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Abstract

We study multivariate $\mathbb{L}_\infty$-approximation for a weighted Korobov space of periodic functions for which the Fourier coefficients decay exponentially fast. The weights are defined, in particular, in terms of two sequences $\boldsymbol{a} = \{a_j\}$ and $\boldsymbol{b} = \{b_j\}$ of positive real numbers bounded away from zero. We study the minimal worst-case error $e^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(n,s)$ of all algorithms that use n information evaluations from a class Λ in the s-variate case. We consider two classes Λ in this paper: the class Λ^{all} of all linear functionals and the class Λ^{std} of only function evaluations.

We study exponential convergence of the minimal worst-case error, which means that $e^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(n,s)$ converges to zero exponentially fast with increasing n. Furthermore, we consider how the error depends on the dimension s. To this end, we define the notions of κ-EC-weak, EC-polynomial and EC-strong polynomial tractability, where EC stands for "exponential convergence". In particular, EC-polynomial tractability means that we need a polynomial number of information evaluations in s and $1+\log \varepsilon^{-1}$ to compute an ε-approximation. We derive necessary and sufficient conditions on the sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ for obtaining exponential error convergence, and also for obtaining the various notions of tractability. The results are the same for both classes Λ.

$\mathbb{L}_2$-approximation for functions from the same function space has been considered in [2]. It is surprising that most results for $\mathbb{L}_\infty$-approximation coincide with their counterparts for $\mathbb{L}_2$-approximation. This allows us to deduce also results for $\mathbb{L}_p$-approximation for $p \in [2,\infty]$.

Keywords: Multivariate $\mathbb{L}_\infty$-approximation, worst-case error, tractability, exponential convergence, Korobov spaces
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1 Introduction

Function approximation is a topic addressed in a huge number of papers and monographs. We study the problem of multivariate $\mathbb{L}_\infty$-approximation of functions which belong to a special class of one-periodic functions defined on $[0,1]^s$. Here, $s \in \mathbb{N} := \{1,2,\dots\}$ and our emphasis is on large s. These functions belong to a weighted Korobov space whose elements share the property that their Fourier coefficients decay exponentially fast.

Korobov spaces are special types of reproducing kernel Hilbert spaces $H(K_s)$ with a reproducing kernel of the form

$$K_s(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \rho(\boldsymbol{h}) \exp(2\pi \mathtt{i} \boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y})) \qquad \text{for all } \boldsymbol{x},\boldsymbol{y} \in [0,1]^s$$

for some function $\rho : \mathbb{Z}^s \to \mathbb{R}_+$.

We approximate functions by algorithms that use n information evaluations, where we allow information evaluations from the class $\Lambda^{\rm all}$ of all continuous linear functionals or, alternatively, from the narrower class $\Lambda^{\rm std}$ of standard information which consists of only function evaluations. The quality of our algorithms is measured by the worst-case approximation error, i.e., by the largest error over the unit ball of the function space.

For large s, it is crucial to study how the errors of algorithms depend not only on n but also on s. The information complexity $n^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(\varepsilon,s)$ is the minimal number n for which there exists an algorithm using n information evaluations from the class $\Lambda \in \{\Lambda^{\rm all},\Lambda^{\rm std}\}$ with an error of at most ε times a constant. If this constant is 1, then we speak about the absolute error criterion. If the constant is equal to the initial error, i.e., the error without using any information evaluations, we speak about the normalized error criterion. The information complexity is proportional to the minimal cost of computing an ε-approximation since linear algorithms are optimal and their cost is proportional to $n^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(\varepsilon,s)$.

In many papers, as for example [7, 8, 9, 10, 11, 19], and also in the recent trilogy [13]–[15], Korobov spaces are studied for functions ρ of the form

$$\rho_{\rm pol}(\boldsymbol{h}) = \rho_{\rm pol}(h_1,h_2,\dots,h_s) = \prod_{j=1}^s \rho_{\rm pol}(h_j), \quad \rho_{\rm pol}(h_j) = \begin{cases} 1 & \text{if } h_j = 0, \\ \gamma_j/\left|h_j\right|^\alpha & \text{if } h_j \neq 0, \end{cases}$$

where $\alpha > 1$ is a smoothness parameter for the elements of the Korobov space (the number of derivatives of the functions is roughly $\alpha/2$). This means that $\rho_{\rm pol}(\boldsymbol{h})$ decays polynomially in $\boldsymbol{h}$. The function $\rho_{\rm pol}$ also depends on weights γ_j which model the influence of the different variables and groups of variables of the problem. For this choice of $\rho_{\rm pol}$, it can be shown that one can achieve polynomial error convergence for $\mathbb{L}_2$- and $\mathbb{L}_\infty$-approximation and, under suitable conditions on the weights, also avoid a curse of dimensionality and achieve different types of tractability. By tractability, we mean that the information complexity does neither depend exponentially on s nor on ε^{-1}. In particular, we speak of polynomial tractability if the information complexity depends at most polynomially on s and ε^{-1} and of strong polynomial tractability if it depends polynomially on ε^{-1} and not on s, see [7, 8, 9, 10, 12] as well as [13]–[15] for further details. We stress that the results for $\mathbb{L}_2$- and $\mathbb{L}_\infty$-approximation for Korobov spaces based on $\rho_{\rm pol}$ are not the same.

Indeed, let

$$\delta_0 := \inf\left\{\delta \geq 0 : \sum_{j=1}^{\infty} \gamma_j^{\delta} < \infty\right\} \quad \text{and} \quad \alpha_0 := \min(\alpha, \delta_0^{-1}).$$

Then the best rate of error convergence with strong polynomial tractability for a Korobov space based on $\rho_{\rm pol}$ is, if we allow information from $\Lambda^{\rm all}$, of order $\alpha_0/2$ for $\mathbb{L}_2$-approximation, and $(\alpha_0 - 1)/2$, if $\alpha_0 > 1$, for $\mathbb{L}_\infty$-approximation. The convergence rates for $\Lambda^{\rm std}$ are not known exactly and the known upper bounds are slightly weaker than for the class $\Lambda^{\rm all}$.

For the Korobov spaces considered in the present paper, we choose ρ as $\rho_{\rm exp}(\boldsymbol{h})$ which decays exponentially in $\boldsymbol{h}$, and again $\rho_{\rm exp}$ depends on weights expressed by two sequences of positive real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$, which model the influence of the variables of the problem. For this choice of $\rho_{\rm exp}$, we obtain exponential error convergence instead of polynomial error convergence. To be more precise, let $e^{\mathbb{L}_\infty-{\rm app},\Lambda}(n,s)$ be the minimal worst-case error among all algorithms that use n information evaluations from a permissible class Λ in the s-variate case. By *exponential convergence* of the nth minimal approximation error we mean that

$$e^{\mathbb{L}_\infty-{\rm app},\Lambda}(n,s) \leq C(s)\, q^{\,(n/C_1(s))^{\,p(s)}} \quad \text{for all} \quad n,s \in \mathbb{N},$$

where, $q \in (0,1)$ is independent of s, whereas C, C_1, and p are allowed to be dependent on s. We have *uniform exponential convergence* if p can be chosen independently of s.

Under suitable conditions on the weight sequences $\boldsymbol{a}$ and $\boldsymbol{b}$, we achieve stronger notions of tractability than for the case of polynomial error convergence, which is then referred to as Exponential Convergence-tractability (or, for short, EC-tractability). Roughly speaking, EC-tractability is defined similarly to the standard notions of tractability, but we replace ε^{-1} by $1 + \log \varepsilon^{-1}$.

The case of $\mathbb{L}_2$-approximation for $\rho(\boldsymbol{h})$ depending exponentially on $\boldsymbol{h}$ was dealt with in the recent paper [2]. The results there can be also obtained as special cases of a more general approach presented in the paper [4].

For the case of $\mathbb{L}_\infty$-approximation, which is considered in the present paper, it turns out that most of the results are the same as for $\mathbb{L}_2$-approximation. Surprising as this may seem, the reason for the similarities between $\mathbb{L}_2$- and $\mathbb{L}_\infty$-approximation may lie in the expression of the worst-case error in terms of the ordered eigenvalues $\lambda_1, \lambda_2, \ldots$ of a certain operator $W_s : H(K) \to H(K)$, see below. For $\mathbb{L}_2$-approximation, the minimal error if we use n evaluations from $\Lambda^{\rm all}$ is $\sqrt{\lambda_{n+1}}$, whereas the minimal error for $\mathbb{L}_\infty$-approximation is $\sqrt{\sum_{k=n+1}^{\infty} \lambda_k}$. For the case of the spaces considered in this paper, the eigenvalues λ_k depend exponentially on k, which means that λ_n and $\sum_{k=n+1}^{\infty} \lambda_k$ behave similarly, which suggests that the errors for $\mathbb{L}_2$-approximation and $\mathbb{L}_\infty$-approximation should also have similar properties. Moreover, as we shall also show in the present paper, there are no differences in the results between the class $\Lambda^{\rm all}$ and the class $\Lambda^{\rm std}$, and no difference between the absolute and the normalized error criterion. However, for one concept of tractability there is a difference in the results between $\mathbb{L}_2$-approximation and $\mathbb{L}_\infty$-approximation.

The rest of the paper is structured as follows. In Section 2, we introduce the weighted Korobov space considered in this paper, and in Section 3 we define precisely what we mean by exponential error convergence and by various notions of Exponential Convergence-tractability. Our main result is stated in Section 4. Furthermore, in Section 5, we outline relations of the $\mathbb{L}_\infty$-approximation problem to the $\mathbb{L}_2$-approximation problem. After some preliminary observations in Section 6, we then outline our main results in Sections 7–9. In Section 10, we summarize and compare our results on $\mathbb{L}_\infty$-approximation to previous results on $\mathbb{L}_2$-approximation, and, in the final Section 11, we give some remarks on $\mathbb{L}_p$-approximation.

2 The Korobov space $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$

The Korobov space $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ discussed in this section is a reproducing kernel Hilbert space. For general information on reproducing kernel Hilbert spaces we refer to [1].

Let $\boldsymbol{a} = \{a_j\}_{j\geq 1}$ and $\boldsymbol{b} = \{b_j\}_{j\geq 1}$ be two sequences of real positive weights such that

$$b_* := \inf_j b_j > 0 \qquad \text{and} \qquad a_* := \inf_j a_j > 0. \tag{1}$$

Throughout the paper we additionally assume that

$$a_* = a_1 \leq a_2 \leq a_3 \leq \dots .$$

Fix $\omega \in (0,1)$ and denote

$$\omega_{\boldsymbol{h}} = \omega^{\sum_{j=1}^s a_j |h_j|^{b_j}} \qquad \text{for all} \qquad \boldsymbol{h} = (h_1, h_2, \dots, h_s) \in \mathbb{Z}^s.$$

We consider a Korobov space of complex-valued one-periodic functions defined on $[0,1]^s$ with a reproducing kernel of the form

$$K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \omega_{\boldsymbol{h}} \exp\big(2\pi \mathtt{i} \boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y})\big) \quad \text{for all} \quad \boldsymbol{x},\boldsymbol{y} \in [0,1]^s$$

with the usual dot product

$$\boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y}) = \sum_{j=1}^s h_j (x_j - y_j),$$

where h_j, x_j, y_j are the jth components of the vectors $\boldsymbol{h}, \boldsymbol{x}, \boldsymbol{y}$, respectively, and $\mathtt{i} = \sqrt{-1}$.

The kernel $K_{s,\boldsymbol{a},\boldsymbol{b}}$ is well defined since

$$|K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{y})| \leq K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x},\boldsymbol{x}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \omega_{\boldsymbol{h}} = \prod_{j=1}^s \left(1 + 2\sum_{h=1}^\infty \omega^{a_j h^{b_j}}\right) < \infty. \tag{2}$$

The last series is indeed finite since

$$\sum_{h=1}^\infty \omega^{a_j h^{b_j}} \leq \sum_{h=1}^\infty \omega^{a_* h^{b_*}} < \infty,$$

and both a_* and b_* are assumed to be strictly greater than zero.

The Korobov space with reproducing kernel $K_{s,\boldsymbol{a},\boldsymbol{b}}$ is a reproducing kernel Hilbert space and is denoted by $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$. We suppress the dependence on ω in the notation since ω will be fixed throughout the paper and $\boldsymbol{a}$ and $\boldsymbol{b}$ will be varied.

Clearly, functions from $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ are infinitely many times differentiable, see [3], and, if $b_* \ge 1$ they are also analytic as shown in [2, Proposition 2] [1].

