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Holonomic Tools for Basic Hypergeometric Functions

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Dedicated to Professor Mourad Ismail
at the occasion of his 70th birthday

Abstract

With the exception of q-hypergeometric summation, the use of computer algebra packages implementing Zeilberger's "holonomic systems approach" in a broader mathematical sense is less common in the field of q-series and basic hypergeometric functions. A major objective of this article is to popularize the usage of such tools also in these domains. Concrete case studies showing software in action introduce to the basic techniques. An application highlight is a new computer-assisted proof of the celebrated Ismail-Zhang formula, an important q-analog of a classical expansion formula of plane waves in terms of Gegenbauer polynomials.

1 Introduction

Quoting Knuth [11, p. 62] the *Concrete Tetrahedron* [10] is "sort of the sequel to *Concrete Mathematics* [3]." Indeed, presenting algorithmic ideas in connection with the symbolic treatment of combinatorical sums, recurrences, and generating functions, it can be viewed as an algorithmic supplement to [3] directed at an audience using computer algebra. Most of the methods under consideration fit into the "holonomic systems approach to special functions identities" notably pioneered by Zeilberger [18].

The authors of this article feel that in contrast to applications in the domain of classical hypergeometric functions, the use of such methods and tools

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is less common in the field of q-series and basic hypergeometric functions. A major objective of this article is to popularize the holonomic systems approach also in these domains. In order to illustrate some of the basic (pun intended!) techniques, concrete case studies of software in action are given. To this end, computer algebra packages written in Mathematica are used. These packages are freely downloadable (upon password request) by following the instructions at

`http://www.risc.uni-linz.ac.at/research/combinat/software`

The article is structured as follows: In Section 2 the objects of a computational case study are q-versions of modified Lommel polynomials introduced by Ismail in [4]. To derive properties of this polynomial family we apply q-holonomic computer algebra tools for guessing, generalized telescoping, and the execution of closure properties.

Section 3 introduces to the algebraic language and concepts needed for an algorithmic treatment of functions defined by mixed (q-)difference-differential equations. Following Zeilberger's holonomic systems approach, special functions are described by (generators of) annihilating ideals in operator algebras. Special function operations like addition, multiplication, integration, or summation are lifted to operations on (the generators of) these ideals. The HolonomicFunctions package implements this algebraic/algorithmic framework. Gegenbauer polynomials are used to show some basic features of the software.

Employing the algorithmic machinery described, Section 4 presents a new computer-assisted proof of the celebrated Ismail-Zhang formula from [7]. This important identity is a q-analog of a classical expansion formula of plane waves in terms of Gegenbauer polynomials and involving Bessel functions. Ingredients of the q-analog are the basic exponential function $\mathcal{E}_q(x; i\omega)$, also introduced by Ismail and Zhang in [7], as well as Jackson's second q-analog of the Bessel functions $J_\nu(x)$ and q-Gegenbauer polynomials.

We want to mention explicitly that the task of computing an annihilating operator for the series side of the Ismail-Zhang formula is leading to the frontiers of what is computationally feasible today. To compute the operator **annSumRHS** in In[43], we had to use recent algorithmic developments [13] as well as human inspection and trial and error to determine suitable denominators in a decisive preprocessing step.

2 Basic Bessel Functions and q-Lommel Polynomials

We begin with basic Bessel functions considered by Ismail in [4]:

$$J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{(q;q)_n (q^{\nu+1};q)_n}, \qquad 0 < q < 1,$$

where

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad (a;q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).$$

After opening a Mathematica session we load Riese's package [15] which implements a q-version of Zeilberger's "fast" Algorithm [17]:

In[1]:= << **RISC´qZeil´**

Package q-Zeilberger version 4.50 written by Axel Riese Copyright 1992-2009, Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

The package provides the q-rising factorials via the **qPochhammer** command, i.e., **qPochhammer**$[a, q, k] := (a; q)_k$ and **qPochhammer**$[a, q] := (a; q)_\infty$. For better readability we set

In[2]:= $\mathbf{(a\_)_{k\_} := \textbf{qPochhammer}[a, q, k]}$

A recurrence for $J_\nu^{(1)}(x; q)$ (=: SUM[ν]) is computed as follows:

In[3]:= $\textbf{qZeil}\left[\frac{(q^{\nu+1})_\infty (-1)^n (x/2)^{\nu+2n}}{(q)_\infty (q)_n (q^{\nu+1})_n}, \{n, 0, \infty\}, \nu, 2\right]$

qZeil::natbounds : Assuming appropriate convergence.

Out[3]= $\mathrm{SUM}[\nu] = q^{1-\nu}\left(-\mathrm{SUM}[\nu-2]\right) - \frac{2\left(1-q^{1-\nu}\right)\mathrm{SUM}[\nu-1]}{x}$

This corresponds exactly to (1.18), $k = 1$, in [4]. Setting $r_1^{(\nu)}(x) := 2(1 - q^\nu)x$ we rewrite the previous output Out[3] as

$$q^\nu\,\mathrm{SUM}[\nu+1] = r_1^{(\nu)}\left(\frac{1}{x}\right)\mathrm{SUM}[\nu] - \mathrm{SUM}[\nu-1]. \tag{1}$$

By iterating this recurrence, one produces a sequence $r_1^{(\nu)}(x), r_2^{(\nu)}(x), r_3^{(\nu)}(x), \ldots$ of polynomials such that

$$q^\nu q^{\nu+1}\,\mathrm{SUM}[\nu+2] = r_2^{(\nu)}\left(\frac{1}{x}\right)\mathrm{SUM}[\nu] - r_1^{(\nu+1)}\left(\frac{1}{x}\right)\mathrm{SUM}[\nu-1], \tag{2}$$

$$q^\nu q^{\nu+1} q^{\nu+2}\,\mathrm{SUM}[\nu+3] = r_3^{(\nu)}\left(\frac{1}{x}\right)\mathrm{SUM}[\nu] - r_2^{(\nu+1)}\left(\frac{1}{x}\right)\mathrm{SUM}[\nu-1], \tag{3}$$

and so on. In other words, setting $r_0^{(\nu)}(x) := 1$, the polynomial sequence $\left(r_n^{(\nu)}(x)\right)_{n\geqslant 0}$ determined this way satisfies the relation

$$q^{n\nu+n(n-1)/2} J_{\nu+n}^{(1)}(x; q) = r_n^{(\nu)}\left(\frac{1}{x}\right) J_\nu^{(1)}(x; q) - r_{n-1}^{(\nu+1)}\left(\frac{1}{x}\right) J_{\nu-1}^{(1)}(x; q), \quad n \geqslant 1. \tag{4}$$

This is recurrence (1.19) for $k = 1$ in [4]. As noted ibid. the polynomials $r_n^{(\nu)}(x)$ are q-versions of the modified Lommel polynomials. The goal of the present case study is to illustrate how computer algebra tools can be used to find out more about the polynomials $r_n^{(\nu)}(x)$.

2.1 Guessing a q-holonomic recurrence

First, by iterating recurrence (1) as in (2) and (3), we compute the seven initial polynomials $1, r_1^{(\nu)}(x), r_2^{(\nu)}(x), \ldots, r_6^{(\nu)}(x)$ and store them in a list (not shown in full detail here for space reasons):

In[4]:= $\mathbf{rpolys = \{1, 2(1-q^\nu)x, (4 - 4q^\nu - 4q^{1+\nu} + 4q^{1+2\nu})x^2 - q^\nu, -2(-1 + q^{1+\nu})x(-q^\nu - q^{1+\nu} + 4x^2 - 4q^\nu x^2 - 4q^{2+\nu}x^2 + 4q^{2+2\nu}x^2), \ldots\};}$

As described in [9], the package

In[5]:= << **RISC´qGeneratingFunctions´**

qGeneratingFunctions Package version 1.9.1 written by Christoph Koutschan Copyright 2006-2015, Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

can be used to guess a recursive pattern for the sequence $\left(r_n^{(\nu)}(x)\right)_{n\geqslant 0}$. To do so, we execute

In[6]:= **srec = QREGuess[rpolys,** $\boldsymbol{s[n]}$**]**

QREGuess::data : Not enough data. The result might be wrong.

