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Dedicated to A. J. Guttmann, for his 70th birthday.

Abstract. We previously reported on a recursive method to generate the expansion of the lattice Green function of the $d$-dimensional face-centred cubic lattice (fcc). The method was used to generate many coefficients for $d = 7$ and the corresponding linear differential equation has been obtained. In this paper, we show the strength and the limit of the method by producing the series and the corresponding linear differential equations for $d = 8, 9, 10, 11, 12$. The differential Galois groups of these linear differential equations are shown to be symplectic for $d = 8, 10, 12$ and orthogonal for $d = 9, 11$. The recursion relation naturally provides a 2-dimensional array $T_d(n,j)$ where only the coefficients $T_d(n,0)$ correspond to the coefficients of the lattice Green function of the $d$-dimensional fcc. The coefficients $T_d(n,j)$ are associated to $D$-finite bivariate series annihilated by linear partial differential equations that we analyze.

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1. Introduction

The lattice Green function (LGF) of the $d$-dimensional face-centred cubic (fcc) lattice is given by a $d$-fold integral whose expansion around the origin is hard to obtain as the dimension goes higher [1, 2, 3, 4]. Only for $d = 3$ a closed form is known [5]. But since the integrand is of a very simple form—a rational function, after an appropriate variable transform—it follows from the theory of holonomic functions [6, 7] that those integrals are $D$-finite or holonomic, i.e. each of them satisfies a linear ordinary differential equation (ODE) with polynomial coefficients. For $d = 4$ the corresponding linear ODE was obtained in [8], for $d = 5$ in [2], and for $d = 6$ in [4], by different methods. In a previous paper [9] we forwarded a recursive method that was efficient
enough to allow us generate many series coefficients for $d = 7$ necessary to obtain the linear ODE. Since the recursion parameter in the method is the dimension $d$, we have obtained many short series for $d$ as high as 45. From these data and the Landau equations method [10] on the integrals, we inferred many properties that we conjecture to be common to all the linear ODEs of $d$-dimensional fcc lattices. The order-eleven linear differential operator, corresponding to the linear differential equation, we have obtained [9] for the lattice Green function of the 7-dimensional face-centred cubic (fcc) has been found to verify a property recently forwarded [11]. This property is a canonical decomposition of irreducible linear differential operators with symplectic or orthogonal differential Galois groups and corresponds to the occurrence of a homomorphism of the operator and its adjoint. This property has been seen to occur [12, 13, 14] for many linear differential operators that emerge in lattice statistical physics and enumerative combinatorics.

In this paper, we show the strength and the limit of the method. With some technical improvements in the computations, we show how much high in dimension $d$ we can go in generating sufficiently many terms of the LGF series in order to obtain the corresponding linear ODE. We find that the conjectures (especially on the singularities) given in [9] are all verified. We also find that the canonical decompositions of the operators follow the scheme given in [11].

Furthermore, the recursion relation gives a 2-dimensional array $T_d(n, j)$ where only the coefficients $T_d(n, 0)$ correspond to the coefficients of the lattice Green function of the $d$-dimensional fcc. We give the integrals whose expansion gives bivariate series with coefficients $T_d(n, j)$, and address the D-finite systems that annihilate these bivariate series.

The paper is organized as follows. Recalls are given in Section 2 where improvements of the method and some computational details are also given. Section 3 deals with our results on the differential equations annihilating the LGF of the $d$-dimensional fcc lattice for $d = 8, 9, 10, 11, 12$. The orders and singularities of all these linear ODEs are seen in agreement with our conjectures and computed Landau singularities in [9]. In Section 4, we show that the differential Galois groups of the operators are symplectic for $d = 8, d = 10, d = 12$ and orthogonal for $d = 9, d = 11$. We give for the operators corresponding to $d = 8$ and $d = 9$ the canonical decomposition that should [11] occur for the operators with symplectic or orthogonal differential Galois groups. In Section 5 and the following sections, we address the multidimensional integral whose expansion generates the double array $T_d(n, j)$ and produce the linear differential equations that annihilate the corresponding bivariate series.

