ÖAW AUSTRIAN ACADEMY OF SCIENCES

Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

RICAM
JOHANN-RADON-INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Topological solvability and DAE-index conditions for mass flow controlled pumps in liquid flow networks

A-K. Baum, M. Kolmbauer, G. Offner

RICAM-Report 2016-33

Topological solvability and DAE-index conditions for mass flow controlled pumps in liquid flow networks

Ann-Kristin Baum*† Michael Kolmbauer† Günter Offner ‡

Thursday 29th September, 2016

Abstract

This work is devoted to the analysis of a model for the thermal management in liquid flow networks consisting of pipes and pumps. The underlying model equation for the liquid flow is not restricted to the equation of motion and the continuity equation, describing the mass transfer through the pipes, but also includes thermodynamic effects in order to cover cooling and heating processes. The resulting model gives rise to a differential-algebraic equation (DAE), for which a proof of unique solvability and an index analysis is presented. For the index analysis, the concepts of the *Strangeness Index* is pursued. Exploring the network structure of the liquid flow network via graph theoretical approaches allows to develop network topological criteria for the existence of solutions and the DAE-index. The topological criteria are explained by various examples.

Keywords: Differential-algebraic equations, topological index criteria, hydraulic network.

AMS(MOS) subject classification: 65L80, 94C15

Introduction

Increasingly demanding emissions legislation specifies the performance requirements for the next generation of products from vehicle manufacturers. Conversely, the increasingly stringent emissions legislation is coupled with the trend in increased power, drivability and safety expectations from the consumer market. Promising approaches to meet these requirements are downsizing the internal combustion engines (ICE), the application of turbochargers, variable valve timing, advanced combustion systems or comprehensive exhaust aftertreatment but also different variants of combinations of the ICE with an electrical engine in terms of hybridization or even a purely electric propulsion. The challenges in the development of future powertrains do not only lie in the design of individual components but in the assessment of the powertrain as a whole. On a system engineering level it is required to optimize individual components globally and to balance the interaction of different sub-systems. A typical system engineering model comprises several sub-systems. For instance in case of a hybrid propulsion these can be the vehicle chassis, the drive line, the air path of the ICE including combustion and exhaust

*Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenberger Str. 69, 4040 Linz, Austria.

†MathConsult GmbH, Altenberger Str. 69, 4040 Linz, Austria.

‡AVL List GmbH, Hans-List-Platz 1, 8020 Graz, Austria.

aftertreatment, the cooling system of the ICE, the electrical propulsion system including the engine and a battery pack and finally according control systems. Similar to the ICE, the battery pack requires a cooling system as well. Both cooling systems are typically represented by an according hydraulic network in the overall model. The simulation and optimization of hydraulic networks has been studied in various works, including [13, 12, 23, 4, 7] and the references therein. The considered models are motivated by drinking water supply systems, where the main target is to circulate an amount of water at any time, assuring a certain pressure at extraction points. The aim of this work is to consider and analyze hydraulic networks used for thermal management systems. Examples in automotive applications are the above mentioned cooling systems.

In contrast to water transportation networks, the primary interest is not the pressure distribution across the whole system, but the temperature distribution. Consequently, the models have to be equipped with energy balance laws in order to model the thermodynamic effects. The intention of this work is to extend the results, which are already available for water transportation networks [13, 12] to cooling and heating systems used for thermal management and to networks including mass flow controlled pumps. Thermal flow networks consisting solely of pipes have been analyzed to the work in [2]. The extension to networks of pipes and pumps is not straight forward, since additional to Kirchhoff's first law also Kirchhoff's second law has to be considered. Especially Kirchhoff's second law restricts the allowed pump constellations for a valid liquid flow models.

The model under consideration is a quasi-stationary pipe network, cp. [13], equipped with energy balance laws. This model is suited to describe circuits, which are filled with incompressible fluids (e.g. water). Here incompressible means, that density changes with respect to temperature changes or pressure changes are neglected.

While general networks consist of various types of elements (pipes, pumps, valves) [23], the model here is restricted to pipes and pumps only. Despite this simplification, the demanding issues are caused by the arbitrary network structure of the underlying model. Since valves can change the topology of the underlying network due to there discrete nature, they have to be treated separately.

State-of-the-art modeling and simulation packages such as Dymola[1], Matlab/Simulink[2], Flowmaster[3], Amesim[4], SimulationX[5] or Cruise M[6] offer many excellent concepts for the automatic generation of dynamic system models, including hydraulic networks. Modeling is done in a modularized way, based on a network of subsystems which again consists of simple standardized sub-components. The network structure (topology) carries the core information of the network properties and therefore is predestinated to be exploited for the analysis and numerical simulation of those. In many applications the network describing equations are differential-algebraic equations (DAEs). Hence the analysis of existence and uniqueness of solutions, as well as rank considerations are a delicate issue.

Topology based index analysis for networks connects the research fields of *Analysis for DAEs* [22] and *Graph Theory* [8] in order to provide the appropriate base to analyze DAEs stemming from automatic generated system models. So far it has been established for various types of

[1] http://www.dynasim.com
[2] http://www.mathworks.com
[3] http://www.mentor.com
[4] http://www.plm.automation.siemens.com
[5] http://www.iti.de
[6] http://www.avl.com

networks, including electric circuits [24], gas supply networks [10] and water supply networks [13, 12, 23]. Although all those networks share some similarities, an individual investigation is required due to their different physical nature. Recently, a unified modeling approach for different types of flow networks has been introduced in [14], aiming for a unified topology based index analysis for the different physical domains on an abstract level.
The structure of this work is the following. In Section 1, the main two concepts required for the basic ingredients of the analysis are introduced. First an introduction to graph theory is given, then the application to equations imposed on networks is described and the core tools for the following analysis are proven. The network model and arising DAE is formulated in Section 2. Furthermore some basic properties are derived, which lead to the full DAE analysis in Section 3. Beside existence and uniqueness results, DAE-index considerations are performed. Throughout the analysis, the sufficient algebraic conditions are linked to necessary conditions imposed on the network structure. Those topological conditions are explained in term of examples. A summary of the results with comments on their practical relevance in commercial simulation software concludes the paper in Section 4.

1 Graphs and their application in network dynamics

In this section, we introduce the notation and graph theoretical concepts that we need in our analysis and prove some additional results in Lemma 1.1 and Lemma 1.2.

For a detailed introduction to graph theory, we refer the reader to, e.g., [8] or [3]. A *graph* $\mathscr{G}$ is a pair $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$ of subsets $\mathscr{V}, \mathscr{E} \subset \mathbb{N}$ such that $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$, i.e., each element $e_j \in \mathscr{E}$ corresponds to a pair $(v_{i_1}, v_{i_2}) \in \mathscr{V} \times \mathscr{V}$, cp. [8, p. 2]. If the pairs $(v_i, v_k) \in \mathscr{E}$ are *ordered*, then $\mathscr{G}$ is called an *oriented* graph, cp. [8, p. 25]. If $\mathscr{G}$ is oriented, then v_i and v_k are called originating and terminating vertex of $e_j = (v_i, v_k)$, respectively, [8, p. 25]. If $\mathscr{G}$ contains no self-loops or parallel edges, then $\mathscr{G}$ is called *simple*, cp. [8, p. 25].
Two vertices $v_i, v_k \in \mathscr{V}$ are called *adjacent* if there exists an edge $e_j \in \mathscr{E}$ such that $e_j = (v_i, v_{i_2})$ [8, p. 13]. The edge e_j is called *incident* to v_i and v_i, respectively [8, p. 13]. Two edges $e_j, e_l \in \mathscr{E}$ are called *adjacent* if they are incident with a common vertex v_i [8, p. 13]. For $v_i \in \mathscr{V}$, the incident edges are summarized in the set

$$\mathscr{E}_{inc}(v_i) := \{e_j \in \mathscr{E} \,|\, \exists\, v_\ell \in \mathscr{V} : e_j = (v_i, v_\ell)\}.$$

If $\mathscr{E}_{inc}(v_i) = \emptyset$, then v_i is *isolated* and if $|\mathscr{E}_{inc}(v_i)| = 1$, then v_i is an *end vertex* [8, p. 2].
The connection structure of $\mathscr{G}$ is described by the *incidence matrix* A, which, if $\mathscr{G}$ is oriented, is defined as

$$A_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is the left vertex of } e_j, \\ -1, & \text{if } v_i \text{ is the right vertex of } e_j, \\ 0, & \text{else.} \end{cases}$$

A subset $\mathscr{G}_s = \{\mathscr{V}_s, \mathscr{E}_s\}$ with $\mathscr{V}_s \subset \mathscr{V}_s$ is a *subgraph* of $\mathscr{G}$ if $\mathscr{E}_s \subset \mathscr{V}_s \times \mathscr{V}_s$ [8, p. 3]. If $\mathscr{V}_s = \mathscr{V}$, then $\mathscr{G}_s$ *spanns* $\mathscr{G}$ [8, p. 3]. The incidence matrix of $\mathscr{G}_s$ is given by $A_s = [A_{ij}]_{(v_i, e_j) \in \mathscr{V}_s \times \mathscr{E}_s}$ [8, p. 3].

In our analysis, we consider a simple, oriented graph $\mathscr{G}$ whose vertices $\mathscr{V}$ and edges $\mathscr{E}$ are composed from subsets $\mathscr{V}_1, ..., \mathscr{V}_{\hat{v}}$ and $\mathscr{E}_1, ..., \mathscr{E}_{\hat{e}}$ such that $\mathscr{V} = \cup_{I=1}^{\hat{v}} \mathscr{V}_{\mathrm{I}}$, $\mathscr{E} = \cup_{J=1}^{\hat{e}} \mathscr{E}_{\mathrm{J}}$. Accordingly, the incidence matrix A is composed from submatrices A_{IJ} describing the connection structure of the subsets $\mathscr{G}_{\mathrm{IJ}} := \{\mathscr{V}_{\mathrm{I}}, \mathscr{E}_{\mathrm{J}}\}$. In general, a set $\mathscr{G}_{\mathrm{IJ}}$ is not a proper subgraph as the edges $\mathscr{E}_{\mathrm{J}}$ may be incident to vertices outside $\mathscr{V}_{\mathrm{I}}$. Then, the connection matrix A_{IJ} does not have the usual pattern of two non-zero entries per column. To characterize the fundamental subspaces of A_{IJ}, we partition the edges into

$$\mathscr{E}_{\mathrm{J}} = \mathscr{E}_{\mathrm{IJ}}^{\mathrm{inner}} \cup \mathscr{E}_{\mathrm{IJ}}^{\mathrm{loose}} \cup \mathscr{E}_{\mathrm{IJ}}^{\mathrm{isolated}},$$

where $\mathscr{E}_{\mathrm{J}}^{\mathrm{inner}}$ contains the edges incident to vertices in $\mathscr{V}_{\mathrm{I}}$, i.e.,

$$\mathscr{E}_{\mathrm{J}}^{\mathrm{inner}} := \{e_j \in \mathscr{E}_{\mathrm{J}} \,|\, e_j = (v_{j_1}, v_{j_2}) \text{ with } v_{j_1}, v_{j_2} \in \mathscr{V}_{\mathrm{I}}\},$$

$\mathscr{E}_{\mathrm{IJ}}^{\mathrm{loose}}$ contains the *loose edges* incident to a vertex in $\mathscr{V}_{\mathrm{I}}$ and a vertex outside $\mathscr{V}_{\mathrm{I}}$, i.e.,

$$\mathscr{E}_{\mathrm{IJ}}^{\mathrm{loose}} := \{e_j \in \mathscr{E}_{\mathrm{J}} \,|\, e_j = (v_{j_1}, v_{j_2}) \text{ with } v_{j_1} \in \mathscr{V}_{\mathrm{I}}, v_{j_2} \in \mathscr{V} \setminus \mathscr{V}_{\mathrm{I}}\},$$

and $\mathscr{E}_{\mathrm{IJ}}^{\mathrm{isolated}}$ contains the *isolated edges* incident to vertices outside $\mathscr{V}_{\mathrm{I}}$, i.e.,

$$\mathscr{E}_{\mathrm{IJ}}^{\mathrm{isolated}} := \{e_j \in \mathscr{E}_{\mathrm{J}} \,|\, e_j = (v_{j_1}, v_{j_2}) \text{ with } v_{j_1}, v_{j_2} \in \mathscr{V} \setminus \mathscr{V}_{\mathrm{I}}\}.$$

For simplicity, we assume that there is at most one loose edge per vertex. Using an equivalence relation, the following results can be extend to the case of multiple loose edges per vertex. Furthermore, we set

$$\mathscr{V}_{\mathrm{IJ}}^{\mathrm{outer}} := \mathscr{V}_{\mathrm{I}} \cup \mathscr{V}_{\mathrm{IJ}}^{\mathrm{c}},$$

where $\mathscr{V}_{\mathrm{IJ}}^{\mathrm{c}}$ contains the vertices outside $\mathscr{V}_{\mathrm{I}}$ that are incident to edges in $\mathscr{E}_{\mathrm{J}}$, i.e.,

$$\mathscr{V}_{\mathrm{IJ}}^{\mathrm{c}} := \{v_i \in \mathscr{V} \setminus \mathscr{V}_{\mathrm{I}} \,|\, \mathscr{E}_{\mathrm{adj}}(v_i) \cap \mathscr{E}_{\mathrm{J}} \neq \emptyset\}.$$

With this notation, we set

$$\mathscr{G}_{\mathrm{IJ}}^{\mathrm{outer}} := \{\mathscr{V}_{\mathrm{IJ}}^{\mathrm{outer}}, \mathscr{E}_{\mathrm{J}}\}, \quad \mathscr{G}_{\mathrm{IJ}}^{\mathrm{inner}} := \{\mathscr{V}_{\mathrm{I}}, \mathscr{E}_{\mathrm{IJ}}^{\mathrm{inner}}\},$$

where $\mathscr{G}_{\mathrm{IJ}}^{\mathrm{outer}}$ is the minimal subgraph containing $\mathscr{G}_{\mathrm{IJ}}$ and $\mathscr{G}_{\mathrm{IJ}}^{\mathrm{inner}}$ the maximal subgraph contained in $\mathscr{G}_{\mathrm{IJ}}$. Using $\mathscr{G}_{\mathrm{IJ}}^{\mathrm{outer}}$, $\mathscr{G}_{\mathrm{IJ}}^{\mathrm{inner}}$ we can straightforward extend the standard definitions, cp., e.g., [6, 8], to the set $\mathscr{G}_{\mathrm{IJ}}$.

A subset $\mathscr{P} := \{\mathscr{V}_{\mathscr{P}}, \mathscr{E}_{\mathscr{P}}\} \subset \mathscr{G}_{\mathrm{IJ},k}$ is called a *path* in $\mathscr{G}_{\mathrm{IJ}}$ if it is a path in $\mathscr{G}_{\mathrm{IJ}}^{\mathrm{outer}}$, i.e., if the vertices in $\mathscr{V}_{\mathscr{P}}$ are pairwise distinct and there exists a numbering such that v_i, e_j are adjacent to v_{i+1}, e_{j+1} for $(i,j) \in \mathbb{N}^{|\mathscr{V}_{\mathscr{P}}|-1} \times \mathbb{N}^{|\mathscr{E}_{\mathscr{P}}|-1}$, respectively. If $\mathscr{G}$ is oriented, with respect to this numbering, we assign a *sign* to every edge $e_j \in \mathscr{P}$ by

$$\mathrm{sgn}_{\mathscr{P}}(e_j) = \begin{cases} 1, & e_j = (v_{j_1}, v_{j_2}), \\ -1, & e_j = (v_{j_2}, v_{j_1}), \end{cases} \tag{1}$$

and define the *path matrix* $P = \sum_{e_j \in \mathscr{E}_{\mathscr{P}}} \mathrm{sgn}_{\mathscr{P}}(e_j) e_j$, where $e_1, ..., e_{|\mathscr{E}|} \in \mathbb{R}^{|\mathscr{E}|}$ denotes the standard canonical basis.

If $\mathrm{sgn}_{\mathscr{P}}(e_j) = \mathrm{sgn}_{\mathscr{P}}(e_l)$ for $e_j, e_l \in \mathscr{E}_{\mathscr{P}}$, then $\mathscr{P}$ is called *directed*. If $v_{i_1}, v_{i_{|\mathscr{E}_{\mathscr{P}}|}} \in \mathscr{V}_{\mathrm{IJ}}^{\mathrm{c}}$, then $\mathscr{P}$ is called a *crossing path*. If $v_{i_1}, v_{i_{|\mathscr{E}_{\mathscr{P}}|}} \in \mathscr{V}_{\mathrm{I}}$ with $v_{i_1} = v_{i_{|\mathscr{E}_{\mathscr{P}}|}}$, then $\mathscr{C} := \mathscr{P}$ is called a *cycle* in $\mathscr{G}_{\mathrm{IJ}}$.

The set $\mathscr{G}_{\mathrm{IJ}}$ is *connected* if $\mathscr{E}_{\mathrm{IJ}}^{\mathrm{isolated}} = \emptyset$ and if $\mathscr{G}_{\mathrm{IJ}}^{\mathrm{inner}}$ is connected, i.e., if every pair of vertices $v_i, v_k \in \mathscr{V}_{\mathrm{I}}$ can be connected by a path. If $\mathscr{G}_{\mathrm{IJ}}$ is not connected, then it is composed of connected components $\mathscr{G}_{\mathrm{IJ},k} = \{\mathscr{V}_{\mathrm{IJ},k}, \mathscr{E}_{\mathrm{IJ},k}\}$, $k = 1, ..., K$, containing the connected components $\mathscr{G}_{\mathrm{IJ},k}^{\mathrm{inner}}$ of $\mathscr{G}_{\mathrm{IJ}}^{\mathrm{inner}}$, respectively, plus loose edges $\mathscr{E}_{\mathrm{IJ},k}^{\mathrm{loose}}$ incident to vertices in $\mathscr{G}_{\mathrm{IJ},k}^{\mathrm{inner}}$.

A subgraph $\mathscr{T}_1 \subset \mathscr{G}$ of a connected graph $\mathscr{G}$ that contains no cycles and spans $\mathscr{G}$ is called a *spanning tree* [8, p. 13]. Every connected graph has at least one spanning tree [8, p. 14]. If $\mathscr{G}_{\mathrm{IJ}}$ is connected, then a subset $\mathscr{T}_1 \subset \mathscr{G}_{\mathrm{IJ}}$ is called a *spanning tree* if $\mathscr{T}_1 = \mathscr{T}_1^{\mathrm{inner}} \cup \{e_0\}$, where $\mathscr{T}_1^{\mathrm{inner}}$ is a spanning tree of $\mathscr{G}_{\mathrm{IJ}}^{\mathrm{inner}}$ and $e_0 \in \mathscr{E}_{\mathrm{IJ}}^{\mathrm{loose}}$ is a *reference loose edge*. The complement $\mathscr{T}_2$ is called the *chord set*. For the incidence matrix, the associated edges are selected by the permutation $\Pi = [\Pi_1, \Pi_2]$ with $\Pi_i = [e_j]_{e_j \in \mathscr{T}_i}$, $i = 1, 2$, where $e_1, ..., e_{|\mathscr{E}_{\mathrm{J}}|} \in \mathbb{R}^{|\mathscr{E}_{\mathscr{J}}|}$ denotes the standard canonical basis. Every interior chord $e_k \in \mathscr{E}_{\mathscr{T}_2} \cap \mathscr{E}_{\mathrm{IJ}}^{\mathrm{inner}}$ defines a unique *fundamental cycle* $\mathscr{C}_k = \{\mathscr{V}_k, \mathscr{E}_k\}$ with $\mathscr{E}_k \setminus \{e_l\} \subset \mathscr{T}_1 \cap \mathscr{E}_{\mathrm{IJ}}^{\mathrm{inner}}$. The *fundamental cycle matrix* is defined as $C = [C_1, ..., C_c]$, where C_k is the cycle matrix of $\mathscr{C}_k$. Similarly, every loose chord $e_k \in \mathscr{E}_{\mathrm{IJ}}^{\mathrm{loose}} \setminus \{e_0\}$ defines a unique *fundamental crossing path* $\mathscr{P}_k$ starting and ending (or vice versa) in e_k and e_0, respectively. The *fundamental crossing path matrix* is defined as $P_{\mathrm{IJ}} = [P_1, ..., P_{|\mathscr{E}_{\mathrm{IJ}}^{\mathrm{loose}}|-1}]$, where P_k denotes the path matrix of $\mathscr{P}_k$.

If $\mathscr{G}_{\mathrm{IJ}}$ is connected and $\mathscr{E}_{\mathrm{IJ}}^{\mathrm{loose}} = \emptyset$, then choosing a *ground node* $v_0 \in \mathscr{V}$, we set $\mathscr{V}_2 := \{v_0\}$ and denote the associated *reduced vertex set* by $\mathscr{V}_1 := \mathscr{V} \setminus \mathscr{V}_2$. If $\mathscr{E}_{\mathrm{IJ}}^{\mathrm{loose}} \neq \emptyset$, then $\mathscr{V}_2 = \emptyset$ and the reduced vertex set is given by $\mathscr{V}_1 = \mathscr{V}_{\mathrm{I}}$. For the incidence matrix, these vertices are selected by the permutation $\Gamma = [\Gamma_1, \Gamma_2]$ with $\Gamma_i = [e_k]_{v_k \in \mathscr{V}_i}$, $i = 1, 2$, where $e_1, ..., e_{|\mathscr{V}|} \in \mathbb{R}^{|\mathscr{V}_{\mathrm{I}}|}$ denotes the standard canonical basis.

For a subset $\mathscr{V}_s \subset \mathscr{V}_{\mathrm{I}}$, the *vertex identification* of $\mathscr{V}_s$ merges the elements of $\mathscr{V}_s$ into a new vertex $\bar{v} := \bigcup_{v_i \in \mathscr{V}_s} v_i$. Removing all edges connecting vertices $v_i, v_k \in \mathscr{V}_s$, we obtain the *contraction* of $\mathscr{G}_{\mathrm{IJ}}$ with respect to $\mathscr{V}_s$, which is the graph $\bar{\mathscr{G}}_{\mathrm{IJ}} := \{\bar{\mathscr{V}}_{\mathrm{I}}, \bar{\mathscr{E}}_{\mathrm{J}}\}$ with $\bar{\mathscr{V}}_{\mathrm{I}} := (\mathscr{V}_{\mathrm{I}} \setminus \mathscr{V}_s) \bigcup \{\bar{v}\}$ and $\bar{\mathscr{E}}_{\mathrm{J}} := \mathscr{E}_{\mathrm{J}} \setminus \{e_j \,|\, e_j = (v_{j_1}, v_{j_2}) \,|\, v_{j_1}, v_{j_2} \in \mathscr{V}_s\}$. Note that $\bar{\mathscr{G}}$ might have multiple edges and self loops even if $\mathscr{G}$ is simple. The associated identification matrix is given by $\mathbf{1}^T\Pi$, where $\mathbf{1} = [1, ..., 1] \in \mathbb{R}^{|\mathscr{V}_s|}$ and $\Pi \in \mathbb{R}^{|\mathscr{V}_s| \times |\mathscr{V}_s|}$ is a permutation such that $[\mathbf{1}^T\Pi]_i = 1$ if and only if $v_i \in \mathscr{V}_s$.