Out[6]= $\{-2qx\, s[n-1]\left(q-q^{n+\nu}\right)+s[n-2]q^{n+\nu}+q^2 s[n]=0,$
$s[0]=1, s[1]=2x\left(1-q^{\nu}\right)\}$

Ignoring the warning, and observing that when using as input more than 7 polynomials the guessed recurrence remains stable, the output (i.e., the automatic guess) can be interpreted as a conjecture (it corresponds to (1.20) for $k=1$ in [4]).

Conjecture 1. *Let* $s_n^{(\nu)}(x), n \geqslant 0$*, be the sequence uniquely defined by the recurrence in* Out[6]*. Then* $r_n^{(\nu)}(x) = s_n^{(\nu)}(x)$ *for all* $n \geqslant 0$.

Definition 2. *A sequence* $(a_n)_{n\geqslant 0}$ *that satisfies a linear recurrence with coefficients being polynomials in* q^n *with coefficients in a field* $\mathbb{K}(q)$ *is called q-holonomic.*

In (computational) applications the coefficient field $\mathbb{K}(q)$ is a rational function field; usually $\mathbb{K}$ is a transcendental extension $\mathbb{K} = \mathbb{Q}(a,b,c,\ldots)$ of $\mathbb{Q}$ containing parameters a, b, c, and so on. In our example, $\mathbb{K} = \mathbb{Q}(q^{\nu})$.

As pointed out in [4] Conjecture 1 can be proved by straightforward induction. In order to introduce various aspects of computer algebra we present an algorithmic proof exploiting *holonomic closure properties.*

2.2 Proof of Conjecture 1 and q-holonomic closure properties

Iterating recurrence (1) as in (2) and (3) uniquely determines the polynomials $r_n^{(\nu)}$. Hence to prove Conjecture 1 it suffices to prove

$$q^{n\nu+n(n-1)/2}J_{\nu+n}^{(1)}(x;q) = s_n^{(\nu)}\left(\frac{1}{x}\right)J_{\nu}^{(1)}(x;q) - s_{n-1}^{(\nu+1)}\left(\frac{1}{x}\right)J_{\nu-1}^{(1)}(x;q), \quad n \geqslant 1, \tag{5}$$

where $s_n^{(\nu)}(x)$ is the sequence uniquely defined by the recurrence Out[6] together with the initial values $s_0^{(\nu)} := 1 = r_0^{(\nu)}$ and $s_1^{(\nu)}(x) := 2(1-q^{\nu})x = r_1^{(\nu)}(x)$.

First we call the **qZeil** package to obtain a recurrence with respect to n for the left side of (5):

In[7]:= **recLHS = qZeil**$\left[q^{n\nu+n((n-1)/2)}\frac{(q^{\nu+n+1})_\infty(-1)^j(x/2)^{\nu+n+2j}}{(q)_j(q^{\nu+n+1})_j},\right.$
$\left.\{j,0,\infty\},n,2,\{\nu\}\right]$ **/.** $n \to n+2$

qZeil::natbounds : Assuming appropriate convergence.

Out[7]= $\mathrm{SUM}[n+2] = \dfrac{2\mathrm{SUM}[n+1]\left(1-q^{n+\nu+1}\right)}{x} - \mathrm{SUM}[n]q^{n+\nu}$

In what follows it will be convenient to work directly with operators. To this end we load

In[8]:= **<< RISC´HolonomicFunctions´**

HolonomicFunctions Package version 1.7.1 (09-Oct-2013) written by Christoph Koutschan Copyright 2007-2013, Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

The following procedure writes the recurrence **recLHS**, which is Out[7], into an operator **opLHS** which annihilates $q^{n\nu+n(n-1)/2}J^{(1)}_{n+\nu}(x;q)$, the left-hand side of (5):

In[9]:= **opLHS = ToOrePolynomial[recLHS, SUM[n], OreAlgebra[QS[N, q^n]]]**

Out[9]= $S^2_{N,q} + \left(\dfrac{2Nq^{\nu+1}}{x} - \dfrac{2}{x}\right)S_{N,q} + Nq^{\nu}$

The notation becomes clear by comparison to **recLHS**: N stands for q^n, and $S_{N,q}$ denotes the shift operator with respect to n. For instance, $S_{N,q}F(n) = F(n+1)$, $S_{N,q}N = qNS_{N,q}$, and $S^2_{N,q}F(n) = F(n+2)$.

Calling the same procedure from the **HolonomicFunctions** package, we obtain operator forms of the recurrences of the two parts on the right-hand side of (5):

In[10]:= **srec1 = srec[[1]] /. {$x \to 1/x, n \to n+2$}**

Out[10]= $-\dfrac{2qs[n+1]\left(q-q^{n+\nu+2}\right)}{x} + s[n]q^{n+\nu+2} + q^2 s[n+2] = 0$

This recurrence for the $s^{(\nu)}_n(1/x)$ then is rewritten as an annihilating operator of $s^{(\nu)}_n\left(\frac{1}{x}\right)J^{(1)}_{\nu}(x;q)$:

In[11]:= **op1RHS = {ToOrePolynomial[srec1, s[n], OreAlgebra[QS[N, q^n]]]}**

Out[11]= $\left\{q^2S^2_{N,q} + \left(\dfrac{2Nq^{\nu+3}}{x} - \dfrac{2q^2}{x}\right)S_{N,q} + Nq^{\nu+2}\right\}$

Analogously we obtain an annihilating operator of $-s^{(\nu+1)}_{n-1}\left(\frac{1}{x}\right)J^{(1)}_{\nu}(x;q)$:

In[12]:= **srec2 = srec[[1]] /. {$x \to 1/x, n \to n+2, \nu \to \nu+1$}**

Out[12]= $-\dfrac{2qs[n+1]\left(q-q^{n+\nu+3}\right)}{x} + s[n]q^{n+\nu+3} + q^2 s[n+2] = 0$

In[13]:= **op2RHS = {ToOrePolynomial[srec2, s[n], OreAlgebra[QS[N, q^n]]]}**

Out[13]= $\left\{q^2S^2_{N,q} + \left(\dfrac{2Nq^{\nu+4}}{x} - \dfrac{2q^2}{x}\right)S_{N,q} + Nq^{\nu+3}\right\}$

By constructively utilizing the holonomic closure properties we compute an operator annihilating $s^{(\nu)}_n\left(\frac{1}{x}\right)J^{(1)}_{\nu}(x;q) - s^{(\nu+1)}_{n-1}\left(\frac{1}{x}\right)J^{(1)}_{\nu}(x;q)$:

In[14]:= **opRHS = DFinitePlus[op1RHS, op2RHS][[1]] // Factor**

Out[14]= $x^2S^4_{N,q} + 2(q+1)x\left(Nq^{\nu+3}-1\right)S^3_{N,q} + q\big(4N^2q^{2\nu+5} + Nx^2q^{\nu+1} + Nx^2q^{\nu+2} - 4Nq^{\nu+2} - 4Nq^{\nu+3} + 4\big)S^2_{N,q} + 2N(q+1)xq^{\nu+2}\left(Nq^{\nu+2}-1\right)S_{N,q} + N^2x^2q^{2\nu+3}$

As often when applying holonomic closure properties this operator is not equal to **opLHS**, but a left multiple of it:

$$\mathbf{opRHS} = \left(x^2 S_{N,q}^2 + 2q(Nq^{\nu+3}-1)xS_{N,q} + Nq^{\nu+3}x^2\right)\mathbf{opLHS}.$$

With the **HolonomicFunctions** package this factorization can be found as follows:

