of the interface conditions, namely, the no-slip condition and the balance of the surface tractions. In addition, it allows the fluid and structure pressure jumps across the interface, that usually occurs in the FSI simulation. A geometrical multigrid method is then used to solve the linearized system of equations with the Jacobian matrix:

$$K = \begin{bmatrix} S_{uu} & S_{uv} & \\ S_{vu} & S_{vv} & B \\ B_s^T & B_f^T & \end{bmatrix}, \tag{3.1}$$

where S represents the discrete reactive, diffusive and convective terms, B the discrete gradient term, B_s^T and B_f^T the discrete divergence term. In particular a Vanka-like smoother [84, 79] is performed on the mesh level l, i.e., on each patch (element) Ω_i, a local 39×39 system matrix K_{Ω_i} (in 2D) is inverted, with 3 pressure degrees of freedom and associated 36 displacement and velocity degrees of freedom:

$$x_l^{k+1} = x_l^k + \omega^k \sum_{\Omega_i} K_{\Omega_i}^{-1} r_{\Omega_i}^n, \; k = 1, ..., \nu, \tag{3.2}$$

where ν denotes the smoothing steps, ω^k the relaxation parameter, $r_{\Omega_i}^n$ the defection on the patch Ω_i, and K_{Ω_i} has the following structure:

$$K_{\Omega_i} = \begin{bmatrix} S_{uu|\Omega_i} & S_{uv|\Omega_i} & \\ S_{vu|\Omega_i} & S_{vv|\Omega_i} & B_{|\Omega_i} \\ B_{s|\Omega_i} & B_{f|\Omega_i} & \end{bmatrix}, \tag{3.3}$$

that is a restriction of the global stiffness matrix K onto the patch Ω_i. This geometric multigrid method shows the accuracy and robustness with respect to the constitutive models.

3.2 A Dirichlet-Neumann smoother

In [66], a geometric mulitigrid preconditioner within the GMRES method is used in solving the linearized FSI system of algebraic equations. A smoothing step consists of approximations of the structure Neumann and fluid Dirichlet sub-problems on the mesh level l:

$$x_l^{k+1} = \mathbf{S}_l(x^k, b_l),\ k = 1, ..., \nu, \tag{3.4}$$

where $\mathbf{S}_l$ denotes the corresponding smoothing operator. In fact, the Neumann and Dirichlet sub-problem approximations correspond to the dynamic (balance of surface traction) and kinematic (no-slip) interface conditions, respectively. For approximation of each sub-problem in the smoothing step, several preconditioned Richardson iterations are applied. In particular, a stabilized incomplete lower-upper decomposition of the stiffness matrix for each sub-problem is used as a corresponding preconditioner. The numerical results have demonstrated the robustness and efficiency with respect to the mesh size and problem parameters, e.g., fluid density, average inflow velocity, shear modulus and Poisson's ratio.

4 Algebraic multigrid methods

4.1 Smoothed aggregation multigrid methods

In [36], the block Gauss-Seidel preconditioner combined with approximate block inverses for respective fluid, mesh and structure sub-problems by algebraic multigrid methods are used to solve the FSI system with the Jacobian matrix:

$$K = \begin{bmatrix} S & & S^{SF} \\ & G & G^{GF} \\ F^{FS} & F^{FG} & F \end{bmatrix}, \tag{4.1}$$

where on the main diagonal, S, G and F represent the stiffness matrix from the structure, mesh and fluid sub-problem, respectively, the off-diagonal blocks the coupling matrix between two different fields indicated with the supscriptions S-structure, F-fluid and G-mesh. In particular, the preconditioner P_{BGS} has the following form

$$P_{BGS} = \begin{bmatrix} S & & \\ & G & \\ F^{FS} & F^{FG} & F \end{bmatrix}. \tag{4.2}$$

As pointed out in [36], that the preconditioner P_{BGS} only models a one-way interaction from the structure to the fluid without feedback from the fluid to the structure. To approximate the inverses of sub-problem in the preconditioning steps, in particular, the so-called smoothed aggregation multigrid for symmetric positive definite elasticity equations [83, 82, 56] and Petrov-Galerkin smoothed algebraic multigrid for the nonsymmetric Navier-Stokes equations [69] are used.