In the following, we assume that $\mathscr{G}_{\mathrm{IJ}}$ is numbered such that

$$\mathscr{E}_{\mathrm{IJ}} = \mathscr{E}_{\mathrm{IJ},1} \cup \ldots \cup \mathscr{E}_{\mathrm{IJ},K} \cup \mathscr{E}_{\mathrm{IJ}}^{isolated}, \quad \mathscr{V}_{\mathrm{IJ}} = \mathscr{V}_{\mathrm{IJ},1} \cup \ldots \cup \mathscr{V}_{\mathrm{IJ},K}, \tag{2}$$

where $\mathscr{G}_{\mathrm{IJ},k} = \{\mathscr{E}_{\mathrm{IJ},k}, \mathscr{V}_{\mathrm{IJ},k}\}$, $k = 1, ..., K$, are the connected components of $\mathscr{G}_{\mathrm{IJ}}$. These are ordered such that, for $k = 1, ..., \hat{k}_1$, $\mathscr{G}_{\mathrm{IJ},k}$ corresponds to proper subgraphs, for $k = \hat{k}_1 + 1, ..., \hat{k}$ to subsets with loose edges and for $k = \hat{k} + 1, ..., K$ to isolated vertices. Accordingly, we denote by $\mathscr{G}_{\mathrm{IJ},k}^{\mathrm{outer}}$, $\mathscr{G}_{\mathrm{IJ},k}^{\mathrm{inner}}$ the corresponding subgraphs of $\mathscr{G}_{\mathrm{IJ},k}$. Then, the connection matrix is given as

$$A_{\mathrm{IJ}} = \begin{bmatrix} A_{\mathrm{IJ},1} & & & \\ & \ddots & & \\ & & A_{\mathrm{IJ},\hat{k}} & \\ & & & 0 \end{bmatrix} \quad \text{with} \quad A_{\mathrm{IJ},k} = \begin{bmatrix} A_{\mathrm{IJ},k}^{\mathrm{inner}} & A_{\mathrm{IJ},k}^{\mathrm{loose}} \end{bmatrix}, \tag{3}$$

where $A_{\mathrm{IJ},k}^{\mathrm{inner}}$ is the incidence matrix of $\mathscr{G}_{\mathrm{IJ},k}^{\mathrm{inner}}$ and $A_{\mathrm{IJ},k}^{\mathrm{loose}}$ denotes the connection matrix of $\{\mathscr{V}_{\mathrm{IJ},k}, \mathscr{E}_{\mathrm{IJ},k}^{\mathrm{loose}}\}$. The zero block row and column correspond to the isolated vertices and edges, respectively. For each $\mathscr{G}_{\mathrm{IJ},k}$, we number $\mathscr{G}_{\mathrm{IJ},k}^{\mathrm{outer}}$ such that $\mathscr{V}_{\mathrm{IJ},k}^{\mathrm{outer}} = \mathscr{V}_{\mathrm{IJ},k} \cup \mathscr{V}_{\mathrm{IJ},k}^{c}$ and the incidence matrix $A_{\mathrm{IJ},k}^{\mathrm{outer}}$ is given by $A_{\mathrm{IJ},k}^{\mathrm{outer}} = [A_{\mathrm{IJ},k}^T, (A_{\mathrm{IJ},k}^c)^T]^T$, where $A_{\mathrm{IJ},k}^c$ describes the connection structure of $\{\mathscr{V}_{\mathrm{IJ},k}^c, \mathscr{E}_{\mathrm{IJ},k}\}$.

Considering the substructures introduced above, we define the *reduced vertex set* $\mathscr{V}_{\mathrm{IJ},k,1} := \cup_{k=1}^K \mathscr{V}_{\mathrm{IJ},k,1}$ with *ground node* $\mathscr{V}_{\mathrm{IJ},k,2} := \cup_{k=1}^K \mathscr{V}_{\mathrm{IJ},k,2}$ with selection matrices

$$\Gamma_{\mathrm{IJ}} = [\Gamma_{\mathrm{IJ},1}, \Gamma_{\mathrm{IJ},2}], \tag{4a}$$

the *spanning tree* $\mathcal{T}_{\mathrm{IJ},k,1} := \cup_{k=1}^{K}\mathcal{T}_{\mathrm{IJ},k,1}$ with *chord set* $\mathcal{T}_{\mathrm{IJ},k,2} := \cup_{k=1}^{K}\mathcal{T}_{\mathrm{IJ},k,2}$ with selection matrices

$$\Pi_{\mathrm{IJ}} = \left[\Pi_{\mathrm{IJ},1}, \Pi_{\mathrm{IJ},2}\right], \tag{4b}$$

the *fundamental cycles* $\mathcal{C}_{\mathrm{IJ},k} := \cup_{k=1}^{K}\mathcal{C}_{\mathrm{IJ},k,l}$ and the crossing paths $\mathcal{P}_{\mathrm{IJ},k} := \cup_{k=1}^{K}\mathcal{P}_{\mathrm{IJ},k,l}$ with selection matrices

$$V_2 = \left[C_{\mathrm{IJ}}, P_{\mathrm{IJ}}, L_{\mathrm{IJ}}\right], \tag{4c}$$

where $\mathcal{V}_{\mathrm{IJ},k,i}$, $\mathcal{T}_{\mathrm{IJ},k,i}$, $i = 1, 2$, $\mathcal{C}_{\mathrm{IJ},k} = \{\mathcal{C}_{\mathrm{IJ},k,1}, ..., \mathcal{C}_{\mathrm{IJ},k,|\mathcal{C}_{\mathrm{IJ},k}|}\}$ and $\mathcal{P}_{\mathrm{IJ},k} = \{\mathcal{P}_{\mathrm{IJ},k,1}, ..., \mathcal{P}_{\mathrm{IJ},k,|\mathcal{P}_{\mathrm{IJ},k}|}\}$ denote the respective structure in each component $\mathcal{G}_{\mathrm{IJ},k}$ with selection matrices $\Gamma_{\mathrm{IJ},k,i}$, $\Pi_{\mathrm{IJ},k,i}$, $C_{\mathrm{IJ},k}$, $P_{\mathrm{IJ},k}$, such that $\Gamma_{\mathrm{IJ},i} = \mathrm{diag}\left(\Gamma_{\mathrm{IJ},k,i}\right)_{k=1,...,K}$, $\Pi_{\mathrm{IJ},i} = \mathrm{diag}\left(\Pi_{\mathrm{IJ},k,i}\right)_{k=1,...,K}$, $i = 1, 2$ and $P_{\mathrm{IJ}} = \mathrm{diag}\left(P_{\mathrm{IJ},k}\right)_{k=1,...,K}$, $C_{\mathrm{IJ}} = \mathrm{diag}\left(C_{\mathrm{IJ},k}\right)_{k=1,...,K}$ and $L_{\mathrm{IJ}} = [0, I_{|\mathcal{E}_{\mathrm{IJ},k}^{\mathrm{loose}}|}]^T$. For the connected components that are proper subgraphs, we further consider the *identification* $\bar{\mathcal{V}}_{\mathrm{IJ}} = \bigcup_{k=1}^{\hat{k}} \bar{v}_{i_k}$ with $\bar{v}_{i_k} := v_{i_{k,0}} \cup (\cup_{v_i \in \mathcal{V}_{\mathrm{I},k}} v_i)$ and identification matrix

$$\mathbf{1}_{\mathrm{IJ}} = \mathrm{diag}\left(\mathbf{1}_{|V_{\mathrm{IJ},k}|}\right)_{k=1,...,\hat{k}}, \tag{4d}$$

Like for a proper graph and its incidence matrix, cp., e.g., [8, pp. 23], we can interpret the fundamental subspaces of a submatrix A_{IJ} as substructures of the set $\mathcal{G}_{\mathrm{IJ}}$.

Lemma 1.1. *Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a simple, oriented graph with $\mathcal{V} = \cup_{I \in I_{\mathcal{V}}}\mathcal{V}_I$, $\mathcal{E} = \cup_{J \in J_{\mathcal{E}}}\mathcal{E}_J$. Consider a subset $\mathcal{G}_{IJ} := \{\mathcal{V}_I, \mathcal{E}_J\}$ with connection matrix A_{IJ}. Then,* $\mathrm{rank}(A_{IJ}) = \sum_{k=1}^{\hat{k}} |\mathcal{V}_{IJ,k}| - \hat{k}$*, where $\mathcal{G}_{IJ,k}$, $k = 1, ..., \hat{k}$ denotes the connected components of $\mathcal{G}_{IJ}$ that itself are subgraphs. For the matrices defined in* (4)*, it holds that* $\ker(A_{IJ}) = \mathrm{span}(V_2)$, $\mathrm{corange}(A_{IJ}) = \mathrm{span}(\Pi_{IJ,1})$ *and* $\mathrm{coker}(A_{IJ}) = \mathrm{span}(\mathbf{1}_{IJ})$, $\mathrm{range}(A_{IJ}) = \mathrm{span}(\Gamma_{IJ,1})$.
The matrices $U := [\Gamma_{IJ,1}, \mathbf{1}_{IJ}]$ and $V := [\Pi_{IJ,1}, V_{IJ,2}]$ are nonsingular with

$$U^{-1} = \begin{bmatrix} U_2^- \\ \Gamma_{IJ,2}^T \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} V_2^- & 0 \\ \Pi_{IJ,2}^T & 0 \\ 0 & I_{|\mathcal{E}_{IJ,k}^{loose}|} \end{bmatrix},$$

where $U_2^- = \Gamma_{IJ,1}^T - \mathbf{1}_{IJ}\Gamma_{IJ,2}^T$ and $V_2^- = \Pi_{IJ,1}^T - \Pi_{IJ,1}^T[C_{IJ}, P_{IJ}]\Pi_{IJ,2}^T$.

Proof. From (3), we get that $\mathrm{rank}(A_{\mathrm{IJ}}) = \sum_{k=1}^{\hat{k}} \mathrm{rank}(A_{\mathrm{IJ},k})$. Noting that $\mathcal{G}_{\mathrm{IJ},k}^{\mathrm{outer}}$ is connected as $\mathcal{G}_{\mathrm{IJ},k}$ is connected and using that $\mathcal{V}_{\mathrm{IJ}}^{\mathrm{outer}} = \mathcal{V}_{\mathrm{I}} \cup \mathcal{V}_{\mathrm{IJ}}^{\mathrm{c}}$, we have that $\mathrm{rank}(A_{\mathrm{IJ},k}^{\mathrm{outer}}) = |\mathcal{V}_{\mathrm{IJ},k}| + |\mathcal{V}_{\mathrm{IJ},k}^{\mathrm{c}}| - 1$, cp. [3, p. 23]. For $k = 1, ..., \hat{k}_1$, we have that $\mathcal{V}_{\mathrm{IJ},k}^{\mathrm{c}} = \emptyset$ implying that $\mathrm{rank}(A_{\mathrm{IJ},k}^{\mathrm{outer}}) = \mathrm{rank}(A_{\mathrm{IJ},k})$ with $\mathrm{rank}(A_{\mathrm{IJ},k}) = |\mathcal{V}_{\mathrm{IJ},k}| - 1$. For $k = \hat{k}_1 + 1, ..., \hat{k}$, we have that $\mathcal{V}_{\mathrm{IJ},k}^{\mathrm{c}} \neq \emptyset$, implying that $\mathrm{rank}(A_{\mathrm{IJ},k}^{\mathrm{outer}}) > |\mathcal{V}_{\mathrm{IJ},k}|$. Thus, we can choose $|\mathcal{V}_{\mathrm{IJ},k}|$ linearly independent rows from $A_{\mathrm{IJ},k}^{\mathrm{outer}}$ and choosing the block row associated with $A_{\mathrm{IJ},k}$, we get $\mathrm{rank}(A_{\mathrm{IJ},k}) = |\mathcal{V}_{\mathrm{IJ},k}|$. In conclusion, we have proven that $\mathrm{rank}(A_{\mathrm{IJ}}) = \sum_{k=1}^{\hat{k}_1}(|\mathcal{V}_{\mathrm{IJ},k}| - 1) + \sum_{k=\hat{k}_1+1}^{\hat{k}} |\mathcal{V}_{\mathrm{IJ},k}|$, i.e., $\mathrm{rank}(A_{\mathrm{IJ}}) = \sum_{k=1}^{\hat{k}} |\mathcal{V}_{\mathrm{IJ},k}| - \hat{k}_1$.

Now, we consider a connected component $\mathcal{G}_{\mathrm{IJ},k}$. With the given numbering, the fundamental cycle and crossing path matrices are given by

$$C_{\mathrm{IJ},k} = \begin{bmatrix} C_{\mathrm{IJ}}^{\mathrm{inner}} \\ 0 \\ 0 \end{bmatrix}, \quad P_{\mathrm{IJ},k} = \begin{bmatrix} * \\ \mathbf{1}_{|\mathcal{E}_{\mathrm{IJ},k}|-1} \\ I_{|\mathcal{E}_{\mathrm{IJ},k}|-1} \end{bmatrix}, \tag{5}$$

where $C^{\text{inner}}_{\text{IJ},k}$ denotes the fundamental cycle matrix of $\mathcal{G}^{\text{inner}}_{\text{IJ},k}$. From (3) and (5) it follows that

$$A_{\text{IJ},k}C_{\text{IJ},k} = A^{\text{inner}}_{\text{IJ},k}C^{\text{inner}}_{\text{IJ},k} = 0,$$

as the fundamental cycles of $\mathcal{G}^{\text{inner}}_{\text{IJ}}$ lie in $\ker(A_{\text{IJ}})$. Similarly, we get from (3) and (5) that

$$A^{\text{outer}}_{\text{IJ},k}P_{\text{IJ},k} = \begin{bmatrix} 0 \\ A^{c}_{\text{IJ},k}P_{\text{IJ},k} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{1}_{|\mathcal{E}_{\text{IJ},k}|-1} \\ I_{|\mathcal{E}_{\text{IJ},k}|-1} \end{bmatrix}$$

as the incidence matrix applied to a path matrix returns exactly the starting and end vertices of the path, cp. [6, p. 157]. Thus, $A_{\text{IJ},k}P_{\text{IJ},k} = 0$, implying that $\operatorname{span}([C_{\text{IJ},k}, P_{\text{IJ},k}]) \subset \ker(A_{\text{IJ},k})$. Considering (5), we find that $\operatorname{rank}([C_{\text{IJ},k}, P_{\text{IJ},k}]) = \operatorname{rank}(C_{\text{IJ},k}) + \operatorname{rank}(P_{\text{IJ},k})$ with $\operatorname{rank}(C_{\text{IJ},k}) = |\mathcal{E}^{\text{inner}}_{\text{IJ},k}| - |\mathcal{V}_{\text{IJ},k}| + 1$ and $\operatorname{rank}(P_{\text{IJ},k}) = |\mathcal{E}^{\text{loose}}_{\text{IJ},k}| - 1$. For $k = 1, ..., \hat{k}_1$, we thus get that

$$\operatorname{rank}([C_{\text{IJ},k}, P_{\text{IJ},k}]) = |\mathcal{E}_{\text{IJ},k}| - |\mathcal{V}_{\text{IJ},k}| + 1 = \dim(\ker(A_{\text{IJ},k})),$$

while for $k = \hat{k}_1 + 1, ..., \hat{k}$, we get that

$$\operatorname{rank}([C_{\text{IJ},k}, P_{\text{IJ},k}]) = |\mathcal{E}^{\text{inner}}_{\text{IJ},k}| - |\mathcal{V}_{\text{IJ},k}| + 1 + |\mathcal{E}^{\text{loose}}_{\text{IJ},k}| - 1 = |\mathcal{E}_{\text{IJ},k}| - |\mathcal{V}_{\text{IJ},k}| = \dim(\ker(A_{\text{IJ},k})).$$

Hence, $\operatorname{span}([C_{\text{IJ},k}, P_{\text{IJ},k}]) = \ker(A_{\text{IJ},k})$ and it follows that $\operatorname{span}(V_2) = \ker(A_{\text{IJ}})$.

For the left nullspace, we note that for $k = 1, ..., \hat{k}$, $\mathcal{G}_{\text{IJ},k}$ is a proper subgraph, implying that every column of $A_{\text{IJ},k}$ contains exactly two nonzero entires $1, -1$. Hence, $\mathbf{1}^T_{|\mathcal{V}_{\text{IJ},k}|}A_{\text{IJ},k} = 0$. As $\operatorname{rank}(A_{\text{IJ},k}) = |\mathcal{V}_{\text{IJ},k}| - 1$, it follows that $\operatorname{span}(\mathbf{1}_{|\mathcal{V}_{\text{IJ},k}|}) = \operatorname{coker}(A_{\text{IJ},k})$ and thus $\operatorname{span}(\mathbf{1}_{|\mathcal{V}_{\text{IJ}}|}) = \operatorname{coker}(A_{\text{IJ}})$.

For $\operatorname{corange}(A_{\text{IJ}})$, we note that for $k = 1, ..., \hat{k}_1$, $\operatorname{rank}(A_{\text{IJ},k}) = |\mathcal{V}_{\text{IJ},k}| - 1$, such that we can select $|\mathcal{V}_{\text{IJ},k}| - 1$ linearly independent columns in $A_{\text{IJ},k}$, i.e., there exists a permutation $\Pi_{\text{IJ},1}, \Pi_{\text{IJ},2} \in \mathbb{R}^{|\mathcal{E}_{\text{IJ},k}| \times |\mathcal{E}_{\text{IJ},k}|}$ such that $A_{\text{IJ},k}\Pi_{\text{IJ},k,1}$ has full rank. For $k = \hat{k}_1 + 1, ..., \hat{k}$, $\operatorname{rank}(A_{\text{IJ},k}) = |\mathcal{V}_{\text{IJ},k}|$ and there exists a permutation $\Pi_{\text{IJ},1}, \Pi_{\text{IJ},2} \in \mathbb{R}^{|\mathcal{E}_{\text{IJ},k}| \times |\mathcal{E}_{\text{IJ},k}|}$ such that $A_{\text{IJ},k}\Pi_{\text{IJ},k,1}$ has full rank, where $\Pi_{\text{IJ},k,1}$ selects $|\mathcal{V}_{\text{IJ},k}| - 1$ linearly independent columns associated with edges on a spanning tree of $|\mathcal{G}^{\text{inner}}_{\text{IJ},k}$ as well as one loose edge $e_{k_0} \in \mathcal{E}_{\text{IJ},k}$ as reference loose edge.

Similarly, for $k = 1, ..., \hat{k}_1$, we can select $|\mathcal{V}_{\text{IJ},k}| - 1$ linearly independent rows in $A_{\text{IJ},k}$, i.e., there exists a permutation $\Gamma_{\text{IJ},k,1}, \Gamma_{\text{IJ},k,2}$ such that $\Gamma^T_{\text{IJ},1}A_{\text{IJ}}$ has full rank. For $k = \hat{k}_1 + 1, ..., \hat{k}$, $\operatorname{rank}(A_{\text{IJ},k}) = |\mathcal{V}_{\text{IJ},k}|$, implying that $\Gamma_{\text{IJ},k,1} = I_{|\mathcal{V}_{\text{IJ},k}|}$.

If $\ker(A_{\text{IJ}}) = \operatorname{span}(V_2)$, $\operatorname{corange}(A_{\text{IJ}}) = \operatorname{span}(\Pi_{\text{IJ},1})$ and $\operatorname{coker}(A_{\text{IJ}}) = \operatorname{span}(\mathbf{1}_{\text{IJ}})$, $\operatorname{range}(A_{\text{IJ}}) = \operatorname{span}(\Gamma_{\text{IJ},1})$, then the matrices $U := [\Gamma_{\text{IJ},1}, \mathbf{1}_{\text{IJ}}]$ and $V := [\Pi_{\text{IJ},1}, V_2]$ are nonsingular. To verify the representation of U^{-1}, V^{-1}, we check the properties of the inverse. For convenience, we drop the index IJ of the subset $\mathcal{G}_{\text{IJ}}$. First, we note that

$$\begin{bmatrix} \Gamma_1 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \Gamma_1^T - \mathbf{1}\Gamma_2^T \\ \Gamma_2^T \end{bmatrix} = \Gamma_1\Gamma_1^T + (\mathbf{1} - \Gamma_1\mathbf{1}_{\text{IJ}})\Gamma_2^T = \Gamma_1\Gamma_1^T + \Gamma_2\mathbf{1}\Gamma_2^T.$$

Noting that $[\Gamma_1\mathbf{1}]_i = 1$ for $v_i \in \mathcal{V}_{\text{IJ},1}$ and $[\Gamma_1\mathbf{1}]_i = 0$ for $v_i \in \mathcal{V}_{\text{IJ},2}$, we have that $\mathbf{1} - \Gamma_1\mathbf{1} = \Gamma_2$, such that

$$\begin{bmatrix} \Gamma_1 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \Gamma_1^T - \mathbf{1}\Gamma_2^T \\ \Gamma_2^T \end{bmatrix} = \Gamma_1\Gamma_1^T + \Gamma_2\Gamma_2^T = I_{|V_{\text{IJ}}|}.$$

On the other hand, we have that

$$\begin{bmatrix}\Gamma_1^T - \mathbf{1}\Gamma_2^T \\ \Gamma_2^T\end{bmatrix}\begin{bmatrix}\Gamma_1 & \mathbf{1}\end{bmatrix} = \begin{bmatrix}I_{|\mathscr{V}_{red}|-1} & (\Gamma_1^T - \mathbf{1}\Gamma_2^T)\mathbf{1} \\ 0 & \Gamma_2^T\mathbf{1}\end{bmatrix}.$$

In $\Gamma_1^T - \mathbf{1}\Gamma_2^T$, every row contains exactly two non-zero entries given by $1, -1$, implying that $(\Gamma_1^T - \mathbf{1}\Gamma_2^T)\mathbf{1} = 0$. With $\Gamma_2^T\mathbf{1}_{\mathrm{IJ}} = I_{|\mathscr{V}_{\mathrm{IJ},2}|}$, we thus get that $[\Gamma_1, \mathbf{1}]^{-1}[\Gamma_1, \mathbf{1}] = I_{|\mathscr{V}_{\mathrm{IJ}}|}$.
As $\mathscr{E}(\mathscr{T}) \cap \mathscr{E}_{\mathrm{IJ}}^{\mathrm{loose}} = \emptyset$, where $\mathscr{E}(\mathscr{T})$ denotes the edges of a spanning tree $\mathscr{T}$, we have that

$$\begin{bmatrix}\Pi_1 & V_2\end{bmatrix} = \begin{bmatrix}\tilde{\Pi}_1 & C & P & \\ 0 & 0 & 0 & I_{|\mathscr{E}_{\mathrm{IJ},k}^{\mathrm{loose}}|}\end{bmatrix}.$$

Thus, it suffices to show that

$$\begin{bmatrix}\tilde{\Pi}_1 & [C, P]\end{bmatrix}^{-1} = \begin{bmatrix}\Pi_1^T - \Pi_1^T[C, P]\Pi_2^T \\ \Pi_2^T\end{bmatrix}.$$

We show the assertion by checking the properties of the inverse. First, using that $I_{|\mathscr{V}_{\mathrm{IJ}}|} - \Pi_1\Pi_1^T = \Pi_2\Pi_2^T$, we get that

$$\begin{aligned}\begin{bmatrix}\Pi_1 & [C, P]\end{bmatrix}\begin{bmatrix}\Pi_1^T - \Pi_1^T[C, P]\Pi_2^T \\ \Pi_2^T\end{bmatrix} &= \Pi_1\Pi_1^T - \Pi_1\Pi_1^T[C, P]\Pi_2^T + [C, P]\Pi_2^T \\ &= \Pi_1\Pi_1^T + \Pi_2\Pi_2^T[C, P]\Pi_2^T.\end{aligned}$$

The matrix $\Pi_2\Pi_2^T$ is a projection onto the edges of the chord set $\mathscr{T}_{\mathrm{IJ},2}$. As the fundamental cycles and crossing paths $\mathscr{C}_{k_l}, \mathscr{P}_{k_m}$ contain exactly one edge $e_{k_l}, e_{k_m} \in \mathscr{T}_{\mathrm{IJ},2}$, we have that $\Pi_2\Pi_2^T[C, P] = \Pi_2$. Hence,

$$\begin{bmatrix}\Pi_1 & C & P\end{bmatrix}\begin{bmatrix}\Pi_1^T - \Pi_1^T[C, P]\Pi_2^T \\ \Pi_2^T\end{bmatrix} = \Pi_1\Pi_1^T + \Pi_2\Pi_2^T = I_{|\mathscr{E}_{\mathrm{IJ}}|}.$$

On the other hand, we have that

$$\begin{bmatrix}\Pi_1^T - \Pi_1^T[C, P]\Pi_2^T \\ \Pi_2^T\end{bmatrix}\begin{bmatrix}\Pi_1 & [C, P]\end{bmatrix} = \begin{bmatrix}I_{|\mathscr{E}(\mathscr{T}_{\mathrm{IJ}})|} & \Pi_1^T[C, P](I_{n_{\mathrm{J}}-n_{\mathrm{I}}+1} - \Pi_2^T[C, P]) \\ 0 & \Pi_2^T[C, P]\end{bmatrix}.$$

Again, as every fundamental cycle and crossing path contains exactly one chord, we have that $\Pi_2^T[C, P] = I_{n_{\mathrm{J}}-n_{\mathrm{I}}+1}$. Then, it follows that $[\Pi_1, [C, P]]^{-1}[\Pi_1, [C, P]] = I_{|\mathscr{E}_{\mathrm{IJ}}|}$. □

Hence, the fundamental cycles, crossing paths and loose edges spann the right nullspace $\ker(A_{\mathrm{IJ}})$, while the edges in the spanning tree build a basis of $\mathrm{corange}(A_{\mathrm{IJ}})$. The identification of connected components with a ground node spanns the left nullspace $\mathrm{coker}(A_{\mathrm{IJ}})$, while the vertices of the reduced vertex set build a basis of $\mathrm{corange}(A_{\mathrm{IJ}})$.