2. LGF generation of series

2.1. Recalls on the recursive method

The lattice Green function (LGF) of the $d$-dimensional face-centred cubic (fcc) lattice reads

\[
LGF_d(x) = \frac{1}{\pi^d} \int_0^\pi \cdots \int_0^\pi \frac{dk_1 \cdots dk_d}{1 - x \cdot \lambda_d},
\]  

(1)
where $\lambda_d$, called the structure function of the lattice, is given by:

$$\lambda_d = \left(\frac{d}{2}\right)^{-1} \cdot \sum_{i=1}^{d} \sum_{j=i+1}^{d} \cos(k_i) \cdot \cos(k_j). \tag{2}$$

The expansion around the origin of $LGF_d(x)$ is given by

$$LGF_d(x) = \sum_{n=0}^{\infty} x^n \cdot T_d(n,0), \quad z = \frac{x}{4} \cdot \left(\frac{d}{2}\right)^{-1}, \tag{3}$$

where the array $T_d(n,j)$ is obtained with the recursive relation

$$T_d(n,j) = \sum_{p=0}^{n} \sum_{q=q_1}^{q_2} \binom{n}{p} \binom{2j}{2q+p-n} \binom{2n+2j-2p-2q}{n+j-p-q} \cdot T_{d-1}(p,q), \tag{4}$$

$$q_1 = \lfloor(n-p+1)/2\rfloor, \quad q_2 = \lfloor(n+p+2j)/2\rfloor, \tag{5}$$

where $[x]$ is the integer part of $x$. To start the recursion, one needs:

$$T_2(n,j) = \sum_{p=p_1}^{p_2} \binom{2j}{2p} \binom{2n+2j-2p}{n+j-p}, \tag{6}$$

$$p_1 = \lfloor(n+1)/2\rfloor, \quad p_2 = \lfloor(n+2j)/2\rfloor. \tag{7}$$

### 2.2. Computational details

The shape of the recurrence (4) suggests to start with the two-dimensional array $T_2(n,j)$, then compute $T_3(n,j)$, and so on. Once $T_3(n,j)$ is completed, the data of $T_2(n,j)$ is not any more needed since the recurrence (4) is of order 1 with respect to $d$. Also from the recurrence it is easy to see that for computing $T_d(n,0)$, $0 \leq n \leq N$, the desired coefficients of the Taylor series, one needs the values of $T_{d-1}(n,j)$ for $0 \leq n \leq N$ and $0 \leq j \leq \lfloor(N-n)/2\rfloor$ (the same range applies to the arrays $T_{d-2}, \ldots, T_2$). The advantage of such an implementation is that it stores only $O(N^2)$ elements, which are integers. The disadvantage is that one has to fix $N$ at the very beginning, but the number of terms needed for constructing the linear differential operator is not known in advance.

If one looks at the recurrence (4) more closely, one discovers the remarkable fact that neither its coefficients, i.e. the product of the three binomials, nor its support, i.e. the summation bounds, depend on the parameter $d$. Hence, in the previous approach, the coefficients are the same in each step, but they are recomputed in each iteration $d = 3, 4, \ldots$, which is clearly a waste of computational resources. In principle, we could collect all the coefficients in a big matrix $A$ that maps the array $T_{d-1}$ to $T_d$, so that $T_d = A^{d-2} \cdot T_2$. For this purpose the two-dimensional arrays $T_d(n,j)$, $0 \leq n \leq N$, $0 \leq j \leq \lfloor(N-n)/2\rfloor$, have to be represented as vectors of dimension $[(N/2+1)^2]$. So the whole computation then boils down to compute the power of some matrix, and then multiply it to the vector that corresponds to $T_2$. The problem is that the matrix $A$ has dimension $[(N/2+1)^2] \times [(N/2+1)^2]$, which already for the 8-dimensional fcc lattice (where we need at least $N = 704$ Taylor coefficients, see Table 1) means a $124609 \times 124609$ square matrix. Of course, $A$ is not dense. A simple calculation reveals that it has $(N+2)(N+4)(N^2+4N+12)/96$ nonzero entries if $N$ is even, and $(N+1)(N+3)(N^2+6N+17)/96$ nonzero entries when $N$ is odd. It follows that $A$ has sparsity $1/6$. Nevertheless it would require a considerable and impractical amount of memory to store the full matrix: for $d = 8$ it has about 2.6
coefficients: one by adding and deleting a few entries, and apply the simple recurrences for binomial improvements, some of which we want to mention briefly here. For example, one the vector does not need to compute \( a \) naive approach (by a factor of approximately \( T \)). The drawback of this approach is that one has to keep the whole three-dimensional array \( A \) single column vector. The vector \( a \) same vector \( T \) that in this way all the data that is required for \( n \) without loss of generality, we will focus on the case of even \( d \) order, for all \( d \) between 2 and the dimension of the lattice. For sake of brevity, and without loss of generality, we will focus on the case of even \( n \) in the following. Note that in this way all the data that is required for \( T_d(2k, n/2 - k) \) is already available. Similarly as before, we can obtain \( T_d(2k, n/2 - k) \) as the scalar product \( a \cdot T_{d-1} \), where \( a \) is a row vector and the two-dimensional array \( T_{d-1} \), again, has to be interpreted as a single column vector. The vector \( a \) corresponds to a single row of the above-mentioned matrix \( A \). Then \( T_3(2k, n/2 - k) \) \( T_4(2k, n/2 - k) \) \( T_5(2k, n/2 - k) \) \( \ldots \) are computed by using always the same vector \( a \), so that any recomputation of coefficients is avoided. The only drawback of this approach is that one has to keep the whole three-dimensional array \( T_d(n, j) \) in memory, and therefore this method is more memory-intensive than the naive approach (by a factor of approximately \( d/2 \)).