In[15]:= $\mathbf{LMultiple = OreReduce[opRHS, \{opLHS\}, Extended \to True]}$

Out[15]= $\left\{0, 1, \left\{x^2 S_{N,q}^2 + 2qx\left(Nq^{\nu+3}-1\right)S_{N,q} + Nx^2q^{\nu+3}\right\}\right\}$

In[16]:= $\mathbf{OreTimes[LMultiple[[3,1]], opLHS]}$

Out[16]= $x^2 S_{N,q}^4 + 2(q+1)x\left(Nq^{\nu+3}-1\right)S_{N,q}^3 + q\big(4N^2q^{2\nu+5} + Nx^2q^{\nu+1} + Nx^2q^{\nu+2} - 4Nq^{\nu+2} - 4Nq^{\nu+3} + 4\big)S_{N,q}^2 + 2N(q+1)xq^{\nu+2}\left(Nq^{\nu+2}-1\right)S_{N,q} + N^2x^2q^{2\nu+3}$

Summarizing, we have shown that both sides of (5) satisfy a recurrence of order 4 with respect to n. Hence the proof of Conjecture 1 is completed by verifying (5) for $n = 1, 2, 3, 4$ which amounts to checking $r_n^{(\nu)}(x) = s_n^{(\nu)}(x)$ for $n = 0, 1, 2, 3$.

2.3 q-holonomic functions and generalized telescoping

In order to gain more insight into the structure of the polynomials $s[n] := s_n^{(\nu)}(x)$ we look at the generating function

$$F(t) := \sum_{n=0}^{\infty} s[n]t^n = \sum_{j=0}^{\infty} s_n^{(\nu)}(x)t^n.$$

Taking as input the recurrence **srec**, which including the initial values is a unique presentation of $\big(s_n^{(\nu)}(x)\big)_{n\geqslant 0}$, we first derive a q-difference equation for $F(t)$ by calling a procedure from the **qGeneratingFunctions** package:

In[17]:= $\mathbf{qDiffEq = QRE2SE[srec}, \boldsymbol{s[n]}, \mathbf{F}[\boldsymbol{t}]]$

Out[17]= $\left\{tq^{\nu}(t+2x)F[qt] + F[t](1-2tx) - 1 = 0, \langle 1\rangle[F[t]] = 1\right\}$

Here $\langle 1\rangle[F[t]] = 1$ stands for $F[0] = 1$. In view of the q-shift operator $S_{t,q}^k F(t) = F(q^k t)$, q-difference equations like this are also called q-shift equations.

Definition 3. *A function $F(t)$ that satisfies a linear q-difference equation with coefficients being polynomials in t with coefficients in a field $\mathbb{K}(q)$ is called q-holonomic.*

As noted in [4] **qDiffEq**, which is Out[17], can be iterated to obtain an explicit series representation for $F(t)$:

$$\begin{aligned}(1-2tx)F(t) &= 1 - 2xtq^{\nu}\Big(1 + \frac{1}{2}\frac{t}{x}\Big)F(qt)\\ &= 1 - 2xtq^{\nu}\frac{1+\frac{1}{2}\frac{t}{x}}{1-2txq} + q(2xtq^{\nu})^2\frac{1+\frac{1}{2}\frac{t}{x}}{1-2txq}\frac{1+\frac{1}{2}\frac{t}{x}q}{1-2txq^2}F(q^2t) + \dots\end{aligned}$$

In the limit this iteration process results in [4, (3.3)]:

$$F(t) = \sum_{j=0}^{\infty} \frac{(-2xtq^{\nu})^j\left(-\frac{1}{2}\frac{t}{x}\right)_j}{\left(2xt\right)_{j+1}} q^{j(j-1)/2}. \qquad (6)$$

The series presentation (6) can be used to derive a q-hypergeometric sum representation for the $s_n^{(\nu)}(x)$ using two versions of the q-binomial theorem, the finite version by Gauß and the infinite one by Heine; see (3.3) and (3.6) in [4].

To illustrate further functionalities of the **HolonomicFunctions** package we derive a q-difference equation for the right-hand side of (6):

In[18]:= $\mathbf{CreativeTelescoping}\left[\frac{(-2xtV)^j\left(-\frac{1}{2}t/x\right)_j}{(2xt)_{j+1}}q^{j(j-1)/2},\right.$
$\left.\mathbf{QS}[J,q^j]-1,\mathbf{QS}[t,q^T]\right]$ /. $V\to q^\nu$ // **Factor**

Out[18]= $\{\{-tq^\nu(t+2x)S_{t,q}+(2tx-1)\},\{2tx-1\}\}$

Setting

$$g_j(t) := \frac{(-2xtq^\nu)^j\left(-\frac{1}{2}\frac{t}{x}\right)_j}{\left(2xt\right)_{j+1}}q^{j(j-1)/2}$$

and $\Delta_j F(j) := F(j+1)-F(j)$, the output constitutes the solution of a generalized telescoping problem and means that

$$q^\nu t(t+2x)g_j(qt)-(2tx-1)g_j(t)=\Delta_j(2tx-1)g_j(t).$$

Summing this from $j=0$ to $j=\infty$, and setting the right-hand side of (6) to $G(t)$, gives

$$q^\nu t(t+2x)G(qt)-(2tx-1)G(t)=(2tx-1)\left(g_\infty(t)-\frac{1}{1-2xt}\right).$$

Noting that $g_\infty(t)=0$ (as a formal power series in q, or analytically taking $|q|<1$) we obtain **qDiffEq**; i.e., $G(t)$ satisfies the same q-difference equation as for $F(t)$.

As in the case $q=1$ [10, Thm. 7.1] there is a simple but important connection between q-holonomic generating functions and their coefficient sequences:

Theorem 4.

$$F(t)=\sum_{n=0}^{\infty}a_nt^n \textit{ is } q\textit{-holonomic} \quad\Longleftrightarrow\quad \left(a_n\right)_{n\geqslant 0} \textit{ is } q\textit{-holonomic.}$$

One direction of the theorem has been exploited above when deriving **qDiffEq** from **srec**. The inverse direction, this means, to compute from the q-difference equation for $F(t)$ a q-recurrence for the coefficient polynomials $s(n)=s_n^{(\nu)}(x)$, is done as follows:

In[19]:= **QSE2RE[qDiffEq, F[t], s[n]]**

Out[19]= $\{q^2s[n]=2qx\,s[n-1]\left(q-q^{n+\nu}\right)-s[n-2]q^{n+\nu},s[0]=1,s[1]=-2x\left(q^\nu-1\right)\}$

The output is **srec**, the q-recurrence Out[6] for the modified q-Lommel polynomials.

3 Interlude: annihilating ideals of operators

In order to state, in an algebraic language, the concepts that are introduced in this section, and for writing mixed (q-)difference-differential equations in a concise way, the following operator notation is employed: let D_x denote the partial derivative operator with respect to x (x is then called a *continuous variable*), S_n the forward shift operator with respect to n (n is then called a *discrete variable*), and $S_{t,q}$ the q-shift operator with respect to t. More precisely, these operator symbols act on a function f by

$$D_x f = \frac{\partial f}{\partial x}, \qquad S_n f = f\big|_{n \to n+1}, \quad \text{and} \quad S_{t,q} f = f\big|_{t \to qt}.$$