Now, we equip the vertices and edges of $\mathscr{G}$ with potentials and flows, respectively. To each vertex $v_i \in \mathscr{V}$, we assign a potential p_i that are summarized as $p = [p_i]_{i=1,\dots,|\mathscr{V}|}$. Similarly, to each edge $e_j \in \mathscr{G}$, we assign a *flow* q_j and set $q = [q_j]_{j=1,\dots,|\mathscr{E}|}$. The flow is directed: A flow q_j is called *positive*, i.e., $q_j > 0$, if q_j agrees with the direction of its associated edge e_j. If $q_j \neq 0$ is opposed to the direction of edge e_j, then q_J is called *negative*, i.e., $q_j < 0$. If $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$ with $\mathscr{V} = \cup_{I \in I_{\mathscr{V}}}\mathscr{V}_{\mathrm{I}}$, $\mathscr{E} = \cup_{J \in I_{\mathscr{E}}}\mathscr{E}_{\mathrm{J}}$, we partition the flows and potential accordingly and write $q_{\mathrm{J},j} \in \mathscr{E}_{\mathrm{J}}$ and $p_{\mathrm{I},i} \in \mathscr{V}_{\mathrm{I}}$.

The flow and potential satisfy the following fundamental relations that generalize Kirchhoff's circuit laws, i.e.,

$$\Gamma_1^T A q = 0, \tag{6a}$$

$$V_2^T A^T p = 0. \tag{6b}$$

The equations (6) allow to give a physical interpretation of Lemma 1.1. The fundamental cycles, crossing paths and loose edges correspond structures on which the potential difference vanishes, while the spanning tree selects a structure on which the potential difference is well-defined. The reduced vertex set denotes those vertices on which the potential is fixed in relation to the reference value given by the ground node. Thus, the identification of the reduced vertex set with its ground node summarizes all vertices on which the potential is not fixed, like in an isolated vertex or a subgraph without connection to a ground node.

We transform the flow and potential with respect to these substructures by setting

$$\tilde{q} := [\Pi_{\mathrm{IJ},1}, V_{\mathrm{IJ},2}]^{-1} q, \quad \tilde{p} := [\Gamma_{\mathrm{IJ},1}, \mathbf{1}_{\mathrm{IJ}}]^{-1} p, \tag{7}$$

such that

$$\tilde{q}_1 = (\Pi_{\mathrm{IJ},1}^T - \Pi_{\mathrm{IJ},1}^T [C_{\mathrm{IJ}}, P_{\mathrm{IJ}}] \Pi_{\mathrm{IJ},2}^T) q, \qquad \tilde{p}_1 = (\Gamma_{\mathrm{IJ},1}^T - \mathbf{1}_{\mathrm{IJ}} \Gamma_{\mathrm{IJ},2}^T) p,$$
$$\tilde{q}_2 = \Pi_{\mathrm{IJ},2}^T q, \qquad \tilde{p}_1 = \Gamma_{\mathrm{IJ},2}^T p.$$

The flows q_2 belong to edges on the *chord set* $\mathcal{T}_{\mathrm{IJ},2}$, while the flows q_1 denote the difference between a branch flow $q_{1,j} \in \mathcal{T}_{\mathrm{IJ}}$ and the chord flows $q_{2,l} \in \mathcal{T}_{\mathrm{IJ},2}$ associated with fundamental cycles and crossing paths $q_{1,j}$ is part of. Similarly, p_2 denote the potentials in the ground nodes $\mathcal{V}_{\mathrm{IJ},2}$ while p_1 correspond the difference between a potential $p_{1,j} \in \mathcal{V}_{\mathrm{IJ},1}$ and to its associated ground node $p_{2,j} \in \mathcal{V}_{\mathrm{IJ},2}$.

Now, we think of the flow as information running through the network. To describe the structure of the subset $\mathcal{G}_{\mathrm{IJ}}$ on this informational level, we partition the set of incident edges $\mathcal{E}_{inc}(v_i)$, $v_i \in \mathcal{V}_{\mathrm{I}}$, into those along which v_i *receives* and *sends* information, respectively, i.e., we set

$$\mathcal{E}_{inc,s}(v_i) := \{e_j \in \mathcal{E}_{inc}(v_i) \mid A_{ij}\,\mathrm{sgn}(q_j(t)) > 0, \text{ i.e., } q_j \text{ starts in } v_i\},$$
$$\mathcal{E}_{inc,e}(v_i) := \{e_j \in \mathcal{E}_{inc}(v_i) \mid A_{ij}\,\mathrm{sgn}(q_j(t)) < 0, \text{ i.e., } q_j \text{ ends in } v_i\}.$$

Defining the *flow matrix*

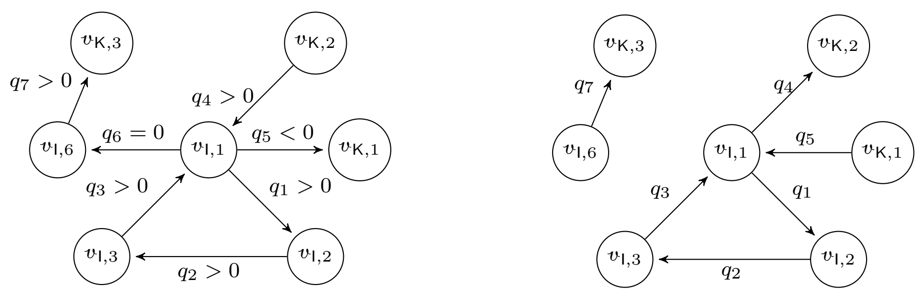
$$B_{\mathrm{IJ},i\ell} = \begin{cases} \sum_{e_j \in \mathcal{E}_{inc,s}(v_i) \cap \mathcal{E}_{\mathrm{J}}} |q_j|, & i = \ell, \\ -|q_j|, & e_j \in \mathcal{E}_{inc,e}(v_i) \cap \mathcal{E}_{inc}(v_\ell) \cap \mathcal{E}_{\mathrm{J}},\ i \neq \ell, \\ 0, & \text{else}, \end{cases} \tag{8}$$

the information flow in $\mathcal{G}_{\mathrm{IJ}}$ is graphically represented by the *flow graph* $\mathcal{G}_{\mathrm{IJ}}^{\mathrm{flow}} := \mathcal{G}(B_{\mathrm{IJ}}^T)$, where the graph $\mathcal{G}(A)$ of a matrix $A \in \mathbb{R}^{n \times n}$ is defined as $\mathcal{G}(A) = \big\{\{v_1, ..., v_n\}, \{(v_i, v_j) | a_{ij} \neq 0\}\big\}$, i.e., whenever the ij-th entry is nonzero, there is an edge from vertex v_i to v_j, cp. [20, p. 528]. Hence,

$$\mathcal{G}_{\mathrm{IJ}}^{\mathrm{flow}} = \{\mathcal{V}_{\mathrm{I}}, \mathcal{E}_{\mathrm{IJ}}^{\mathrm{flow}}\} \tag{9}$$
$$\mathcal{E}_{\mathrm{IJ}}^{\mathrm{flow}} := \{e_j := (v_\ell, v_i)\} \mid \mathcal{E}_{inc,e}(v_i) \cap \mathcal{E}_{inc}(v_\ell) \neq \emptyset, v_\ell \neq v_i \ \vee\ \mathcal{E}_{inc,s}(v_i) \neq \emptyset, v_\ell = v_i\}.$$

The graph $\mathscr{G}_{\text{IJ}}^{\text{flow}}$ basically has the same connection structure as the set $\mathscr{G}_{\text{IJ}}$. At vertices $v_i \in \mathscr{V}_{\text{I}}$ sending a non-zero mass flow into $\mathscr{G}_{\text{IJ}}$, however, $\mathscr{G}_{\text{IJ}}^{\text{flow}}$ has self loops and edges $e_j \in \mathscr{E}_{\text{J}}$ equipped with a zero mass flow $q_j = 0$ are absent in $\mathscr{E}_{\text{IJ}}^{\text{flow}}$. The orientation of $\mathscr{G}_{\text{IJ}}^{\text{flow}}$ is determined by the direction of the mass flows, where $e_j \in \mathscr{E}_{\text{IJ}}^{\text{flow}}$ is directed from v_ℓ to v_i if v_i *receives* a mass flow from v_ℓ, cp. Figure 1.

Figure 1: A graph (left) and its flow graph (right).



The connectivity of $\mathscr{G}_{\text{IJ}}^{\text{flow}}$ on this informational level is described by the concept of *strong connectivity*, cp. [20, p. 528], i.e., we assume that $\mathscr{G}_{\text{IJ}}^{\text{flow}}$ is composed from *strongly connected components* $\mathscr{G}_{flow,\text{IJ},k} := \{\mathscr{E}_{flow,\text{IJ},k}, \mathscr{V}_{flow,\text{IJ},k}\}$, i.e., every pair of vertices $v_i, v_k \in \mathscr{V}_{flow,\text{IJ},k}$ is connected by a directed path from v_i to v_k and a directed path from v_k to v_i, cp. [20, p. 528]. For each $\mathscr{G}_{flow,\text{IJ},k}$, we denote the interior subgraph by $\mathscr{G}_{flow,\text{IJ},k}^{\text{inner}} = \{\mathscr{E}_{flow,\text{IJ},k}^{\text{inner}}, \mathscr{V}_{flow,\text{IJ},k}^{\text{inner}}\}$ for $k = 1, ..., K$.
The flow matrix B_{IJ} is nonsingular, if in every strongly connected component $\mathscr{G}_{\text{IJ},k}$ of $\mathscr{G}_{\text{IJ}}$ there exists at least one vertex $v_{\hat{i}}$ sending a nonzero flow into $\mathscr{G}_{\text{IJ}} \setminus \mathscr{G}_{\text{IJ},k}$, i.e., outside the strongly connected component.

Lemma 1.2. *Let $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$ be a simple, oriented graph with $\mathscr{V} = \cup_{I \in I_{\mathscr{V}}} \mathscr{V}_I$, $\mathscr{E} = \cup_{J \in J_{\mathscr{E}}} \mathscr{E}_J$. Consider a subset $\mathscr{G}_{IJ} := \{\mathscr{V}_I, \mathscr{E}_J\}$ with connection matrix A_{IJ}. Then,*

$$B_{IJ} = \frac{1}{2} A_{IJ} \operatorname{diag}(q_{J,j})_j \left(\operatorname{diag}(\operatorname{sgn}(q_{J,j}))_j A_{IJ}^T + |A_{IJ}^T| \right).$$

For $v_i \in \mathscr{V}_I$, if $\sum_{e_j \in \mathscr{E}_{inc,s}(v_i) \cap \mathscr{E}_J} |q_j| > 0$ and

$$\sum_{e_j \in \mathscr{E}_{inc,s}(v_i) \cap (\mathscr{E}_J \setminus \mathscr{E}_{IJ,flow,k}^{inner})} |q_j| \geq 0, \qquad k = 1, ..., K, \tag{10}$$

where $\mathscr{G}_{flow,IJ,k} := \{\mathscr{E}_{flow,IJ,k}, \mathscr{V}_{flow,IJ,k}\}$ denotes the strongly connected components of $\mathscr{G}_{IJ}^{flow}$ with interior subgraphs $\mathscr{G}_{flow,IJ,k}^{inner} = \{\mathscr{E}_{flow,IJ,k}^{inner}, \mathscr{V}_{flow,IJ,k}^{inner}\}$, and for every $k = 1, ..., K$ there exists $v_{\hat{i}} \in \mathscr{V}_{flow,IJ,k}$, such that (10) is strictly satisfied, then B_{IJ} is nonsingular.

Proof. To prove the representation of B_{IJ}, we set $\tilde{A}_{\text{IJ}} := A_{\text{IJ}} \operatorname{diag}(q_{J,j})_j \left(\operatorname{diag}(\operatorname{sgn}(q_{J,j}))_j A_{\text{IJ}}^T + |A_{\text{IJ}}^T| \right)$ and note that

$$B_{\text{IJ},i\ell} := \sum_{e_j \in \mathscr{E}_J} |q_j| A_{\text{IJ},ij} \left(A_{\text{IJ},\ell j} + |A_{\text{IJ},\ell j}| \operatorname{sgn}(q_j) \right).$$

For $\mathscr{v}_i, \mathscr{v}_\ell \in I_\mathscr{V}$ and $j \in J_\mathscr{E}$, the entries of the incidence matrix A satisfy

$$A_{ij}A_{\ell j} = \begin{cases} 1, & e_j \in \mathscr{E}_{inc}(\mathscr{v}_i),\ i = \ell, \\ -1, & e_j \in \mathscr{E}_{inc}(\mathscr{v}_i) \cap \mathscr{E}_{inc}(\mathscr{v}_\ell),\ i \neq \ell, \\ 0, & \text{else}, \end{cases}$$

$$A_{ij}|A_{\ell j}| = \begin{cases} A_{ij}, & e_j \in \mathscr{E}_{inc}(\mathscr{v}_i) \cap \mathscr{E}_{inc}(\mathscr{v}_\ell), \\ 0, & \text{else} \end{cases}$$

and together with the definition of $\mathscr{E}_{inc,s}(\mathscr{v}_i), \mathscr{E}_{inc,e}(\mathscr{v}_i)$, we verify that $2B_{\mathrm{IJ}} = \tilde{A}_{\mathrm{IJ}}$.

If $\mathscr{G}_{flow,\mathrm{IJ}}$ is composed from K strongly connected components $\mathscr{G}_{flow,\mathrm{IJ},k} := \{\mathscr{V}_{flow,\mathrm{IJ},k}, \mathscr{E}_{flow,\mathrm{IJ},k}\}$ it follows that B_{IJ} is congruent to a block upper triangular matrix with irreducible diagonal blocks $B_{\mathrm{IJ},kk}$, $k = 1, ..., K$, i.e., there exists a permutation Π, such that $\Pi^T B_{\mathrm{IJ}} \Pi = [B_{\mathrm{IJ},kl}]_{kl}$ with $B_{\mathrm{IJ},kl} = 0$, $k > l$ and $B_{\mathrm{IJ},kk}$ irreducible, cp. [20]. We show that under the given conditions, each $B_{\mathrm{IJ},kk}$ is irreducibly diagonally dominant and hence nonsingular for $k = 1, ..., K$, cp., e.g., [11, p. 403] or [5, p. 67]. Then, B_{IJ} is nonsingular.

The i-th column of $B_{\mathrm{IJ},kk}$ is given by

$$[B_{\mathrm{IJ},kk}e_i]_\ell = \begin{cases} 2\sum_{e_j \in \mathscr{E}_{inc,s}(\mathscr{v}_i)\cap\mathscr{E}_{\mathrm{J}}} |q_j|, & i = l, \\ -2|q_j|, & e_j \in \mathscr{E}_{inc,s}(\mathscr{v}_i) \cap \mathscr{E}_{inc}(\mathscr{v}_\ell) \cap \mathscr{E}_{\mathrm{J}},\ i \neq \ell, \quad \ell = 1, ..., |\mathscr{V}_{flow,\mathrm{IJ},k}|, \\ 0, & \text{else}, \end{cases}$$

and noting that

$$\bigcup_{\mathscr{v}_\ell \in V_{\mathrm{IJ},flow,k}\setminus\{\mathscr{v}_i\}} \Big(\mathscr{E}_{inc,s}(\mathscr{v}_i) \cap \mathscr{E}_{inc}(\mathscr{v}_\ell) \cap \mathscr{E}_{\mathrm{J}}\Big) = \mathscr{E}_{inc,s}(\mathscr{v}_i) \cap \mathscr{E}^{\mathrm{inner}}_{\mathrm{IJ},flow,k}$$

for $\mathscr{v}_i \in \mathscr{V}_{\mathrm{IJ},flow,k}$, the i-th column sum of $B_{\mathrm{IJ},kk}$ is given by

$$\sum_{\mathscr{v}_\ell \in \mathscr{V}_{\mathrm{IJ},flow,k}} |B_{\mathrm{IJ},kk,i\ell}| = \sum_{e_j \in \mathscr{E}_{inc,s}(\mathscr{v}_i)\cap(\mathscr{E}_{\mathrm{J}}\setminus\mathscr{E}^{\mathrm{inner}}_{\mathrm{IJ},flow,k})} |q_j|.$$

Hence, if condition (10) is satisfied for $\mathscr{v}_i \in \mathscr{V}_{flow,\mathrm{IJ},k}$ and $k = 1, ..., K$, then $B_{\mathrm{IJ},kk}$ is diagonally dominant for $k = 1, ..., K$. For $k = 1, ..., K$, if there exists $\mathscr{v}_{\hat{i}} \in \mathscr{V}_{flow,\mathrm{IJ},k}$, such that (10) is strictly satisfied, then, together with the irreducibility, it follows that $B_{\mathrm{IJ},kk}$ is irreducible diagonally dominant for $k = 1, ..., K$. Then, B_{IJ} is nonsingular. □

Example 1.1. *For the graph of Figure 1, the flow matrix is given by*

$$B_{IJ} = \begin{array}{c} \\ \mathscr{v}_{I,1} \\ \mathscr{v}_{I,2} \\ \mathscr{v}_{I,3} \\ \mathscr{v}_{I,6} \end{array} \begin{array}{c} \begin{array}{cccc} \mathscr{v}_{I,1} & \mathscr{v}_{I,2} & \mathscr{v}_{I,3} & \mathscr{v}_{I,6} \end{array} \\ \begin{bmatrix} |q_1| + |q_4| & 0 & -|q_3| & 0 \\ -|q_1| & |q_2| & 0 & 0 \\ 0 & -|q_2| & |q_3| & 0 \\ 0 & 0 & 0 & |q_7|. \end{bmatrix} \end{array}$$

The strongly connected components of $\mathscr{G}^{flow}_{IJ}$ are given by $\mathscr{E}^{inner}_{flow,IJ,1} = \{\{\mathscr{v}_{I,1}, \mathscr{v}_{I,2}, \mathscr{v}_{I,3}\}, \{e_1, e_2, e_3\}\}$ and $\mathscr{E}^{inner}_{flow,IJ,2} = \{\{\mathscr{v}_{I,4}\}\}$. The matrix B_{IJ} is irreducible as there exists no permutation that *would transform this matrix to an upper triangular matrix and we observe that B_{IJ} is non-singular only if $|q_4|, |q_7| > 0$, i.e., only if the flows $|q_4|, |q_7| > 0$ start in $\mathcal{v}_{I,1}$, $\mathcal{v}_{I,6}$. This agrees with the conditions of Lemma 1.2, claiming that this is reflected in the solvability condition, claiming that $\sum_{e_j \in \mathscr{E}_{inc,s}(\mathcal{v}_{I,1}) \cap (\mathscr{E}_J \setminus \mathscr{E}^{inner}_{IJ,flow,k})} |q_j| = |q_4| > 0,$, $\sum_{e_j \in \mathscr{E}_{inc,s}(\mathcal{v}_{I,6}) \cap (\mathscr{E}_J \setminus \mathscr{E}^{inner}_{IJ,flow,k})} |q_j| = |q_7| > 0$, whereas $\sum_{e_j \in \mathscr{E}_{inc,s}(\mathcal{v}_{I,i}) \cap (\mathscr{E}_J \setminus \mathscr{E}^{inner}_{IJ,flow,k})} |q_j| = 0$ for $i = 1, 2, 3$.*

Besides these graph theoretical results, we frequently use the following identity for the rank of a block matrix $A = [A_{ij}]_{i,j=1,2}$, cp., e.g., [11, p. 25]. If A_{11}, A_{22} are nonsingular, then

$$\operatorname{rank}(A) = \operatorname{rank}(A_{11}) + \mathcal{S}_{A_{11}}(A) = \operatorname{rank}(A_{22}) + \mathcal{S}_{A_{22}}(A). \tag{11}$$

where $\mathcal{S}_{A_{11}}(A) := A_{22} - A_{21}A_{11}^{-1}A_{12}$, $\mathcal{S}_{A_{22}}(A) = A_{11} - A_{12}A_{22}^{-1}A_{21}$ denotes the Schur complements.