Furthermore, a truly monolithic algebraic multigrid method was developed for solving the FSI system of equations in [36]. The combined level transfer operators P_l^{FSI} and R_l^{FSI} using the individual prolongation and restriction operators for each sub-problem on a level l are constructed as

$$P_l^{FSI} = \begin{bmatrix} P_l^S & & \\ & P_l^G & \\ & & P_l^F \end{bmatrix}, \; R_l^{FSI} = \begin{bmatrix} R_l^S & & \\ & R_l^G & \\ & & R_l^F \end{bmatrix}, \tag{4.3}$$

where P_l^S, P_l^G and P_l^F are prolongation operators on level l for the structure, mesh and fluid sub-problem, respectively; R_l^S, R_l^G and R_l^F are corresponding restriction operators. The matrix on the coarse level $l+1$ is then built as:

$$\begin{aligned} K_{l+1} &= \begin{bmatrix} R_l^S & & \\ & R_l^G & \\ & & R_l^F \end{bmatrix} \begin{bmatrix} S & & S^{SF} \\ & G & G^{GF} \\ F^{FS} & F^{FG} & F \end{bmatrix}_l \begin{bmatrix} P_l^S & & \\ & P_l^G & \\ & & P_l^F \end{bmatrix} \\ &= \begin{bmatrix} R_l^S S_l P_l^S & & R_l^S S_l^{SF} P_l^F \\ & R_l^G G_l P_l^G & R_l^G G_l^{GF} P_l^F \\ R_l^F F_l^{FS} P_l^S & R_l^F F_l^{FG} P_l^G & R_l^F F_l P_l^F \end{bmatrix}. \end{aligned} \tag{4.4}$$

The damped Richardson iteration with block Gauss-Seidel preconditioner $\tilde{P}_{BGS}$ is used as a smoother: For $k = 0$,

$$x^{k+1} = x^k + \omega_k \tilde{P}_{BGS}^{-1}(b - Kx^k) \tag{4.5}$$

with the Richardson iteration index k and a level-dependent damping parameter ω_k. In the preconditioning step, the inverse approximation for each sub-problem are realized by simple smoothing steps, e.g., ILU or Gauss-Seidel. The monolithic FSI solution method combined with the new monolithic algebraic multigrid shows the best performance.

In [58], a novel generalized monolithic solution procedure is proposed, that keeps consistency in the nonconforming temporal and spatial discretizations. The above algebraic multigrid methods are utilized in solving the condensed monolithic linear system by eliminating the unknown Lagrange multipliers λ discretized by dual mortar shape functions [90].

In [35], these block Gauss-Seidel and monolithic algebraic multigrid preconditioners are employed in the preconditioners for the coupled system of the lung model problem, which contains the standard FSI system and additional algebraic constraints [95]. Using the idea of semi-implicit methods for the pressure-linked equations (SIMPLE, see, e.g., [22]) of the coupled system, the predictor equation, arising there, is actually corresponding to the FSI system and approximated by these FSI preconditioners.

4.2 Matrix-graph based algebraic multigrid methods

In [54], a similar technique has been employed in developing algebraic multigrid methods for solving the linearized FSI system using three hyperelastic structure models, namely Neo-Hookean, Mooney-Rivlin and anisotropic two-layer thick-walled artery (see [8, 46, 1]). In particular, the W-cycles of the special matrix-graph based algebraic multigrid methods for individual fields have been applied in each smoothing step [50, 86, 87, 40, 94]. Furthermore, the algebraic multilevel methods has been developed for solving the coupled FSI system [2, 3, 85, 51]. These methods show robustness with respect to the mesh size and different structure models.

5 Domain decomposition preconditioners

5.1 A restricted additive Schwarz preconditioner

In [92], a fully implicit domain decomposition based Newton-Krylov-Schwarz method is proposed for solving the nonlinear coupled FSI system of algebraic equations in 3D. In solving the right-preconditioned Jacobian system with the GMRES method, a restricted additive Schwarz preconditioner [21] is used, that is an overlapping variant of the classical additive Schwarz preconditioner for general sparse linear systems:

$$P_{RAS}^{-1} = \sum_{l=1}^{N} (R_l^0)^T B_l^{-1} R_l, \tag{5.1}$$

where R_l denotes the restriction operator (a mapping of global unknowns to the corresponding ones on the local overlapping sub-domain), $B_l = R_l K R_l^T$ the local sub-domain Jacobian, R_l^0 the restrictions to the unknowns on the non-overlapping sub-domain. Such a preconditioner requires a particular partition of the finite element meshes into overlapping sub-domains containing both the fluid and structure elements. Note that the overlapping regions are only used to construct the sub-domain Jacobians and provide information to solve the local sub-problems, but not considered in the prolongation procedure. Although increasing overlapping size will reduce iteration numbers of the GMRES method, the computational time may increase due to higher local sub-problem solving cost. The preconditioner has shown parallel scalability up to 2048 processors and robustness with respect to material parameters, e.g., fluid density, structure density, Young's modulus and Poisson ratio.

The same technique has been used in earlier work [11] for solving the FSI problems in 2D, that shows the robustness with respect to some material parameters, and parallel scalability up to hundreds of processors.