The operator notation allows us to translate linear homogeneous (q-)difference-differential equations into polynomials in the operator symbols D_x, S_n, $S_{t,q}$, etc., with coefficients in some field $\mathbb{F}$. Typically, $\mathbb{F}$ is a rational function field in the variables x, n, t, q, etc. For example, the equation

$$\frac{\partial}{\partial x} f(k, n+1, x, y) + n \frac{\partial}{\partial y} f(k, n, x, y) + x f(k+1, n, x, y) - f(k, n, x, y) = 0$$

turns into $P f(k, n, x, y) = 0$, where P is the operator $D_x S_n + n D_y + x S_k - 1$. An example in the q-case is the annihilating operator **opLHS**, given in Out[9], for $q^{n\nu + n(n-1)/2} J^{(1)}_{n+\nu}(x; q)$,

$$J := S_{N,q}^2 + \left(-\frac{2}{x} + \frac{2Nq^{\nu+1}}{x} \right) S_{N,q} + Nq^{\nu}, \tag{7}$$

which is a polynomial in the q-shift operator $S_{N,q}$ whose coefficients are elements of the rational function field $\mathbb{Q}(q, q^{\nu}, N, x)$ where $N = q^n$. Note that in general the ring $\mathbb{F}[D_x, S_n, S_{t,q}, \ldots]$ is not commutative: coefficients from $\mathbb{F}$ do not commute with the "variables" D_x, S_n, $S_{t,q}$, etc. For instance, for some $a(x, n, t) \in \mathbb{F}$ one has

$$\begin{aligned} D_x \cdot a(x, n, t) &= a(x, n, t) \cdot D_x + \frac{\partial}{\partial x} a(x, n, t), \\ S_n \cdot a(x, n, t) &= a(x, n+1, t) \cdot S_n, \\ S_{t,q} \cdot a(x, n, t) &= a(x, n, qt) \cdot S_{t,q}. \end{aligned}$$

Such non-commutative rings of operators are called *Ore algebras*; more precise definitions and properties of such algebras can be found in [12].

Example 5. We demonstrate how arithmetic operations in an Ore algebra can be used to compute the polynomials $r_i^{(\nu)}$ for $i = 1, 2, 3, \ldots$. For this purpose let us convert the recurrence Out[3] into an operator:

In[20]:= $\mathbf{op = Factor[ToOrePolynomial[SUM[\nu] == -q^{1-\nu}\, SUM[\nu - 2] - 2(1 - q^{1-\nu}\, SUM[\nu - 1])/x, SUM[\nu], OreAlgebra[QS[V, q^{\nu}]]]]}$

Out[20]= $S_{V,q}^2 + \frac{2(qV - 1)}{qVx} S_{V,q} + \frac{1}{qV}$

Then iterating this recurrence according to (2) and (3) corresponds to reducing the operator $V(qV) \cdots (q^{i-1}V) S_{V,q}^i$, which encodes the left-hand side, with the previously defined **op**; the result is an operator that corresponds to the right-hand side. Its leading coefficient is precisely the desired $r_i^{(\nu+1)}(1/x)$ (note that the symbol ** stands for noncommutative multiplication).

In[21]:= $\textbf{Table[LeadingCoefficient[OreReduce[}(q^{i(i-1)/2}V^{i-1}) ** \textbf{QS}[V, q^\nu]^i,$
$\{\textbf{op} \ /. \ x \to 1/x\}]] \ /. \ V \to V/q, \{i, 1, 4\}]$

Out[21]= $\{1, -2(V-1)x, 4qV^2x^2 - 4qVx^2 - 4Vx^2 - V + 4x^2,$
$-2x(qV-1)(4q^2V^2x^2 - 4q^2Vx^2 - qV - 4Vx^2 - V + 4x^2)\}$

We define the *annihilator* (with respect to some Ore algebra $\mathbb{O}$) of a function f by:

$$\mathrm{Ann}_{\mathbb{O}}(f) := \{P \in \mathbb{O} \mid Pf = 0\}.$$

It can easily be seen that $\mathrm{Ann}_{\mathbb{O}}(f)$ is a left ideal in $\mathbb{O}$. Every left ideal $I \subseteq \mathrm{Ann}_{\mathbb{O}}(f)$ is called an *annihilating ideal* for f. For example, the operator J given in (7) is an element of $\mathrm{Ann}_{\mathbb{O}}\big(q^{n\nu+n(n-1)/2}J^{(1)}_{n+\nu}(x;q)\big)$ with $\mathbb{O} = \mathbb{F}[S_{N,q}] = \mathbb{Q}(q, q^\nu, N, x)[S_{N,q}]$. Actually, it is the unique (up to multiplication by elements from $\mathbb{F}$) generator of that principal left ideal.

Definition 6. *Let $\mathbb{O} = \mathbb{F}[\dots]$ be an Ore algebra. A function f is called* ∂-finite *with respect to $\mathbb{O}$ if $\mathbb{O}/\mathrm{Ann}_{\mathbb{O}}(f)$ is a finite-dimensional $\mathbb{F}$-vector space. The dimension of this vector space is called the* (holonomic) rank *of f with respect to $\mathbb{O}$.*

Example 7. The following **HolonomicFunctions** procedure delivers the annihilator of the Gegenbauer (also called ultraspherical) polynomials $C_m^{(\nu)}(x)$:

In[22]:= $\textbf{annG = Annihilator[GegenbauerC[}m, \nu, x\textbf{], \{S[}m\textbf{], S[}\nu\textbf{], Der[}x\textbf{]\}]}$

Out[22]= $\{2\nu S_\nu - xD_x + (-m - 2\nu),\ (m+1)S_m + (1-x^2)D_x + (-mx - 2\nu x),$
$(x^2-1)D_x^2 + (2\nu x + x)D_x + (-m^2 - 2m\nu)\}$

This means, these three elements generate $I := \mathrm{Ann}_{\mathbb{O}}\big(C_m^{(\nu)}(x)\big)$ as a left ideal in the operator algebra $\mathbb{O} = \mathbb{F}[S_m, S_\nu, D_x]$ with $\mathbb{F} = \mathbb{Q}(m, \nu, x)$. Their leading monomials are S_ν, S_m, and D_x^2, which shows that the function $C_m^{(\nu)}(x)$ is ∂-finite with respect to $\mathbb{O}$ and in particular:

$$\mathrm{rank}_{\mathbb{O}}\big(C_m^{(\nu)}(x)\big) = \dim_{\mathbb{F}}\big(\mathbb{O}/I\big) = 2.$$

In the holonomic systems approach, the data structure for representing functions is an annihilating ideal (given by a finite set of generators) plus initial values. When working with (left) ideals, we make use of *(left) Gröbner bases* [1, 8] which are an important tool for executing certain operations (e.g., the ideal membership test) in an algorithmic way.

For functions annilated by univariate operators from the Ore algebras $\mathbb{F}[S_n]$ or $\mathbb{F}[D_x]$ or $\mathbb{F}[S_{N,q}]$, the notions of ∂-finite and (q-)holonomic coincide. Despite being closely related to being ∂-finite, for functions annihilated by multivariate Ore operators the definition of holonomic is much more technical. In general, the holonomic property reflects certain elimination properties of annihlating operators which are required for summation and integration of special functions.

Without proof we state the following theorem about *closure properties* of ∂-finite functions; its proof can be found in [12, Chap. 2.3]. We remark that all of them are algorithmically executable, and the algorithms work with the above mentioned data structure.

Theorem 8. *Let $\mathbb{O}$ be an Ore algebra and let f and g be ∂-finite with respect to $\mathbb{O}$ of rank r and s, respectively. Then* *(i)* $f+g$ *is* ∂*-finite of rank* $\leqslant r+s$.

(ii) $f\cdot g$ *is* ∂*-finite of rank* $\leqslant rs$.

(iii) f^2 *is* ∂*-finite of rank* $\leqslant r(r+1)/2$.

(iv) Pf *is* ∂*-finite of rank* $\leqslant r$ *for any* $P\in\mathbb{O}$.

(v) $f|_{x\to A(x,y,\dots)}$ *is* ∂*-finite of rank* $\leqslant rd$ *if* $x,y,\dots$ *are continuous variables and if the algebraic function* A *satisfies a polynomial equation of degree* d.

(vi) $f|_{n\to A(n,k,\dots)}$ *is* ∂*-finite of rank* $\leqslant r$ *if* A *is an integer-linear expression in the discrete variables* $n,k,\dots$.