2 A network model for incompressible flow networks

We consider a network

$$\mathcal{N} = \{\mathcal{Pi}, \mathcal{Pu}, \mathcal{De}, \mathcal{Jc}, \mathcal{Re}\} \tag{12}$$

that is composed of pipes, pumps, demands, junctions and reservoirs and that is filled by an incompressible fluid, e.g. water. The pipes $\mathcal{Pi} := \{\mathrm{Pi}_1, ..., \mathrm{Pi}_{n_{\mathrm{Pi}}}\}$ and pumps $\mathcal{Pu} := \{\mathrm{Pu}_1, ..., \mathrm{Pu}_{n_{\mathrm{Pu}}}\}$ are connected by junctions $\mathcal{Jc} := \{\mathrm{Jc}_1, ..., \mathrm{Jc}_{n_{\mathrm{Jc}}}\}$, in which the mass flow of the fluid is split or merged. We distinguish between virtual connection points $\mathcal{Jc}_0 := \{\mathrm{Jc}_{0,1}, ..., \mathrm{Jc}_{0,n_{\mathrm{Jc},0}}\}$ and connection points $\mathcal{Jc}_V := \{\mathrm{Jc}_{V,1}, ..., \mathrm{Jc}_{V,n_{\mathrm{Jc},V}}\}$ posessing a volume $V_i > 0$. Those virtual connections point have a certain importance in the design of system simulation software, since they allow to connect standardized sub-components without introducing additional volumes (and as a consequence additional thermal inertia). The connection to the environment is modeled by reservoirs $\mathcal{Re} := \{\mathrm{Re}_1, ..., \mathrm{Re}_{n_{\mathrm{Re}}}\}$ and demand branches $\mathcal{De} := \{\mathrm{De}_1, ..., \mathrm{De}_{n_{\mathrm{De}}}\}$ that impose predefined pressures enthalpies as well as mass and enthalpy flows into the network. The number of each element in $\mathcal{N}$ is denoted by $n_{\mathrm{Pi}}, n_{\mathrm{Pu}}, n_{\mathrm{Jc}}$, where $n_{\mathrm{Jc}} = n_{\mathrm{Jc},0} + n_{\mathrm{Jc},V}$ and n_{De} and n_{Re}, respectively, and we set $n := n_{\mathrm{Pi}} + n_{\mathrm{Pu}} + n_{\mathrm{Jc}} + n_{\mathrm{Re}} + n_{\mathrm{De}}$.
Given boundary conditions $\bar{p}_{\mathrm{Re}} = [\bar{p}_{\mathrm{Re}_i}]_{i=1,...,n_{\mathrm{Re}}}$, $\bar{h}_{\mathrm{Re}} = [\bar{h}_{\mathrm{Re}_i}]_{i=1,...,n_{\mathrm{Re}}}$ and $\bar{q}_{\mathrm{De}} = [\bar{q}_{\mathrm{De}_i}]_{i=1,...,n_{\mathrm{De}}}$, $\bar{H}_{\mathrm{De}} = [\bar{H}_{\mathrm{De}_i}]_{i=1,...,n_{\mathrm{De}}}$, the task is to compute the mass and enthalpy flows $q_{\mathrm{Pi}} = [q_{\mathrm{Pi}_i}]_{i=1,...,n_{\mathrm{Pi}}}$, $q_{\mathrm{Pu}} = [q_{\mathrm{Pu}_i}]_{i=1,...,n_{\mathrm{Pu}}}$, $q_{\mathrm{De}} = [q_{\mathrm{De}_i}]_{i=1,...,n_{\mathrm{De}}}$ and $H_{\mathrm{Pi}} = [H_{\mathrm{Pi}_i}]_{i=1,...,n_{\mathrm{Pi}}}$, $H_{\mathrm{Pu}} = [H_{\mathrm{Pu}_i}]_{i=1,...,n_{\mathrm{Pu}}}$, $H_{\mathrm{De}} = [H_{\mathrm{De}_i}]_{i=1,...,n_{\mathrm{De}}}$ in the pipes, pumps and demand branches as well as the pressures and specific enthalpies $p_{\mathrm{Jc}} = [p_{\mathrm{Jc}_i}]_{i=1,...,n_{\mathrm{Jc}}}$, $p_{\mathrm{Re}} = [p_{\mathrm{Re}_i}]_{i=1,...,n_{\mathrm{Re}}}$ and $h_{\mathrm{Jc}} = [h_{\mathrm{Jc}_i}]_{i=1,...,n_{\mathrm{Jc}}}$, $h_{\mathrm{Re}} = [h_{\mathrm{Re}_i}]_{i=1,...,n_{\mathrm{Re}}}$ in the junctions and reservoirs.
To set up the governing equations for the network, we consider the characteristic relation that every element imposes on the enthalpy flow and specific enthalpy as well as on the mass flow and pressure. In a pipe Pi_j, the enthalpy flow H_j agrees with the product of the mass flow q_j and the specific enthalpy h_i in the originating vertex $\mathcal{v}_i$. Hence, if the pipe Pi_j is directed from $\mathcal{v}_{j_1}$ to $\mathcal{v}_{j_2}$, we have that

$$H_j = \frac{q_{\mathrm{Pi},j}}{2}((sgn(q_j) + 1)h_{j_1} - (sgn(q_j) - 1)h_{j_2}) =: f_{\mathrm{Pi}^*}(q_{\mathrm{Pi},j}, h_{j_1}, h_{j_2}). \tag{13a}$$

We have that $f_{\mathrm{Pi}^*,j} \in C(\Omega_{\mathrm{Pi}_j} \times (-\infty,\infty), \mathbb{R})$ with

$$\mathrm{D}_1\, f_{\mathrm{Pi}^*,j}(q_{\mathrm{Pi},j}, h_{j_1}, h_{j_2}) = \begin{cases} h_{j_1}, & q_{\mathrm{Pi},j} > 0, \\ 0, & q_{\mathrm{Pi},j} = 0, \\ h_{j_2}, & q_{\mathrm{Pi},j} < 0, \end{cases}$$

where $\Omega_{\mathrm{Pi}_j} \subset (-\infty,\infty)$ denotes the domains of admissible mass flows in Pi_j. Similarly, in a pump Pu_j that is directed from $\mathscr{v}_{j_1}$ to $\mathscr{v}_{j_2}$, the enthalpy flow is given by

$$\begin{aligned} H_j &= f_{\mathrm{Pu}^*}(q_j, h_{j_1}, h_{j_2}) \\ &:= \frac{q_j}{2}((sgn(q_j)+1)h_{j_1} - (sgn(q_j)-1)h_{j_2}) + \delta h_j = f_{\mathrm{Pu}^*}(q_j, h_{j_1}, h_{j_2}), \end{aligned} \tag{13b}$$

where δh_j is heat induced by the pump. For simplicity, in the following we assume that $\delta h = 0$. Due to energy conservation, in a junction $\mathrm{Jc}_{V,i}$, the sum of all enthalpy fluxes H_j entering or leaving Jc_i equals the product of the volume V_i and the change of the specific enthalpy $h_{\mathrm{Jc}_V,i}$, i.e.,

$$\sum_{H_j \in \mathscr{E}_{inc}(\mathrm{Jc}_{V,i})} H_j = V_i \dot{h}_{\mathrm{Jc}_V,i}. \tag{13c}$$

In a virtual connection point $\mathrm{Jc}_{0,i}$, we have

$$\sum_{H_j \in \mathscr{E}_{inc}(\mathrm{Jc}_{0,i})} H_j = 0. \tag{13d}$$

In a demand branch De_i, the enthalpy flow $H_{\mathrm{De},j}$ is set to a described value $\bar{H}_{\mathrm{De},j}$, i.e.,

$$H_{\mathrm{De},j} = \bar{H}_{\mathrm{De},j}. \tag{13e}$$

Similarly, in a reservoir Re_i, the specific enthalpy $h_{\mathrm{Re},i}$ is kept at a described value $\bar{h}_{\mathrm{Re},i}$, i.e.,

$$h_{\mathrm{Re},i} = \bar{h}_{\mathrm{Re},i}. \tag{14}$$

To include the connection structure of the network $\mathscr{N}$, we represent $\mathscr{N}$ as a graph $\mathscr{G}$. The pipes, pumps and demand branches correspond to the edges of $\mathscr{G}$ while the junctions and reservoirs serve as vertices, i.e., we consider

$$\mathscr{G} = \{\mathscr{V}, \mathscr{E}\} \quad \text{with} \quad \mathscr{E} = \{\mathscr{Pi}, \mathscr{Pu}, \mathscr{De}\} \quad \text{and} \quad \mathscr{V} = \{\mathscr{Jc}_0, \mathscr{Jc}_V, \mathscr{Re}\}. \tag{15}$$

We impose the following assumptions on the connection structure of $\mathscr{N}$.

Assumption 2.1. *Consider a network $\mathscr{N}$ as in* (12).

(i) Two junctions are connected at most by one pipe or one pump. Each pipe, pump and demand has an assigned direction.

(ii) The network is connected, i.e., every pair of junctions and/or reservoirs can be reached by a sequence of pipes and pumps.

(iii) Every junction is adjacent to at most one demand branch. Every reservoir is connected at most to one pipe or pump.

Given Assumption 2.1, the graph $\mathscr{G}$ given in (15) is simple and connected and the reservoirs are end vertices. Assigning a direction to each pipe, pump and demand, $\mathscr{G}$ is oriented, allowing to speak of a positive or negative mass flow. Note that the orientation of the pipes and pumps is arbitrary and only serves as a reference condition, it is not necessarily related with the true or expected direction of the fluid flow.

Representing the network as simple, oriented graph, the structure of $\mathscr{N}$ is fully described by the incidence matrix A associated with $\mathscr{G}$. According to $\mathscr{G}$, we partition the incidence matrix as

$$A = \begin{bmatrix} A_{\mathrm{Jc}_V,\mathrm{Pi}} & A_{\mathrm{Jc}_V,\mathrm{Pu}} & A_{\mathrm{Jc}_V,\mathrm{De}} \\ A_{\mathrm{Jc}_0,\mathrm{Pi}} & A_{\mathrm{Jc}_0,\mathrm{Pu}} & A_{\mathrm{Jc}_0,\mathrm{De}} \\ A_{\mathrm{Re},\mathrm{Pi}} & A_{\mathrm{Re},\mathrm{Pu}} & A_{\mathrm{Re},\mathrm{De}} \end{bmatrix} = \begin{bmatrix} A_{\mathrm{Jc}} \\ A_{\mathrm{Re}} \end{bmatrix}$$

and summarize the enthalpy fluxes, specific enthalpies and their differences as

$$H = \begin{bmatrix} H_{\mathrm{Pi}} \\ H_{\mathrm{Pu}} \\ H_{\mathrm{De}} \end{bmatrix}, \quad h = \begin{bmatrix} h_{\mathrm{Jc}_V} \\ h_{\mathrm{Jc}_0} \\ h_{\mathrm{Re}} \end{bmatrix}, \quad \Delta h = \begin{bmatrix} \Delta h_{\mathrm{Jc}_V} \\ \Delta h_{\mathrm{Jc}_0} \\ \Delta h_{\mathrm{Re}} \end{bmatrix},$$

where $h_{\mathrm{Jc}_V}, h_{\mathrm{Jc}_0}$ refer to the enthalpies associated with junctions of positive and zero volume, respectively. Furthermore, we consider the matrix

$$|A| = \big[|A_{ij}|\big]_{(i,j)\in\mathscr{V}\times\mathscr{E}}$$

containing the element-wise absolute values of incidence matrix A and set

$$B_\star(q_*) = \frac{1}{2}\operatorname{diag}(q_*(t))\,\big(\operatorname{diag}(\operatorname{sgn}(q_*(t)))\,A^T_{\star,*} + |A^T_{\star,*}|\big), \tag{16}$$

for $\star = \mathrm{Jc}_0, \mathrm{Jc}_V, \mathrm{Re}$, $* = \mathrm{Pi}, \mathrm{Pu}$.

By the definition of A, the enthalpy drop $\Delta h_j = h_{j_1} - h_{j_2}$ along a given edge $e_j = (v_{j_1}, v_{j_2})$ is given by $e_j^T A^T h = \Delta h_j$, such that the pipe equation (13a) for enthalpy flow H_{Pi} can be summarized as

$$H_{\mathrm{Pi}} = B_{\mathrm{Jc}_0}(q_{\mathrm{Pi}})h_{\mathrm{Jc}_0} + B_{\mathrm{Jc}_V}(q_{\mathrm{Pi}})h_{\mathrm{Jc}_V} + B_{\mathrm{Re}}(q_{\mathrm{Pi}})h_{\mathrm{Re}} =: f_{\mathrm{Pi}^*}(q_{\mathrm{Pi}}, h_{\mathrm{Jc}_V}, h_{\mathrm{Jc}_0}, h_{\mathrm{Re}}), \tag{17a}$$

with $f_{\mathrm{Pi}^*} \in C^1(\Omega_{\mathrm{Pi}} \times \Omega_{\mathrm{Jc}^*} \times \Omega_{\mathrm{Re}^*}, \mathbb{R}^{n_{\mathrm{Pi}}})$, where $\Omega_{\mathrm{Pi}} = \times_{j=1}^{n_{\mathrm{Pi}}} \Omega_{\mathrm{Pi}_j}$, Ω_{Jc^*} and Ω_{Re^*} denotes the domains of admissible mass flows and enthalpies in $\mathscr{Pi}$ and $\mathscr{Jc}^*, \mathscr{Re}^*$, respectively. Similarly, the pump equation (13b) for the full network reads

$$H_{\mathrm{Pu}} = B_{\mathrm{Jc}_0}(q_{\mathrm{Pu}})h_{\mathrm{Jc}_0} + B_{\mathrm{Jc}_V}(q_{\mathrm{Pu}})h_{\mathrm{Jc}_V} + B_{\mathrm{Re}}(q_{\mathrm{Pu}})h_{\mathrm{Re}} + \delta h =: f_{\mathrm{Pu}^*}(q_{\mathrm{Pu}}, h_{\mathrm{Jc}_V}, h_{\mathrm{Jc}_0}, h_{\mathrm{Re}}), \tag{17b}$$

with $f_{\mathrm{Pu}^*} \in C^1(\Omega_{\mathrm{Pu}} \times \Omega_{\mathrm{Jc}^*} \times \Omega_{\mathrm{Re}^*}, \mathbb{R}^{n_{\mathrm{Pi}}})$, where $\Omega_{\mathrm{Pu}} = \times_{j=1}^{n_{\mathrm{Pu}}} \Omega_{\mathrm{Pu}_j}$ denotes the domain of admissible mass flow in $\mathscr{Pu}$ and $\delta h := [\delta h_j]_{j=1,\dots,n_{\mathrm{Pu}}}$.

The sum of all flows entering or leaving a junction Jc_i is given by $e_i^T A H = \sum_{e_j\in\mathscr{E}_{inc}(\mathrm{Jc}_i)} H_j$, such that the junction equations (13c), (13d) can be summarized as

$$A_{\mathrm{Jc,Pi}}H_{\mathrm{Pi}} + A_{\mathrm{Jc,De}}H_{\mathrm{De}} = V_{\mathrm{Jc}}\dot{h}_{\mathrm{Jc}}, \tag{17c}$$

$$A_{\mathrm{Jc,Pi}}H_{\mathrm{Pi}} + A_{\mathrm{Jc,De}}H_{\mathrm{De}} = 0. \tag{17d}$$

For the demand branches and reservoirs, we obtain the simple relations

$$H_{\mathrm{De}} = \bar{H}_{\mathrm{De}}, \tag{17e}$$
$$h_{\mathrm{Re}} = \bar{h}_{\mathrm{Re}}, \tag{17f}$$

where we assume that $\bar{H}_{\mathrm{De}} \in C^1(\mathcal{I}_{\mathrm{De}}, \mathbb{R})$, $\bar{h}_{\mathrm{Re}} \in C^1(\mathcal{I}_{\mathrm{Re}}, \mathbb{R})$ for $\mathcal{I}_{\mathrm{De}} = \cap_{j=1}^{n_{\mathrm{De}}} \mathcal{I}_{\mathrm{De}_j}$, $\mathcal{I}_{\mathrm{Re}} = \cap_{j=1}^{n_{\mathrm{Re}}} \mathcal{I}_{\mathrm{Re}_i}$.

For the mass flows q and pressures p, we proceed similarly and summarize the flows, pressures and pressure differences as

$$q = \begin{bmatrix} q_{\mathrm{Pi}} \\ q_{\mathrm{Pu}} \\ q_{\mathrm{De}} \end{bmatrix}, \quad p = \begin{bmatrix} p_{\mathrm{Jc}} \\ p_{\mathrm{Re}} \end{bmatrix}, \quad \Delta p = \begin{bmatrix} \Delta p_{\mathrm{Pi}} \\ \Delta p_{\mathrm{Pu}} \\ \Delta p_{\mathrm{De}} \end{bmatrix}.$$

For the pipes, we set $C_1 = \mathrm{diag}\,(c_{1,j})_j$, $C_2 = \mathrm{diag}\,(c_{2,j})_j$, $C_3 = [c_{3,j}]_j$ for $j = 1, ...n_{\mathrm{Pi}}$, where $c_{i,j}$ characterizes the physical properties of Pi_j. Including thermal effects, properties like the density of the mass flow q_j typically depend on the specific enthalpy h_{j_1} in the originating vertex $\mathcal{v}_{j_1}$, implying that $c_{2,j} = c_{2,j}(h_{j_1})$, in particular. Defining the *pipe function*

$$\begin{aligned} f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}, h_{\mathrm{Jc}_0}, h_{\mathrm{Jc}_V}, h_{\mathrm{Re}}) := {} & C_1(A_{\mathrm{Jc,Pi}}^T p_{\mathrm{Jc}} + A_{\mathrm{Re,Pi}}^T p_{\mathrm{Re}}) \\ & + C_2(h_{\mathrm{Jc}_V}, h_{\mathrm{Jc}_0}, h_{\mathrm{Re}})\,\mathrm{diag}\,(|q_{\mathrm{Pi},j}|)_j\, q_{\mathrm{Pi}} + C_3, \end{aligned}$$

with $f_{\mathrm{Pi}} \in C^1(\Omega_{\mathrm{Pi}} \times \Omega_{\mathrm{Jc}} \times \Omega_{\mathrm{Re}}, \mathbb{R}^{n_{\mathrm{Pi}}})$, where Ω_{Jc} and Ω_{Re} denotes the domains of admissible pressures in $\mathcal{P}i$ and $\mathcal{J}c, \mathcal{R}e$, respectively, the pipe flows are specified by the differential equation

$$\dot{q}_{\mathrm{Pi}} = f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}, h_{\mathrm{Jc}_0}, h_{\mathrm{Jc}_V}, h_{\mathrm{Re}}). \tag{17g}$$

For the pumps, the relation between mass flow and pressure drop is described by the *pump function*

$$f_{\mathrm{Pu}} := [f_{\mathrm{Pu},j}]_{j=1,\ldots,n_{\mathrm{Pu}}},$$

where we assume that $f_{\mathrm{Pu}} \in C^1(\Omega_{\mathrm{Pu}}, \mathbb{R}^{n_{\mathrm{Pu}}})$, cp. e.g., [9]. Then, we get the pump equation

$$A_{\mathrm{Jc,Pu}}^T p_{\mathrm{Jc}} + A_{\mathrm{Re,Pu}}^T p_{\mathrm{Re}} = f_{\mathrm{Pu}}(q_{\mathrm{Pu}}). \tag{17h}$$

Due to the conservation of mass, the sum of all flows entering or leaving a junction Jc_i is equal and we get that

$$A_{\mathrm{Jc,Pi}} q_{\mathrm{Pi}} + A_{\mathrm{Jc,Pu}} q_{\mathrm{Pu}} + A_{\mathrm{Jc,De}} q_{\mathrm{De}} = 0. \tag{17i}$$

For the demand branches and reservoirs, we obtain the simple relations

$$q_{\mathrm{De}} = \bar{q}_{\mathrm{De}}, \tag{17j}$$
$$p_{\mathrm{Re}} = \bar{p}_{\mathrm{Re}}, \tag{17k}$$

where we assume that $\bar{q}_{\mathrm{De}} \in C^1(\mathcal{I}_{\mathrm{De}}, \mathbb{R})$, $\bar{p}_{\mathrm{Re}} \in C^1(\mathcal{I}_{\mathrm{Re}}, \mathbb{R})$ for $\mathcal{I}_{\mathrm{De}} = \cap_{j=1}^{n_{\mathrm{De}}} \mathcal{I}_{\mathrm{De}_j}$, $\mathcal{I}_{\mathrm{Re}} = \cap_{j=1}^{n_{\mathrm{Re}}} \mathcal{I}_{\mathrm{Re}_i}$.

In conclusion, the dynamic of the network $\mathcal{N}$ is modeled by the differential-algebraic system (17). Each equation of (17) and each entry of the state has a direct physical counterpart in the network. We use this relation to find conditions when (17) is uniquely solvable and to re-interpret these conditions as conditions on the structure and the elements of the network $\mathcal{N}$.

As prerequisites, we analyze the following substructures of the network $\mathcal{N}$. We consider the subset of junctions and pumps $\mathcal{G}_{\mathrm{Jc,Pu}} := \{\mathcal{Jc}, \mathcal{Pu}\}$ with connection matrix $A_{\mathrm{Jc,Pu}}$. We assume that $\mathcal{G}_{\mathrm{Jc,Pu}}$ is composed of K connected components $\mathcal{G}_{\mathrm{Jc,Pu},k} = \{\mathcal{Jc}_k, \mathcal{Pu}_k\}$, that are numbered such that $\mathcal{G}_{\mathrm{Jc,Pu},k}$ corresponds to proper subgraphs for $k = 1, ..., \hat{k}$ and to subsets with loose edges for $k = \hat{k}+1, ..., K$. The connection matrix is partitioned accordingly into $A_{\mathrm{Jc,Pu}} = \mathrm{diag}\left(A_{\mathrm{Jc,Pu},k}\right)_k$.
According to Section 1 , we partition $\mathcal{G}_{\mathrm{Jc,Pu}}$ into a *reduced vertex set* $\mathcal{Jc}_1$ with *ground nodes* $\mathcal{Jc}_2$ as well as a *pump spanning tree* $\mathcal{Pu}_1$ with *chord set* $\mathcal{Pu}_2$. The associated selection matrices are given by

$$\Gamma = [\Gamma_1, \Gamma_2], \quad \Pi_{\mathrm{Pu}} = [\Pi_{\mathrm{Pu},1}, \Pi_{\mathrm{Pu},2}],$$

cp. (4a). We consider the *fundamental cycles* $\mathcal{C}_{\mathrm{Jc,Pu}} = \cup_{k=1}^{K} \mathcal{C}_{\mathrm{Jc,Pu},k}$, *crossing paths* $\mathcal{P}_{\mathrm{Jc,Pu}} = \cup_{k=1}^{K} \mathcal{P}_{\mathrm{Jc,Pu},k}$ and loose pumps $\mathcal{Pu}^{loose}$ with selection matrix

$$V_2 := [C_{\mathrm{Jc,Pu}},\ P_{\mathrm{Jc,Pu}},\ L_{\mathrm{Jc,Pu}}],$$

cp. (4c). For $k = 1, ..., \hat{k}$, we consider the componentwise vertex identification of $\mathcal{Jc}_k$ and set $\bar{\mathcal{Jc}}_k := \{\bar{\mathrm{Jc}}_k\}$ where $\bar{\mathrm{Jc}}_k := \cup_{\mathrm{Jc}_i \in \mathcal{Jc}_k} \mathrm{Jc}_i$. The associated identification matrix is given by

$$U_2 := \mathbf{1}_{\mathrm{Jc,Pu}},$$

cp. (4d). According to Lemma 1.1, then $\mathrm{rank}(A_{\mathrm{Jc,Pu}}) = n_{\mathrm{Jc}} - \hat{k}$, where $\hat{k}$ denotes the number of connected components in $\mathcal{G}_{\mathrm{Jc,Pu}}$ that itself are subgraphs and $\ker(A_{\mathrm{Jc,Pu}}) = \mathrm{span}(V_2)$, $\mathrm{corange}(A_{\mathrm{Jc,Pu}}) = \mathrm{span}(\Pi_{\mathrm{Pu}_1})$ as well as $\mathrm{coker}(A_{\mathrm{Jc,Pu}}) = \mathrm{span}(\mathbf{1}_{\mathrm{Jc,Pu}})$ and $\mathrm{range}(A_{\mathrm{Jc,Pu}}) = \mathrm{span}(\Gamma_1)$. From these matrices, we define the transformations

$$U := [\Gamma_1,\, U_2], \quad V := [\Pi_{\mathrm{Pu}_1},\, V_2], \tag{18a}$$

of which the inverses are given by

$$U^{-1} = \begin{bmatrix} U_2^- \\ \Gamma_2^T \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} V_2^- \\ \Pi_{\mathrm{Pu}_2}^T \end{bmatrix}, \tag{18b}$$

where $U_2^- = \mathrm{diag}(U_{2,k}^-)_{k=1,...,\hat{k}}$, $V_2^- = [\mathrm{diag}(V_{2,k}^-)_{k=1,...,\hat{k}},\, 0]$ with $U_{2,k}^- = \Gamma_{1,k}^T - \mathbf{1}_{n_{\mathcal{V}_{\mathcal{G}_{\mathrm{Jc,Pu},k}}}-1} \Gamma_{2,k}^T$, $V_{2,k}^- = \Pi_{\mathrm{Pu},k,1}^T - \Pi_{\mathrm{Pu},k,1}^T V_2 \Pi_{\mathrm{Pu},k,2}^T$.
From the vertex identification $\bar{\mathcal{Jc}}$, we define the set $\mathcal{G}_{\bar{\mathcal{Jc}},\mathrm{Pi}} := \{\bar{\mathcal{Jc}}, \mathcal{Pi}\}$ composed of L connected components $\mathcal{G}_{\mathrm{Jc,Pi},k} = \{\bar{\mathcal{Jc}}_k, \mathcal{Pi}_k\}$. We partition $\mathcal{G}_{\bar{\mathcal{Jc}},\mathrm{Pi}}$ into a *pipe spanning tree* $\mathcal{Pi}_1$ with *pipe chord set* $\mathcal{Pi}_2$, where $\bar{\mathcal{Jc}}_i = \cup_{k=1}^{L} \bar{\mathcal{Jc}}_{k,i}$ and $\mathcal{Pi}_i = \cup_{k=1}^{L} \mathcal{Pi}_{k,i}$ for $i = 1, 2$. The selection matrices are given by

$$\Pi_{\mathrm{Pi}} = [\Pi_{\mathrm{Pi},2}, \Pi_{\mathrm{Pi},1}]. \tag{18c}$$

Then, $\operatorname{rank}(A_{\bar{\mathrm{Jc}},\mathrm{Pi}}) = \hat{k} - \hat{l}$, where $\hat{l}$ denotes the number of connected components in $\mathscr{G}_{\bar{\mathrm{Jc}},\mathrm{Pi}}$ that itself are subgraphs, and $\operatorname{corange}(A_{\bar{\mathrm{Jc}},\mathrm{Pi}}) = \operatorname{span}(\Pi_{\mathrm{Pi}_1})$, where $\Pi_{\mathrm{Pi}} = [\Pi_{\mathrm{Pi}_1}, \Pi_{\mathrm{Pi}_2}]$ is a permutation.