The described computational scheme allows for lots of further (technical) improvements, some of which we want to mention briefly here. For example, one does not need to compute the vector \( a \) from scratch for each \( k \), but reuse the previous one by adding and deleting a few entries, and apply the simple recurrences for binomial coefficients:

\[
\binom{n+1}{k} = \frac{n+1}{n-k+1} \cdot \binom{n}{k} \quad \text{and} \quad \binom{n}{k+1} = \frac{n-k}{k+1} \cdot \binom{n}{k}.
\tag{8}
\]

Multiplying by a simple rational number is much cheaper than calculating a binomial coefficient.

With a little effort there is also the possibility to parallelize the computation. This can be done by splitting the sequence \( T_d(0, n/2), T_d(2, n/2 - 1), \ldots, T_d(n, 0) \) into parts, each of which is done by a single processor. The only caveat is the contribution of \( T_d(0, n/2), \ldots, T_d(2k - 2, n/2 - k + 1) \) to the computation of \( T_d(2k, n/2 - k) \), which has to be postponed until all processors have finished their task. This causes some synchronization overhead at each iteration of \( n \), which prevents us from using an excessive amount of processors. For example, the computation time for the required \( N = 999 \) terms for the 9-dimensional fcc lattice dropped from 60 hours to 7.5 hours by using 10 parallel processors.

For the interested reader we also mention the timings for the other dimensions of the lattice considered here: in the 10-dimensional case we obtained the necessary \( N = 1739 \) terms in 3 days using 20 parallel processes, for \( d = 11 \) the same number of processes was running for 18 days to compute the 2464 terms mentioned in Table 1. For the 12-dimensional fcc lattice we only computed modulo \( p = 2^{31} - 1 \), and found that the minimal number of terms necessary for constructing the linear differential operator is 3618: these were obtained in 10 days using 25 parallel processes.
Lattice Green Functions

Table 1. The number of terms $N_m$ (and $N_0$) needed to obtain the minimal-order linear ODE of order $Q_{\text{min}}$ (and the optimal-order linear ODE of order $Q_{\text{opt}}$) annihilating $LG_{d}(x)$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$N_m$</th>
<th>$N_0$</th>
<th>$N_m - N_0$</th>
<th>$Q_{\text{opt}} - Q_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>40</td>
<td>40</td>
<td>0</td>
<td>4 - 4 = 0</td>
</tr>
<tr>
<td>5</td>
<td>98</td>
<td>88</td>
<td>10</td>
<td>7 - 6 = 1</td>
</tr>
<tr>
<td>6</td>
<td>342</td>
<td>228</td>
<td>114</td>
<td>11 - 8 = 3</td>
</tr>
<tr>
<td>7</td>
<td>732</td>
<td>391</td>
<td>341</td>
<td>16 - 11 = 5</td>
</tr>
<tr>
<td>8</td>
<td>1740</td>
<td>704</td>
<td>1036</td>
<td>21 - 14 = 7</td>
</tr>
<tr>
<td>9</td>
<td>2964</td>
<td>999</td>
<td>1965</td>
<td>26 - 18 = 8</td>
</tr>
<tr>
<td>10</td>
<td>6509</td>
<td>1739</td>
<td>4770</td>
<td>36 - 22 = 14</td>
</tr>
<tr>
<td>11</td>
<td>10864</td>
<td>2464</td>
<td>8400</td>
<td>43 - 27 = 16</td>
</tr>
<tr>
<td>12</td>
<td>19503</td>
<td>3618</td>
<td>15885</td>
<td>53 - 32 = 21</td>
</tr>
</tbody>
</table>