The bounds on the ranks are generically sharp. For example, the operator **opRHS** annihilating the right-hand side of (5) has been computed by exploiting ∂-finite closure properties in the spirit of Theorem 8. We continue with Theorem 9 which establishes the closure of holonomic functions with respect to sums and integrals; for its proof, we once again refer to [18, 12].

Theorem 9. *Let the function* f *be holonomic with respect to* D_x *(resp.* S_n*). Then also* $\int_a^b f\,\mathrm{d}x$ *(resp.* $\sum_{n=a}^b f$*) is holonomic.*

Example 10. We continue the discussion from Example 7 by again considering the Gegenbauer polynomials

$$C_m^{(\nu)}(x) := \sum_{k=0}^{m} F[x,m,\nu,k] \tag{8}$$

where

In[23]:= $\mathbf{F[x\_, m\_, \nu\_, k\_] := \frac{P[2\nu, m]}{m!}\frac{P[-m, k]\,P[m+2\nu, k]}{P[\nu+1/2, k]\,k!)}\left(\frac{1-x}{2}\right)^{k}}$

with

In[24]:= $\mathbf{P[x\_, k\_] := Pochhammer[x, k]}$

This time we want to derive the annihilator of $C_m^{(\nu)}(x)$ from its definition (8). For this purpose, we compute annihilating operators of the hypergeometric term $F[x,m,\nu,k]$ in telescoping form:

In[25]:= **CreativeTelescoping[F[x,m,ν,k], S[k] − 1, {Der[x], S[m]}] // Factor**

Out[25]= $\Big\{\{-(x-1)(x+1)D_x+(m+1)S_m+x(-(m+2\nu)),$
$(m+2)S_m^2-2x(m+\nu+1)S_m+(m+2\nu)\},$
$\Big\{-\frac{k(2k+2\nu-1)}{k-m-1}, \frac{2k(2k+2\nu-1)(m+\nu+1)}{(k-m-2)(k-m-1)}\Big\}\Big\}$

The output has to be interpreted as follows:

$$\big(-(-1+x)(1+x)D_x+(m+1)S_m-(m+2\nu)x\big)F[x,m,\nu,k] = (S_k-1)\frac{k(2k+2\nu-1)}{k-m-1}F[x,m,\nu,k] \tag{9}$$

and

$$\big((m+2)S_m^2 - 2(m+\nu+1)xS_m + (m+2\nu)\big)F[x,m,\nu,k] = \\ -(S_k-1)\frac{2k(m+\nu+1)(2k+2\nu-1)}{(k-m-2)(k-m-1)}F[x,m,\nu,k]. \quad (10)$$

Note that the relations (9) and (10) can be easily verified (even without using a computer). To compute them, the package **HolonomicFunctions** employs non-commutative Gröbner bases; the monomial order is deduced from the order in which the operators are given. Indeed, by changing in the input the order of **Der**[x] and **S**[m], one obtains a different result:

In[26]:= **CreativeTelescoping[F[x, m, ν, k], S[k] − 1, {S[m], Der[x]}] // Factor**

Out[26]= $\Big\{\{(m+1)S_m - (x-1)(x+1)D_x + x(-(m+2\nu)),$

$-(x-1)(x+1)D_x^2 - (2\nu+1)xD_x + m(m+2\nu)\},$

$\Big\{-\frac{k(2k+2\nu-1)}{k-m-1}, -\frac{k(2k+2\nu-1)}{x-1}\Big\}\Big\}$

This repeats (9), but computes another purely differential relation

$$\big(-(-1+x)(1+x)D_x^2 - (2\nu+1)xD_x + m(m+2\nu)\big)F[x,m,\nu,k] = \\ (S_k-1)\frac{k(2k+2\nu-1)}{x-1}F[x,m,\nu,k]. \quad (11)$$

Summing (9), (10), and (11) with respect to k from 0 to ∞ gives the well-known shift/differential relations for the Gegenbauer polynomials.

These computations were done in the operator algebra $\mathbb{O} = \mathbb{F}[S_m, S_k, D_x]$ with $\mathbb{F} = \mathbb{Q}(\nu, m, k, x)$. Let us include in addition the shift operator S_ν:

In[27]:= **CreativeTelescoping[F[x, m, ν, k], S[k] − 1, {S[m], S[ν], Der[x]}]**

Out[27]= $\Big\{\{2\nu S_\nu - xD_x + (-m-2\nu), (m+1)S_m + (1-x^2)D_x + (-mx-2\nu x),$

$(1-x^2)D_x^2 + (-2\nu x - x)D_x + (m^2+2m\nu)\},$

$\Big\{-\frac{k}{x-1}, \frac{-2k^2-2k\nu+k}{k-m-1}, \frac{-2k^2-2k\nu+k}{x-1}\Big\}\Big\}$

We see that summing the resulting telescoping relations with respect to k from 0 to ∞, gives the generators of the annihilating ideal **annG** computed in Out[22].

Finally we note that for the q-case we need to consider the Gegenbauer polynomials in the (equivalent) form:

$$C_m^{(\nu)}\big(\cos(\theta)\big) := \sum_{k=0}^{m} G[x,m,\nu,k]$$

where

In[28]:= $\boldsymbol{G[x\_, m\_, \nu\_, k\_] := \frac{\mathrm{P}[\nu,k]\,\mathrm{P}[\nu,m-k]}{k!(m-k)!}A^{m-2k}}$

with $x = \cos(\theta)$, $A = e^{i\theta}$ and $P[\nu,k]$ being defined as the Pochhammer symbol $(\nu)_k$, as in In[6]. In the q-context we will be interested to compute annihilating operators containig shifts in m and ν, and, as above, in telescoping form with respect to $S_k - 1$. More precisely, in the next section we will compute a q-version of

In[29]:= **CreativeTelescoping[G[x, m, ν, k], S[k] − 1, {S[m], S[ν]}] // Factor**

Out[29]= $\Big\{\{A(A^2+1)(m+1)S_m - (A-1)^2(A+1)^2\nu S_\nu - 2A^2(m+2\nu),$

$-(A-1)^2(A+1)^2\nu(\nu+1)S_\nu^2 + \nu\big(A^4m + A^4\nu + A^4 - 2A^2m - 6A^2\nu -$

$4A^2 + m + \nu + 1\big)S_\nu + A^2(m+2\nu)(m+2\nu+1)\},$

$\Big\{\dfrac{A^2k(k-m-\nu)\big(A^2k - A^2m + A^2\nu - A^2 - k + m + \nu + 1\big)}{\nu(k-m-1)},$

$\Big(A^2k(-k+m+\nu)\big(A^2k^2 - A^2km + A^2k\nu - A^2m\nu - k^2 + km + k\nu +$

$2k - m - \nu - 1\big)\Big) \Big/ \Big(\nu(\nu+1)\Big)\Big\}\Big\}$

4 The Ismail-Zhang Formula

An important classical expansion formula is the expansion of the plane wave in terms of ultraspherical polynomials $C_m^{(\nu)}(x)$, also called Gegenbauer polynomials:

$$e^{irx} = \left(\frac{2}{r}\right)^\nu \Gamma(\nu) \sum_{m=0}^\infty i^m (\nu+m)\, J_{\nu+m}(r)\, C_m^{(\nu)}(x).$$

Ismail and Zhang [7, (3.32)] had found the following q-analog of this formula:

$$\mathcal{E}_q(x; i\omega) = \frac{(q;q)_\infty\, \omega^{-\nu}}{(q^\nu;q)_\infty\, (-q\omega^2;q^2)_\infty} \times \sum_{m=0}^\infty i^m (1-q^{\nu+m})\, q^{m^2/4} J_{\nu+m}^{(2)}(2\omega;q)\, C_m(x;q^\nu|q), \qquad (12)$$

where $J_{\nu+m}^{(2)}(2\omega;q)$ is Jackson's q-Bessel function defined by

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^\infty q^{(\nu+n)n} \frac{(-1)^n (z/2)^{\nu+2n}}{(q;q)_n (q^{\nu+1};q)_n}.$$

In the Ismail-Zhang formula (12), Jackson's second q-analog of the Bessel function $J_\nu(z)$ appears; the remaining ingredients, the basic exponential function $\mathcal{E}_q(x; i\omega)$ and the q-Gegenbauer polynomials $C_m(x; q^\nu|q)$, are explained subsequently. There are several proofs of the Ismail-Zhang formula; see the books [5] and [16] for references and for the embedding of the formula in a broader context. In this section we present a new, computer-assisted proof of (12).