We partition and transform the variables according to these substructures by setting

$$\begin{aligned} p_{\mathrm{Jc}_1} &:= U_2^- p_{\mathrm{Jc}}, \quad q_{\mathrm{Pu}_1} := V_2^- q_{\mathrm{Pu}}, \quad\;\; q_{\mathrm{Pi}_1} := \Pi_{\mathrm{Pi}_1}^T q_{\mathrm{Pi}}, \\ p_{\mathrm{Jc}_2} &:= \Gamma_2^T p_{\mathrm{Jc}}, \quad q_{\mathrm{Pu}_2} := \Pi_{\mathrm{Pu}_2}^T q_{\mathrm{Pu}}, \quad q_{\mathrm{Pi}_2} := \Pi_{\mathrm{Pi}_2}^T q_{\mathrm{Pi}}. \end{aligned} \tag{19}$$

The components q_{Pu_2}, q_{Pi_2} and p_{Jc_2} belong to mass flows in pumps on the *pump chord set* $\mathscr{P}u_2$, pipes on the *pipe chord set* $\mathscr{P}i_2$ and pressures in *ground nodes* $\mathscr{J}c_2$, respectively. The mass flows q_{Pi_1} and q_{Pu_1} denote the mass flows in pipes or pumps on the pipe or pump spanning tree compared to the mass flow in the fundamental cycles and crossing paths this transfer element is part of. The pressure $p_{\mathrm{Jc}_1,i}$ denotes the pressure difference between a junction Jc_i in the reduced vertex set and the associated ground node.
We denote the associated connection matrices accordingly, e.g., by setting $A_{\mathrm{Jc}_1,\mathrm{Pu}_1} := \Gamma_1^T A_{\mathrm{Jc},\mathrm{Pu}} \Pi_{\mathrm{Pu}_1}$.

On these substructures, we consider the matrices

$$\begin{aligned} C &:= A_{\bar{\mathrm{Jc}},\mathrm{Pi}} C_2 A_{\bar{\mathrm{Jc}},\mathrm{Pi}}^T, \qquad D(q_{\mathrm{Pu}}) := V_2^T \,\mathrm{D}\, f_{\mathrm{Pu}}(q_{\mathrm{Pu}}) V_2, \\ B(q_{\mathrm{Pi}}, q_{\mathrm{Pu}}) &:= A_{\mathrm{Jc}_0,\mathrm{Pi}} B_{\mathrm{Jc}_0}(q_{\mathrm{Pi}}) + A_{\mathrm{Jc}_0,\mathrm{Pu}} B_{\mathrm{Jc}_0}(q_{\mathrm{Pu}}), \end{aligned}$$

where C is the Jacobian of the pipe function f_{Pi} restricted to the contraction $\bar{\mathscr{J}c}$ and with respect to the pressure p_{Jc_2}, B is the Jacobian of the pipe and enthalpy function $f_{\mathrm{Pi}^*}, f_{\mathrm{Pu}^*}$ with respect to h_{Jc_0}, and D is the Jacobian of the pump function f_{Pu^*} restricted to the virtual connection points $\mathscr{J}c_0$ and with respect to the pump flows q_{Pu_2}.
We give topological conditions when B, C are nonsingular. For B, in particular, we consider the flow graph $\mathscr{G}^{\mathrm{flow}}_{\mathrm{Jc}_0,(\mathrm{Pi},\mathrm{Pu})}$ of the subset $\mathscr{G}_{\mathrm{Jc}_0,(\mathrm{Pi},\mathrm{Pu})}$. As the directions of the mass flows may change with $t \in \mathscr{I}$, the flow graph is state-dependent in general and we write $\mathscr{G}^{\mathrm{flow}}_{\mathrm{Jc}_0,(\mathrm{Pi},\mathrm{Pu})}(q_{\mathrm{Pi}}, q_{\mathrm{Pu}})$. Accordingly, the sets $\mathscr{E}_{inc,s}(\mathrm{Jc}_{0,i})$, $\mathscr{E}_{inc,e}(\mathrm{Jc}_{0,i})$ are state-dependent and we write $\mathscr{E}_{inc,s}(\mathrm{Jc}_{0,i}; q_{\mathrm{Pi}}, q_{\mathrm{Pu}})$, $\mathscr{E}_{inc,e}(\mathrm{Jc}_{0,i}; q_{\mathrm{Pi}}, q_{\mathrm{Pu}})$.

Lemma 2.1. *Consider the network* (12) *with graph* $\mathscr{G}$ *and incidence matrix* A. *Consider the subsets* $\mathscr{G}_{Jc,Pu}, \mathscr{G}_{\bar{J}c,Pi}$ *with submatrices* $A_{Jc,Pu}, A_{\bar{J}c,Pi}$.

(i) If $n_{Re} > 0$, *then* $\operatorname{rank}(A_{\bar{J}c,Pi}) = \hat{k}$ *and* C *is nonsingular.*

(ii) For $t \in \mathscr{I}$, *let* $\mathscr{G}^{flow}_{Jc_0,(Pi,Pu);k}(t)$, $k = 1, ..., K$, *denote the strongly connected components in the flow graph* $\mathscr{G}^{flow}_{Jc_0,(Pi,Pu)}(t)$ *at time* t. *For* $Jc_{0,i} \in \mathscr{J}c_{0,k}$ *and* $k = 1, ..., K$, *if*

$$\sum_{e_j \in \mathscr{E}_{inc,s}(Jc_{0,i}; q_{Pi}, q_{Pu}) \cap \{\mathscr{P}i \cup \mathscr{P}u\}} |q_j| > 0, \tag{20a}$$

$$\sum_{e_j \in \mathscr{E}_{inc,s}(Jc_{0,i}; q_{Pi}, q_{Pu}) \cap (\{\mathscr{P}i \cup \mathscr{P}u\} \setminus \mathscr{E}^{inner}_{Jc_0,(Pi,Pu),flow,k}(t))} |q_j| \geq 0 \qquad k = 1, ..., K \geq 0, \tag{20b}$$

and for every $k = 1, ..., K$ *there exists* $\hat{J}c_{0,k} \in \mathscr{J}c_{0,k}$, *such that* (20b) *is strictly satisfied, then* $B(q_{Pi}, q_{Pu})$ *is nonsingular.*

Proof. Neglecting the demand branches $\mathrm{De}_1, ..., \mathrm{De}_{n_{\mathrm{De}}}$ from $\mathcal{G}_{\mathcal{N}}$, we obtain the subgraph $\mathcal{G}_{\mathcal{N}\setminus\mathcal{D}e} := \{\{\mathcal{J}c, \mathcal{R}e\}, \{\mathcal{P}i, \mathcal{P}u\}\}$ whose incidence matrix is given by

$$A_{\mathcal{G}_{\mathcal{N}\setminus\mathcal{D}e}} = \begin{bmatrix} A_{\mathrm{Jc,Pi}} & A_{\mathrm{Jc,Pu}} \\ A_{\mathrm{Re,Pi}} & A_{\mathrm{Re,Pu}} \end{bmatrix}.$$

As $\mathcal{G}_{\mathcal{N}\setminus\mathcal{D}e}$ is a connected subgraph, it follows that $\mathrm{rank}(A_{\mathcal{G}_{\mathcal{N}\setminus\mathcal{D}e}}) = n_{\mathcal{V}} - 1$, cp. Lemma 1.1. For $[A_{\mathrm{Jc,Pi}}, A_{\mathrm{Jc,Pu}}]$, this implies that $\mathrm{rank}([A_{\mathrm{Jc,Pi}}, A_{\mathrm{Jc,Pu}}]) = n_{\mathrm{Jc}}$ if $n_{\mathrm{Re}} > 0$. Considering the transformations U, V defined in (18), we thus have that

$$\begin{aligned} n_{\mathrm{Jc}} &= \mathrm{rank}\left(U^T [A_{\mathrm{Jc,Pi}}, A_{\mathrm{Jc,Pu}}] V\right) = \mathrm{rank}\left(\begin{bmatrix} A_{\mathrm{Jc_1,Pi}} & A_{\mathrm{Jc_1,Pu_1}} & 0 \\ A_{\bar{\mathrm{J}}\mathrm{c,Pi}} & 0 & 0 \end{bmatrix}\right) \\ &= \mathrm{rank}(A_{\mathrm{Jc_1,Pu_1}}) + \mathrm{rank}(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}) = n_{\mathrm{Jc}} - \hat{k} + \mathrm{rank}(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}), \end{aligned}$$

and noting that $\mathrm{rank}(A_{\mathrm{Jc,Pu}}) = n_{\mathrm{Jc}} - \hat{k}$, where $\hat{k}$ denotes the number of connected components in $\mathcal{G}_{\mathrm{Jc,Pu}}$ that itself are subgraphs, cp. Lemma 1.1, it follows that $\mathrm{rank}(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}) = \hat{k}$.
Noting that $C_1 = \mathrm{diag}\,(c_{1,j})_{j=1,...n_{\mathrm{Pi}_j}}$ is positive definite because $c_{1,j} > 0$, $j = 1, ...n_{\mathrm{Pi}_j}$, we can factorize the matrix C into $C = (A_{\bar{\mathrm{J}}\mathrm{c,Pi}}\sqrt{C_1})(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}\sqrt{C_1})^T$, where $\sqrt{C_1} = \mathrm{diag}\left(\sqrt{c_{1,j}}\right)_{j=1,...n_{\mathrm{Pi}_j}}$ is nonsingular. Then, $\mathrm{rank}(C) = \mathrm{rank}(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}\sqrt{C_1}) = \mathrm{rank}(A_{\bar{\mathrm{J}}\mathrm{c,Pi}}) = \hat{k}$, implying that $C \in \mathbb{R}^{\hat{k}\times\hat{k}}$ is nonsingular.
(ii) For convenience, we omit the argument t. Noting that

$$\begin{aligned} B(q_{\mathrm{Pi}}, q_{\mathrm{Pu}}) = \frac{1}{2}\left[A_{\mathrm{Jc_0,(Pi)}}, A_{\mathrm{Jc_0,(Pu)}}\right] &\begin{bmatrix} \mathrm{diag}\,(q_{\mathrm{Pi}}) & 0 \\ 0 & \mathrm{diag}\,(q_{\mathrm{Pu}}) \end{bmatrix} \\ \left(\begin{bmatrix} \mathrm{diag}\,(\mathrm{sgn}(q_{\mathrm{Pi}})) & 0 \\ 0 & \mathrm{diag}\,(\mathrm{sgn}(q_{\mathrm{Pu}})) \end{bmatrix}\right. &\left.\left[A_{\mathrm{Jc_0,(Pi)}}, A_{\mathrm{Jc_0,(Pu)}}\right]^T + \left[|A_{\mathrm{Jc_0,(Pi)}}|, |A_{\mathrm{Jc_0,(Pu)}}|\right]^T\right), \end{aligned}$$

we find that $B(q_{\mathrm{Pi}}, q_{\mathrm{Pu}})$ corresponds to the sum of the flow matrix $B_{\mathrm{Jc_0},\{\mathcal{P}i,\mathcal{P}u\}}$ of $\{\mathcal{J}c_0, \{\mathcal{P}i, \mathcal{P}u\}\}$, i.e., $B = B_{\mathrm{Jc_0},\{\mathcal{P}i,\mathcal{P}u\}}$. Under the given assertions, $B(q_{\mathrm{Pi}}, q_{\mathrm{Pu}})$ is nonsingular, cp. Lemma 1.2. □

We call the set of virtual connection points $\mathcal{J}c_0$ *enthalpy reachable in* $t \in \mathcal{I}$, if the assertions of Lemma 2.1, (ii) are satisfied in $q_{\mathrm{Pi}}, q_{\mathrm{Pu}} \in \Omega_{\mathrm{Pi}} \times \Omega_{\mathrm{Pu}}$.

3 Topological solvability conditions for the pressure and temperature model

To analyze the solvability of (17), we define the *network function* $F = [F_{pres}^T, F_{enth}^T, F_{bound}^T]^T \in C^1(\mathbb{D}, \mathbb{R}^{2n})$ with

$$F_1 := \begin{bmatrix} F_{\mathrm{Pi}} \\ F_{\mathrm{Jc}_V^*} \end{bmatrix}, \quad F_{pres} := \begin{bmatrix} F_{\mathrm{Pu}} \\ F_{\mathrm{Jc}} \end{bmatrix}, \quad F_{enth} := \begin{bmatrix} F_{\mathrm{Jc}_0^*} \\ F_{\mathrm{Pi}^*} \\ F_{\mathrm{Pu}^*} \end{bmatrix}, \quad F_{bound} := \begin{bmatrix} F_{\mathrm{De}} \\ F_{\mathrm{Re}} \\ F_{\mathrm{De}^*} \\ F_{\mathrm{Re}^*} \end{bmatrix}, \tag{21}$$

for $x = [q^T, H^T, p^T, h^T]^T$, domain of definition $\mathbb{D} := \mathscr{I} \times \Omega_x \times \dot{\Omega}_x$ with $\mathscr{I} = \mathscr{I}_{\mathrm{De}} \cap \mathscr{I}_{\mathrm{Re}}$, $\Omega_x := (\Omega_{\mathrm{Pi}} \times \Omega_{\mathrm{Pu}} \times \Omega_{\mathrm{De}})^2 \times (\Omega_{\mathrm{Jc}} \times \Omega_{\mathrm{Re}})^2$, $\Omega_{\dot{x}} \subset \mathbb{R}^{2n}$, and

$$F_{\mathrm{Pi}}(t, x, \dot{x}) = \dot{q}_{\mathrm{Pi}} - f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}, h_{\mathrm{Jc}}, h_{\mathrm{Re}}) \tag{22a}$$

$$F_{\mathrm{Pu}}(t, x) = A^T_{\mathrm{Jc,Pu}} p_{\mathrm{Jc}} + A^T_{\mathrm{Re,Pu}} p_{\mathrm{Re}} - f_{\mathrm{Pu}}(q_{\mathrm{Pu}}) \tag{22b}$$

$$F_{\mathrm{Jc}}(t, x) = A_{\mathrm{Jc,Pi}} q_{\mathrm{Pi}} + A_{\mathrm{Jc,Pu}} q_{\mathrm{Pu}} + A_{\mathrm{Jc,De}} q_{\mathrm{De}} \tag{22c}$$

$$F_{\mathrm{Jc}^*_V}(t, x, \dot{x}) = \hat{V} \dot{h}_{\mathrm{Jc}_V} - A_{\mathrm{Jc}_V,\mathrm{Pi}} H_{\mathrm{Pi}} - A_{\mathrm{Jc}_V,\mathrm{Pu}} H_{\mathrm{Pu}} - A_{\mathrm{Jc}_V,\mathrm{De}} H_{\mathrm{De}} \tag{22d}$$

$$F_{\mathrm{Jc}^*_0}(t, x) = -A_{\mathrm{Jc,Pi}} H_{\mathrm{Pi}} - A_{\mathrm{Jc,Pu}} H_{\mathrm{Pu}} - A_{\mathrm{Jc,De}} H_{\mathrm{De}} \tag{22e}$$

$$F_{\mathrm{Pi}^*}(t, x) = H_{\mathrm{Pi}} - f_{\mathrm{Pi}^*}(q_{\mathrm{Pi}}, h_{\mathrm{Jc}_V}, h_{\mathrm{Jc}_0}, h_{\mathrm{Re}}) \tag{22f}$$

$$F_{\mathrm{Pu}^*}(t, x) = H_{\mathrm{Pu}} - f_{\mathrm{Pu}^*}(q_{\mathrm{Pu}}, h_{\mathrm{Jc}_V}, h_{\mathrm{Jc}_0}, h_{\mathrm{Re}}) \tag{22g}$$

$$F_{\mathrm{De}}(t, x) = q_{\mathrm{De}} - \bar{q}_{\mathrm{De}}, \qquad F_{\mathrm{Re}}(t, x) = p_{\mathrm{Re}} - \bar{p}_{\mathrm{Re}}, \tag{22h}$$

$$F_{\mathrm{De}^*}(t, x) = H_{\mathrm{De}} - \bar{H}_{\mathrm{De}}, \qquad F_{\mathrm{Re}^*}(t, x) = h_{\mathrm{Re}} - \bar{h}_{\mathrm{Re}}. \tag{22i}$$

To keep the smoothness assumptions on F as relaxed as possible, we partition the state into differential and algebraic variables x_d, x_a and set

$$x = [{x_d}^T, {x_a}^T]^T, \qquad x_d = [q^T_{\mathrm{Pi}}, h^T_{\mathrm{Jc}_V}]^T, \ x_a = [q^T_{\mathrm{Pu}}, p^T_{\mathrm{Jc}}, H^T_{\mathrm{Pi}^*}, H^T_{\mathrm{Pu}^*}, h^T_{\mathrm{Jc}_0}, q^T_{\mathrm{De}}, p^T_{\mathrm{Re}}, q^T_{\mathrm{De}^*}, p^T_{\mathrm{Re}^*}]^T.$$

In addition to the network function, we define the *surrogate network function* $\hat{F} = [\hat{F}_1^T, \hat{F}_2^T, F^T_{bound}]^T \in C^1(\mathbb{D}, \mathbb{R}^{2n})$ with

$$\hat{F}_1 := \begin{bmatrix} F_{\mathrm{Pi}_2} \\ F_{\mathrm{Jc}_V} \end{bmatrix}, \quad \hat{F}_{2,pres} := \begin{bmatrix} F_{\dot{\mathrm{Jc}}} \\ F_{\mathrm{Pu}} \\ F_{\mathrm{Jc}} \end{bmatrix}, \quad \hat{F}_{2,enth} := \begin{bmatrix} F_{\mathrm{Pi}^*} \\ F_{\mathrm{Pu}^*} \\ F_{\mathrm{Jc}^*_0} \end{bmatrix}, \tag{23}$$

where $F_{\mathrm{Jc}_V}, F_{\mathrm{Pu}}, F_{\mathrm{Jc}}, F_{\mathrm{Pi}^*}, F_{\mathrm{Pu}^*}, F_{\mathrm{Jc}^*_0}$ are given as in (22) and

$$F_{\mathrm{Pi}_2}(t, x, \dot{x}) := \Pi^T_{\mathrm{Pi}_2} \dot{q}_{\mathrm{Pi}} - \Pi^T_{\mathrm{Pi}_2} f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}, h_{\mathrm{Jc}}, h_{\mathrm{Re}}), \tag{24a}$$

$$F_{\dot{\mathrm{Jc}}}(t, x) := A_{\bar{\mathrm{J}}\mathrm{c,Pi}} f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}, h_{\mathrm{Jc}}, h_{\mathrm{Re}} p) - A_{\bar{\mathrm{J}}\mathrm{c,De}} \dot{\bar{q}}_{\mathrm{De}}. \tag{24b}$$

From the surrogate function, we define the set of *consistent initial values* by

$$\mathscr{C}_{IV} := \hat{F}_2^{-1}(0).$$

Using the concept of derivative arrays and the strangeness index as developed in [15, 16, 17, 18], we characterize the unique solvability of the DAE model (17).

Theorem 3.1. *Let $\mathscr{N}$ be a network given by* (12) *that satisfies Assumptions 2.1 and let $F \in C^1(\mathbb{D}, \mathbb{R}^n)$ be the associated network function. If $n_{Re} > 0$ and, on $\mathscr{C}_{IV}$, the set $\mathscr{J}c_0$ is enthalpy reachable and the matrix $D(q_{Pu})$ is pointwise nonsingular, then the following assertions hold.*

1. *For every $(t_0, x_0) \in \mathscr{C}_{IV}$, there exists an interval $(t_0^-, t_0^+) \subset \mathscr{I}$, such that the initial value problem*

$$F(t, x, \dot{x}) = 0, \tag{25a}$$

$$x(t_0) = x_0, \tag{25b}$$

   is uniquely solvable with $x \in C^1((t_0^-, t_0^+), \mathbb{R}^{2n})$.