3. The differential equations of the LGF of the $d$-dimensional fcc lattice, $d = 8, \cdots, 12$

With the generated series, we obtain the corresponding linear ODE by the guessing method. The linear ODE for $d = 8, 9, 10, 11$ are obtained in exact arithmetic and the linear ODE for $d = 12$ is obtained modulo one prime. These linear ODEs are given in electronic form in [15].

The linear differential operators corresponding to $d = 8, 9, 10, 11, 12$ are called, respectively, $G_{14}^{8\text{Dfcc}}, G_{18}^{9\text{Dfcc}}, G_{22}^{10\text{Dfcc}}, G_{27}^{11\text{Dfcc}}, G_{32}^{12\text{Dfcc}}$, where the subscript refers to the order of the linear ODE. These orders are in agreement with the conjecture given in [9]:

$$q = \frac{d^2}{4} - \frac{d}{2} + \frac{17}{8} - \frac{(-d)^4}{8}. \quad (9)$$

We have also agreement with the regular singularities $x_s$ obtained by the Landau equations method, which read [9]

$$x_s = \left(\frac{d}{2}\right) \cdot \frac{1}{\xi(d,k,j)}, \quad (10)$$

where

$$\xi(d,k,j) = \frac{d^2 - (k + 4j + 1) \cdot d + 4j^2 + k + 4jk}{2 \cdot (1 - k)}, \quad (11)$$

with $k = 0, 2, 3, \cdots, d - 1$, $j = 0, \cdots, [(d - k)/2]$, and where $[x]$ is the integer part of $x$.

The singularities as they occur in front of the head derivative of respectively $G_{14}^{8\text{Dfcc}}, G_{18}^{9\text{Dfcc}}, G_{22}^{10\text{Dfcc}}, G_{27}^{11\text{Dfcc}}, G_{32}^{12\text{Dfcc}}$ are given in Appendix A.

As far as the local exponents at the singularities are concerned, one remarks that the regular pattern seen [9] at $x = 0$ continues. Note that this is the pattern from which we inferred the order of the linear ODE. The local exponents, at each singularity, are given in Table 2. The singularities $x_d$, which read $x_3 = -3$, $x_5 = -5$, $x_6 = -15$, $x_7 = -7$, $x_8 = -14$, $x_9 = -9$, $x_{10} = -15$, $x_{11} = -11$, $x_{12} = -33/2$, seem to be
given by:

\[ x_d = -\frac{2 d \cdot (d - 1)}{2 d - 5 - 3 \cdot (-1)^d}. \]  

(12)

For \( d = 11 \), there is also the singularity \( x = -55 \) not included in "others" (not shown in Table 2) with local exponents \( 9/2, 11/2 \).

<table>
<thead>
<tr>
<th>( d )</th>
<th>( x = 0 )</th>
<th>( x = \infty )</th>
<th>( x = 1 )</th>
<th>( x_d )</th>
<th>others</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( 0^3 )</td>
<td>3/2</td>
<td>1/2</td>
<td>( 0^3 )</td>
<td>1/2</td>
</tr>
<tr>
<td>4</td>
<td>( 0^4 )</td>
<td>2^2</td>
<td>1^2</td>
<td>( 2^2 )</td>
<td>3/2</td>
</tr>
<tr>
<td>5</td>
<td>( 0^5, 1 )</td>
<td>5/2</td>
<td>3/2</td>
<td>( 3/4, 5/4 )</td>
<td>3/2</td>
</tr>
<tr>
<td>6</td>
<td>( 0^6, 1^2 )</td>
<td>3^2</td>
<td>2^2</td>
<td>( 2^2, 3^2 )</td>
<td>2^2</td>
</tr>
<tr>
<td>7</td>
<td>( 0^7, 1^3, 2 )</td>
<td>7/2</td>
<td>5/2</td>
<td>( 3/2, 5/2, 2^3 )</td>
<td>5/2</td>
</tr>
<tr>
<td>8</td>
<td>( 0^8, 1^4, 2^2 )</td>
<td>4^2</td>
<td>3^2</td>
<td>( 3^2, 4^2 )</td>
<td>3^2</td>
</tr>
<tr>
<td>9</td>
<td>( 0^9, 1^5, 3^3, 2 )</td>
<td>9/2, 11/2</td>
<td>7/2</td>
<td>( 9/4, 11/4, 13/4, 15/4 )</td>
<td>7/2</td>
</tr>
<tr>
<td>10</td>
<td>( 0^{10}, 1^6, 2^4, 3^2 )</td>
<td>5^2</td>
<td>4^2</td>
<td>( 4^2, 5^2 )</td>
<td>4^2</td>
</tr>
<tr>
<td>11</td>
<td>( 0^{11}, 1^7, 2^5, 3^3, 4 )</td>
<td>11/2</td>
<td>9/2</td>
<td>( 7/2, 9/2, 3^2, 4^3, 5^2 )</td>
<td>9/2</td>
</tr>
<tr>
<td>12</td>
<td>( 0^{12}, 1^8, 2^6, 3^4, 4^2 )</td>
<td>6^2</td>
<td>5^2</td>
<td>( 5^2, 6^2 )</td>
<td>5^2</td>
</tr>
</tbody>
</table>