For $f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ we have

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \widehat{f}(\boldsymbol{h}) \, \exp(2\pi \mathtt{i} \boldsymbol{h}\cdot\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x} \in [0,1]^s,$$

where $\widehat{f}(\boldsymbol{h}) = \int_{[0,1]^s} f(\boldsymbol{x}) \exp(-2\pi\mathtt{i}\boldsymbol{h}\cdot\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$ is the $\boldsymbol{h}$th Fourier coefficient. The inner product of f and g from $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ is given by

$$\langle f, g\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \widehat{f}(\boldsymbol{h}) \, \overline{\widehat{g}(\boldsymbol{h})} \, \omega_{\boldsymbol{h}}^{-1},$$

where $\overline{z}$ means the complex conjugate of $z \in \mathbb{C}$, and the norm of f from $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ by

$$\|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} = \left(\sum_{\boldsymbol{h}\in\mathbb{Z}^s} |\widehat{f}(\boldsymbol{h})|^2 \omega_{\boldsymbol{h}}^{-1}\right)^{1/2} < \infty.$$

Integration of functions from $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ was already considered in [5] and, in the case $a_j = b_j = 1$ for all $j \in \mathbb{N}$, also in [3]. Furthermore, multivariate approximation of functions from $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ in the $\mathbb{L}_2$ norm was considered in the recent papers [2, 4]. A survey of these results can be found in [6]. In the present paper we consider the problem of multivariate approximation in the $\mathbb{L}_\infty$ norm which we shortly call $\mathbb{L}_\infty$-approximation.

3 $\mathbb{L}_\infty$-approximation

In this section we consider $\mathbb{L}_\infty$-approximation of functions from $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$. This problem is defined as an approximation of the embedding from the Korobov space $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ to the space $\mathbb{L}_\infty([0,1]^s)$, i.e.,

$$\mathrm{EMB}_{s,\infty} : H(K_{s,\boldsymbol{a},\boldsymbol{b}}) \to \mathbb{L}_\infty([0,1]^s) \qquad \text{given by} \qquad \mathrm{EMB}_{s,\infty}(f) = f.$$

This embedding is continuous since for $f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ we have $f(x) = \langle f, K_{s,\boldsymbol{a},\boldsymbol{b}}(\cdot,x)\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})}$ and

$$\begin{aligned}
\|\mathrm{EMB}_{s,\infty}(f)\|_{\mathbb{L}_\infty([0,1]^s)} &= \|f\|_{\mathbb{L}_\infty([0,1]^s)} = \sup_{x\in[0,1]^s} |f(x)| = \sup_{x\in[0,1]^s} |\langle f, K_{s,\boldsymbol{a},\boldsymbol{b}}(\cdot,x)\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})}| \\
&\le \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \sup_{x\in[0,1]^s} \sqrt{K_{s,\boldsymbol{a},\boldsymbol{b}}(x,x)} \\
&= \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \prod_{j=1}^{s} \left(1 + 2\sum_{h=1}^{\infty} \omega^{a_j h^{b_j}}\right)^{1/2}.
\end{aligned}$$

[1] The assumption $b_* \ge 1$ is not explicit but it is needed in the proof of [2, last line of p.27].

Here, we use the supremum instead of the essential supremum since f is continuous. Furthermore, the last inequality is sharp for $f = K_{s,\boldsymbol{a},\boldsymbol{b}}(\cdot, x)$ for any $x \in [0,1]^s$. This proves that

$$\|\mathrm{EMB}_{s,\infty}\| = \prod_{j=1}^{s} \left(1 + 2\sum_{h=1}^{\infty} \omega^{a_j h^{b_j}}\right)^{1/2}.$$

Without loss of generality, see e.g., [18], we approximate $\mathrm{EMB}_{s,\infty}$ by linear algorithms $A_{n,s}$ of the form

$$A_{n,s}(f) = \sum_{k=1}^{n} \alpha_k L_k(f) \quad \text{for} \quad f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \tag{3}$$

where each α_k is a function from $\mathbb{L}_\infty([0,1]^s)$ and each L_k is a continuous linear functional defined on $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ from a permissible class Λ of information. We consider two classes:

- $\Lambda = \Lambda^{\mathrm{all}}$, the class of all continuous linear functionals defined on $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$. Since $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ is a Hilbert space, for every $L_k \in \Lambda^{\mathrm{all}}$ there exists a function f_k from $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ such that $L_k(f) = \langle f, f_k \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})}$ for all $f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}})$.
- $\Lambda = \Lambda^{\mathrm{std}}$, the class of standard information consisting only of function evaluations. That is, $L_k \in \Lambda^{\mathrm{std}}$ iff there exists $\boldsymbol{x}_k \in [0,1]^s$ such that $L_k(f) = f(\boldsymbol{x}_k)$ for all $f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}})$.

Since $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ is a reproducing kernel Hilbert space, function evaluations are continuous linear functionals and therefore $\Lambda^{\mathrm{std}} \subseteq \Lambda^{\mathrm{all}}$. More precisely,

$$L_k(f) = f(\boldsymbol{x}_k) = \langle f, K_{s,\boldsymbol{a},\boldsymbol{b}}(\cdot, \boldsymbol{x}_k) \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})}$$

and

$$\|L_k\| = \|K_{s,\boldsymbol{a},\boldsymbol{b}}(\cdot, \boldsymbol{x}_k)\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} = \sqrt{K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x}_k, \boldsymbol{x}_k)} = \prod_{j=1}^{s} \left(1 + 2\sum_{h=1}^{\infty} \omega^{a_j h^{b_j}}\right)^{1/2}.$$

The *worst-case error* of the algorithm (3) is defined as

$$e^{\mathbb{L}_\infty-\mathrm{app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), A_{n,s}) := \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \le 1}} \|f - A_{n,s}(f)\|_{\mathbb{L}_\infty([0,1]^s)},$$

where $\|f - A_{n,s}(f)\|_{\mathbb{L}_\infty([0,1]^s)}$ is defined in terms of the essential supremum.

Let $e^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(n,s)$ be the nth minimal worst-case error,

$$e^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(n,s) = \inf_{A_{n,s}} e^{\mathbb{L}_\infty-\mathrm{app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), A_{n,s}),$$

where the infimum is taken over all linear algorithms $A_{n,s}$ of the form (3) using n information evaluations from the class Λ. For $n = 0$ the best we can do is to approximate f by zero, and the initial error is

$$e^{\mathbb{L}_\infty-\mathrm{app}}(0,s) = \|\mathrm{EMB}_{s,\infty}\| = \prod_{j=1}^{s} \left(1 + 2\sum_{h=1}^{\infty} \omega^{a_j h^{b_j}}\right)^{1/2}. \tag{4}$$

Note that the initial error may be arbitrarily large for large s. For example, take $a_j = b_j = 1$ for all $j \geq 1$. Then

$$\|\mathrm{EMB}_{s,\infty}\| = \left(1 + \frac{2\omega}{1-\omega}\right)^{s/2}$$

is exponentially large in s. This means that $\mathbb{L}_\infty$-approximation may be *not* properly normalized. On the other hand, if $\sum_{j,h=1}^{\infty} \omega^{a_j h^{b_j}} < \infty$ then $\|\mathrm{EMB}_{s,\infty}\|$ is of order 1 for all s, and $\mathbb{L}_\infty$-approximation is properly normalized. In particular, this holds for $a_j = j$ and $b_j = 1$ since then $\sum_{j,h=1}^{\infty} \omega^{a_j h^{b_j}} < (1-\omega)^{-2}$ and $\|\mathrm{EMB}_{s,\infty}\| \leq \exp((1-\omega)^{-2})$.

We study *exponential convergence* in this paper, which is abbreviated as EXP. As in [2, 5, 6], this means that there exist a number $q \in (0,1)$ and functions $p, C, C_1 : \mathbb{N} \to (0,\infty)$ such that

$$e^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(n,s) \leq C(s)\, q^{\,(n/C_1(s))^{\,p(s)}} \qquad \text{for all} \qquad n \in \mathbb{N}. \tag{5}$$

If (5) holds we would like to find the largest possible rate $p(s)$ of exponential convergence defined as

$$p^*(s) = \sup\{\, p(s) \ : \ \ p(s) \ \text{ satisfies (5)}\,\}. \tag{6}$$

Uniform exponential convergence, abbreviated as UEXP, means that the function p in (5) can be taken as a constant function, i.e., $p(s) = p > 0$ for all $s \in \mathbb{N}$. Similarly, let

$$p^* = \sup\{\, p \ : \ \ p(s) = p > 0 \ \text{ satisfies (5) for all } s \in \mathbb{N}\,\}$$

denote the largest rate of uniform exponential convergence.

We consider the absolute and normalized error criteria. For $\varepsilon \in (0,1)$, $s \in \mathbb{N}$, and $\Lambda \in \{\Lambda^{\mathrm{all}}, \Lambda^{\mathrm{std}}\}$, the *information complexity for the absolute error criterion* is defined as

$$n_{\mathrm{abs}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(\varepsilon,s) := \min\left\{n \ : \ e^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(n,s) \leq \varepsilon\right\}.$$

Hence, $n_{\mathrm{abs}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(\varepsilon,s)$ is the minimal number of information evaluations from Λ which is required to achieve an error of at most ε.

For $\varepsilon \in (0,1)$, $s \in \mathbb{N}$, and $\Lambda \in \{\Lambda^{\mathrm{all}}, \Lambda^{\mathrm{std}}\}$, the *information complexity for the normalized error criterion* is defined as

$$n_{\mathrm{norm}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(\varepsilon,s) := \min\left\{n \ : \ e^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(n,s) \leq \varepsilon e^{\mathbb{L}_\infty-\mathrm{app}}(0,s)\right\}.$$

Thus, $n_{\mathrm{norm}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(\varepsilon,s)$ is the minimal number of information evaluations from Λ which is required to reduce the initial error $e^{\mathbb{L}_\infty-\mathrm{app}}(0,s)$ by a factor of $\varepsilon \in (0,1)$.

In this paper, we study four different cases, namely

- the absolute error criterion with information from Λ^{std},
- the absolute error criterion with information from Λ^{all},

- the normalized error criterion with information from $\Lambda^{\rm std}$,
- the normalized error criterion with information from $\Lambda^{\rm all}$.

There are several relations between these cases which will be helpful in the analysis. First, note that clearly

$$e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm all}}(n,s) \,\le\, e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm std}}(n,s), \tag{7}$$

and therefore

$$n_{\rm setting}^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm all}}(\varepsilon,s) \,\le\, n_{\rm setting}^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm std}}(\varepsilon,s) \tag{8}$$

where setting $\in \{{\rm abs},{\rm norm}\}$. Furthermore, since $e^{\mathbb{L}_\infty-{\rm app}}(0,s)>1$ we have

$$n_{\rm norm}^{\mathbb{L}_\infty-{\rm app},\Lambda} \le n_{\rm abs}^{\mathbb{L}_\infty-{\rm app},\Lambda} \qquad \text{for } \Lambda\in\{\Lambda^{\rm all},\Lambda^{\rm std}\}. \tag{9}$$

We are ready to define tractability concepts similarly to [2, 3, 5, 6], and we use the name *Exponential Convergence (EC) Tractability* for these concepts, as introduced in [6]. Following the recent paper of Petras and Papageorgiou [16], we also study κ-EC-WT which is defined for $\kappa\ge 1$. We stress again that all these concepts correspond to the standard concepts of tractability with ε^{-1} replaced by $1+\log\varepsilon^{-1}$.

For $\Lambda\in\{\Lambda^{\rm all},\Lambda^{\rm std}\}$ and setting $\in\{{\rm abs},{\rm norm}\}$, we say that we have:

- *κ-Exponential Convergence-Weak Tractability (κ-EC-WT)* for $\kappa\ge 1$ if

  $$\lim_{s+\log\varepsilon^{-1}\to\infty} \frac{\log\, n_{\rm setting}^{\mathbb{L}_\infty-{\rm app},\Lambda}(\varepsilon,s)}{s+[\log\varepsilon^{-1}]^\kappa}=0.$$

  Here we set $\log 0=0$ by convention. For $\kappa=1$ we say that we have *Exponential Convergence-Weak Tractability (EC-WT)*.