2. For every $(t_0, x_0) \in \mathscr{C}_{IV}$, there exists an interval $(t_0^-, t_0^+) \subset \mathscr{I}$, such that a function $x \in C^1((t_0^-, t_0^+), \mathbb{R}^{2n})$ solves (25) if and only if x solves

$$\hat{F}(t, x, \dot{x}) = 0, \tag{26a}$$

$$x(t_0) = x_0. \tag{26b}$$

Proof. We structure our proof in the following way. First, we show that that every solution of (26) solves (25). Using the transformations (18), we show that (26) can be decoupled to an explicit system, whose unique solvability is covered by classical ODE theory and the Implicit Function Theorem. Using the concept of derivative arrays and the strangeness index, we finally derive the surrogate model (26) and show that every solution of (25) solves (26).
To prove that every solution of (26) solves (25), let $x \in C^1(\hat{\mathscr{I}}, \mathbb{R}^{2n})$ solve (26) with $(t_0, x_0) \in \mathscr{C}_{IV}$. Using a nonsingular matrix $S \in \mathbb{R}^{2n\times 2n}$, we transform the states according to (44) and set

$$\tilde{x} := S^{-1}x = \begin{bmatrix} q_{\mathrm{Pi}_2}^T, & p_{\mathrm{Jc}_V}^T, & q_{\mathrm{Pi}_1}^T, & q_{\mathrm{Pu}_1}^T, & q_{\mathrm{Pu}_2}^T, & p_{\mathrm{Jc}_1}^T, & p_{\mathrm{Jc}_2}^T, & q_{\mathrm{De}}^T, & p_{\mathrm{Re}}^T, & H_{\mathrm{De}}^T, & h_{\mathrm{Re}}^T \end{bmatrix}^T.$$

We transform the domain of definition accordingly and set $\Omega_{\tilde{x}} := S^{-1}\cdot\Omega_x$, $\Omega_{\dot{\tilde{x}}} := S^{-1}\cdot\Omega_{\dot{x}}$, $\tilde{\mathbb{D}} := \mathscr{I}\times\Omega_{\tilde{x}}\times\Omega_{\dot{\tilde{x}}}$, and partition the state into $\tilde{x} = [\tilde{x}_d^T, \tilde{x}_a^T]^T$ with $\tilde{x}_a = [p_{\mathrm{Jc}_2}^T, p_{\mathrm{Jc}_1}^T, q_{\mathrm{Pu}_2}^T, q_{\mathrm{Pu}_1}^T, q_{\mathrm{Pi}_1}^T, q_{\mathrm{De}}^T, p_{\mathrm{Re}}^T]^T$, $\tilde{x}_d = q_{\mathrm{Pi}_2}$.
For the equations, we choose a nonsingular matrix $\tilde{S} \in \mathbb{R}^{2n\times 2n}$ and set $\tilde{F}(t, \tilde{x}, \dot{\tilde{x}}) := \tilde{S}^T F(t, S\tilde{x}, S\dot{\tilde{x}})$ such that $\tilde{F} := [\tilde{F}_1^T, \tilde{F}_{2,pres}^T, \tilde{F}_{2,enth}^T, F_{bound}^T]^T \in C^1(\tilde{\mathbb{D}}, \mathbb{R}^n)$ is given by

$$\tilde{F}_1 := \begin{bmatrix} \tilde{F}_{\mathrm{Pi}_2} \\ \tilde{F}_{\mathrm{Jc}_V} \end{bmatrix}, \quad \tilde{F}_{2,pres} := \begin{bmatrix} \tilde{F}_{\mathrm{Pu}_1} \\ \tilde{F}_{\bar{\mathrm{Pu}}} \\ \tilde{F}_{\mathrm{Jc}_1} \\ \tilde{F}_{\bar{\mathrm{Jc}}} \\ \tilde{F}_{\dot{\bar{\mathrm{Jc}}}} \end{bmatrix}, \quad \tilde{F}_{2,enth} := \begin{bmatrix} \tilde{F}_{\mathrm{Pi}^*} \\ \tilde{F}_{\mathrm{Pu}^*} \\ \tilde{F}_{\mathrm{Jc}_0^*} \\ \tilde{F}_{\tilde{\mathrm{Jc}}^*} \end{bmatrix},$$

with, noting that $\dot{q}_{\mathrm{Pi}_2} = \Pi_{\mathrm{Pi}_2}^T q_{\mathrm{Pi}}$ as Π_{Pi_2} is constant,

$$\begin{aligned} \tilde{F}_1 &= \hat{F}_1 \circ (S^{-1} \times S^{-1}), \\ \tilde{F}_{\dot{\bar{\mathrm{Jc}}}} &= F_{\dot{\bar{\mathrm{Jc}}}} \circ S^{-1}, & \tilde{F}_{\mathrm{Pu}_1} &= \Pi_{\mathrm{Pu}_1}^T F_{\mathrm{Pu}} \circ S^{-1}, \\ \tilde{F}_{\mathrm{Pi}^*} &= F_{\mathrm{Pi}^*} \circ S^{-1}, & \tilde{F}_{\mathrm{Pu}^*} &= F_{\mathrm{Pu}^*} \circ S^{-1}, \\ \tilde{F}_{\tilde{\mathrm{Jc}}^*} &= (A_{\mathrm{Jc,Pi}} F_{\mathrm{Pi}^*} + A_{\mathrm{Jc,Pu}} F_{\mathrm{Pu}^*} + F_{\mathrm{Jc}^*}) \circ S^{-1}, & \tilde{F}_{\bar{\mathrm{Pu}}} &= V_2^T F_{\mathrm{Pu}} \circ S^{-1}, \\ \tilde{F}_{\mathrm{Jc}_1} &= \Gamma_1^T F_{\mathrm{Jc}} \circ S^{-1}, & \tilde{F}_{\bar{\mathrm{Jc}}} &= U_2^T F_{\mathrm{Jc}} \circ S^{-1}. \end{aligned}$$

Then, the transformation $\tilde{x} = S^{-1}x$ of the solution x solves

$$\tilde{F}(t, \tilde{x}, \dot{\tilde{x}}_d) = 0, \quad \tilde{x}(t_0) = \tilde{x}_0, \tag{27}$$

Differentiating the mass balance in (27), i.e., $\tilde{F}_{\mathrm{Jc}}(t, \tilde{x}) = 0$, and noting that $A_{\bar{\mathrm{Jc}},\mathrm{Pi}_1}$ is nonsingular, we find that $\tilde{x}$ also solves

$$\dot{q}_{\mathrm{Pi}_1} = -A_{\bar{\mathrm{Jc}},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{Jc}},\mathrm{Pi}_2} \dot{q}_{\mathrm{Pi}_2} - A_{\bar{\mathrm{Jc}},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{Jc}},\mathrm{De}} \dot{q}_{\mathrm{De}}. \tag{28}$$

From the pipe and the demand equation in (27), i.e., $\tilde{F}_{\mathrm{Pi}_2}(t,\tilde{x}) = 0$, $\tilde{F}_{\mathrm{De}}(t,\tilde{x}) = 0$, we further find that $\tilde{x}$ solves

$$\dot{q}_{\mathrm{Pi}_2} = \Pi_{\mathrm{Pi}_2}^T f_{\mathrm{Pi}}\big(\Pi_{\mathrm{Pi}_1} q_{\mathrm{Pi}_1} + \Pi_{\mathrm{Pi}_2} q_{\mathrm{Pi}_2},\ A_{\mathrm{Jc}_1,\mathrm{Pi}}^T p_{\mathrm{Jc}_1} + A_{\bar{\mathrm{Jc}},\mathrm{Pi}}^T p_{\mathrm{Jc}_2} + A_{\mathrm{Re},\mathrm{Pi}}^T \bar{p}_{\mathrm{Re}}\big), \tag{29}$$

$$A_{\bar{\mathrm{Jc}},\mathrm{De}} \dot{q}_{\mathrm{De}} = A_{\bar{\mathrm{Jc}},\mathrm{Pi}} f_{\mathrm{Pi}}\big(\Pi_{\mathrm{Pi}_1} q_{\mathrm{Pi}_1} + \Pi_{\mathrm{Pi}_2} q_{\mathrm{Pi}_2},\ A_{\mathrm{Jc}_1,\mathrm{Pi}}^T p_{\mathrm{Jc}_1} + A_{\bar{\mathrm{Jc}},\mathrm{Pi}}^T p_{\mathrm{Jc}_2} + A_{\mathrm{Re},\mathrm{Pi}}^T \bar{p}_{\mathrm{Re}}\big) \dot{q}_{\mathrm{De}}. \tag{30}$$

Inserting (29), (30) into (28), it follows that $\tilde{x}$ solves the differential equation

$$\begin{aligned} 0 &= F_{\mathrm{Pi}_1}(t,\tilde{x},\dot{\tilde{x}}) \\ &:= \dot{q}_{\mathrm{Pi}_1} - A_{\bar{\mathrm{Jc}},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{Jc}},\mathrm{Pi}_1} \Pi_{\mathrm{Pi}_1}^T f_{\mathrm{Pi}}\big(\Pi_{\mathrm{Pi}_1} q_{\mathrm{Pi}_1} + \Pi_{\mathrm{Pi}_2} q_{\mathrm{Pi}_2}, A_{\mathrm{Jc}_1,\mathrm{Pi}}^T p_{\mathrm{Jc}_1} + A_{\bar{\mathrm{Jc}},\mathrm{Pi}}^T p_{\mathrm{Jc}_2} + A_{\mathrm{Re},\mathrm{Pi}}^T \bar{p}_{\mathrm{Re}}\big). \end{aligned} \tag{31}$$

Replacing the equation $\tilde{F}_{\dot{\bar{\mathrm{Jc}}}}(t,\tilde{x}) = 0$ in (27) by (31), we find that the solution of (26) solves

$$\bar{F}(t,\tilde{x},\dot{\tilde{x}}) = 0, \quad \tilde{x}(t_0) = \tilde{x}_0, \tag{32}$$

where $\bar{F} = [\bar{F}_1^T, \bar{F}_{2,pres}^T, \bar{F}_{2,enth}^T, F_{bound}^T]^T$ is given by

$$\bar{F}_1 := \begin{bmatrix} \tilde{F}_{\mathrm{Pi}_1} \\ \tilde{F}_{\mathrm{Pi}_2} \\ \tilde{F}_{\mathrm{Jc}_V} \end{bmatrix}, \quad \bar{F}_{2,pres} := \begin{bmatrix} \tilde{F}_{\mathrm{Pu}_1} \\ \tilde{F}_{\bar{\mathrm{Pu}}} \\ \tilde{F}_{\mathrm{Jc}_1} \\ \tilde{F}_{\bar{\mathrm{Jc}}} \end{bmatrix}, \quad \bar{F}_{2,enth} := \begin{bmatrix} \tilde{F}_{\mathrm{Pi}^*} \\ \tilde{F}_{\mathrm{Pu}^*} \\ \tilde{F}_{\mathrm{Jc}_0^*} \end{bmatrix}.$$

Reverting the variable transformation and combining the pump and junction equations by V, U using a nonsingular transformation $\bar{S}$, however, we verify that x solves

$$\bar{S}^{-1} \bar{F}(t, S^{-1}x, S^{-1}\dot{x}) = F(t,x,\dot{x}). \tag{33}$$

Hence, if $x \in C^1(\hat{\mathscr{J}}, \mathbb{R}^{2n})$ solves (26) with $(t_0, x_0) \in \mathscr{C}_{IV}$, then x solves (25).

To prove that (26) possesses a unique solution for every $(t_0, x_0) \in \mathscr{C}_{IV}$, we decouple (26) using the transformations (18) to an explicit system to which we can apply classical ODE theory and the Implicit Function Theorem. Considering again the transformed system (27), we observe that the Jacobian $\partial_{\tilde{x}_a} \tilde{F}_2$ of (27) with respect to $\tilde{x}_a$ is given by

$$\partial_{\tilde{x}_a} \tilde{F}_2 = \begin{bmatrix} \partial_{\tilde{x}_a} \tilde{F}_{2,11} & * & * \\ 0 & \partial_{\tilde{x}_a} \tilde{F}_{2,12} & * \\ 0 & 0 & I_{2n_{\mathrm{De}}+2n_{\mathrm{Re}}} \end{bmatrix},$$

where

$$\partial_{\tilde{x}_a} \tilde{F}_{2,11} = \begin{array}{c} \\ \mathrm{D}F_{\dot{\bar{\mathrm{Jc}}}} \\ \mathrm{D}F_{\mathrm{Pu}_1} \\ \mathrm{D}F_{\mathrm{Pi}^*} \\ \mathrm{D}F_{\mathrm{Pu}^*} \end{array} \begin{array}{c} \begin{array}{cccc} p_{\mathrm{Jc}_2} & p_{\mathrm{Jc}_1} & H_{\mathrm{Pi}} & H_{\mathrm{Pu}} \end{array} \\ \begin{bmatrix} A_{\bar{\mathrm{Jc}},\mathrm{Pi}} \mathrm{D}_2 f_{\mathrm{Pi}} A_{\bar{\mathrm{Jc}},\mathrm{Pi}}^T & A_{\bar{\mathrm{Jc}},\mathrm{Pi}} \mathrm{D}_2 f_{\mathrm{Pi}} A_{\mathrm{Jc}_1,\mathrm{Pi}}^T & 0 & 0 \\ 0 & A_{\mathrm{Jc}_1,\mathrm{Pu}_1}^T & 0 & 0 \\ 0 & 0 & I_{n_{\mathrm{Pi}}} & 0 \\ 0 & 0 & 0 & I_{n_{\mathrm{Pu}}} \end{bmatrix} \end{array},$$

$$\partial_{\tilde{x}_a} \tilde{F}_{2,12} = \begin{array}{c} \\ \mathrm{D}F_{\tilde{\mathrm{Jc}}^*} \\ \mathrm{D}F_{\bar{\mathrm{Pu}}} \\ \mathrm{D}F_{\mathrm{Jc}_1} \\ \mathrm{D}F_{\bar{\mathrm{Jc}}} \end{array} \begin{array}{c} \begin{array}{cccc} h_{\mathrm{Jc}_0} & q_{\mathrm{Pu}_2} & q_{\mathrm{Pu}_1} & q_{\mathrm{Pi}_1} \end{array} \\ \begin{bmatrix} B & A_{\mathrm{Jc},\mathrm{Pu}} \mathrm{D}_1 f_{\mathrm{Pu}^*} V_2 & A_{\mathrm{Jc},\mathrm{Pu}} \mathrm{D}_1 f_{\mathrm{Pu}^*} \Pi_{\mathrm{Pu}_1} & A_{\mathrm{Jc},\mathrm{Pi}} \mathrm{D}_1 f_{\mathrm{Pi}^*} \Pi_{\mathrm{Pi}_1} \\ 0 & -D & -V_2^T \mathrm{D} f_{\mathrm{Pu}} \Pi_{\mathrm{Pu}_1} & 0 \\ 0 & 0 & A_{\mathrm{Jc}_1,\mathrm{Pu}_1} & A_{\mathrm{Jc}_1,\mathrm{Pi}_1} \\ 0 & 0 & 0 & A_{\bar{\mathrm{Jc}},\mathrm{Pi}_1} \end{bmatrix} \end{array}.$$

On $\mathscr{C}_{IV}$, the diagonal entries of $\partial_{\tilde{x}_a}\tilde{F}_{2,11}, \partial_{\tilde{x}_a}\tilde{F}_{2,12}$ are pointwise nonsingular, such that $\partial_{\tilde{x}_a}\tilde{F}_2$ is pointwise nonsingular on $\mathscr{C}_{IV}$. For $(t_0, \tilde{x}_0)$ with $(t_0, S\tilde{x}_0) \in \mathscr{C}_{IV}$, we can thus solve the algebraic equation in (27) locally for $\tilde{x}_a$ as a function of $\tilde{x}_d$, cp. [21]. With $\tilde{F} \in C^1(\tilde{\mathbb{D}}, \mathbb{R}^n)$, there exist neighborhoods $\mathscr{I}_0 \times \mathscr{U}(q_{2,0}) \times \mathscr{U}(\tilde{x}_{a,0}) \subset \mathscr{I} \times \Omega_{\tilde{x}}$ and a function $g \in C^1(\mathscr{I}_0 \times \mathscr{U}(q_{2,0}), \mathscr{U}(\tilde{x}_{a,0}))$, such that $(t, \tilde{x})$ solves $\tilde{F}_2(t, \tilde{x}) = 0$ if and only if $\tilde{x}_a = g(t, \tilde{x}_d)$. Setting

$$f(t, x_d) := \tilde{F}_1(t, [\tilde{x}_d^T, g^T(t, \tilde{x}_d)]^T, \dot{x}_d) + \dot{x}_d,$$

it follows that a function $\tilde{x} \in C^1(\hat{\mathscr{J}}, \mathbb{R}^{2n})$ solves (27) and if and only if $\tilde{x}$ solves the explicit system

$$\dot{\tilde{x}}_d = f(t, \tilde{x}_d), \quad \tilde{x}_d(t_0) = \tilde{x}_{d,0} \tag{34a}$$

$$\tilde{x}_a = g(t, x_d). \tag{34b}$$

As $g \in C^1(\mathscr{I}_0 \times \mathscr{U}(q_{2,0}), \mathscr{U}(\tilde{x}_{a,0}))$ and $\tilde{F}_1 \in C^1(\mathscr{I} \times \tilde{\Omega}_x \times \mathbb{R}^d, \mathbb{R}^d)$, the composition satisfies $f \in C^1(\mathscr{I}_0 \times \mathscr{U}(x_{d,0}), \mathbb{R}^{n_{\mathrm{Pi}}})$. Hence, for every initial value $(t_0, x_{d,0}) \in \mathscr{I}_0\mathscr{U}(x_{d,0})$, (34a) has a unique, maximally extended solution $x_d \in C^2((t^-_{0,x_d}, t^+_{0,x_d}), \mathbb{R}^d)$, cp. [1]. Then, also (34b) has a unique solution $\tilde{x}_a \in C^1(\mathscr{I}_{x_a}, \mathbb{R}^d)$, where $\mathscr{I}_{x_a} := \mathscr{I}_0 \cap (t^-_{0,x_d}, t^+_{0,x_d})$. In t_0, in particular, we have $\tilde{x}_a(t_0) = g(t_0, q_{\mathrm{Pi},2,0})$. Setting

$$\mathscr{C}_{sexp} := \{(t_0, \tilde{x}_d, \tilde{x}_a) \in \mathscr{I}_0 \times \mathscr{U}(q_{2,0}) \times \mathscr{U}(\tilde{x}_{a,0}) \,|\, q_{\mathrm{Pi},2,0} \in \mathscr{U}(q_{2,0}), \tilde{x}_a(t_0) = g(t_0, q_{\mathrm{Pi},2,0})\},$$

and $(t_0^-, t_0^+) := (t^-_{0,x_d}, t^+_{0,x_d}) \cap \mathscr{I}_{x_a}$, it follows that (34) is uniquely solvable for every $(t_0, \tilde{x}_0) \in \mathscr{C}_{sexp}$ with $x = [{x_d}^T, {x_a}^T]^T$ such that $x_d \in C^2(\mathscr{J}, \mathbb{R}^{n_{\mathrm{Pi}}})$, $x_a \in C^1(\mathscr{J}, \mathbb{R}^{n-n_{\mathrm{Pi}}})$. As a function $x \in C^1(\mathscr{J}, \mathbb{R}^n)$ solves the surrogate model (26) if and only if its transformation $\tilde{x} = S^{-1}x$ solves the explicit system (34) and noting that

$$\mathscr{C}_{IV} = \{(t_0, x_0) \in \mathscr{I}_0 \times \mathbb{D}_x \,|\, (t_0, S^{-1}x_0) \in \mathscr{C}_{sexp}\}$$

by the construction of (34), it follows that the surrogate model is uniquely solvable on $\mathscr{C}_{IV}$ with $x \in C^1((t_0^-, t_0^+), \mathbb{R}^{2n}$.

Now, we show that every solution $x_d \in C^2(\mathscr{J}, \mathbb{R}^d)$, $x_a \in C^1(\mathscr{J}, \mathbb{R}^a)$, $\mathscr{J} \subset \mathscr{I}$, of (25) with $(t_0, x_0) \in \mathscr{C}_{IV}$ also solves the surrogate model (26) on $\mathscr{J}$. We consider the *derivative array* $\mathscr{F} := [F^T, \dot{F}^T]^T$ of size $\mu = 1$ with F given by (21) and

$$\dot{F}_{pres} := \frac{d}{dt}F_{pres} = \begin{bmatrix} \ddot{q}_{\mathrm{Pi}} - \mathrm{D}_1 f_{\mathrm{Pi}}\dot{q}_{\mathrm{Pi}} - \mathrm{D}_2\, f_{\mathrm{Pi}}(A^T_{\mathrm{Jc,Pi}}\dot{p}_{\mathrm{Jc}} + A^T_{\mathrm{Re,Pi}}\dot{p}_{\mathrm{Re}}) \\ A^T_{\mathrm{Jc,Pu}}\dot{p}_{\mathrm{Jc}} + A^T_{\mathrm{Re,Pu}}\dot{p}_{\mathrm{Re}} - \mathrm{D}_1 f_{\mathrm{Pu}}\dot{q}_{\mathrm{Pu}} \\ A_{\mathrm{Jc,Pi}}\dot{q}_{\mathrm{Pi}} + A_{\mathrm{Jc,Pu}}\dot{q}_{\mathrm{Pu}} + A_{\mathrm{Jc,De}}\dot{q}_{\mathrm{De}} \end{bmatrix},$$

$$\dot{F}_{enth} := \frac{d}{dt}F_{enth} = \begin{bmatrix} V_{\mathrm{Jc}}\ddot{h}_{\mathrm{Jc}} - A_{\mathrm{Jc,Pi}}\dot{H}_{\mathrm{Pi}} - A_{\mathrm{Jc,Pu}}\dot{H}_{\mathrm{Pu}} - A_{\mathrm{Jc,De}}\dot{H}_{\mathrm{De}} \\ -A_{\mathrm{Jc,Pi}}\dot{H}_{\mathrm{Pi}} - A_{\mathrm{Jc,Pu}}\dot{H}_{\mathrm{Pu}} - A_{\mathrm{Jc,De}}\dot{H}_{\mathrm{De}} \\ \dot{H}_{\mathrm{Pi}} - \mathrm{D}_1 f_{\mathrm{Pi}^*}\dot{q}_{\mathrm{Pi}} - \mathrm{D}_2\, f_{\mathrm{Pi}^*}\dot{h} \\ \dot{H}_{\mathrm{Pu}} - \mathrm{D}f_{\mathrm{Pu}}\dot{q}_{\mathrm{Pu}} - \mathrm{D}_1 f_{\mathrm{Pu}}\dot{h} \end{bmatrix},$$

$$\dot{F}_{enth} := \frac{d}{dt}F_{enth} = \begin{bmatrix} \dot{q}_{\mathrm{De}} - \dot{\bar{q}}_{\mathrm{De}} \\ \dot{p}_{\mathrm{Re}} - \dot{\bar{p}}_{\mathrm{Re}} \\ \dot{H}_{\mathrm{De}} - \dot{\bar{H}}_{\mathrm{De}} \\ \dot{h}_{\mathrm{Re}} - \dot{\bar{h}}_{\mathrm{Re}} \end{bmatrix}.$$

We consider the *algebraic solution set* $\mathscr{F}^{-1}(0) = \{z \in \mathbb{R}^{6n+1} \,|\, \mathscr{F}(z) = 0\}$, i.e., the set of all vectors $z = (t, x, v, w)$ that satisfy $\mathscr{F}(z) = 0$ in the algebraic sense without a differential relation between the components and denote the set of initial values (t_0, x_0) that are part of a vector $(t_0, x_0, v_0, w_0) \in \mathscr{F}^{-1}(0)$ by

$$\mathscr{C}_1 := \{(t_0, x_0) \in \Omega_x \,|\, \exists (v_0, w_0) \in \Omega_x \times \mathbb{R}^n \colon (t_0, x_0, v_0, w_0) \in \mathscr{F}^{-1}(0)\}.$$

As every solution $x_d \in C^2(\mathcal{J}, \mathbb{R}^d)$, $x_a \in C^1(\mathcal{J}, \mathbb{R}^a)$, $\mathcal{J} \subset \mathcal{I}$, of (26) with $(t_0, x_0) \in \mathscr{C}_{IV}$ solves the (25) and hence the derivative array $\mathscr{F}(t, x, \dot{x}, \ddot{x}) = 0$, it follows that, considering $x(t_0)$, $\mathscr{C}_{IV} \subset \mathscr{C}_1$. In particular, this implies that $\mathscr{F}^{-1}(0) \neq \emptyset$.
Considering the Jacobians $M(z) := \partial_{v,w}\mathscr{F}(z)$, $N(z) := \partial_x \mathscr{F}(z)$, where $z = (t, x, v, w) \in \mathscr{F}^{-1}(0)$, we first show that $\partial_{v,w} M(z) = 0$, $\partial_{v,w} N(z) = 0$, implying that $M(z) = M(x)$ and $N(z) = N(x)$. For $(t, x) \in \mathscr{C}_{IV}$, then we prove that $M(x), N(x)$ satisfy the following rank assumptions:

(i) $a := \operatorname{corank}(M(x)) = n_{\mathrm{Pi}} + 2n_{\mathrm{Pu}} + 2n_{\mathrm{Jc}_V} + 3n_{\mathrm{Jc}_0} + 2n_{\mathrm{De}} + 2n_{\mathrm{Re}} - \hat{k}$,

(ii) $\operatorname{rank}(Z_2^T N(z)) = a$, where $Z_2 \in \mathbb{R}^{n \times a}$ is a basis of $\operatorname{coker}(M(x))$,

(iii) $\operatorname{rank}(\partial_v F(z) T_2) = d$, where $T_2 \in \mathbb{R}^{a \times d}$ is a basis of $\ker(N)$ and $d := 2n - a$.