5.2 A two-level Schwarz preconditioner

In [12], a two-level Newton-Krylov-Schwarz algorithm is used to solve the nonlinear FSI system at each time step, that is given in an abstract form:

$$G_h(x_h) = 0. \tag{5.2}$$

The solution of the analogous coarse grid problem

$$G_H(x_H) = 0 \tag{5.3}$$

is taken as a good initial guess for solving the nonlinear problem (5.2) on the fine mesh. We mention that the nonlinear system (5.3) on the coarse mesh is solved by an inexact Newton method. To solve the linearized system on the coarse and fine meshes, the one-level restricted additive Schwarz preconditioner P_H as (5.1) and the two-level hybrid preconditioner P_h [72, 77] are employed, respectively. In particular, on the fine mesh level, a multiplicative preconditioner combined both the coarse-level and fine-level ones is proposed:

$$P_h^{-1} = I_H^h B_0^{-1} I_h^H + \sum_{l=1}^{N} (R_l^0)^T B_l^{-1} R_l, \tag{5.4}$$

where the second term is the restricted additive Schwarz preconditioner on the fine level, I_h^H denotes the restriction from the fine grid to the coarse one, $I_H^h = (I_h^H)^T$ the corresponding interpolation operator from the coarse grid to the fine one, and B_0^{-1} the solution procedure on the coarse grid. The preconditioner has demonstrated robustness with respect to the structure's Young's modulus and the time step size, and the nearly perfect weak scaling up to some thousand processors.

6 Some numerical studies

In this section, we test some of monolithic FSI solvers that are developed in our previous work [54, 55]. We solve a 3D benchmark problem using block Gauss-Seidel, complete LDU factorization and AMG preconditioned GMRES solvers. The computational FSI domain and the mesh, generated by Netgen [70], are illustrated in Fig. 2. Three mesh levels are used to test the robustness of the preconditioners; see mesh information in Table 1, where we provide the number of nodes (#Nod), tetrahedral elements (#Tet) and degrees of freedom (#Dof).

In this experiments, we extend the FSI2 test, that was originally proposed in [80] as a two-dimensional FSI test, to three dimensions. The geometry configuration is

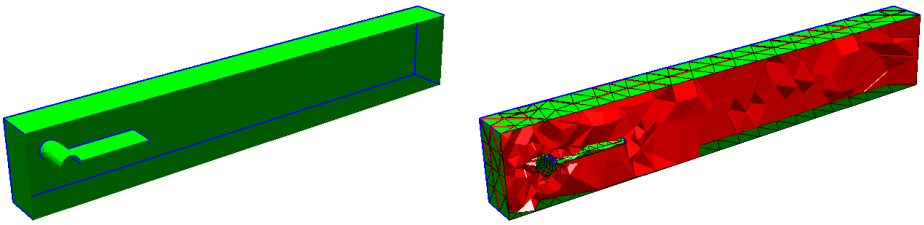

Figure 2. Geometry (left) and mesh (right) for half of the computational FSI domain.

Level	#Nod	#Tet	#Dof
L_1	1913	8400	14073
L_2	13166	67200	93114
L_3	97292	537600	670944

Table 1. Three finite element meshes.

illustrated in Fig. 3. The structure domain is an elasticity plate with length 40 cm, 30 cm and 2 cm in the x-, y- and z-direction, respectively, attached to a cylinder. The fluid domain is a cuboid with length 250 cm, 41 cm and 41 cm in the three directions, respectively, cut by the cylinder along the y-direction. The cylinder has length 41 cm in the y-direction and radius 5 cm at the rest, which is fixed as in the Turek's benchmark problem. We mention that the elasticity plate is a bit away from the center of the cuboid in either y- or z-direction.

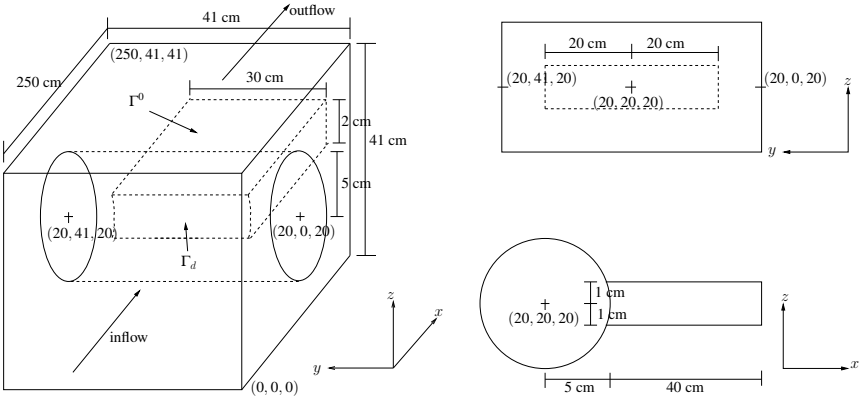

Figure 3. A schematic illustration of the geometry and boundary configuration.