- *Exponential Convergence-Polynomial Tractability (EC-PT)* if there exist non-negative numbers c,τ_1 and τ_2 such that

  $$n_{\rm setting}^{\mathbb{L}_\infty-{\rm app},\Lambda}(\varepsilon,s)\le c\, s^{\,\tau_1}\,(1+\log\varepsilon^{-1})^{\,\tau_2} \qquad \text{for all} \quad s\in\mathbb{N},\ \varepsilon\in(0,1).$$

- *Exponential Convergence-Strong Polynomial Tractability (EC-SPT)* if there exist non-negative numbers c and τ such that

  $$n_{\rm setting}^{\mathbb{L}_\infty-{\rm app},\Lambda}(\varepsilon,s)\le c\,(1+\log\varepsilon^{-1})^{\,\tau} \qquad \text{for all} \quad s\in\mathbb{N},\ \varepsilon\in(0,1).$$

  The exponent τ^* of EC-SPT is defined as the infimum of τ for which EC-SPT holds.

Let us state some remarks about these definitions.

Note that for $\kappa=1$ we obtain EC-WT, whereas for $\kappa>1$, the notion of EC-WT is relaxed. The results for $\kappa=1$ and $\kappa>1$ can be quite different.

It is easy to see that if EC-PT holds for $\Lambda\in\{\Lambda^{\rm all},\Lambda^{\rm std}\}$ and for the absolute or normalized error criterion, then UEXP holds as well. Indeed, due to (8) and (9), it is sufficient to show this result for $\Lambda^{\rm all}$ and the normalized setting. Then EC-PT means that

$$n_{\rm norm}^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm all}}(\varepsilon,s)\le c s^{\tau_1}(1+\log\varepsilon^{-1})^{\tau_2},$$

which implies

$$e_{\text{norm}}^{\mathbb{L}_\infty-\text{app},\Lambda^{\text{all}}}(n,s) \le \mathrm{e}^{1-((n/(cs^{\tau_1}))^{1/\tau_2}} \prod_{j=1}^{s} \left(1+2\sum_{h=1}^{\infty} \omega^{a_j h^{b_j}}\right).$$

Hence, we have UEXP, as claimed.

For the *absolute error criterion*, we note, as in [2, 3], that if (5) holds then

$$n_{\text{abs}}^{\mathbb{L}_\infty-\text{app},\Lambda}(\varepsilon,s) \le \left\lceil C_1(s)\left(\frac{\log C(s)+\log\varepsilon^{-1}}{\log q^{-1}}\right)^{1/p(s)}\right\rceil \qquad \text{for all} \quad s\in\mathbb{N} \ \text{ and } \ \varepsilon\in(0,1). \tag{10}$$

Furthermore, if (10) holds then

$$e^{\mathbb{L}_\infty-\text{app},\Lambda}(n+1,s) \le C(s)\, q^{\,(n/C_1(s))^{\,p(s)}} \qquad \text{for all} \quad s,n\in\mathbb{N}.$$

This means that (5) and (10) are practically equivalent. Note that $1/p(s)$ determines the power of $\log\,\varepsilon^{-1}$ in the information complexity, whereas $\log\,q^{-1}$ affects only the multiplier of $\log^{1/p(s)}\varepsilon^{-1}$. From this point of view, $p(s)$ is more important than q. That is why we would like to have (5) with the largest possible $p(s)$. We shall see how to find such $p(s)$ for the parameters $(\boldsymbol{a},\boldsymbol{b})$ of the weighted Korobov space.

For the *normalized error criterion*, we replace ε by ε multiplied by the initial error. If the initial error is of order one for all s we obtain the same results for both error criteria. On the other hand, if the initial error is badly normalized this may change tractability results. Note, however, that exponential convergence is independent of the error criteria.

For both error criteria, exponential convergence implies that asymptotically, with respect to ε tending to zero, we need $\mathcal{O}(\log^{1/p(s)}\varepsilon^{-1})$ information evaluations to compute an ε-approximation to functions from the Korobov space. However, it is not clear how long we have to wait to see this nice asymptotic behavior especially for large s. This, of course, depends on how $C(s), C_1(s)$ and $p(s)$ depend on s. This is the subject of tractability which is extensively studied in many papers. So far tractability has been usually studied in terms of s and ε^{-1}. The current state of the art on tractability can be found in [13, 14, 15]. In this paper we follow the approach of [2, 3, 5, 6] and we study tractability in terms of s and $1+\log\varepsilon^{-1}$.

4 Main result

In this section we present results for $\mathbb{L}_\infty$-approximation. The proofs of these results will be given in the subsequent sections.

Theorem 1. *Consider $\mathbb{L}_\infty$-approximation defined over the Korobov space with kernel $K_{s,\boldsymbol{a},\boldsymbol{b}}$ with arbitrary sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying* (1)*. The following results hold for $\Lambda\in\{\Lambda^{\text{all}},\Lambda^{\text{std}}\}$ and for the absolute and normalized error criterion.*

1. *EXP holds for arbitrary $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying* (1) *and*

$$p^*(s) = 1/B(s) \qquad \textit{with} \qquad B(s) := \sum_{j=1}^{s}\frac{1}{b_j}.$$

2. *UEXP holds iff* $\boldsymbol{a}$ *is an arbitrary sequence and* $\boldsymbol{b}$ *such that*

$$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty.$$

If so then $p^* = 1/B$.

3. κ*-EC-WT for* $\kappa \ge 1$ *holds iff* $\lim_{j\to\infty} a_j = \infty$.

4. *EC-WT+UEXP holds iff* $B < \infty$ *and* $\lim_{j\to\infty} a_j = \infty$.

5. *The following notions are equivalent:*

$$\begin{aligned} &\text{EC-PT} \Leftrightarrow \text{EC-PT+EXP} \Leftrightarrow \text{EC-PT+UEXP} \\ &\qquad \Leftrightarrow \text{EC-SPT} \Leftrightarrow \text{EC-SPT+EXP} \Leftrightarrow \text{EC-SPT+UEXP}. \end{aligned}$$

6. *EC-SPT+UEXP holds iff*

$$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty \quad \text{and} \quad \alpha^* := \liminf_{j\to\infty} \frac{\log a_j}{j} > 0. \tag{11}$$

If so, then $\tau^* \in \left[B, B + \frac{\log 3}{\alpha^*}\right]$. *In particular, if* $\alpha^* = \infty$, *then* $\tau^* = B$.

□

We now briefly comment on Theorem 1. We find it surprising that the results are the same for Λ^{all} and Λ^{std} and they do not depend on the error criteria.

Exponential convergence holds for all $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying (1). What is more, the rate $p^*(s)$ is independent of $\boldsymbol{a}$ and depends only on $\boldsymbol{b}$. Note that $B(s) \le s/b_*$ and therefore $p^*(s) \ge b_*/s$, and the last bound is sharp if $b_j = b_*$ for all $j \in \mathbb{N}$. In this case $p^*(s)$ is small for large s and tends to zero as s approaches infinity. On the other hand, uniform exponential convergence holds independently of $\boldsymbol{a}$ and only for summable b_j^{-1}. Obviously, B can be arbitrarily large and p^* arbitrarily small.

The notion of κ-EC-WT for $\kappa \ge 1$ is independent of $\boldsymbol{b}$ and holds iff a_j goes to infinity. We stress that the rate how fast a_j goes to infinity is irrelevant. We shall see later that for $\mathbb{L}_2$-approximation the result is different since for $\kappa > 1$ and the class Λ^{all}, the notion of κ-EC-WT holds for all $\boldsymbol{a}$ and $\boldsymbol{b}$.

The notion of EC-WT does not necessarily imply uniform exponential convergence since EC-WT holds for all $\boldsymbol{b}$. To guarantee EC-WT and UEXP we must assume summable b_j^{-1} and a_j converging to infinity.

The next point of Theorem 1 shows that a number of tractability notions are equivalent for $\mathbb{L}_\infty$-approximation. Probably, the most interesting one is that EC-PT is equivalent to EC-SPT+UEXP. In particular, there is no difference between EC-PT and EC-SPT.

Based on these equivalences, it is therefore enough to find necessary and sufficient conditions for EC-SPT+UEXP. It turns out that this holds iff b_j^{-1}'s are summable and a_j's are exponentially large in j.

5 Relations to $\mathbb{L}_2$-approximation

In [2] we studied $\mathbb{L}_2$-approximation of functions from $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$. This problem

$$\mathrm{EMB}_{s,2} : H(K_{s,\boldsymbol{a},\boldsymbol{b}}) \to \mathbb{L}_2([0,1]^s) \qquad \text{given by} \qquad \mathrm{EMB}_{s,2}(f) = f$$

is defined as an approximation of the embedding from the Korobov space $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ to the space $\mathbb{L}_2([0,1]^s)$. Again for this problem it is enough to use linear algorithms $A_{n,s}$ of the form

$$A_{n,s}(f) = \sum_{k=1}^{n} \alpha_k L_k(f) \qquad \text{for} \qquad f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}),$$

where each α_k is a function from $\mathbb{L}_2([0,1]^s)$ and each L_k is a continuous linear functional defined on $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ from the class $\Lambda \in \{\Lambda^{\mathrm{all}}, \Lambda^{\mathrm{std}}\}$.

In the same vein as for the $\mathbb{L}_\infty$-case the *worst-case error* of the algorithm $A_{n,s}$ is now defined as

$$e^{\mathbb{L}_2-\mathrm{app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), A_{n,s}) := \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \leq 1}} \|f - A_{n,s}(f)\|_{\mathbb{L}_2([0,1]^s)},$$

and the nth minimal worst-case error is defined by

$$e^{\mathbb{L}_2-\mathrm{app},\Lambda}(n,s) = \inf_{A_{n,s}} e^{\mathbb{L}_2-\mathrm{app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), A_{n,s}),$$

where the infimum is taken over all linear algorithms $A_{n,s}$ using n information evaluations from the class Λ. For $n = 0$ we obtain the initial error

$$e^{\mathbb{L}_2-\mathrm{app},\Lambda}(0,s) = 1,$$

as shown in [2]. Hence, there is no difference between the absolute and normalized error criteria for $\mathbb{L}_2$-approximation.

For $\varepsilon \in (0,1)$, $s \in \mathbb{N}$, and $\Lambda \in \{\Lambda^{\mathrm{all}}, \Lambda^{\mathrm{std}}\}$, the *information complexity* (for both the absolute and normalized error criteria) is defined as

$$n^{\mathbb{L}_2-\mathrm{app},\Lambda}(\varepsilon, s) := \min\left\{n \,:\, e^{\mathbb{L}_2-\mathrm{app},\Lambda}(n,s) \leq \varepsilon\right\}.$$

It is easy to show that $\mathbb{L}_2$-approximation is not harder than $\mathbb{L}_\infty$-approximation for the absolute error criterion. Namely we have the following lemma.

Lemma 1. *For $\Lambda \in \{\Lambda^{\mathrm{all}}, \Lambda^{\mathrm{std}}\}$ we have*

$$e^{\mathbb{L}_2-\mathrm{app},\Lambda}(n,s) \leq e^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(n,s) \tag{12}$$

and therefore

$$n^{\mathbb{L}_2-\mathrm{app},\Lambda}(\varepsilon,s) \leq n_{\mathrm{abs}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(\varepsilon,s). \tag{13}$$

□

Proof. Note that any algorithm $A_{n,s} = \sum_{k=1}^n \alpha_k L_k$ with $\alpha_k \in \mathbb{L}_\infty = \mathbb{L}_\infty([0,1]^s)$ is also an algorithm $A_{n,s} = \sum_{k=1}^n \alpha_k L_k$ with $\alpha_k \in \mathbb{L}_2 = \mathbb{L}_2([0,1]^s)$. Thus, the class of admissible linear algorithms for $\mathbb{L}_\infty$-approximation is contained in the class of admissible linear algorithms for $\mathbb{L}_2$-approximation. Furthermore,

$$\begin{aligned}
e^{\mathbb{L}_2-\mathrm{app},\Lambda}(n,s) &= \inf_{\substack{A_{n,s} \\ \alpha_k \in \mathbb{L}_2}} \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \le 1}} \|f - A_{n,s}(f)\|_{\mathbb{L}_2([0,1]^s)} \\
&\le \inf_{\substack{A_{n,s} \\ \alpha_k \in \mathbb{L}_\infty}} \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \le 1}} \|f - A_{n,s}(f)\|_{\mathbb{L}_2([0,1]^s)} \\
&\le \inf_{\substack{A_{n,s} \\ \alpha_k \in \mathbb{L}_\infty}} \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \le 1}} \|f - A_{n,s}(f)\|_{\mathbb{L}_\infty([0,1]^s)} \\
&= e^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(n,s),
\end{aligned}$$

which proves (12) and implies (13). □

The notions of (U)EXP, κ-EC-WT, EC-WT, EC-PT, EC-SPT for $\mathbb{L}_2$-approximation in $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ are defined in the same way as for $\mathbb{L}_\infty$-approximation in $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ but with $e^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(n,s)$ replaced by $e^{\mathbb{L}_2-\mathrm{app},\Lambda}(n,s)$ and $n_{\mathrm{setting}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda}(\varepsilon,s)$ by $n^{\mathbb{L}_2-\mathrm{app},\Lambda}(\varepsilon,s)$.