By [18, Thm. 4.11], it follows that every solution $x \in C^1(\hat{\mathcal{J}}, \mathbb{R}^{2n})$, $\hat{\mathcal{J}} \subset \mathcal{I}$, of (25) with $(t_0, x_0) \in \mathscr{C}_1$ solves the surrogate model (26) on $\hat{\mathcal{J}}$.
To check the items (i)-(iii), we transform the Jacobians M, N by nonsingular transformations build from the matrices U, V, Π_{Pi} defined in (18). To keep track of the single operations and permutations, we indicate for every block row and column the applied transformation. For (i), we transform the Jacobian M by nonsingular transformations $\bar{\Pi}_M, \Pi_M \in \mathbb{R}^{4n \times 4n}$ such that

$$\bar{\Pi}_M^T M \Pi_M = \begin{bmatrix} I_{n_{\mathrm{Pi}}+n_{\mathrm{Jc}_V}+2n_{\mathrm{De}}+2n_{\mathrm{Re}}} & 0 & 0 & 0 \\ * & \tilde{M}_{22} & \tilde{M}_{24} & 0 \\ * & \tilde{M}_{32} & 0 & 0 \\ \tilde{M}_{41} & 0 & 0 & 0 \end{bmatrix},$$

where

$$
\tilde{M}_{22} = \begin{array}{c}
\\ U_1^T \dot{\mathrm{D}} F_{\mathrm{Jc}} \\ V_2^T \dot{\mathrm{D}} F_{\mathrm{Pu}} \\ V_1^T \dot{\mathrm{D}} F_{\mathrm{Pu}} \\ \dot{\mathrm{D}} F_{\mathrm{Pi}} \\ \dot{\mathrm{D}} F_{\mathrm{Pu}}^* \\ \dot{\mathrm{D}} F_{\mathrm{Pi}}^* \\ \dot{\mathrm{D}} F_{\mathrm{Jc}_V^*}
\end{array}
\begin{array}{ccccccc}
\dot{q}_{\mathrm{Pu}} V_1 & \dot{q}_{\mathrm{Pu}} V_2 & \dot{p}_{\mathrm{Jc}} U_1 & \ddot{q}_{\mathrm{Pi}} & \dot{H}_{\mathrm{Pu}} & \dot{H}_{\mathrm{Pi}} & \ddot{h}_{\mathrm{Jc},V} \\ \hline
A_{\mathrm{Jc}_1,\mathrm{Pu}_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
-V_2^T \,\mathrm{D}\, f_{\mathrm{Pu}} V_1 & -D & 0 & 0 & 0 & 0 & 0 \\
-V_1^T \,\mathrm{D}\, f_{\mathrm{Pu}} V_1 & -V_1^T \,\mathrm{D}\, f_{\mathrm{Pu}} V_2 & A_{\mathrm{Jc}_1,\mathrm{Pu}_1}^T & 0 & 0 & 0 & 0 \\
0 & 0 & -\,\mathrm{D}_2\, f_{\mathrm{Pi}} A_{\mathrm{Jc}_1,\mathrm{Pi}}^T & I_{\mathrm{Pi}} & 0 & 0 & 0 \\
-\,\mathrm{D}\, f_{\mathrm{Pu}^*} V_1 & -\,\mathrm{D}\, f_{\mathrm{Pu}^*} V_2 & 0 & 0 & I_{\mathrm{Pu}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{\mathrm{Pi}} & 0 \\
0 & 0 & 0 & 0 & -A_{\mathrm{Jc}_V,\mathrm{Pu}} & -A_{\mathrm{Jc}_V,\mathrm{Pi}} & V_{\mathrm{Jc}}
\end{array},
$$

$$
\tilde{M}_{24} = \begin{array}{c}
\\ U_1^T \dot{\mathrm{D}} F_{\mathrm{Jc}} \\ V_2^T \dot{\mathrm{D}} F_{\mathrm{Pu}} \\ V_1^T \dot{\mathrm{D}} F_{\mathrm{Pu}} \\ \dot{\mathrm{D}} F_{\mathrm{Pi}} \\ \dot{\mathrm{D}} F_{\mathrm{Pu}}^* \\ \dot{\mathrm{D}} F_{\mathrm{Pi}}^* \\ \dot{\mathrm{D}} F_{\mathrm{Jc}_V^*}
\end{array}
\begin{array}{cccc}
\dot{h}_{\mathrm{Jc},0} & \ddot{h}_{\mathrm{Jc},0} & \dot{p}_{\mathrm{Jc}} U_2 & \ddot{p}_{\mathrm{Jc}} U_2 \\ \hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\,\mathrm{D}_3\, f_{\mathrm{Pi}} & 0 & \mathrm{D}_2\, f_{\mathrm{Pi}} A_{\mathrm{Jc}_2,\mathrm{Pi}}^T & 0 \\
-B_{\mathrm{Jc}_0}(q_{\mathrm{Pu}}) & 0 & 0 & 0 \\
-B_{\mathrm{Jc}_0}(q_{\mathrm{Pi}}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array},
$$

$$
\tilde{M}_{32} = \begin{array}{c} \\ \dot{\mathrm{D}} F_{\mathrm{Jc}_0^*} \end{array}
\begin{array}{ccccccc}
\dot{q}_{\mathrm{Pu}} V_1 & \dot{q}_{\mathrm{Pu}} V_2 & \dot{p}_{\mathrm{Jc}} U_1 & \ddot{q}_{\mathrm{Pi}} & \dot{H}_{\mathrm{Pu}} & \dot{H}_{\mathrm{Pi}} & \ddot{h}_{\mathrm{Jc},V} \\ \hline
0 & 0 & 0 & 0 & -A_{\mathrm{Jc}_0,\mathrm{Pu}} & -A_{\mathrm{Jc}_0,\mathrm{Pi}} & 0
\end{array},
$$

$$
\tilde{M}_{41} = \begin{array}{c}
\\ U_2^T \dot{\mathrm{D}} F_{\mathrm{Jc}} \\ \mathrm{D}\, F_{\mathrm{Pu}}, \mathrm{D}\, F_{\mathrm{Pu}}^* \\ \mathrm{D}\, F_{\mathrm{Pi}}^* \\ \mathrm{D}\, F_{\mathrm{Jc}}, \mathrm{D}\, F_{\mathrm{Jc}_0^*} \\ \mathrm{D}\, F_{bound}
\end{array}
\begin{array}{cccccc}
\dot{q}_{\mathrm{De}} & \dot{H}_{\mathrm{De}} & \dot{p}_{\mathrm{Re}} & \dot{h}_{\mathrm{Re}} & \dot{q}_{\mathrm{Pi}} & \dot{h}_{\mathrm{Jc}_V} \\ \hline
A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{De}} & 0 & 0 & 0 & A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}.
$$

By the choice of U_1, V_1 and the assumption on D, the diagonal block $\tilde{M}_{22}(x)$ is nonsingular on $\mathscr{C}_{IV}$, implying that

$$
\operatorname{rank}(M(x)) = n_{\mathrm{Pi}} + n_{\mathrm{Jc}_V} + 2n_{\mathrm{De}} + 2n_{\mathrm{Re}} + \operatorname{rank}(\tilde{M}_{22}) + \operatorname{rank}(\mathcal{S}_{11}(\bar{\Pi}_M^T M \Pi_M)(z)),
$$

on $\mathscr{C}_{IV}$, cp. (11), where the Schur complement is given by

$$
\mathcal{S}_{11}(\bar{\Pi}_M^T M \Pi_M)(z) = -(\tilde{M}_{32}\tilde{M}_{22}^{-1}\tilde{M}_{24})(z) = \begin{array}{c} \\ \dot{\mathrm{Jc}}_0^* \end{array}
\begin{array}{cccc}
\dot{h}_{\mathrm{Jc},0} & \ddot{h}_{\mathrm{Jc},0} & \dot{p}_{\mathrm{Jc}} U_2 & \ddot{p}_{\mathrm{Jc}} U_2 \\ \hline
B & 0 & 0 & 0
\end{array}.
$$

As $\mathcal{J}c_0$ is enthalpy reachable on $\mathscr{C}_{IV}$, the matrix B is pointwise nonsingular on $\mathscr{C}_{IV}$ with $\operatorname{rank}(B(z)) = n_{\mathrm{Jc}_0}$, cp. Lemma 2.1, and it follows that

$$
\operatorname{rank}(M(x)) = 3(n_{\mathrm{Pi}} + n_{\mathrm{Jc}_V}) + (n_{\mathrm{Pu}} + n_{\mathrm{Jc}_0} + n_{\mathrm{De}} + n_{\mathrm{Re}}) - \hat{k}
$$

and $a = \operatorname{corank}(M(x))$ on $\mathscr{C}_{IV}$. For (ii), we exploit the structure of $\bar{\Pi}_M^T M \Pi_M$ to construct a basis $Z_2 \in \mathbb{R}^{4n\times a}$ of $\operatorname{corange}(M(x))$. Setting

$$Z_2^T = \begin{bmatrix} -\tilde{M}_{41} & 0 & 0 & I_a \end{bmatrix} \bar{\Pi}_M^T, \tag{35}$$

we find that $\operatorname{span}(Z_2) = \operatorname{corange}(M(x))$ for every $x \in \mathscr{C}_{IV}$. Applying Z_2 and a suitable transformation $\Pi_N \in \mathbb{R}^{2n\times 2n}$ to the Jacobian N, we get that

$$\bar{\Pi}_N^T Z_2^T N \Pi_N(z) = \begin{bmatrix} I_{2n_{\mathrm{De}}+2n_{\mathrm{Re}}} & 0 & 0 \\ * & \tilde{N}_{22} & \tilde{N}_{23} \\ * & \tilde{N}_{32} & 0 \end{bmatrix}, \tag{36}$$

where

$$\tilde{N}_{22} = \begin{array}{c} \\ U_2^T \,\mathrm{D}F_{\mathrm{Jc}} \\ U_1^T \,\mathrm{D}F_{\mathrm{Jc}} \\ V_2^T \,\mathrm{D}F_{\mathrm{Pu}} \\ V_1^T \,\mathrm{D}F_{\mathrm{Pu}} \\ \star \\ \mathrm{D}F_{\mathrm{Pu}}^* \\ \mathrm{D}F_{\mathrm{Pi}}^* \end{array} \begin{array}{c} \begin{array}{ccccccc} q_{\mathrm{Pi}}\Pi_1 & q_{\mathrm{Pu}}V_1 & q_{\mathrm{Pu}}V_2 & p_{\mathrm{Jc}}U_1 & p_{\mathrm{Jc}}U_2 & H_{\mathrm{Pu}} & H_{\mathrm{Pi}} \end{array} \\ \begin{bmatrix} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\mathrm{Jc}_1,\mathrm{Pi}_1} & A_{\mathrm{Jc}_1,\mathrm{Pu}_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & V_2^T \,\mathrm{D}f_{\mathrm{Pu}}V_1 & D & 0 & 0 & 0 & 0 \\ 0 & V_1^T \,\mathrm{D}f_{\mathrm{Pu}}V_1 & V_1^T \,\mathrm{D}f_{\mathrm{Pu}}V_2 & A_{\mathrm{Jc}_1,\mathrm{Pu}_1}^T & 0 & 0 & 0 \\ A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}}\,\mathrm{D}_1 f_{\mathrm{Pi}}\Pi_1 & 0 & 0 & \tilde{\mathrm{D}}_2 f_{\mathrm{Pi}} & C & 0 & 0 \\ 0 & -\mathrm{D}\, f_{\mathrm{Pu}^*}V_1 & -\mathrm{D}\, f_{\mathrm{Pu}^*}V_2 & 0 & 0 & I_{\mathrm{Pu}} & 0 \\ -\mathrm{D}\, f_{\mathrm{Pi}^*}\Pi_1 & 0 & 0 & 0 & 0 & 0 & I_{\mathrm{Pi}} \end{bmatrix} \end{array},$$

$$\tilde{N}_{23} = \begin{array}{c} \\ U_2^T \,\mathrm{D}F_{\mathrm{Jc}} \\ U_1^T \,\mathrm{D}F_{\mathrm{Jc}} \\ V_2^T \,\mathrm{D}F_{\mathrm{Pu}} \\ V_1^T \,\mathrm{D}F_{\mathrm{Pu}} \\ \star \\ \mathrm{D}F_{\mathrm{Pu}}^* \\ \mathrm{D}F_{\mathrm{Pi}}^* \end{array} \begin{array}{c} \begin{array}{ccc} q_{\mathrm{Pi}}\Pi_2 & h_{\mathrm{Jc},V} & h_{\mathrm{Jc},0} \end{array} \\ \begin{bmatrix} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_2} & 0 & 0 \\ A_{\mathrm{Jc}_1,\mathrm{Pi}_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}}\,\mathrm{D}_1 f_{\mathrm{Pi}}\Pi_2 & A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}}\partial_{h_{\mathrm{Jc},V}} f_{\mathrm{Pi}} & A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}}\,\mathrm{D}_3\, f_{\mathrm{Pi}} \\ 0 & -B_{\mathrm{Jc}_V}(q_{\mathrm{Pu}}) & -B_{\mathrm{Jc}_0}(q_{\mathrm{Pu}}) \\ -\mathrm{D}\, f_{\mathrm{Pi}^*}\Pi_2 & -B_{\mathrm{Jc}_V}(q_{\mathrm{Pi}}) & -B_{\mathrm{Jc}_0}(q_{\mathrm{Pi}}) \end{bmatrix} \end{array},$$

$$\tilde{N}_{32} = \begin{array}{c} \\ \mathrm{D}F_{\mathrm{Jc}_0^*} \end{array} \begin{array}{c} \begin{array}{ccccccc} q_{\mathrm{Pi}}\Pi_1 & q_{\mathrm{Pu}}V_1 & q_{\mathrm{Pu}}V_2 & p_{\mathrm{Jc}}U_1 & p_{\mathrm{Jc}}U_2 & H_{\mathrm{Pu}} & H_{\mathrm{Pi}} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -A_{\mathrm{Jc}_0,\mathrm{Pu}} & -A_{\mathrm{Jc}_0,\mathrm{Pi}} \end{bmatrix} \end{array},$$

with $\star := -A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{De}}\,\mathrm{D}\,F_{\mathrm{De}} - A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}}\,\mathrm{D}\,F_{\mathrm{Pi}} + \dot{\mathrm{D}}F_{\mathrm{Pi}}$. As $n_{\mathrm{Re}} > 0$, the matrix C is nonsingular, cp. Lemma 2.1. By the choice of $\Gamma_1, \mathrm{Pi}_{\mathrm{Pu}_1}$ and the assumptions on D, then the diagonal block $\tilde{N}_{22}(z)$ is pointwise nonsingular on $\mathscr{C}_{IV}$ with $\operatorname{rank}(\tilde{N}_{22}(z)) = n_{\mathrm{Jc}} + 2n_{\mathrm{Pu}} + n_{\mathrm{Pi}} + \hat{k}$. Hence,

$$\operatorname{rank}(Z_2^T N) = 2(n_{\mathrm{De}} + n_{\mathrm{Re}}) + \operatorname{rank}(\tilde{N}_{22}) + \operatorname{rank}(\tilde{N}_{32}\tilde{N}_{22}^{-1}\tilde{N}_{23}).$$

Noting that $\tilde{N}_{32}\tilde{N}_{22}^{-1}\tilde{N}_{23} = [*, *, B^T]^T$, we have verified that $\operatorname{rank}(Z_2^T N(x)) = a$ on $\mathscr{C}_{IV}$. For (iii), we exploit the structure of $Z_2^T N(x)\Pi_N$ and construct a basis $T_2 \in C(\mathbb{D}, \mathbb{R}^{n\times d})$ of $\ker(Z_2^T N(x))$, where $d = 2n - a = n_{\mathrm{Pi}} - n_{\mathrm{Jc}} + \hat{k}_2$. Choosing $X_3 \in C^1(\mathscr{I} \times \Omega_x, \mathbb{R}^{n_{\mathrm{Pi}}\times d})$ with $\operatorname{span}(X_3) = \ker(\tilde{N}_{32}\tilde{N}_{22}^{-1}\tilde{N}_{23})$ and setting

$$T_2 = \Pi_N^T \begin{bmatrix} 0 & -\tilde{N}_{22}^{-1}\tilde{N}_{23} & I_{n_{\mathrm{Pi}}} \end{bmatrix}^T X_3,$$

we have $\operatorname{span}(T_2(x)) = \ker(Z_2^T N(x))$ for every $x \in \mathscr{C}_{IV}$. Then, we find that

$$F_{\dot{q},\dot{p}} T_2 = \begin{bmatrix} (\Pi_{\mathrm{Pi}_2} - \Pi_{\mathrm{Pi}_1} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_2})^T & 0 & 0 & 0 \\ 0 & 0 & I_{n_{\mathrm{Jc}_V}} & 0 \end{bmatrix}^T$$

and noting that

$$\Pi_{\mathrm{Pi}_2} - \Pi_{\mathrm{Pi}_1} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_2} = \Pi_{\mathrm{Pi}} \begin{bmatrix} -A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_1}^{-1} A_{\bar{\mathrm{J}}\mathrm{c},\mathrm{Pi}_2} \\ I_d \end{bmatrix},$$

it follows that $\operatorname{rank}((F_{\dot{x}} T_2)(z)) = d$ for $z = (t, x, v) \in \mathscr{F}^{-1}(0)$ with $(t, x) \in \mathscr{C}_{IV}$. Setting

$$Z_1 := \begin{bmatrix} \Pi_{\mathrm{Pi}_2}^T & 0 & 0 & 0 \\ 0 & I_{n_{\mathrm{Jc}_V}} & 0 & 0 \end{bmatrix}^T \tag{37}$$

we have verified that $\operatorname{rank}((Z_1^T F_{\dot{x}} T_2)(z)) = d$ for $z = (t, x, v) \in \mathscr{F}^{-1}(0)$ with $(t, x) \in \mathscr{C}_{IV}$. Hence, the network model (25) satisfies the assumptions (i)-(iii), implying that every sufficiently smooth solution of (25) with $(t_0, x_0) \in \mathscr{C}_1$ solves the surrogate model $Z_1^T F(t, x, \dot{x}) = 0$, $Z_2^T \mathscr{F}(t, x, \dot{x}, \ddot{x}) = 0$, $x(t_0) = x_0$, cp. [19, Thm. 4.11]. With Z_1, Z_2 given by (37), (35), we have that $\hat{F}_1 := Z_1^T F$ and $\hat{F}_2 := Z_2^T \mathscr{F}$.
Noting that, as $f_{\mathrm{Pi}} \in C^1(\Omega_{\mathrm{Pi}} \times (-\infty, \infty)^{n_{\mathrm{Jc}} + n_{\mathrm{Re}}}, \mathbb{R}^{n_{\mathrm{Pi}}})$, every solution $x \in C^1((t_0^-, t_0^+), \mathbb{R}^{2n})$ of (26) satisfies $x_d \in C^2(\mathscr{J}, \mathbb{R}^{n_{\mathrm{Pi}}})$, $x_a \in C^1(\mathscr{J}, \mathbb{R}^{n - n_{\mathrm{Pi}}})$, it follows that, for $(t_0, x_0) \in \mathscr{C}_{IV}$, a function $x \in C^1(\mathscr{J}, \mathbb{R}^{2n})$ solves (25) if and only if x solves (26). □

Note that the smoothness of the algebraic components x_a depends on the smoothness of the pump function.

Translated as conditions on the network structure and its elements, the solvability conditions of Theorem 3.1 mean that a reservoir is needed as reference value for the pressure p_{Jc} and for the enthalpy h_{Jc_0} in the virtual connection points $\mathscr{J}c_0$ and that, on fundamental cycles and crossing paths as well as in isolated pumps of $\mathscr{G}_{\mathrm{Jc,Pu}}$, the pumps must be able to adjust the mass flow to a given pressure difference.
As the transfer elements (the pipes and pumps) only specify the pressure difference, a reservoir is needed as reference value for the pressure p_{Jc}, so in every connected component there needs to be a reservoir. Similarly, in the virtual connection points $\mathscr{J}c_0$, the enthalpy h_{Jc_0} is computed by inserting the pipe and pump equations into the energy balance. Here as well, only the enthalpy difference is specified, so in order to obtain a unique solution, we need a reference value. As the enthalpy flow depends on the direction of the mass flow, these virtual connection points need to be *strongly connected* to a reservoir.
Usually, pumps return a pressure difference for given mass flow. On structures of $\mathscr{G}_{\mathrm{Jc,Pu}}$ where the pressure difference vanishes, however, the pumps have to work the other way round, which, mathematically, is reflected by the nonsingularity condition on D. We illustrate this by an example.

Example 3.1. *We consider pumps Pu_1, Pu_2, Pu_3 connected to a cycle that is connected to a demand De. The network model* (25a) *reads*

$$\begin{aligned} p_{Jc,2} - p_{Jc,1} &= f_{Pu,1}(q_{Pu,1}), \quad q_{Pu,1} = q_{Pu,2}, \\ p_{Jc,3} - p_{Jc,2} &= f_{Pu,2}(q_{Pu,2}), \quad q_{Pu,2} = q_{Pu,3}, \\ p_{Jc,1} - p_{Jc,3} &= f_{Pu,3}(q_{Pu,3}), \quad q_{Pu,3} = q_{Pu,1} + q_{De}, \quad q_{De} = \bar{q}_{De}. \end{aligned} \tag{38}$$

From the mass balances, we get that $\bar{q}_{De} \equiv 0$ *and* $q_{Pu,1} = q_{Pu,2} = q_{Pu,3}$. *In combination with the pump equations, it follows that*

$$f_{Pu,1}(q_{Pu,1}) + f_{Pu,2}(q_{Pu,1}) + f_{Pu,3}(q_{Pu,1}) = 0.$$

Hence, the input $\bar{q}_{De}$ *is not freely choosable and* (38) *is locally solvable for* $q_{Pu,1,0} \in \mathbb{R}$ *if and only if* $\sum_{j=1}^{3} \mathrm{D}\, f_{Pu,j}(q_{Pu,1,0})$ *is nonsingular. However, as the pump equations only specify the pressure difference, the DAE* (38) *will not be uniquely solvable unless the model is connected to a reservoir.*
Similarly, coupling two pumps Pu_1, Pu_2 *between two reservoirs* Re_1, Re_2, *we obtain the system*

$$\begin{aligned} p_{Jc,1} - p_{Re,1} &= f_{Pu,1}(q_{Pu,1}), \quad q_{Pu,1} = q_{Pu,2}, \\ p_{Re,2} - p_{Jc,1} &= f_{Pu,2}(q_{Pu,2}), \end{aligned} \tag{39}$$

and observe that (39) *is locally solvable if and only if* $\sum_{j=1}^{2} \mathrm{D}\, f_{Pu,j}(q_{Pu,1,0})$ *is nonsingular for* $q_{Pu,1,0} \in \mathbb{R}$.