4.1 The basic exponential function

The basic exponential function $\mathcal{E}_q(x; i\omega)$, as well as its more general version $\mathcal{E}_q(x, y; i\omega)$, was introduced by Ismail and Zhang [7] and satisfies numerous important and also beautiful properties. For illustrative reasons we choose to introduce $\mathcal{E}_q(x;i\omega)$ via the basic cosine and sine functions: For $x=\cos(\theta)$ and $|\omega|<1$ we define:

$$\mathcal{E}_q(x;i\omega) := C_q(x;\omega) + i\,S_q(x;\omega)$$

where the basic cosine function $C_q(x;\omega)$ is defined as

$$C_q(x;\omega) := \frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\sum_{j=0}^{\infty}\frac{(-qe^{2i\theta};q^2)_j\,(-qe^{-2i\theta};q^2)_j}{(q;q^2)_j\,(q^2;q^2)_j}(-\omega^2)^j,$$

and the basic sine function $S_q(x;\omega)$ as

$$S_q(x;\omega) := \frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\frac{2q^{1/4}\omega}{1-q}\cos(\theta)\sum_{j=0}^{\infty}\frac{(-qe^{2i\theta};q^2)_j\,(-qe^{-2i\theta};q^2)_j}{(q^3;q^2)_j\,(q^2;q^2)_j}(-\omega^2)^j.$$

It is not difficult to check that

$$\lim_{q\to 1-} C_q\big(x;\omega(1-q)/2\big) = \cos(\omega x)$$
$$\lim_{q\to 1-} S_q\big(x;\omega(1-q)/2\big) = \sin(\omega x)$$

In the following we shall use the abbreviation $A=e^{i\theta}$, as before, and the following short-hand notation for the **qPochhammer** command:

```
In[30]:= qP = qPochhammer;
```

Consequently, the input $\mathbf{qP}[-w^2,q^2]$ stands for $(-\omega^2;q^2)_\infty$, and $\mathbf{qP}[-qA^2,q^2,j]$ for $(-qe^{2i\theta};q^2)_j$. The continuous q-ultraspherical (q-Gegenbauer) polynomials $C_m(x;q^\nu|q)$, $x=\cos(\theta)$, are defined as

$$C_m(\cos\theta;\beta|q) := \sum_{k=0}^{m}\frac{(\beta;q)_k\,(\beta;q)_{m-k}}{(q;q)_k\,(q;q)_{m-k}}e^{i(m-2k)\theta}.$$

To prove (12) we compute annihilating operators representing q-difference equations for the left- and right-hand sides, respectively. First we derive a q-shift equation for the q-Cosine. This is done analogously to the treatment of the right-hand side of (6):

In[31]:= $\mathbf{CreativeTelescoping}\Big[\frac{\mathbf{qP}[-\omega^2,q^2]}{\mathbf{qP}[-q\omega^2,q^2]}\frac{\mathbf{qP}[-qA^2,q^2,j]\,\mathbf{qP}[-q/A^2,q^2,j]}{\mathbf{qP}[q,q^2,j]\,\mathbf{qP}[q^2,q^2,j]}(-\omega^2)^j,$
$\mathbf{QS}[J,q^j]-1,\mathbf{QS}[\omega,q^w]\Big]$ // **Factor**

Out[31]= $\Big\{\big\{A^2\big(q^2\omega^2+1\big)S_{\omega,q}^2+(A^4q^2\omega^2-A^2q-A^2+q^2\omega^2)S_{\omega,q}+A^2q\big(q\omega^2+1\big)\big\},$
$\Big\{\frac{A^2(J-1)(J+1)\big(J^2-q\big)\big(q\omega^2+1\big)}{\omega^2+1}\Big\}\Big\}$

Denoting by $c_j(\omega)$ the summand in the q-cosine series and in view of $S_{\omega,q}^j f(\omega) = f(q^j\omega)$, this output means that

$$A^2\big(q^2\omega^2+1\big)c_j(q^2\omega)+(A^4q^2\omega^2-A^2q-A^2+q^2\omega^2)c_j(q\omega)+A^2q\big(q\omega^2+1\big)c_j(\omega) = -\Delta_j\frac{A^2(J-1)(J+1)\big(J^2-q\big)\big(q\omega^2+1\big)}{\omega^2+1}c_j(\omega),$$

where $J = q^j$. Summing the right-hand side from $j = 0$ to $j = \infty$ gives

$$-\frac{qA^2(1+q\omega^2)}{1+\omega^2}c_\infty(\omega) + \frac{A^2(-1+q^0)(1+q^0)(q^0-q)(1+q\omega^2)}{1+\omega^2}c_0(\omega) = 0.$$

Hence

In[32]:= **annCos = %[[1]]**

Out[32]= $\{A^2(q^2\omega^2+1)S_{\omega,q}^2 + (A^4q^2\omega^2 - A^2q - A^2 + q^2\omega^2)S_{\omega,q} + A^2q(q\omega^2+1)\}$

annihilates $C_q(x;\omega)$. An annihilator, resp. q-difference equation, for the q-Sine is derived analogously:

In[33]:= **Factor**$\Big[$**CreativeTelescoping**$\Big[\mathbb{i}\frac{\mathbf{qP}[-\omega^2,q^2]\,2q^{1/4}\omega}{\mathbf{qP}[-q\omega^2,q^2](1-q)}\mathbf{Cos}[\theta]$
$\frac{\mathbf{qP}[-q^2A^2,q^2,j]\,\mathbf{qP}[-q^2/A^2,q^2,j]}{\mathbf{qP}[q^3,q^2,j]\,\mathbf{qP}[q^2,q^2,j]}(-\omega^2)^j, \mathbf{QS}[J,q^j]-1, \mathbf{QS}[\omega,q^w]\Big]\Big]$

Out[33]= $\Big\{\{A^2(q^2\omega^2+1)S_{\omega,q}^2 + (A^4q^2\omega^2 - A^2q - A^2 + q^2\omega^2)S_{\omega,q} + A^2q(q\omega^2+1)\},$
$\Big\{\frac{A^2(J-1)(J+1)q(J^2q-1)(q\omega^2+1)}{\omega^2+1}\Big\}\Big\}$

In[34]:= **annSin = %[[1]]**

Out[34]= $\{A^2(q^2\omega^2+1)S_{\omega,q}^2 + (A^4q^2\omega^2 - A^2q - A^2 + q^2\omega^2)S_{\omega,q} + A^2q(q\omega^2+1)\}$

Finally we exploit the q-holonomic closure properties; more precisely, in view of Theorem 8(i) we "add" the q-difference equations for the q-cosine function and i times the q-sine function to obtain a q-difference equation for $\mathcal{E}_q(x;i\omega)$. The latter is the generator of the annihilating ideal of $\mathcal{E}_q(x;i\omega)$:

In[35]:= **annLHS = DFinitePlus[annCos, annSin]**

Out[35]= $\{(A^2q^2\omega^2 + A^2)S_{\omega,q}^2 + (A^4q^2\omega^2 - A^2q - A^2 + q^2\omega^2)S_{\omega,q} + (A^2q^2\omega^2 + A^2q)\}$

The result is not surprising: since $C_q(x;\omega)$ and $S_q(x;\omega)$ satisfy the same q-difference equation (compare **annSin** with **annCos**), also their linear combination satisfies the same equation. Conversely, the operator above annihilates any linear combination

$$c_1C_q(x;\omega) + c_2S_q(x;\omega)$$

where c_1 and c_2 are constants, i.e., independent of ω. The order of **annLHS** is 2, hence one derives explicit expressions for the c_i by picking the coefficients of ω^0 and ω^1 in $\mathcal{E}_q(x;i\omega)$, respectively:

$$c_1 = 1, \quad c_2 = \frac{2q^{1/4}}{1-q}\cos(\theta). \tag{13}$$

Summarizing, $\mathcal{E}_q(x;i\omega)$ is uniquely determined by the q-difference operator **annLHS** and the initial values (13).