We will be using the results for $\mathbb{L}_2$-approximation proved in [2].

Theorem 2 ([2, Theorem 1])**.** *Consider $\mathbb{L}_2$-approximation defined over the Korobov space $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ with weight sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying* (1)*. The following results hold for both classes Λ^{all} and Λ^{std}.*

- *EXP holds for all considered $\boldsymbol{a}$ and $\boldsymbol{b}$ with $p^*(s) = 1/B(s)$, where $B(s) = \sum_{j=1}^s b_j^{-1}$.*
- *UEXP holds iff $\boldsymbol{a}$ is an arbitrary sequence and $\boldsymbol{b}$ is such that $B = \sum_{j=1}^\infty b_j^{-1} < \infty$. If so then $p^* = 1/B$.*
- *EC-WT holds iff $\lim_{j\to\infty} a_j = \infty$.*
- *The notions of EC-PT and EC-SPT are equivalent, and hold iff*

$$B = \sum_{j=1}^\infty b_j^{-1} < \infty \quad \text{and} \quad \alpha^* = \liminf_{j\to\infty} \frac{\log a_j}{j} > 0.$$

If so then $\tau^* \in \left[B, B + \min\left(B, \frac{\log 3}{\alpha^*}\right)\right]$. In particular, if $\alpha^* = \infty$ then $\tau^* = B$.

□

For the class Λ^{all} we have the full characterization of $\mathbb{L}_2$- and $\mathbb{L}_\infty$-approximation in terms of the eigenpairs of the operator $W_s = \mathrm{EMB}_{s,2}^* \mathrm{EMB}_{s,2} : H(K_{s,\boldsymbol{a},\boldsymbol{b}}) \to H(K_{s,\boldsymbol{a},\boldsymbol{b}})$, which is given by

$$W_s f = \int_{[0,1]^s} f(\boldsymbol{t}) K_{s,\boldsymbol{a},\boldsymbol{b}}(\cdot, \boldsymbol{t}) \,\mathrm{d}\boldsymbol{t}.$$

For $\mathbb{L}_2$-approximation this result is standard and may be found for instance in [13] and [18], whereas for $\mathbb{L}_\infty$-approximation it was proved in [8, Theorem 4 in Section 3] (with $\rho \equiv 1$).

More precisely, for $\boldsymbol{h} \in \mathbb{Z}^s$, let a function $e_{\boldsymbol{h}}$ be defined by

$$e_{\boldsymbol{h}}(\boldsymbol{x}) = \exp(2\pi \mathtt{i}\, \boldsymbol{h} \cdot \boldsymbol{x})\, \omega_{\boldsymbol{h}}^{1/2} \qquad \text{for all} \qquad \boldsymbol{x} \in [0,1]^s.$$

Then $\{e_{\boldsymbol{h}}\}_{\boldsymbol{h}\in\mathbb{Z}^s}$ is a complete orthonormal basis of the Korobov space $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$. It is easily checked that the eigenpairs of W_s are $(\omega_{\boldsymbol{h}}, e_{\boldsymbol{h}})$, i.e.,

$$W_s e_{\boldsymbol{h}} = \omega_{\boldsymbol{h}} e_{\boldsymbol{h}} = \omega^{\sum_{j=1}^s a_j |h_j|^{b_j}}\, e_{\boldsymbol{h}} \qquad \text{for all} \qquad \boldsymbol{h} \in \mathbb{Z}^s,$$

see also [2, Section 5]. Let the ordered eigenvalues of W_s be $\{\lambda_{s,k}\}_{k\in\mathbb{N}}$ with

$$\lambda_{s,1} \ge \lambda_{s,2} \ge \lambda_{s,3} \ge \dots .$$

Obviously, $\{\lambda_{s,k}\}_{k\in\mathbb{N}} = \{\omega_{\boldsymbol{h}}\}_{\boldsymbol{h}\in\mathbb{Z}^s}$ and $\lambda_{s,1} = 1$. Then

$$\begin{aligned} e^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{all}}}(n,s) &= \lambda_{s,n+1}^{1/2} \\ e^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{all}}}(n,s) &= \left(\sum_{k=n+1}^{\infty} \lambda_{s,k} \right)^{1/2}. \end{aligned}$$

Let

$$\mathrm{CRI}_{\mathrm{abs}} = 1 \quad \text{and} \quad \mathrm{CRI}_{\mathrm{norm}} = e^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{all}}}(0,s) = \prod_{j=1}^{s} \left(1 + 2 \sum_{h=1}^{\infty} \omega^{a_j h^{b_j}} \right)^{1/2}.$$

Then for setting $\in \{\mathrm{abs}, \mathrm{norm}\}$ we have

$$n^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s) = \min \left\{ n \ :\ \lambda_{s,n+1} \le \varepsilon^2 \right\}, \tag{14}$$

$$n_{\mathrm{setting}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s) = \min \left\{ n \ :\ \sum_{k=n+1}^{\infty} \lambda_{s,k} \le \varepsilon^2\, \mathrm{CRI}_{\mathrm{setting}}^2 \right\}. \tag{15}$$

Furthermore the nth minimal errors are attained for both $\mathbb{L}_2$- and $\mathbb{L}_\infty$-approximation by the same algorithm

$$A_{n,s}(f) = \sum_{k=1}^{n} \langle f, \eta_{s,k} \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})}\ \eta_{s,k}, \tag{16}$$

where the $\eta_{s,k}$'s are the eigenfunctions $e_{\boldsymbol{h}}$ corresponding to the ordered eigenvalues $\lambda_{s,k}$. That is, $\eta_{s,k} = e_{\boldsymbol{h}(k)}$ and $\lambda_{s,k} = \omega_{\boldsymbol{h}(k)}$ for some $\boldsymbol{h}(k) \in \mathbb{Z}^s$. Note that for any $f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ we have

$$\begin{aligned} \langle f, \eta_{s,k} \rangle_{L_2([0,1]^s)} &= \langle \mathrm{EMB}_{s,2} f, \mathrm{EMB}_{s,2}\, \eta_{s,k} \rangle_{L_2([0,1]^s)} \\ &= \langle f, W_s \eta_{s,k} \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} = \lambda_{s,k}\ \langle f, \eta_{s,k} \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})}. \end{aligned}$$

Therefore, (16) can be equivalently rewritten as

$$A_{n,s}(f) = \sum_{k=1}^{n} \langle f, \eta_{s,k} \rangle_{L_2([0,1]^s)}\, \lambda_{s,k}^{-1}\, \eta_{s,k} = \sum_{k=1}^{n} \langle f, \widetilde{e}_k \rangle_{L_2([0,1]^s)}\ \widetilde{e}_k \tag{17}$$

where

$$\widetilde{e}_k(\boldsymbol{x}) = \frac{\eta_{s,k}(\boldsymbol{x})}{\sqrt{\lambda_{s,k}}} = \frac{e_{\boldsymbol{h}(k)}(\boldsymbol{x})}{\sqrt{\omega_{\boldsymbol{h}(k)}}} = \exp(2\pi\,\mathtt{i}\,\boldsymbol{h}(k)\cdot\boldsymbol{x}).$$

Clearly, $\widetilde{e}_k$'s are orthonormal in $\mathbb{L}_2([0,1]^s$ and $\|\widetilde{e}_k\|_{\mathbb{L}_\infty([0,1]^s)} = 1$ for all $k \in \mathbb{N}$.

We now find an estimate on the nth minimal error for $\mathbb{L}_\infty$-approximation and the class $\Lambda^{\rm std}$ in terms of the nth minimal errors for $\mathbb{L}_\infty$-approximation and the class $\Lambda^{\rm all}$, and for $\mathbb{L}_2$-approximation and the class $\Lambda^{\rm std}$.

Lemma 2. *We have*

$$e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm std}}(n,s) \ \le\ e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm all}}(n,s) \ +\ n\,e^{\mathbb{L}_2-{\rm app},\Lambda^{\rm std}}(n,s).$$

□

Proof. Consider a linear algorithm $B_{n,s}$ that uses n function values for $\mathbb{L}_2$-approximation,

$$B_{n,s}(f) = \sum_{j=1}^{n} \alpha_j\, f(\boldsymbol{x}_j) \qquad \text{for} \ \ f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}),$$

where $\alpha_j \in \mathbb{L}_2([0,1]^s)$ and $\boldsymbol{x}_j \in [0,1]^s$.

We now approximate the algorithm $A_{n,s}$ given by (17) by replacing f in the inner product of $\mathbb{L}_2([0,1]^s)$ by $B_{n,s}(f)$,

$$\widetilde{A}_{n,s}(f) = \sum_{k=1}^{n} \langle B_{n,s}(f), \widetilde{e}_k\rangle_{\mathbb{L}_2([0,1]^s)}\ \widetilde{e}_k = \sum_{j=1}^{n} f(\boldsymbol{x}_j) \left(\sum_{k=1}^{n} \langle \alpha_j, \widetilde{e}_k\rangle_{\mathbb{L}_2([0,1]^s)}\ \widetilde{e}_k \right).$$

This means that the algorithm $\widetilde{A}_{n,s}$ uses at most n function values. Furthermore,

$$A_{n,s}(f) - \widetilde{A}_{n,s}(f) = \sum_{k=1}^{n} \langle f - B_{n,s}(f), \widetilde{e}_k\rangle_{\mathbb{L}_2([0,1]^s)}\ \widetilde{e}_k,$$

which implies

$$\begin{aligned} \|A_{n,s}(f) - \widetilde{A}_{n,s}(f)\|_{\mathbb{L}_\infty([0,1]^s)} &\le\ n\,\|f - B_{n,s}(f)\|_{\mathbb{L}_2([0,1]^s)} \\ &\le\ n\,\|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})}\, e^{\mathbb{L}_2-{\rm app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), B_{n,s}). \end{aligned}$$

Hence,

$$\begin{aligned} \|f - \widetilde{A}_{n,s}\|_{\mathbb{L}_\infty([0,1]^s)} &\le\ \|f - A_{n,s}(f)\|_{\mathbb{L}_\infty([0,1]^s)} + \|A_{n,s}(f) - \widetilde{A}_{n,s}(f)\|_{\mathbb{L}_\infty([0,1]^s)} \\ &\le\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \left(e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm all}}(n,s) \,+\, n\,e^{\mathbb{L}_2-{\rm app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), B_{n,s}) \right). \end{aligned}$$

Choosing $B_{n,s}$ as an optimal algorithm for $\mathbb{L}_2$-approximation and the class $\Lambda^{\rm std}$ we obtain

$$e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm std}}(n,s) \,\le\, e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm all}}(n,s) \,+\, n\,e^{\mathbb{L}_2-{\rm app},\Lambda^{\rm std}}(n,s),$$

as claimed. □

Lemma 2 and known estimates on $\mathbb{L}_2$-approximation for the class $\Lambda^{\rm std}$ allow us to find an estimate on the nth minimal error of $\mathbb{L}_\infty$-approximation for the class $\Lambda \in \{\Lambda^{\rm all}, \Lambda^{\rm std}\}$ in terms of the eigenvalues $\lambda_{s,n}$.

Lemma 3. *Assume that for all $s \in \mathbb{N}$ there are positive numbers β_s and $M_s > 0$ such that*

$$\lambda_{s,n} \le \frac{M_s^2}{n^{2\,\beta_s}} \qquad \text{for all } \; n \in \mathbb{N}.$$

We assume for the class $\Lambda^{\rm all}$ that $\beta_s > \frac{1}{2}$, and for the class $\Lambda^{\rm std}$ that $\beta_s > \frac{3}{2}$. Then

$$\begin{aligned} e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm all}}(n,s) &\le \frac{M_s}{\sqrt{2\,\beta_s - 1}}\,\frac{1}{n^{\beta_s - 1/2}},\\ e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm std}}(n,s) &\le M_s\left(\frac{\sqrt{2}}{2} + C(\beta_s)\right)\frac{1}{n^{\beta_s-3/2}}, \end{aligned}$$

where $C(x) = 2^{2x(2x+1)+x-1/2}((2x+1)/(2x-1))^{1/2}(1+1/(2x))^x$. □

Proof. For the class $\Lambda^{\rm all}$, we easily have

$$\begin{aligned} \left[e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm all}}(n,s)\right]^2 &= \sum_{k=n+1}^{\infty} \lambda_{s,k} \le M_s^2 \sum_{k=n+1}^{\infty} \frac{1}{k^{2\,\beta_s}}\\ &\le M_s^2 \int_n^\infty \frac{{\rm d}x}{x^{2\,\beta_s}} = \frac{M_s^2}{2\,\beta_s - 1}\,\frac{1}{n^{2\,\beta_s - 1}}, \end{aligned}$$

as claimed.