In order to avoid the check if the pump function satisfies this solvability condition, i.e., to avoid the check if D is nonsingular, the considered network can be restricted to those in which pumps are coupled into a cycle or between two reservoirs do not occur.

Lemma 3.1. *Let* $\mathcal{N}$ *be a network given by* (12) *that satisfies Assumptions 2.1. Let* $F \in C^1(\mathbb{D}, \mathbb{R}^n)$ *be the associated network function. If* $n_{Re} > 0$, $\mathcal{J}\!\!\!-\!\mathcal{C}_0$ *is enthalpy reachable and* $\ker(A_{Jc,Pu}) = \{0\}$, *then the assertions of Theorem 3.1 are satisfied.*

Proof. If $\ker(A_{\mathrm{Jc,Pu}}) = \{0\}$, then V_2 is the empty matrix and the solvability condition of Theorem 3.1 is automatically satisfied. □

On the structural level, condition $\ker(A_{\mathrm{Jc,Pu}}) = \{0\}$ means that in every cycle of pumps and every path of pumps between two reservoirs, there is at least one pipe.

Example 3.2. *In Example 3.1, replacing e.g. the pump* Pu_3 *by a pipe* Pi_3, *we obtain the network DAE*

$$\begin{aligned} p_{Jc,2} - p_{Jc,1} &= f_{Pu,1}(q_{Pu,1}), & q_{Pu,1} &= q_{Pu,2}, \\ p_{Jc,3} - p_{Jc,2} &= f_{Pu,2}(q_{Pu,2}), & q_{Pu,2} &= q_{Pi,3}, \\ \dot{q}_{Pi,3} &= f_{Pi,3}(q_{Pi,3}, p_{Jc,1} - p_{Jc,3}), & q_{Pi,3} &= q_{Pu,1} + q_{De}, \quad q_{De} = \bar{q}_{De}. \end{aligned} \tag{40}$$

System (40) *can be solved for* $q_{Pu,1}, q_{Pu,2}, q_{Pi,3}$ *and e.g.,* $p_{Jc,1}, p_{Jc,2}$ *in dependency of the reference pressure* $p_{Jc,3}$ *by simply evaluating the pump equations, there is no need to invert the pump functions. Similarly, in the second example, replacing, e.g., the pump* Pu_2 *by a pipe* Pi_2, *we obtain a solvable system.*

In conclusion, if pumps are present in the network, the solvability condition can be either imposed on the element level, claiming that D is pointwise nonsingular on $\mathcal{C}_{IV}$, or in order to ensure that the model works for every pump specification they can be imposed on the structural level. Depending on the desired modeling freedom, one can choose between these two options.

If the solvability conditions are satisfied and the network is plausible, the next step is to simulate the dynamics of $\mathcal{N}$. The DAE (25a) assembled by glueing together the element

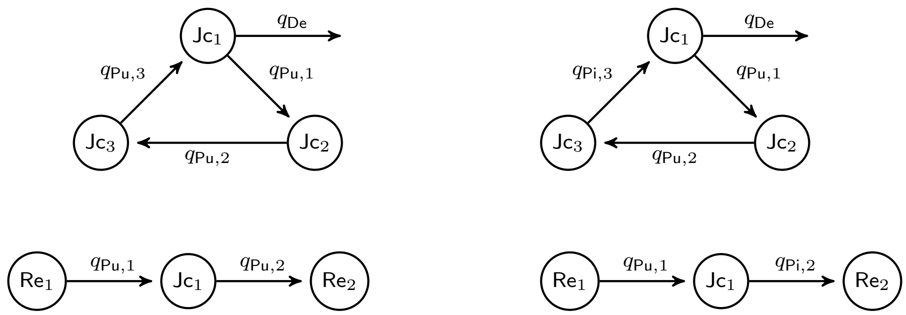


Figure 2: Pump constellations of Example 3.1 and Example 3.2 that trigger the solvability condition "*D is nonsingular*" (left) and constellations that avoid this condition (right).

equations (13) using the incidence matrix, however, is not suitable for a numerical simulation as it contains hidden equations and does not reflect the number of differential and algebraic variables correctly.
While the pressure differences p_{Jc_1} associated with $\mathrm{range}(A_{\mathrm{Jc},\mathrm{Pu}})$ are uniquely specified from the pump equations, the pressures p_{Jc_2} in the ground nodes $\mathscr{Jc}_2$ are associated with $\mathrm{coker}(A_{\mathrm{Jc},\mathrm{Pu}})$ and thus do not receive a pressure value from a pump. Instead, the pressures p_{Jc_2} are specified by the *hidden constraint*

$$A_{\bar{\mathrm{Jc}},\mathrm{Pi}} f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}, h_{\mathrm{Jc}_0}, h_{\mathrm{Jc}_V}, h_{\mathrm{Re}}) - A_{\bar{\mathrm{Jc}},\mathrm{De}} \dot{\bar{q}}_{\mathrm{De}} = 0, \tag{41}$$

arising from inserting the pipe equation, i.e., a differential equation, into the mass balance in the junctions. Claiming that $A_{\bar{\mathrm{Jc}},\mathrm{Pi}} f_{\mathrm{Pi}} A^T_{\bar{\mathrm{Jc}},\mathrm{Pi}}$ is nonsingular, this equation uniquely specifies the pressure p_{Jc_2}. To compensate for the additional equations, the surrogate model (26a) specifies only the pipe flows on the chord set $\mathscr{Pi}_2$ by a differential equation, while the mass flows in pipes on the spanning tree $\mathscr{Pi}_1$ are given by the mass balance $F_{\mathrm{Jc}}(t, x) = 0$, cp. (22c). We illustrate this again by an example.

Example 3.3. *We consider two pipes Pi_1, Pi_2 that are coupled by a junction Jc_1, cp. Figure 3. For simplicity, we assume that the pipes are connected to reservoirs Re_1, Re_2. Then, we obtain the network DAE*

$$\dot{q}_{Pi,1} = f_{Pi,1}(q_{Pi,1}, p_{Re,1} - p_{Jc,1}), \qquad q_{Pi,1}(t_0) = q_{Pi,1,0}, \tag{42a}$$
$$\dot{q}_{Pi,2} = f_{Pi,2}(q_{Pi,2}, p_{Jc,1} - p_{Re,2}), \qquad q_{Pi,2}(t_0) = q_{Pi,2,0}, \tag{42b}$$
$$q_{Pi,1} = q_{Pi,2}. \tag{42c}$$

The pipes specify the mass flows differentially, while the junction relates the flows algebraically. Consequently, only one mass flow evolves dynamically, the other one is fixed algebraically by the mass balance. In particular, only one initial value can be chosen. The pressure only occurs implicitly in the differential equations. Differentiating the algebraic equation and inserting the pipe equations for the derivatives of the mass flows, however, we discover the algebraic equation

$$f_{Pi,1}(q_{Pi,1}, \bar{p}_{Re,1} - p_{Jc,1}) = f_{Pi,2}(q_{Pi,2}, p_{Jc,1} - \bar{p}_{Re,2}). \tag{43}$$

As $\mathrm{D}_2(f_{Pi,2} - f_{Pi,1}) = c_{1,1} + c_{1,2}$ *is nonsingular,* (43) *can be solved for the pressure* $p_{Jc,1}$ *and* (42) *is uniquely solvable. Hence, coupling two pipes by a junction, the network model* (25a) *contains a hidden algebraic equation that is needed to specify the pressure in the coupling junction. Also,* (25a) *does not correctly reflect the number of differential and algebraic variables as only one mass flow evolves dynamically. Thus, we consider the surrogate model*

$$\begin{aligned} \dot{q}_{Pi,1} &= f_{Pi,1}(q_{Pi,1}, p_{Re,1} - p_{Jc,1}), \qquad & q_{Pi,1}(t_0) = q_{Pi,1,0}, \\ f_{Pi,1}(q_{Pi,1}, p_{Re,1} - p_{Jc,1}) &= f_{Pi,2}(q_{Pi,2}, p_{Jc,1} - p_{Re,2}), & \\ q_{Pi,1} &= q_{Pi,2}. & \end{aligned}$$

which corresponds to (26).

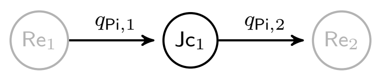


Figure 3: Network of Example 3.3.

From the proof of Theorem 3.1, we see that the solution of (25) can be computed from the explicit system (34). Exploiting the linearity and the triagonal structure of J, we explicitly compute the function g. Using a nonsingular matrix $S \in \mathbb{R}^{2n \times 2n}$, we transform the states according to (19) and set

$$\tilde{x} := S^{-1}x := \left[q_{\mathrm{Pi}_2}^T, \quad p_{\mathrm{Jc}_V}^T, \quad q_{\mathrm{Pi}_1}^T, \quad q_{\mathrm{Pu}_1}^T, \quad q_{\mathrm{Pu}_2}^T, \quad p_{\mathrm{Jc}_1}^T, \quad p_{\mathrm{Jc}_2}^T, \quad q_{\mathrm{De}}^T, \quad p_{\mathrm{Re}}^T, \quad H_{\mathrm{De}}^T, \quad h_{\mathrm{Re}}^T\right]^T \tag{44}$$

We transform the domain of definition accordingly by setting $\Omega_{\tilde{x}} := S^{-1} \cdot \Omega_x$, $\Omega_{\dot{\tilde{x}}} := S^{-1} \cdot \Omega_{\dot{x}}$ and $\tilde{\mathbb{D}} := \mathcal{J} \times \Omega_{\tilde{x}} \times \Omega_{\dot{\tilde{x}}}$ We transform the domain of definition accordingly and set $\Omega_{\tilde{x}} := S^{-1} \cdot \Omega_x$, $\Omega_{\dot{\tilde{x}}} := S^{-1} \cdot \Omega_{\dot{x}}$, $\tilde{\mathbb{D}} := \mathcal{J} \times \Omega_{\tilde{x}} \times \Omega_{\dot{\tilde{x}}}$, and partition the state into $\tilde{x} = [\tilde{x}_d^T, \tilde{x}_a^T]^T$ with $\tilde{x}_a = [p_{\mathrm{Jc}_2}^T, p_{\mathrm{Jc}_1}^T, q_{\mathrm{Pu}_2}^T, q_{\mathrm{Pu}_1}^T, q_{\mathrm{Pi}_1}^T, q_{\mathrm{De}}^T, p_{\mathrm{Re}}^T]^T$, $\tilde{x}_d = q_{\mathrm{Pi}_2}$.

Corollary 3.1. *Let* $\mathcal{N}$ *be a network given by* (12) *that satisfies Assumptions 2.1. Let* $F \in C^1(\mathbb{D}, \mathbb{R}^n)$ *be the associated network function. If* $n_{Re} > 0$, D *pointwise nonsingular on* $\mathscr{C}_{IV}$, *where* $\mathrm{span}(V_2) = \ker(A_{Jc,Pu})$, *and* $\mathcal{J}c_0$ *is enthalpy reachable, then a function* $x \in C^1((t_0^-, t_0^+), \mathbb{R}^{2n})$ *solves* (25a) *if and only if its transformation* $\tilde{x} = S^{-1}x$ *solves the explicit system*

$$\dot{\tilde{x}}_d = f(t, \tilde{x}_d), \quad \tilde{x}_d(t_0) = x_{d,0} \tag{45a}$$

$$\tilde{x}_a = g(t, \tilde{x}_d), \tag{45b}$$

where

$$
\begin{aligned}
f_{Pi_2} &= f_{Pi_2}\big(g_{Pi}(q_{Pi_2}), A^T_{Jc,Pi} g_{Jc}(q_{Pi_2}) + A^T_{Re,Pi}\bar{p}_{Re}\big)\\
f_{Jc_V} &= V^{-1}\big(A_{Jc_V,Pi} f_{Pi^*}(g_{Pi}, g_{Jc_V}, g_{Jc_0}, g_{Re}) + A_{Jc_V,Pu} f_{Pu^*}(g_{Pi}, g_{Jc_V}, g_{Jc_0}, g_{Re}) + A_{Jc_V,De} g_{De^*}\big)\\
g_{Pi^*} &= f_{Pi^*}(g_{Pi}(q_{Pi_2}), h_{Jc_V}, g_{Jc_0}(q_{Pi_2}, h_{Jc_V}), \bar{h}_{Re})\\
g_{Pu^*} &= f_{Pu^*}(g_{Pu}(q_{Pi_2}), h_{Jc_V}, g_{Jc_0}(q_{Pi_2}, h_{Jc_V}), \bar{h}_{Re})\\
g_{Jc_2}(q_{Pi_2}) &= -C^{-1} A_{\bar{J}c,Pi}\Big(C_2(h_{Jc_V}, g_{Jc_0}(q_{Pi_2}, h_{Jc_V}), \bar{h}_{Re})\,\mathrm{diag}\,(g_{Pi,j}(q_{Pi_2}))\, g_{Pi}(q_{Pi_2})\\
&\quad + C_1 A^T_{Jc_1,Pi} g_{Jc_1}(q_{Pi_2}) + C_1 A^T_{Re,Pi}\bar{p}_{Re} + C_3\Big) - C^{-1} A_{\bar{J}c,De}\dot{\bar{q}}_{De}\\
g_{Jc_0}(q_{Pi_2}, h_{Jc_V}) &= -B^{-1}\big(A_{Jc,Pi} B_{Jc_V}\big(g_{Pi}(q_{Pi_2})\big) + A_{Jc,Pu} B_{Jc_V}\big(g_{Pu}(q_{Pi_2})\big)\big) h_{Jc_V} - B^{-1}\big(A_{Jc,Pi} B_{Re}(q_{Pi})\\
&\quad + A_{Jc,Pu} B_{Re}(g_{Pu}(q_{Pi_2}))\big)\bar{h}_{Re} - B^{-1} A_{Jc,De}\bar{H}_{De}\\
g_{Jc_1}(q_{Pi_2}) &= -A^{-T}_{Jc_1,\bar{P}u} A^T_{\bar{J}c,\bar{P}u} p_{Jc_2} + A^{-T}_{Jc_1,\bar{P}u}\Pi^T_{Pu_1} f_{Pu}\big(g_{Pu}(q_{Pi_2})\big) - A^{-T}_{Jc_1,\bar{P}u} A^T_{Re,\bar{P}u}\bar{p}_{Re}\\
g_{Pu_2}(q_{Pi_2}) &= g_{Pu_2}(g_{Pu_1}(q_{Pi_2}))\\
g_{Pu_1}(q_{Pi_2}) &= -A^{-1}_{Jc_1,Pu_1} A_{Jc_1,Pi_1} g_{Pi_1}(q_{Pi_2}) - A^{-1}_{Jc_1,Pu_1} A_{Jc_1,Pi_2} q_{Pi_2} - A^{-1}_{Jc_1,Pu_1} A_{Jc_1,De}\bar{q}_{De}\\
g_{Pi_1}(q_{Pi_2}) &= -A^{-1}_{\bar{J}c,Pi_1} A_{\bar{J}c,Pi_2} q_{Pi_2} - A^{-1}_{\bar{J}c,Pi_1} A_{\bar{J}c,De}\bar{q}_{De}
\end{aligned}
$$

with $g_{De} = \bar{q}_{De}$, $g_{Re} = \bar{p}_{Re}$, $g_{De^*} = \bar{H}_{De}$, $g_{Re^*} = \bar{h}_{Re}$ *and* $g_{Pi}(q_{Pi_2}) := \Pi_{Pi_1} g_{Pi_1}(q_{Pi_2}) + \Pi_{Pi_2} q_{Pi_2}$, $g_{Pu}(q_{Pi_2}) := \Pi_{Pu_1} g_{Pu_1}(q_{Pi_2}) + V_2 g_{Pu_2}(q_{Pi_2})$, $g_{Jc}(q_{Pi_2}) := \Gamma_1 g_{Jc_1}(q_{Pi_2}) + U_2 g_{Jc_2}(q_{Pi_2})$. *The function* $g_{Pu_2} \in C^1(\mathcal{U}(q_{Pi_2,0}) \times \mathcal{U}(p_{\mathfrak{R},0}), \mathbb{R}^{n_{Pu}})$ *is implicitly defined as solution of* $F_{\bar{P}u}(t, \tilde{x}) = 0$.

Note that the algebraic equations (45b) can be solved from bottom to top such that the algebraic variables can be expressed as functions of the chord flows q_{Pi_2} and the input functions $\bar{q}_{\mathrm{De}}, \bar{p}_{\mathrm{Re}}, \bar{H}_{\mathrm{De}}, \bar{H}_{\mathrm{Re}}$.

Remark 3.1. *The solvability conditions of Theorem 3.1 are formulated on the connection structure and the elements of the network. This allows to check the plausibility of the network in a preprocessing step using information about the incidence matrix A and the pump function* f_{Pu}*. If the solvability conditions are violated, the critical structures can be located in* $\mathcal{N}$ *and advice can be given how to modify the model to obtain a physically reasonable system.*
The surrogate model (26a) *can be assembled based on network information only. There is no need to compute* (26a) *from* (25a) *by symbolic or numerical manipulation, as it is necessary for example in a general modeling language like Modelica. In a simulation, this saves computational time as the system-to-solve* (26a) *can be assembled directly from the network. Furthermore, the physical meaning of the equations and the states is preserved, i.e., in* (26a) *DAE, each equation and each variable still has a physical counterpart. Thus, errors in the simulation can be located in the network, allowing constructive error detection and handling.*

Remark 3.2. *The assertions of Theorem 3.1 can be verified by showing that* (25) *has* regular *strangeness index* $\mu = 1$*. Therefore, we show that the rank assumptions (i) - (iii) are not only satisfied by the Jacobians M, N but also by the Jacobians* $\tilde{M}(z) := \partial_{v,w}\tilde{\mathcal{F}}(z)$, $\tilde{N}(z) := \partial_x \tilde{\mathcal{F}}(z)$, *where* $\tilde{\mathcal{F}} = [F^T, \dot{F}^T, \ddot{F}^T]^T$ *denotes the derivative array of size* $\mu = 2$*, cp. [19]. This, however, requires to restrict the interval* $\mathcal{I}$ *such that* $\mathrm{sgn}(q) = const$ *to provide the required smoothness of* f_{Pi} *as well as stricter smoothness assumptions on* $f_{Pu}, \bar{q}_{De}, \bar{p}_{Re}$.

4 Conclusion and Outlook

This work provides a full analysis of a thermal fluid network, which is an extension of the well studied water networks consisting of pipes solely. The analysis is based on a topological network approach, which allows to impose conditions on the underlying network structure, represented by a graph. The provided topological solvability and index criteria in combination with efficient graph algorithm provide a powerful tool for the development of system simulation software. Anyhow, for the practical application it is important to extend those results to networks including valves and tanks, cp. the classification in [14], in order to be able to capture the whole cooling circuit. We mention, that further models for system simulation in automotive application (e.g. waste heat recovery, mobile air conditioning, lubrication systems), show up a similar network structure (with slightly modified equations). Therefore the presented analysis is representative for the latter mentioned.

Acknowledgment

We would like to thank our colleague Roman Wahl for fruitful and enlightening discussions.

References

[1] H. Amann. *Ordinary Differential Equations: An Introduction to Nonlinear Analysis*. De Gruyter studies in Mathematics. de Gruyter, Berlin, DE, 1990.

[2] A-K. Baum and M. Kolmbauer. Solvability and topological index criteria for thermal management in liquid flow networks. Technical Report RICAM-Report 2015-21, Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, 2015.

[3] Norman Biggs. *Algebraic Graph Theory*. Cambridge Mathematical Library. Cambridge University Press, 1974.

[4] J. Burgschweiger, B. Gnädig, and M.C. Steinbach. Optimization models for operative planning in drinking water networks. *Optimization and Engineering*, 10(1):43–73, 2009.

[5] R.W. Cottle, J.S. Pang, and R.E. Stone. *The Linear Complementarity Problem*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 2009.

[6] N. Deo. *Graph Theory with Applications to Engineering and Computer Science*. Prentice-Hall series in automatic computation. Prentice-Hall of India, 2004.

[7] J.W. Deuerlein. Decomposition model of a general water supply network graph. *Journal of Hydraulic Engineering*, 134(6):822–832, 2008.

[8] R. Diestel. *Graduate Texts in Mathematics: Graph Theory*. Springer, Heidelberg, DE, 2000.

[9] EPANET. Software that models the hydraulic and water quality behaviour of water distribution piping systems. 2014. `http://www.epa.gov/nrmrl/wswrd/dw/epanet.html`.

[10] S. Grundel, L. Jansen, N. Hornung, T. Clees, C. Tischendorf, and P. Benner. In *Progress in Differential-Algebraic Equations*, pages 183–205. Springer, 2014.

[11] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, UK, 2012.

[12] C. Huck, L. Jansen, and C. Tischendorf. A topology based discretization of PDAEs describing water transportation networks. *PAMM*, 14(1):923–924, 2014.

[13] L. Jansen and J. Pade. Global unique solvability for a quasi-stationary water network model. *Preprint*, 11, 2013.

[14] L. Jansen and C. Tischendorf. A unified (p) dae modeling approach for flow networks. In *Progress in Differential-Algebraic Equations*, pages 127–151. Springer, 2014.

[15] P. Kunkel and V. Mehrmann. Canonical forms for linear differential-algebraic equations with variable coefficients. *J. Comput. Appl. Math.*, 56:225–251, 1994.

[16] P. Kunkel and V. Mehrmann. Local and global invariants of linear differential algebraic equations and their relation. *Electr. Trans. Numer. Anal.*, 4:138–157, 1996.

[17] P. Kunkel and V. Mehrmann. Regular solutions of nonlinear differential-algebraic equations and their numerical determination. *Numer. Math.*, 79:581–600, 1998.

[18] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution.* EMS Publishing House, Zürich, CH, 2006.

[19] P. Kunkel and V. Mehrmann. Stability properties of differential-algebraic equations and spin-stabilized discretizations. *Electr. Trans. Num. Anal.*, 26:385–420, 2007.

[20] P. Lancaster and M. Tismenetsky. *The Theory of Matrices.* Academic Press, New York, NY, 1985.

[21] J.M. Ortega and W.C. Rheinboldt. *Iterative Solution of Nonlinear Equations in Several Variables.* Classics in Applied Mathematics. SIAM, Philadelphia, PA, 2000.

[22] M.N. Spijker. Contractivity in the numerical solution of initial-value-problems. *Numer. Math.*, 42:271–290, 1983.

[23] M.C. Steinbach. *Topological Index Criteria in DAE for Water Networks.* Konrad-Zuse-Zentrum für Informationstechnik, 2005.

[24] C. Tischendorf. Topological index calculation of differential-algebraic equations in circuit simulation. *Surv. Math. Ind.*, 8:187–199, 1999.