4.2 An annihilator for the Ismail-Zhang series

To compute an annihilating operator for the right-hand side of (12), we algorithmically exploit ∂-finite, resp. q-holonomic, closure properties as described in Section 3. Let us first compute generators of the ideal of operators annihilating

$C_m(\cos\theta; q^\nu|q)$. Recall $x = \cos(\theta)$ and $A = e^{i\theta}$. In addition, we will use the abbreviations

$$K = q^k, \quad M = q^m, \quad N = q^n, \quad V = q^\nu, \text{ and } \quad \omega = q^w.$$

In[36]:= **annqGegenbauer =**

$$\mathbf{CreativeTelescoping}\Big[\frac{\mathbf{qP}[q^\nu, q, k]\,\mathbf{qP}[q^\nu, q, m-k]}{\mathbf{qP}[q, q, k]\,\mathbf{qP}[q, q, m-k]} A^{m-2k}, \mathbf{QS}[\mathbf{K}, q^k] - 1, \{\mathbf{QS}[M, q^m], \mathbf{QS}[V, q^\nu], \mathbf{QS}[\omega, q^w]\}\Big][[1]] \text{ // } \mathbf{Factor}$$

Out[36]= $\{S_{\omega,q} - 1, -A(A^2+1)V(Mq-1)S_{M,q} + (V-1)(A^2-V)(A^2V-1)S_{V,q} + A^2(V+1)(MV^2-1), (V-1)(qV-1)(A^2-qV)(A^2qV-1)S_{V,q}^2 - (V-1)(A^4Mq^2V^2 - A^4qV - A^2Mq^3V^3 - A^2Mq^2V^3 + A^2q + A^2 + Mq^2V^2 - qV)S_{V,q} - A^2q(MV^2-1)(MqV^2-1)\}$

The algorithmic method to compute **annqGegenbauer** follows the creative telescoping strategy described in Section 3. In contrast to the $q = 1$ case, here we also include the shift with respect to ω which in the output gives rise to an additional, trivial generator $S_{\omega,q} - 1$. This is done in order to be able to execute all required closure property computations in one common operator algebra. For further details see [12, 14]; there is also an on-line description of the **CreativeTelescoping** procedure in the **HolonomicFunctions** package:

In[37]:= **?CreativeTelescoping**

> CreativeTelescoping[f, delta, {op1, ..., opk}] or CreativeTelescoping[ann, delta, {op1, ..., opk}] computes creative telescoping relations for the given function f (resp. the given ∂-finite ideal ann annihilating some function f). In particular it returns {{q1, ..., qm}, {r1, ..., rm}}, two lists of OrePolynomials such that qj + delta*rj is in the annihilator of f for all $1 \leqslant j \leqslant m$. The polynomials qj form a Groebner basis in the rational Ore algebra with generators op1, ..., opk whereas the rj's live in the Ore algebra with generators Join[OreOperators[delta], {op1, ..., opk}] (resp. the Ore algebra of ann). For summation (w.r.t. n) set delta to S[n]–1 or Delta[n], and in the q-case to QS[qn,q^n]–1; for integration (w.r.t. x) set delta to Der[x].

In an analogous fashion we compute generators of the annihilating ideal of the q-Bessel function $J_\nu^{(2)}(2\omega; q)$:

In[38]:= **annqBesselJ =**

$$\mathbf{CreativeTelescoping}\Big[q^{(\nu+n)n} \frac{\mathbf{qP}[q^{\nu+1}, q]\,(-1)^n \omega^{\nu+2n}}{\mathbf{qP}[q, q]\,\mathbf{qP}[q, q, n]\,\mathbf{qP}[q^{\nu+1}, q, n]}, \mathbf{QS}[\mathbf{N}, q^n] - 1, \{\mathbf{QS}[M, q^m], \mathbf{QS}[V, q^\nu], \mathbf{QS}[\omega, q^w]\}\Big][[1]]$$

Out[38]= $\{(-V\omega - \omega)S_{V,q} + (q\omega^4 + q\omega^2 + \omega^2 + 1)S_{\omega,q} + (\omega^2 - V), S_{M,q} - 1, (q^5V\omega^4 + q^3V\omega^2 + q^2V\omega^2 + V)S_{\omega,q}^2 + (q^2V\omega^2 + qV\omega^2 - V^2 - 1)S_{\omega,q} + V\}$

The annihilating ideal of $h_1(\omega, m, n) := i^m(1 - q^{\nu+m})$ is obtained as follows:

In[39]:= **annh1 = Annihilator**$[\mathbb{i}^m(1 - q^{\nu+m}), \{\mathbf{QS}[M, q^m], \mathbf{QS}[V, q^\nu], \mathbf{QS}[\omega, q^w]\}]$

Out[39]= $\{S_{\omega,q} - 1, (MV-1)S_{V,q} + (1 - MqV), (MV-1)S_{M,q} + (i - iMqV)\}$

Note that in fact it is trivial to compute the generators of this ideal, just consider the quotients

$$\frac{h_1(q\omega,m,n)}{h_1(\omega,m,n)}, \quad \frac{h_1(\omega,m+1,n)}{h_1(\omega,m,n)}, \quad \text{and} \quad \frac{h_1(\omega,m,n+1)}{h_1(\omega,m,n)},$$

which, after simplification, yield rational functions, whose numerators and denominators appear as coefficients in the first-order operators of Out[39]. The reason for this is that $h_1(\omega,m,n)$ is actually a q-hypergeometric term. When trying to compute the annihilating ideal of $h_2(m) := q^{m^2/4}$ by the **Annihilator** command, the package is trapped by the factor $\frac{1}{4}$ in the exponent and delivers the fourth-order operator $S_{M,q}^4 - q^4M^2$. Although this is correct, in the sense that it is a left multiple of the minimal-order annihilating operator, it is not the operator we wish to work with. Instead, we write down the annihilator of $h_2(m)$ by hand, and convert its generators to Ore polynomials that live in the same Ore algebra as **annh1**:

In[40]:= **annh2 = ToOrePolynomial[{QS[V, q^ν] − 1, QS[ω, q^w] − 1,
QS[M, q^m]^2 − qM}, OreAlgebra[QS[M, q^m], QS[V, q^ν], QS[ω, q^w]]]**

Out[40]= $\{S_{V,q}-1, S_{\omega,q}-1, S_{M,q}^2 - Mq\}$

The annihilating ideal of $h_1(\omega,m,n)h_2(m) = i^m(1-q^{\nu+m})q^{m^2/4}$, is obtained by applying the closure property "multiplication", see Theorem 8(ii):

In[41]:= **annh1h2 = DFiniteTimes[annh1, annh2]**

Out[41]= $\{S_{\omega,q}-1, (MV-1)S_{V,q} + (1-MqV), (MV-1)S_{M,q}^2 + (M^2q^3V - Mq)\}$

We continue by applying the same closure property again, in order to obtain an annihilating ideal of $i^m(1-q^{\nu+m})q^{m^2/4}J_{\nu+m}^{(2)}(2\omega;q)C_m(\cos\theta;q^\nu|q)$; here we use the previously computed annihilators **annqBesselJ** and **annqGegenbauer** of the q-Bessel function and the q-Gegenbauer polynomials, respectively, together with discrete substitution as described in Theorem 8(vi):

In[42]:= **annSmnd = DFiniteTimes[annh1h2,
DFiniteSubstitute[annqBesselJ, {ν → ν + m}], annqGegenbauer];**