For the class $\Lambda^{\rm std}$, we use [15, Theorem 26.15] which states that

$$e^{\mathbb{L}_2-{\rm app},\Lambda^{\rm std}}(n,s) \le \frac{M_s C(\beta_s)}{n^{\beta_s - 1/2}}.$$

From Lemma 2 we then have

$$e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm std}}(n,s) \le \frac{M_s}{\sqrt{2\,\beta_s-1}}\,\frac{1}{n^{\beta_s-1/2}} + \frac{M_s C(\beta_s)}{n^{\beta_s-3/2}} \le M_s\left(\frac{\sqrt{2}}{2} + C(\beta_s)\right)\frac{1}{n^{\beta_s-3/2}},$$

as claimed. □

6 Preliminaries for $\Lambda^{\rm std}$

Before we proceed to prove our main results, we state some preliminary observations that we need for $\mathbb{L}_\infty$-approximation using the information class $\Lambda^{\rm std}$.

We follow [19] in our arguments and present a particular choice of a linear approximation algorithm based on function evaluations that allows us to obtain error bounds. Given a set of points $\mathcal{P} = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$, and function evaluations, $\{f(\boldsymbol{x}_1), \dots, f(\boldsymbol{x}_n)\}$, we define a spline σ as

$$\sigma(f;\mathcal{P}) := \operatorname{argmin}\{\|g\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b}})} \;:\; g \in H(K_{s,\boldsymbol{a},\boldsymbol{b}}), g(\boldsymbol{x}_k) = f(\boldsymbol{x}_k), k = 1,2,\dots,n\}.$$

We would like to use $\sigma(f;\mathcal{P})$ for approximating $f \in H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ in the $\mathbb{L}_\infty$ norm. The first part of the analysis in [19] holds for reproducing kernels K of Hilbert spaces of 1-periodic functions, and it is required that the kernels K take the form

$$K(\boldsymbol{x},\boldsymbol{y}) = \widetilde{K}(\{\boldsymbol{x}-\boldsymbol{y}\}),$$

where $\{\cdot\}$ denotes the fractional part of a real number (defined component-wise). These assumptions are fulfilled for the kernels $K_{s,\boldsymbol{a},\boldsymbol{b}}$ considered here. Therefore the preliminaries outlined in [19] apply to our case as well, and we restrict ourselves to summarizing the most crucial facts from [19]. In fact, the paper [19] discusses approximation algorithms that use lattice points, but the theory also applies to the case where we consider approximation by s-dimensional grids $\mathcal{G}_{n,s}$ as in this paper. Such regular grids have already been studied in [2, 3, 5, 6]. We now recall their definition.

For $s \in \mathbb{N}$, a regular grid with mesh-sizes $m_1, \ldots, m_s \in \mathbb{N}$ is defined as the point set

$$\mathcal{G}_{n,s} = \{(k_1/m_1, \ldots, k_s/m_s) : \quad k_j = 0, 1, \ldots, m_j - 1 \ \text{ for all } j = 1, 2, \ldots, s\},$$

where $n = \prod_{j=1}^s m_j$ is the cardinality of $\mathcal{G}_{n,s}$. By $\mathcal{G}_{n,s}^\perp$ we denote the dual of $\mathcal{G}_{n,s}$, i.e.,

$$\mathcal{G}_{n,s}^\perp = \{\boldsymbol{l} = (l_1, \ldots, l_s) \in \mathbb{Z}^s : l_j \equiv 0 \ (\mathrm{mod}\, m_j) \ \text{ for all } \ j = 1, 2, \ldots, s\}.$$

For $\mathcal{G}_{n,s}$ with mesh-sizes $m_1, \ldots, m_s \in \mathbb{N}$, and cardinality $n = m_1 \cdots m_s$, we write the set $\mathbb{Z}^s$ as a direct sum of $\mathcal{G}_{n,s}^\perp$ and the set

$$\mathcal{V}_n = \mathbb{Z}^s \cap \prod_{j=1}^s \left(-\frac{m_j}{2}, \frac{m_j}{2}\right], \tag{18}$$

i.e.,

$$\mathbb{Z}^s = \mathcal{V}_n \oplus \mathcal{G}_{n,s}^\perp = \{\boldsymbol{v} + \boldsymbol{l} : \boldsymbol{v} \in \mathcal{V}_n \text{ and } \boldsymbol{l} \in \mathcal{G}_{n,s}^\perp\}.$$

Note that $\mathcal{V}_n$ has the property that any two distinct vectors in $\mathcal{V}_n$ differ by a vector that is not in the dual set $\mathcal{G}_{n,s}^\perp$, i.e.,

$$\boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}_n, \boldsymbol{v} \neq \boldsymbol{w} \ \Rightarrow \ \boldsymbol{v} - \boldsymbol{w} \notin \mathcal{G}_{n,s}^\perp \setminus \{\boldsymbol{0}\}.$$

Furthermore, $\boldsymbol{0} \in \mathcal{V}_n$ and

$$\omega_{\boldsymbol{v}}^{-1} \leq \omega_{\boldsymbol{v}+\boldsymbol{l}}^{-1} \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}_n \text{ and all } \boldsymbol{l} \in \mathcal{G}_{n,s}^\perp. \tag{19}$$

This follows from the fact that for $\boldsymbol{v} = (v_1, \ldots, v_s) \in \mathcal{V}_n$ and for $\boldsymbol{l} = (l_1, \ldots, l_s) \in \mathcal{G}_{n,s}^\perp$ we have $|v_j| \leq |v_j + l_j|$ for all $j = 1, \ldots, s$.

Given $\mathcal{G}_{n,s}$ with points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, it is known, see [19] and the references therein, that the spline $\sigma(f;\mathcal{P})$ can be expressed in terms of so-called cardinal functions, ϕ_k, $k = 1, 2, \ldots, n$, where each ϕ_k is a linear combination of the $K_{s,\boldsymbol{a},\boldsymbol{b}}(\cdot, \boldsymbol{x}_r)$. To be more precise,

$$\sigma(f;\mathcal{G}_{n,s})(\boldsymbol{x}) = \sum_{k=1}^n f(\boldsymbol{x}_k)\phi_k(\boldsymbol{x}),$$

$$\phi_k(\boldsymbol{x}) = \sum_{r=1}^n K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x}, \boldsymbol{x}_r)\xi_{r,k},$$

where the $\xi_{r,k}$ are given by a condition expressed by the Kronecker delta function δ,

$$\delta_{j,k} = \sum_{r=1}^{n} K_{s,\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x}_j, \boldsymbol{x}_r)\xi_{r,k}.$$

Going through analogous steps as in [19, Section 3.1], we arrive at an estimate similar to one formulated for lattice points in [19, Theorem 1],

$$\left[e^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}_{n,s})\right]^2 \leq 4 \sum_{\boldsymbol{h}\notin\mathcal{V}_n} \omega_{\boldsymbol{h}} = 4 \sum_{\boldsymbol{v}\in\mathcal{V}_n} \sum_{\boldsymbol{l}\in\mathcal{G}_{n,s}^{\perp}\setminus\{\boldsymbol{0}\}} \omega_{\boldsymbol{v}+\boldsymbol{l}}, \tag{20}$$

where here and in the following we just write $e^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}_{n,s})$ instead of $e^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \sigma(\cdot;\mathcal{G}_{n,s}))$.

It is easy to see that

$$|l|^b \leq 2^b \left(|v+l|^b + |v|^b\right)$$

for any $v, l \in \mathbb{Z}$ and any $b > 0$. From (19) we get for all $\boldsymbol{v} \in \mathcal{V}_n$ and all $\boldsymbol{l} \in \mathcal{G}_{n,s}^{\perp}$ that

$$\begin{aligned} \omega_{\boldsymbol{v}+\boldsymbol{l}} &= \omega^{\sum_{j=1}^s a_j|v_j+l_j|^{b_j}} \leq \omega^{\sum_{j=1}^s 2^{-b_j} a_j |l_j|^{b_j}} \omega^{-\sum_{j=1}^s a_j|v_j|^{b_j}} \\ &= \omega^{\sum_{j=1}^s 2^{-b_j} a_j |l_j|^{b_j}} \omega_{\boldsymbol{v}}^{-1} \leq \omega^{\sum_{j=1}^s 2^{-b_j} a_j |l_j|^{b_j}} \omega_{\boldsymbol{v}+\boldsymbol{l}}^{-1}. \end{aligned}$$

This implies

$$\omega_{\boldsymbol{v}+\boldsymbol{l}} \leq (\omega^{1/2})^{\sum_{j=1}^s 2^{-b_j} a_j |l_j|^{b_j}}.$$

Inserting this estimate into (20) we arrive at

$$\left[e^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}_{n,s})\right]^2 \leq 4 \sum_{\boldsymbol{v}\in\mathcal{V}_n} \sum_{\boldsymbol{l}\in\mathcal{G}_{n,s}^{\perp}\setminus\{\boldsymbol{0}\}} (\omega^{1/2})^{\sum_{j=1}^s 2^{-b_j} a_j |l_j|^{b_j}} = 4nF_n, \tag{21}$$

where

$$F_n = \sum_{\boldsymbol{l}\in\mathcal{G}_{n,s}^{\perp}\setminus\{\boldsymbol{0}\}} \overline{\omega}^{\sum_{j=1}^s 2^{-b_j} a_j |l_j|^{b_j}} = -1 + \prod_{j=1}^{s} \left(1 + 2\sum_{h=1}^{\infty} \overline{\omega}^{a_j 2^{-b_j}(m_j h)^{b_j}}\right), \tag{22}$$

and where we write $\overline{\omega} := \omega^{1/2}$.

7 (Uniform) exponential convergence

In this section, we prove Points 1 and 2 of Theorem 1 for EXP and UEXP.

Let us first consider the result for the class Λ^{std}. We now show how to choose a regular grid in the sense of Section 6 to obtain the desired result.

Let $\omega_1 \in (\overline{\omega}, 1)$. For $s \in \mathbb{N}$ and $\varepsilon \in (0,1)$ define

$$m = \max_{j=1,2,\ldots,s} \left\lceil \left(\frac{4^{b_j}}{a_j} \, \frac{\log\left(1 + \frac{RC_j 2s}{\log(1+\varepsilon^2/4)}\right)}{\log\, \omega_1^{-1}} \right)^{B(s)} \right\rceil,$$

where

$$C_j = \sup_{m\in\mathbb{N}} m^{1/s}(\overline{\omega}/\omega_1)^{m^{1/B(s)}a_* 4^{-b_j}} < \infty,$$

and

$$R = \max_{1\le j\le s} \sum_{h=1}^{\infty} \omega_1^{a_j 4^{-b_j}(h^{b_j}-1)} < \infty.$$

Let $\mathcal{G}^*_{n,s}$ be a regular grid with mesh-sizes $m_1, m_2, \ldots, m_s$ given by

$$m_j := \left\lfloor m^{1/(B(s)\cdot b_j)} \right\rfloor \qquad \text{for} \quad j = 1, 2, \ldots, s \quad \text{and} \quad n = \prod_{j=1}^{s} m_j.$$

We are now going to show that

$$e^{\mathbb{L}_\infty-\text{app},\Lambda^{\text{std}}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}^*_{n,s}) \le \varepsilon, \quad \text{and} \quad n = \mathcal{O}\left(\log^{B(s)}\left(1+\varepsilon^{-1}\right)\right) \tag{23}$$

with the factor in the $\mathcal{O}$ notation independent of ε^{-1} but dependent on s.