For the structure, we use the nonlinear hyperelastic model of the St. Venant Kirchhoff material with density $\rho_s = 1$ g/cm^3 and Lamé constants $\mu_s = 1.15 \times 10^6$ dyn/cm^2

and $\lambda_s = 1.73 \times 10^6$ dyn/cm^2. The structure is fixed on the touching part with the cylinder, i.e., Γ_d in Fig. 3. The rest of the boundary is the interface Γ^0.

We use the incompressible Navier-Stokes equations to model the fluid. The fluid has density $\rho_f = 1.0$ g/cm^3 and kinematic viscosity $\nu = 0.035$ cm^2/s. A smooth parabolic velocity profile in time is prescribed at the inflow:

$$u(0,y,z,t) = \begin{cases} 1.5\bar{U}\frac{yz(H-y)(H-z)}{(\frac{H}{2})^2(\frac{H}{2})^2}\frac{1-\cos(\frac{\pi}{2}t)}{2} \text{ cm/s if } t < 2.0 \\ 1.5\bar{U}\frac{yz(H-y)(H-z)}{(\frac{H}{2})^2(\frac{H}{2})^2} \text{ cm/s if } t \geq 2.0 \end{cases} \tag{6.1}$$

with $H = 41$ and $\bar{U} = 100$. At the outflow, we use the homogeneous Neumann boundary condition. The no-slip condition is prescribed on the rest of the fluid domain boundary.

The structure is initially at rest. The initial condition for the fluid is homogeneous zero.

For spatial and temporal discretization, and linearization of the nonlinear coupled FSI system, we refer to our work in [54, 55]. The time step size is set to 0.625 ms. The velocity fields and structure deformation (enlarged by a factor 16) at time $t = 1.625$ s are plotted in Fig. 4.

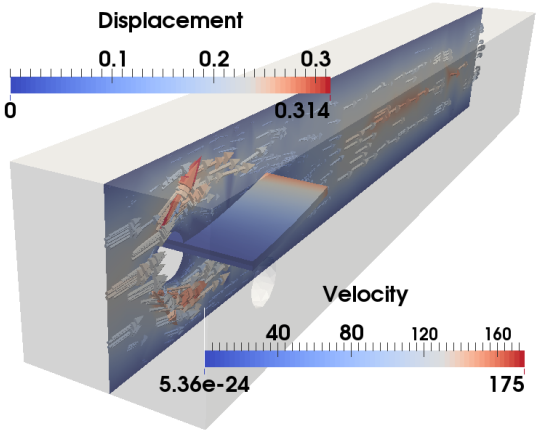

Figure 4. Visualization of fluid velocity field and structure displacement.

For illustration purpose, we only show the iteration numbers and cost in second (s) for solving one linearized FSI system of equations. We stop solving the system when the relative residual error of the preconditioned system in the GMRES method is reduced by a factor 10^7. For the performance of the complete simulation of some 3D benchmarks, we refer to the detailed numerical studies in [55]. In Table 2, we record the number of iterations and computational time in second (the value in brackets) using three preconditioned GMRES solvers on the three mesh levels. It is easy to see from our numerical experiments that, the LDU factorization preconditioned GMRES solver

	GS_GMRES	AMG_GMRES	LDU_GMRES
L_1	23 (26.1 s)	15 (262.1 s)	4 (7.5 s)
L_2	54 (493.6 s)	12 (461.6 s)	7 (183.3 s)
L_3	164 (12133.3 s)	16 (3107.9 s)	9 (2426.3 s)

Table 2. Number of iterations and cost in second (s) of the solvers: GS_GMRES (GMRES solver preconditioned by block Gauss-Seidel), AMG_GMRES (GMRES solver preconditioned by AMG) and LDU_GMRES (GMRES solver preconditioned by complete LDU factorization.

shows better performance than the other two. It is worth mentioning that the cost of both GS_GMRES and LDU_GMRES solvers scales well with respect to the number of degrees of freedom and iteration numbers. For the AMG preconditioned GMRES solver, we observe better scaling factor than expected. That is because we have the same problem size on the coarsest AMG level for all the three mesh levels, which is relatively large (around 2500×2500) and dominates computational cost on the level L_1. The problem size on the coarsest level is not easily reduced further, since in the benchmark, the structure has much fewer Dofs than the fluid one.

7 Conclusions

In this work, we present a short overview on recent development of monolithic FSI solvers in the last decade, that are developed from classical methods for solving individual field problems. Several well developed preconditioners, based on block factorization, geometric and algebraic multigrid, domain decomposition methods, have been discussed. These methods have shown their robustness and efficiency with respect to temporal and spatial discretization parameters as well as material parameters for a certain class of FSI problems. The topic is relatively new and is still under further investigation, although a lot of progress has been made by many research groups.

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