The output list consists of three annihilating operators, and it would require about two pages to display them. The output **annSmnd** is of the form:

$$\begin{aligned}&\big\{(1+q^2\omega^2+q^3\omega^2+q^5\omega^4)MVS_{\omega,q}^2+(qMV\omega^2+q^2MV\omega^2-M^2V^2-1)S_{\omega,q}+MV,\\
&(-A^2MV\omega^2+\ldots-q^5A^2M^5V^9\omega^2)S_{V,q}^2+(A^2MV\omega+\ldots+q^6A^2M^5V^8\omega^5)S_{V,q}S_{\omega,q}+\\
&(-qA^2M^2V^2\omega-\ldots+q^5A^2M^5V^8\omega^3)S_{V,q}+(-A^2+\ldots+q^7A^2M^6V^8\omega^4)S_{\omega,q}+\\
&\qquad(A^2MV-\ldots+q^6A^2M^6V^8\omega^2),\\
&(-A^2V^2\omega^2+\ldots+q^4A^2M^5V^5\omega^2)S_{M,q}^2+(qA^2MV\omega-\ldots-q^4A^2M^4V^7\omega^5)S_{V,q}S_{\omega,q}+\\
&(-q^2A^2M^2V^2\omega+\ldots-q^3A^2M^4V^7\omega^3)S_{V,q}+(-A^2+\ldots-q^6A^2M^5V^7\omega^4)S_{\omega,q}+\\
&\qquad(A^2MV-\ldots-q^5A^2M^5V^7\omega^2)\big\}\end{aligned} \tag{14}$$

The next step in the process of constructing an annihilating ideal of the right-hand side of (12) consists in "doing the sum"

$$\sum_{m=0}^{\infty} i^m(1-q^{\nu+m})\,q^{m^2/4}J_{\nu+m}^{(2)}(2\omega;q)\,C_m(\cos\theta;q^\nu|q).$$

However, applying the **CreativeTelescoping** command, as we did before, does not deliver any result within a reasonable amount of time. By inspecting the leading monomials of **annSmnd**—they are $S_{\omega,q}^2$, $S_{V,q}^2$, and $S_{M,q}^2$—we find that the holonomic rank of **annSmnd** is 8, which is relatively large and which explains the failure of the first attempt. Luckily there exists an algorithm [13] that is more efficient in such situations, but whose drawback is that sometimes it is not able to deduce the correct denominator of the output. The current summation problem is such an example, and therefore we give the correct denominator with an additional option (it can be found by looking at the leading coefficients of **annSmnd** plus some trial and error). The computation then takes about 20 seconds and for better readability we suppress parts of the output.

In[43]:= $\mathbf{annSumRHS = FindCreativeTelescoping[annSmnd, QS[}\boldsymbol{M, q^m}\mathbf{] - 1,}$
$\mathbf{Denominator \to (}\boldsymbol{M^2V^2}\mathbf{ - 1)(}\boldsymbol{q^2M^2V^2}\mathbf{ - 1)(}\boldsymbol{q^4M^2V^2}\mathbf{ - 1)]}$

Out[43]= $\Big\{\big\{(V-1)S_{V,q}+\omega, (-q^5A^2\omega^4-q^3A^2\omega^2-q^2A^2\omega^2-A^2)S_{\omega,q}^2 + (-q^2A^4V\omega^2+qA^2V+A^2V-q^2V\omega^2)S_{\omega,q}-qA^2V^2\big\}, \big\{\cdots\big\}\Big\}$

Finally we obtain the annihilating ideal of the right-hand side of (12):

In[44]:= $\mathbf{annRHS = DFiniteTimes[annSumRHS[[1]], Annihilator[}$
$\mathbf{qP[}\boldsymbol{q,q}\mathbf{]/(}\boldsymbol{\omega^\nu}\mathbf{qP[}\boldsymbol{q^\nu,q}\mathbf{]\,qP[(}\boldsymbol{-q)\omega^2,q^2}\mathbf{]), \{QS[}\boldsymbol{V,q^\nu}\mathbf{], QS[}\boldsymbol{\omega,q^w}\mathbf{]\}]]}$

Out[44]= $\big\{S_{V,q}-1, (q^2A^2\omega^2+A^2)S_{\omega,q}^2+(q^2A^4\omega^2-qA^2-A^2+q^2\omega^2)S_{\omega,q}+(q^2A^2\omega^2+qA^2)\big\}$

Comparison with the left-hand side of (12):

In[45]:= **annLHS**

Out[45]= $\big\{(q^2A^2\omega^2+A^2)S_{\omega,q}^2+(q^2A^4\omega^2-qA^2-A^2+q^2\omega^2)S_{\omega,q}+(q^2A^2\omega^2+qA^2)\big\}$

To complete the proof we have to incorporate initial conditions. To this end we convert the q-shift equation to an equivalent version which is in the format of a q-differential equation. This is supported by the command **QSE2DE** from the **qGeneratingFunctions** package; in order to invoke it, we need to convert the operator **annLHS** in Out[45] to a standard q-shift equation:

In[46]:= $\mathbf{qSeq = ApplyOreOperator[annLHS, }\boldsymbol{f[\omega]}\mathbf{]\ /.\ }\boldsymbol{q\hat{}(a\_.m+b\_.) \to q^b\omega^a}$

Out[46]= $\big\{(q^2A^2\omega^2+qA^2)f[\omega]+(q^2A^4\omega^2-qA^2-A^2+q^2\omega^2)f[q\omega]+(q^2A^2\omega^2+A^2)f[q^2\omega]\big\}$

In[47]:= $\mathbf{QSE2DE[qSeq, }\boldsymbol{f[\omega]}\mathbf{]}$

Out[47]= $\big\{q(A^2+1)^2f[\omega]+q(q-1)(A^4+qA^2+A^2+1)\omega f'[\omega]+(q-1)^2(q^2\omega^2+1)A^2f''[\omega]=0\big\}$

In this equivalent form, $f'(\omega) := D_q f(\omega)$ refers to the q-derivative defined on (formal) power series as

$$D_q \sum_{n=0}^{\infty} a_n\omega^n := \sum_{n=1}^{\infty} a_n \frac{q^n-1}{q-1}\omega^{n-1}.$$

Now our proof can be completed as follows: let $l(\omega)$ and $r(\omega)$ denote the left and right sides of (12), respectively. Above we have shown that both $l(\omega)$ and $r(\omega)$ satisfy the q-differential equation Out[47]. So what is left to show is that

$$l(0) = r(0) = 1 \quad \text{and} \quad l'(0) = r'(0) = \frac{2iq^{1/4}\cos(\theta)}{1-q}.$$

But this, in view of the definition of D_q, amounts to comparing the coefficients of ω^0 and ω^1, respectively, in the Taylor expansions of $l(\omega)$ and $r(\omega)$. Owing to the definitions of the functions involved and noticing that $\omega^{-\nu}$ is cancelling out, this task is an easy verification.

5 Conclusion

As expressed in the Introduction, a major objective of this article is to popularize the holonomic systems approach in the field of q-series and basic hypergeometric functions. In the case studies we presented, RISC software written in Mathematica was used. With respect to q-summation one can find various packages written in Maple or in other computer algebra systems; with regard to the more general q-holonomic setting (operator algebras, non-commutative Gröbner basis methods, etc.) we point explicitly to the Maple package **Mgfun** by F. Chyzak [2].

We want to conclude with a few remarks on the fact that computing an annihilating operator for the series side of the Ismail-Zhang formula (12) is leading to the frontiers of what is computationally feasible today. As already pointed out, ongoing research is trying to push frontiers further by the design of new constructive methods like [13]. Formulas like (12) or the very well-poised basic hypergeometric series ${}_{10}W_9$ in [6], which has been successfully treated by the **HolonomicFunctions** package, provide excellent challenges and inspirations for such algorithmic developments.

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