From (22) we have

$$F_n = -1 + \prod_{j=1}^{s}\left(1 + 2\sum_{h=1}^{\infty} \overline{\omega}^{a_j 2^{-b_j}(m_j h)^{b_j}}\right).$$

Since $\lfloor x \rfloor \ge x/2$ for all $x \ge 1$, we have

$$(m_j h)^{b_j} \ge (h/2)^{b_j}\, m^{1/B(s)} \qquad \text{for all} \qquad j = 1, 2, \ldots, s.$$

Hence,

$$F_n \le -1 + \prod_{j=1}^{s}\left(1 + 2\sum_{h=1}^{\infty} \overline{\omega}^{m^{1/B(s)}a_j 4^{-b_j}\, h^{b_j}}\right),$$

and similarly

$$nF_n \le -1 + \prod_{j=1}^{s}\left(1 + 2\, n^{1/s} \sum_{h=1}^{\infty} \overline{\omega}^{m^{1/B(s)}a_j 4^{-b_j}\, h^{b_j}}\right).$$

Note that

$$n = \prod_{j=1}^{s} m_j = \prod_{j=1}^{s} \left\lfloor m^{1/(B(s)\cdot b_j)} \right\rfloor \le m^{\frac{1}{B(s)}\sum_{j=1}^{s} 1/b_j} = m.$$

Now with $q := \overline{\omega}/\omega_1$ we have $q \in (0,1)$, and hence, for $h \ge 1$,

$$n^{1/s}\, q^{m^{1/B(s)}a_j 4^{-b_j}\, h^{b_j}} \le m^{1/s}\, q^{m^{1/B(s)}a_* 4^{-b_j}} \le \sup_{m\in\mathbb{N}} m^{1/s}\, q^{m^{1/B(s)}a_* 4^{-b_j}} = C_j.$$

Therefore,

$$nF_n \le -1 + \prod_{j=1}^{s}\left(1 + 2C_j \sum_{h=1}^{\infty} \omega_1^{m^{1/B(s)}a_j 4^{-b_j}\, h^{b_j}}\right).$$

We further estimate

$$\sum_{h=1}^{\infty} \omega_1^{m^{1/B(s)}a_j 4^{-b_j}\, h^{b_j}} \quad = \quad \omega_1^{m^{1/B(s)}a_j 4^{-b_j}} \sum_{h=1}^{\infty} \omega_1^{m^{1/B(s)}a_j 4^{-b_j}\,(h^{b_j}-1)} \tag{24}$$

$$\leq \omega_1^{m^{1/B(s)} a_j 4^{-b_j}} \sum_{h=1}^{\infty} \omega_1^{a_j 4^{-b_j} (h^{b_j} - 1)} \tag{25}$$

$$\leq \omega_1^{m^{1/B(s)} a_j 4^{-b_j}} R. \tag{26}$$

From the definition of m we have

$$\omega_1^{m^{1/B(s)} a_j 4^{-b_j}} R \leq \frac{\log(1+\varepsilon^2/4)}{C_j 2s} \qquad \text{for all} \qquad j = 1, 2, \ldots, s.$$

This proves

$$4nF_n \leq 4\left(-1 + \left(1 + \frac{\log(1+\varepsilon^2/4)}{s}\right)^s\right) \leq 4\left(-1 + \exp(\log(1+\varepsilon^2/4))\right) = \varepsilon^2. \tag{27}$$

Now, plugging this into (21) and taking the square root, we obtain

$$e^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}^*_{n,s}) \leq \varepsilon. \tag{28}$$

Hence the first point in (23) is shown, and it remains to verify that n is of the order stated in the proposition. We already noted above that $n \leq m$. However, as pointed out in [5],

$$m = \mathcal{O}\left(\log^{B(s)}\left(1+\varepsilon^{-1}\right)\right),$$

where the factor in the $\mathcal{O}$ notation is independent of ε^{-1} but dependent on s. This completes the proof of (23).

Now for the class Λ^{std}, we conclude from above that

$$n_{\mathrm{abs}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(\varepsilon, s) = \mathcal{O}\left(\log^{B(s)}\left(1+\varepsilon^{-1}\right)\right).$$

This implies that we indeed have EXP for Λ^{std} for all $\boldsymbol{a}$ and $\boldsymbol{b}$, with $p(s) = 1/B(s)$, and thus $p^*(s) \geq 1/B(s)$. On the other hand, according to Lemma 1 the rate of exponential convergence for $\mathbb{L}_\infty$-approximation cannot be larger than for $\mathbb{L}_2$-approximation which was shown to be $1/B(s)$ in [2, Theorem 1, Point 1], see Theorem 2. Thus, we have $p^*(s) = 1/B(s)$.

We turn to UEXP for the class Λ^{std}. Suppose that $\boldsymbol{b}$ is such that

$$B = \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty.$$

Then we can replace $B(s)$ by B in the above argument, and we obtain, in exactly the same way,

$$n_{\mathrm{abs}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(\varepsilon, s) = \mathcal{O}\left(\log^{B}\left(1+\varepsilon^{-1}\right)\right).$$

Hence, we have UEXP with $p^* \geq 1/B$. On the other hand, if we have UEXP for $\mathbb{L}_\infty$-approximation, this implies by Lemma 1 UEXP for $\mathbb{L}_2$-approximation, which in turn, again by the results in [2, Theorem 1, Point 2], see Theorem 2, implies that $B < \infty$ and that $p^* \leq 1/B$.

Regarding the class Λ^{all}, note that we can combine (7) and (12) to

$$e^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{all}}}(n,s) \leq e^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{all}}}(n,s) \leq e^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(n,s). \tag{29}$$

We remark that the conditions in Points 1 and 2 of Theorem 1 exactly match those in [2, Theorem 1, Points 1 and 2]. Hence we can use the results for the class Λ^{std} combined with the respective results in [2] to show EXP and UEXP for the class Λ^{all}. □

8 κ-EC-weak tractability

In this section we first prove Point 3 of Theorem 1. Then we consider the case of $\mathbb{L}_2$-approximation for $\kappa > 1$ since this case has not yet been studied.

For $\mathbb{L}_\infty$-approximation with $\kappa \ge 1$, we now prove that κ-EC-WT implies $\lim_j a_j = \infty$. Due to (9) it is enough to consider the class $\Lambda^{\rm all}$ and the normalized error criterion. Assume that $\alpha = \sup_j a_j < \infty$.

From (15) and the fact that $\lambda_{s,k} \le 1$ for all integer k, we have for $n = n_{\rm norm}^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm all}}(\varepsilon, s)$,

$$\sum_{k=1}^\infty \lambda_{s,k} - n \le \sum_{k=n+1}^\infty \lambda_{s,k} \le \varepsilon^2 \sum_{k=1}^\infty \lambda_{s,k}.$$

Hence,

$$n \ge (1-\varepsilon^2) \sum_{k=1}^\infty \lambda_{s,k} = (1-\varepsilon^2) \prod_{j=1}^s \left(1 + 2\sum_{h=1}^\infty \omega^{a_j h^{b_j}}\right) \ge (1-\varepsilon^2) \prod_{j=1}^s (1+2\omega^{a_j}). \quad (30)$$

This yields that

$$\frac{\log n}{s + [\log \varepsilon^{-1}]^\kappa} \ge \frac{\log(1-\varepsilon^2) + \sum_{j=1}^s \log(1+2\omega^{a_j})}{s + [\log \varepsilon^{-1}]^\kappa} \ge \frac{\log(1-\varepsilon^2) + s\log(1+2\omega^{\alpha})}{s + [\log \varepsilon^{-1}]^\kappa}.$$

Clearly, for a fixed $\varepsilon < 1$ and s tending to infinity, the right hand side of the last formula does not tend to zero. This contradicts κ-EC-WT.

We now show that $\lim_j a_j = \infty$ implies κ-EC-WT. Due to (9) it is enough to consider the class $\Lambda^{\rm std}$ and the absolute error criterion. For any positive η we have

$$\sum_{h=1}^\infty \omega^{\eta\, a_j h^{b_j}} \le \sum_{h=1}^\infty \omega^{\eta\, a_j h^{b_*}} \le \omega^{\eta\, a_j} \sum_{h=1}^\infty \omega^{\eta\, a_j (h^{b_*}-1)} \le D_\eta\, \omega^{\eta\, a_j},$$

where $D_\eta = \sum_{h=1}^\infty \omega^{\eta\, a_*(h^{b_*}-1)} < \infty$. Therefore for any integer n we can estimate

$$n\lambda_{s,n}^\eta \le \sum_{k=1}^\infty \lambda_{s,k}^\eta = \prod_{j=1}^s \left(1 + 2\sum_{h=1}^\infty \omega^{\eta\, a_j h^{b_j}}\right) \le \prod_{j=1}^s (1 + 2D_\eta\, \omega^{\eta\, a_j}).$$

Hence, for any positive η

$$\lambda_{s,n} \le \frac{1}{n^{1/\eta}} \prod_{j=1}^s (1+2D_\eta\, \omega^{\eta\, a_j})^{1/\eta} \qquad \text{for all} \quad s, n \in \mathbb{N}. \quad (31)$$

Thus the assumption of Lemma 3 holds with

$$\beta_s = 1/(2\eta) \quad \text{ and } \quad M_s^2 = \prod_{j=1}^s (1+2D_\eta\, \omega^{\eta\, a_j})^{1/\eta}.$$

For $\eta < \frac{1}{3}$ we have $\beta_s > \frac{3}{2}$ and

$$e^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm std}}(n,s) \le M_s \left(\frac{\sqrt{2}}{2} + C(1/(2\eta))\right) \frac{1}{n^{(1/\eta-3)/2}}.$$

Hence

$$e^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(n,s) \le \varepsilon$$

for

$$n \ge 1 + \left(\left(\frac{\sqrt{2}}{2} + C(1/(2\eta)) \right) \frac{1}{\varepsilon} \prod_{j=1}^{s} (1 + 2D_\eta\, \omega^{\eta a_j})^{1/(2\eta)} \right)^{2\eta/(1-3\eta)} .$$

and therefore we have

$$n_{\mathrm{abs}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s) \le 1 + \left(\left(\frac{\sqrt{2}}{2} + C(1/(2\eta)) \right) \frac{1}{\varepsilon} \prod_{j=1}^{s} (1 + 2D_\eta\, \omega^{\eta a_j})^{1/(2\eta)} \right)^{2\eta/(1-3\eta)} .$$

Hence, using $\log(1+x) \le x$ for all $x \ge 0$, we obtain

$$\log(n_{\mathrm{abs}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s)-1) \le \frac{2\eta}{1-3\eta} \left(\log\left(\frac{\sqrt{2}}{2} + C(1/(2\eta)) \right) + \log \varepsilon^{-1} \right) + \frac{2\,D_\eta}{1-3\eta} \sum_{j=1}^{s} \omega^{\eta\, a_j}.$$

Note that $\lim_j a_j = \infty$ implies that $\lim_j \omega^{\eta\, a_j} = 0$, and $\lim_s \sum_{j=1}^{s} \omega^{\eta\, a_j}/s = 0$. Hence

$$\limsup_{s+\log\varepsilon^{-1}\to\infty} \frac{\log n_{\mathrm{abs}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s)}{s + [\log\varepsilon^{-1}]^{\kappa}} \le \frac{2\eta}{1-3\eta}.$$

Since η can be arbitrarily small, this proves that

$$\lim_{s+\log\varepsilon^{-1}\to\infty} \frac{\log n_{\mathrm{abs}}^{\mathbb{L}_\infty-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s)}{s + [\log\varepsilon^{-1}]^{\kappa}} = 0,$$

and completes the proof of Point 3 of Theorem 1.

Point 4 of Theorem 1 easily follows by combining Point 2 and Point 3 with $\kappa = 1$. □

We now turn to κ-EC-WT for $\mathbb{L}_2$-approximation. The case $\kappa = 1$ corresponds to EC-WT and is covered in Theorem 2 and holds iff $\lim_j a_j = \infty$. We now assume that $\kappa > 1$ and show that the last condition on $\boldsymbol{a}$ is not needed for the class Λ^{all}. The case of $\kappa > 1$ for the class Λ^{std} is open.

Theorem 3. *Consider $\mathbb{L}_2$-approximation defined over the Korobov space $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$ with weight sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying* (1) *and the class Λ^{all}. Then for $\kappa > 1$*

$\mathbb{L}_2$-approximation is κ-EC-WT for all considered $\boldsymbol{a}$ and $\boldsymbol{b}$.

□

Proof. From (31) we conclude that $[e^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{all}}}(n,s)]^2 = \lambda_{s,n+1} \le \varepsilon^2$ for

$$n \ge \frac{(1 + 2D_\eta\, \omega^{\eta a_*})^s}{\varepsilon^{2\eta}}$$

and hence

$$n^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s) \le \frac{(1 + 2D_\eta\, \omega^{\eta a_*})^s}{\varepsilon^{2\eta}}.$$

Therefore for any positive η we have

$$\log n^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s) \le 2\eta \log \varepsilon^{-1} + 2s\, D_\eta \omega^{\eta a_*}.$$

Hence

$$\frac{\log n^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s)}{s+[\log \varepsilon^{-1}]^\kappa} \le \frac{2\eta \log \varepsilon^{-1}}{s+[\log \varepsilon^{-1}]^\kappa} + \frac{2s\, D_\eta\, \omega^{\eta\, a_*}}{s+[\log \varepsilon^{-1}]^\kappa} \le \frac{2\eta \log \varepsilon^{-1}}{s+[\log \varepsilon^{-1}]^\kappa} + 2\, D_\eta\, \omega^{\eta\, a_*}.$$

The first term of the last bound goes to zero as $s + \log \varepsilon^{-1}$ goes to infinity since $\kappa > 1$, whereas the second term is arbitrarily small for large η. Therefore

$$\lim_{s+\log \varepsilon^{-1}\to\infty} \frac{\log n^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s)}{s+[\log \varepsilon^{-1}]^\kappa} = 0.$$

This means that κ-EC-WT holds, for $\kappa > 1$, for all considered $\boldsymbol{a}$ and $\boldsymbol{b}$. □

9 EC-(strong) polynomial tractability

We now prove Points 5 and 6 of Theorem 1. For this we need the following proposition.

Proposition 1. *Assume that*

$$B := \sum_{j=1}^\infty \frac{1}{b_j} < \infty \quad \textit{and} \quad \alpha^* := \liminf_{j\to\infty} \frac{\log a_j}{j} > 0.$$

Let $\mathcal{G}^*_{n,s}$ be a regular grid with mesh-sizes $m_1, m_2, \ldots, m_s$ given by

$$m_j = 2 \left\lceil \left(\frac{\log \varepsilon^{-2}}{a_j^\beta \log \omega^{-1}} \right)^{1/b_j} \right\rceil - 1 \qquad \textit{for all} \qquad j = 1, 2, \ldots, s,$$

with $\beta \in (0,1)$.

Then for any $\eta \in (0, \min(a_*^{1-\beta}, 1))$ and any $\delta \in (0, \alpha^*)$ there exists a positive $\overline{C}_{\beta,\delta,\eta}$ such that

$$e^{\mathbb{L}_\infty-\mathrm{app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}), \mathcal{G}^*_{n,s}) \le \overline{C}_{\beta,\delta,\eta}\ \varepsilon^{\min(a_*^{1-\beta},1)-\eta}$$

and

$$n = \mathcal{O}\left(\left(1 + \log \varepsilon^{-1}\right)^{B + (\log 3)/(\beta\,\delta)} \right),$$

with the factor in the $\mathcal{O}$ notation independent of ε^{-1} and s, and dependent only on β and δ.

Proof. We first note that $m_j \ge 1$ and is always an odd number. Furthermore $m_j = 1$ iff $a_j \ge ((\log \varepsilon^{-2})/(\log \omega^{-1}))^{1/\beta}$. Since for all $\delta \in (0, \alpha^*)$ there exists an integer j^*_δ such that

$$a_j \ge \exp(\delta j) \quad \text{for all} \quad j \ge j^*_\delta,$$

we conclude that

$$j \ge j^*_{\beta,\delta} := \max\left(j^*_\delta, \frac{\log(((\log \varepsilon^{-2})/(\log \omega^{-1}))^{1/\beta})}{\delta} \right) \quad \text{implies} \quad m_j = 1.$$

From (20) we know that

$$\left[e^{\mathbb{L}_\infty-\mathrm{app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}),\mathcal{G}^*_{n,s})\right]^2 \le 4\sum_{\boldsymbol{v}\in\mathcal{V}_n}\sum_{\boldsymbol{l}\in\mathcal{G}^{*\perp}_{n,s}\setminus\{\boldsymbol{0}\}} \omega_{\boldsymbol{v}+\boldsymbol{l}}.$$

We now consider

$$\sum_{\boldsymbol{l}\in\mathcal{G}^{*\perp}_{n,s}\setminus\{\boldsymbol{0}\}} \omega_{\boldsymbol{v}+\boldsymbol{l}} = \sum_{\emptyset\ne\mathfrak{u}\subseteq\{1,\dots,s\}}\prod_{j\in\mathfrak{u}}\left(\sum_{h_j\in\mathbb{Z}\setminus\{0\}}\omega^{a_j|v_j+m_jh_j|^{b_j}}\right)\prod_{j\notin\mathfrak{u}}\omega^{a_j|v_j|^{b_j}},$$

where we separated the cases for $h_j\in\mathbb{Z}\setminus\{0\}$ and $h_j=0$. We bound the second product by one such that

$$\sum_{\boldsymbol{l}\in\mathcal{G}^{*\perp}_{n,s}\setminus\{\boldsymbol{0}\}} \omega_{\boldsymbol{v}+\boldsymbol{l}} \le \sum_{\emptyset\ne\mathfrak{u}\subseteq\{1,\dots,s\}}\prod_{j\in\mathfrak{u}}\left(\sum_{h\in\mathbb{Z}\setminus\{0\}}\omega^{a_j|v_j+m_jh|^{b_j}}\right).$$

Note that for $\boldsymbol{v}\in\mathcal{V}_n$ we have from (18) that $|v_j|<(m_j+1)/2$ for $j=1,2,\dots,s$. In particular, if $m_j=1$ then $v_j=0$ and

$$\sum_{h\in\mathbb{Z}\setminus\{0\}}\omega^{a_j|v_j+m_jh|^{b_j}} = 2\sum_{h=1}^\infty\omega^{a_jh^{b_j}} \le 2\sum_{h=1}^\infty\omega^{a_jh^{b_*}} = 2\omega^{a_j}\sum_{h=1}^\infty\omega^{a_j(h^{b_*}-1)} \le 2\omega^{a_j}D, \tag{32}$$

where $D:=D_1=\sum_{h=1}^\infty\omega^{a_*(h^{b_*}-1)}$.

Let $m_j\ge 3$. Since $|v_j|<(m_j+1)/2$, we conclude that $|v_j|\le(m_j+1)/2-1=(m_j-1)/2$, and $h\ne0$ implies

$$|v_j+m_jh|\ge m_j|h|-|v_j|\ge\frac{m_j+1}{2}|h|.$$

Therefore

$$\begin{aligned}\sum_{h\in\mathbb{Z}\setminus\{0\}}\omega^{a_j|v_j+m_jh|^{b_j}} &\le 2\sum_{h=1}^\infty\omega^{a_j[(m_j+1)/2]^{b_j}h^{b_j}}\\ &= 2\omega^{a_j[(m_j+1)/2]^{b_j}}\sum_{h=1}^\infty\omega^{a_j[(m_j+1)/2]^{b_j}(h^{b_j}-1)}\\ &\le 2\omega^{a_j[(m_j+1)/2]^{b_j}}\sum_{h=1}^\infty\omega^{a_*(h^{b_*}-1)}\\ &= 2\omega^{a_j[(m_j+1)/2]^{b_j}}D. \end{aligned}\tag{33}$$

The inequalities (32) and (33) can be combined as

$$\beta_j:=\sum_{h\in\mathbb{Z}\setminus\{0\}}\omega^{a_j|v_j+m_jh|^{b_j}}\le 2\omega^{a_j[(m_j+1)/2]^{b_j}}D.$$

Note that

$$\sum_{\emptyset\ne\mathfrak{u}\subseteq\{1,\dots,s\}}\prod_{j\in\mathfrak{u}}\left(\sum_{h\in\mathbb{Z}\setminus\{0\}}\omega^{a_j|v_j+m_jh|^{b_j}}\right) = -1+\sum_{\mathfrak{u}\subseteq\{1,\dots,s\}}\prod_{j\in\mathfrak{u}}\beta_j = -1+\prod_{j=1}^s(1+\beta_j).$$

Consequently,

$$\begin{aligned}\left[e^{\mathbb{L}_\infty\text{-app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}),\mathcal{G}^*_{n,s})\right]^2 &\le 4\sum_{\boldsymbol{v}\in\mathcal{V}_n}\left[-1+\prod_{j=1}^s\left(1+2\,\omega^{\,a_j[(m_j+1)/2]^{b_j}}D\right)\right]\\ &= 4n\left[-1+\prod_{j=1}^s\left(1+2\,\omega^{\,a_j[(m_j+1)/2]^{b_j}}D\right)\right].\end{aligned}$$

Using $\log(1+x)\le x$ we obtain

$$\log\left[\prod_{j=1}^s\left(1+2\,\omega^{\,a_j[(m_j+1)/2]^{b_j}}D\right)\right]\le 2D\sum_{j=1}^s\omega^{\,a_j[(m_j+1)/2]^{b_j}}.$$

From the definition of m_j we have $a_j[(m_j+1)/2]^{b_j}\ge a_j^{1-\beta}\,(\log\varepsilon^{-2})/\log\omega^{-1}$. Therefore,

$$\omega^{a_j[(m_j+1)/2]^{b_j}}\le\omega^{a_j^{1-\beta}\,(\log\varepsilon^{-2})/\log\omega^{-1}}=\varepsilon^{2\,a_j^{1-\beta}}.$$

Since $a_j\ge a_*$ for $j\le j^*_{\beta,\delta}-1$ and $a_j\ge\exp(\delta j)$ for $j\ge j^*_{\beta,\delta}$ we obtain

$$\begin{aligned}\gamma &:= 2D\sum_{j=1}^s\omega^{\,a_j[(m_j+1)/2)]^{b_j}}\le 2D\left((j^*_{\beta,\delta}-1)\varepsilon^{2a_*^{1-\beta}}+\varepsilon^2\sum_{j=j^*_{\beta,\delta}}^\infty\varepsilon^{2[\exp((1-\beta)\delta j)-1]}\right)\\ &\le 2D\varepsilon^{2\min(a_*^{1-\beta},1)}\left(j^*_{\beta,\delta}-1+\sum_{j=j^*_{\beta,\delta}}^\infty\varepsilon^{2[\exp((1-\beta)\delta j)-1]}\right).\end{aligned}$$

Without loss of generality, we now choose ε such that $\varepsilon^{-2}\ge 2$. Then,

$$2D\left(j^*_{\beta,\delta}-1+\sum_{j=j^*_{\beta,\delta}}^\infty\varepsilon^{2[\exp((1-\beta)\delta j)-1]}\right)\le C_{\beta,\delta},$$

where

$$C_{\beta,\delta}:=2D\left(j^*_{\beta,\delta}-1+\sum_{j=j^*_{\beta,\delta}}^\infty\left(\frac{1}{2}\right)^{\exp((1-\beta)\delta j)-1}\right)<\infty.$$

Hence we obtain $\gamma\le C_{\beta,\delta}\varepsilon^{2\min(a_*^{1-\beta},1)}$, and by choosing, without loss of generality, $\varepsilon^{-2\min(a_*^{1-\beta},1)}\ge C_{\beta,\delta}$, we have $\gamma\le 1$.

Using convexity we easily check that $-1+\exp(\gamma)\le(\mathrm{e}-1)\gamma$ for all $\gamma\in[0,1]$. Thus for $\varepsilon^{-2\min(a_*^{1-\beta},1)}\ge C_{\beta,\delta}$ we obtain

$$\begin{aligned}-1+\prod_{j=1}^s\left(1+2\,\omega^{\,a_j[(m_j+1)/2]^{b_j}}D\right) &\le -1+\exp\left(2D\sum_{j=1}^s\omega^{\,a_j[(m_j+1)/2]^{b_j}}\right)\\ &= -1+\exp(\gamma)\le(\mathrm{e}-1)\gamma\\ &\le C_{\beta,\delta}\,(\mathrm{e}-1)\varepsilon^{2\min(a_*^{1-\beta},1)}\end{aligned}$$

and hence

$$e^{\mathbb{L}_\infty-\mathrm{app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}),\mathcal{G}^*_{n,s}) \le 2\sqrt{nC_{\beta,\delta}\,(\mathrm{e}-1)}\;\varepsilon^{\min(a_*^{1-\beta},1)}. \qquad (34)$$

We now estimate the number n of function values used by the algorithm. We have

$$n=\prod_{j=1}^{s} m_j=\prod_{j=1}^{\min(s,j^*_{\beta,\delta})} m_j \le \prod_{j=1}^{j^*_{\beta,\delta}}\left(1+2\left(\frac{\log\,\varepsilon^{-2}}{a_j^{\beta}\,\log\,\omega^{-1}}\right)^{1/b_j}\right).$$

We bound $j^*_{\beta,\delta}$ by the sum of the two terms defining it, and obtain

$$\begin{aligned} n &\le 3^{j^*_{\beta,\delta}} a_*^{-B\beta}\left(\frac{\log\,\varepsilon^{-2}}{\log\,\omega^{-1}}\right)^{B} \le 3^{j^*_{\delta}} a_*^{-B\beta}\left(\frac{\log\,\varepsilon^{-2}}{\log\,\omega^{-1}}\right)^{B+(\log 3)/(\beta\,\delta)} \\ &= \mathcal{O}\left(\left(1+\log\,\varepsilon^{-1}\right)^{B+(\log 3)/(\beta\,\delta)}\right). \end{aligned}$$

Inserting this into (34) we obtain for any $\eta>0$ that

$$e^{\mathbb{L}_\infty-\mathrm{app}}(H(K_{s,\boldsymbol{a},\boldsymbol{b}}),\mathcal{G}^*_{n,s}) \le \overline{C}_{\beta,\delta,\eta}\;\varepsilon^{\min(1,a_*^{1-\beta})-\eta},$$

where the positive quantity $\overline{C}_{\beta,\delta,\eta}$ depends on β,δ and η, but not on ε^{-1} and s. This completes the proof of the proposition. □

We are ready to prove Points 5 and 6 of Theorem 1. We consider four cases depending on the information class and the error criterion.

- Case 1: Λ^{std} and the absolute error criterion.

  We already showed that EC-PT implies EC-PT + EXP and EC-PT +UEXP. Therefore the chain of implications from EC-SPT+UEXP to EC-PT is trivial.

  Hence, it is enough to show that EC-PT implies EC-SPT+UEXP. Note that EC-PT for $\mathbb{L}_\infty$-approximation implies by Lemma 1 EC-PT for $\mathbb{L}_2$-approximation which in turn by [2, Theorem 1, Point 5], see also Theorem 2, implies EC-SPT+UEXP for $\mathbb{L}_2$-approximation. This, however, by [2, Theorem 1, Point 6] implies that $B<\infty$ and $\alpha^*>0$, where α^* is defined as in (11). We will show below that these conditions on $\boldsymbol{a}$ and $\boldsymbol{b}$ imply EC-SPT+UEXP for $\mathbb{L}_\infty$-approximation. This ends the proof of Point 5 for this case.

  We now prove Point 6. The necessity of the conditions for EC-SPT+UEXP on $\boldsymbol{a}$ and $\boldsymbol{b}$ for $\mathbb{L}_\infty$-approximation and the class Λ^{std} follows from the same conditions for $\mathbb{L}_2$-approximation shown in [2, Theorem 1, Point 6], and the fact that the information complexity for the $\mathbb{L}_\infty$-case cannot be smaller than for the $\mathbb{L}_2$-case.

  The sufficiency of the conditions is shown by the use of Proposition 1, under the assumption of (11), which states that

  $$n_{\mathrm{abs}}^{L_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(\overline{C}_{\beta,\delta,\eta}\,\varepsilon^{\min(a_*^{1-\beta},1)-\eta},s)=\mathcal{O}\left(\left(1+\log\,\varepsilon^{-1}\right)\right)^{B+(\log\,3)/(\beta\delta)}.$$

  By replacing $\overline{C}_{\beta,\delta,\eta}\,\varepsilon^{\min(a_*^{1-\beta},1)-\eta}$ by ε we obtain

  $$n_{\mathrm{abs}}^{L_\infty-\mathrm{app},\Lambda^{\mathrm{std}}}(\varepsilon,s) \;=\; \mathcal{O}\left(1+\log\,[\overline{C}_{\beta,\delta,\eta}\,\varepsilon^{\min(a_*^{1-\beta},1)-\eta}]^{-1}\right)^{B+(\log\,3)/(\beta\delta)}$$

$$= \mathcal{O}\left(1+\log\,\varepsilon^{-1}\right)^{B+(\log\,3)/(\beta\delta)}$$

with the factor in the $\mathcal{O}$ notation independent of ε^{-1} and s. This proves EC-SPT+UEXP with exponent

$$\tau = B + \frac{\log 3}{\beta\,\delta}.$$

Since β can be arbitrarily close to one, and δ can be arbitrarily close to α^*, the exponent τ^* of EC-SPT is at most

$$B + \frac{\log 3}{\alpha^*},$$

where for $\alpha^* = \infty$ we have $\frac{\log 3}{\alpha^*} = 0$. This completes the proof of Point 6 for $\Lambda^{\rm std}$ and the absolute error criterion.

- Case 2: $\Lambda^{\rm std}$ and the normalized error criterion.

  To prove Point 5, it is clear that EC-SPT+UEXP implies EC-PT. Let us now assume we have EC-PT. Then we also have EC-PT for $\Lambda^{\rm all}$ and the normalized error criterion. This, by what we will show below, implies (11). As we know, (11) implies EC-SPT+UEXP for $\Lambda^{\rm std}$ and the absolute error criterion. However, by (9), the latter implies EC-SPT+UEXP for $\Lambda^{\rm std}$ and the normalized error criterion.

  To prove Point 6, the sufficiency of the conditions follows from the corresponding results for $\Lambda^{\rm std}$ and the absolute error criterion. The necessary conditions for $\Lambda^{\rm std}$ follow from the necessary conditions for $\Lambda^{\rm all}$ and the normalized error criterion that we will prove below.

- Case 3: $\Lambda^{\rm all}$ and the absolute error criterion.

  Let us again start with Point 5. As before, it is enough to show that EC-PT implies EC-SPT+UEXP. EC-PT for $\mathbb{L}_\infty$-approximation and $\Lambda^{\rm all}$ implies EC-PT for $\mathbb{L}_2$-approximation for $\Lambda^{\rm all}$. Then it follows from [2, Theorem 1, Points 5 and 6] that (11) holds. This condition, however, implies EC-SPT+UEXP for $\Lambda^{\rm std}$ and the absolute error criterion, and hence also EC-SPT+UEXP for $\Lambda^{\rm all}$ and the absolute error criterion. Point 5 is therefore shown.

  For Point 6, the sufficient conditions for EC-SPT+UEXP follow from (8), and from the results for $\Lambda^{\rm std}$ and the absolute error criterion. On the other hand, the necessary conditions for EC-SPT+UEXP follow from (13), and from the results for $\mathbb{L}_2$-approximation in [2, Theorem 1, Point 6].

- Case 4: $\Lambda^{\rm all}$ and the normalized error criterion.

  Let us start with Point 5. Again it is obvious that EC-SPT+UEXP implies EC-PT. Conversely, assume now that we have EC-PT. Then by (30), we obtain for $n = n_{\rm norm}^{\mathbb{L}_\infty-{\rm app},\Lambda^{\rm all}}(\varepsilon,s)$,

$$n \ge (1-\varepsilon^2)\prod_{j=1}^{s}(1+2\omega^{a_j}).$$

Since we assumed EC-PT, this means that $\prod_{j=1}^{s}(1+2\omega^{a_j})$ may at most depend polynomially on s. However, due to results from [17], this can only happen if

$$\limsup_{s\to\infty}\sum_{j=1}^{s}\omega^{a_j}/\log s<\infty. \tag{35}$$

So, let us assume that (35) is fulfilled.

Next, consider the square of the initial error,

$$\begin{aligned}\left[e^{\mathbb{L}_\infty-\mathrm{app}}(0,s)\right]^2 &= \prod_{j=1}^{s}\left(1+2\sum_{h=1}^{\infty}\omega^{a_j h^{b_j}}\right)\le\prod_{j=1}^{s}\left(1+2\sum_{h=1}^{\infty}\omega^{a_j h^{b_*}}\right)\\ &= \prod_{j=1}^{s}\left(1+2\omega^{a_j}\sum_{h=1}^{\infty}\omega^{a_j(h^{b_*}-1)}\right)\le\prod_{j=1}^{s}\left(1+2\omega^{a_j}\sum_{h=1}^{\infty}\omega^{a_*(h^{b_*}-1)}\right)\\ &= \prod_{j=1}^{s}\left(1+\omega^{a_j}A\right),\end{aligned}$$

where $A:=2D_1=2\sum_{h=1}^{\infty}\omega^{a_*(h^{b_*}-1)}<\infty$. Due to (35) we see that $e^{\mathbb{L}_\infty-\mathrm{app}}(0,s)$ is bounded by an expression that depends at most polynomially on s. Hence it follows that the conditions for EC-PT regarding the normalized and the absolute error criteria are equivalent. For the absolute error criterion, we already know that EC-PT implies (11). This implies EC-SPT+UEXP due to Point 6 that we show below.

Let us come to Point 6. Suppose that we have EC-SPT+UEXP. This implies EC-PT, which, by the previous argument implies (11).

Suppose now that (11) holds. Then we know from above that EC-SPT+UEXP for Λ^{all} and the absolute error criterion holds. This implies EC-SPT+UEXP for Λ^{all} and the normalized error criterion. □

10 Comparison of $\mathbb{L}_\infty$- and $\mathbb{L}_2$-approximation

We briefly compare the results for $\mathbb{L}_\infty$- and $\mathbb{L}_2$-approximation. As before,

$$B=\sum_{j=1}^{\infty}\frac{1}{b_j}\quad\text{and}\quad\alpha^*=\liminf_{j\to\infty}\frac{\log a_j}{j}.$$

Unless noted otherwise, the conditions in Table 1 are valid for Λ^{all} and Λ^{std} and, in the $\mathbb{L}_\infty$-case, for both error criteria.

We see that the only difference between $\mathbb{L}_\infty$- and $\mathbb{L}_2$-approximation is for the property κ-EC-WT for $\kappa>1$ for the information class Λ^{all}. The condition for κ-EC-WT for $\kappa>1$ for $\mathbb{L}_2$-approximation and the information class Λ^{std} remains an open question.

Property	conditions ($\mathbb{L}_\infty$)	conditions ($\mathbb{L}_2$)
EXP	for all considered $\boldsymbol{a}$ and $\boldsymbol{b}$	for all considered $\boldsymbol{a}$ and $\boldsymbol{b}$
UEXP	iff $\boldsymbol{b}$ such that $B < \infty$	iff $\boldsymbol{b}$ such that $B < \infty$
κ-EC-WT, $\kappa > 1$ for $\Lambda^{\rm all}$	iff $\lim_j a_j = \infty$	for all considered $\boldsymbol{a}$ and $\boldsymbol{b}$
κ-EC-WT, $\kappa > 1$ for $\Lambda^{\rm std}$	iff $\lim_j a_j = \infty$	open
EC-WT	iff $\lim_j a_j = \infty$	iff $\lim_j a_j = \infty$
EC-PT	iff EC-SPT	iff EC-SPT
EC-SPT	iff $B < \infty$ and $\alpha^* > 0$	iff $B < \infty$ and $\alpha^* > 0$

Table 1: Comparison of results for $\mathbb{L}_\infty$- and $\mathbb{L}_2$-approximation

11 Remarks on $\mathbb{L}_p$-approximation

Let us, finally, briefly comment on the case of $\mathbb{L}_p$-approximation for $p \in [2, \infty]$. Let us consider $\mathbb{L}_p$-approximation of functions in $H(K_{s,\boldsymbol{a},\boldsymbol{b}})$, and the absolute error criterion. Let $e^{\mathbb{L}_p-{\rm app},\Lambda}(n,s)$ denote the nth minimal worst case error, and let $n_{\rm abs}^{\mathbb{L}_p-{\rm app},\Lambda}(\varepsilon,s)$ be the information complexity of this problem.

Then, similarly to the proof of Lemma 1, we see that

$$e^{\mathbb{L}_2-{\rm app},\Lambda}(n,s) \le e^{\mathbb{L}_p-{\rm app},\Lambda}(n,s) \le e^{\mathbb{L}_\infty-{\rm app},\Lambda}(n,s) \quad \text{for all} \ \ n,s \in \mathbb{N},$$

and

$$n^{\mathbb{L}_2-{\rm app},\Lambda}(\varepsilon,s) \le n_{\rm abs}^{\mathbb{L}_p-{\rm app},\Lambda}(\varepsilon,s) \le n_{\rm abs}^{\mathbb{L}_\infty-{\rm app},\Lambda}(\varepsilon,s) \quad \text{for all} \ \ \varepsilon \in (0,1), \ \ s \in \mathbb{N}.$$

Hence, we can conclude that for all situations mentioned in Table 1, except for κ-EC-WT with $\kappa > 1$, the results for $\mathbb{L}_p$-approximation and the absolute error criterion are the same as those for $\mathbb{L}_2$-approximation and $\mathbb{L}_\infty$-approximation. Whether a similar observation is also true for the normalized error criterion and for $p \in [1, 2)$ remain an open question.

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