Table 2. The local exponents at the regular singularities. Only the exponents giving a singular behavior are shown.

4. The differential Galois groups of \( G_q^{d_{fcc}} \), \( d = 8, 9, 10, 11, 12 \)

The equivalence of two properties, namely the homomorphism of the operator with its adjoint, and either the occurrence of a rational solution for the symmetric (or exterior) square of that operator, or the drop of order of these squares, have been seen for many linear differential operators [12]. The operators with these properties are such that their differential Galois groups are included in the symplectic or orthogonal differential groups. We have also shown that such operators have a “canonical decomposition” [11], which means that they can be written in terms of “tower of intertwiners”. These properties hold also for the (non-Fuchsian) operators emerging in the square Ising model at the scaling limit [13].

For the linear differential operators annihilating \( LGF_d(x) \), \( d = 5, 6, 7 \), of the fcc lattice, these properties hold [9, 12]. For instance, the order-eleven operator \( G_{11}^{7_{fcc}} \) (corresponding to \( LGF_7(x) \)) has the following canonical decomposition [9]

\[
G_{11}^{7_{fcc}} = (A_1 \cdot B_1 \cdot C_1 \cdot D_1 \cdot E_7 + A_1 \cdot B_1 \cdot E_7 + A_1 \cdot D_1 \cdot E_7 + A_1 \cdot B_1 \cdot C_1 \cdot D_1 \cdot E_7 + E_7 + C_1 + A_1) \cdot r(x),
\]  

(13)

where \( r(x) \) is a rational function, and the factors (the indices correspond to their orders) are all self-adjoint linear differential operators.

The decomposition (13) occurs because \( G_{11}^{7_{fcc}} \) is non-trivially homomorphic to its adjoint, and the decomposition is obtained through a sequence of Euclidean right divisions (see section 5 in [9]).

From this decomposition one understands easily why the symmetric square of \( G_{11}^{7_{fcc}} \) is of order 65, instead of the generically expected order 66. The symmetric square of the self-adjoint order-seven linear differential operator \( E_7 \) is of order 27,
instead of the generically expected order 28 (see [9, 11] for details). From the decomposition (13) one immediately deduces the decomposition of the adjoint of $G_{11}^{D_{fcc}}$ (because the factors are self-adjoint). The symmetric square of the adjoint $G_{11}^{D_{fcc}}$ will annihilate a rational solution which is the square of the solution of the order-one operator $A_1$. The differential Galois group of $G_{11}^{D_{fcc}}$ is included in $SO(11, \mathbb{C})$.

In the sequel, we show that the operators $G_{14}^{D_{fcc}}$, $G_{18}^{D_{fcc}}$, $G_{22}^{D_{fcc}}$, $G_{27}^{D_{fcc}}$ and $G_{32}^{D_{fcc}}$ verify the same properties as the operators $G_{11}^{D_{fcc}}$ (and $G_6^{D_{fcc}}$ for $d = 5$, see [2], $G_9^{D_{fcc}}$ for $d = 6$, see [4]). For $d$ odd (resp. $d$ even), one must consider the symmetric square (resp. exterior square) of the operator.

\subsection{4.1. The differential Galois group of $G_{14}^{D_{fcc}}$}

Producing the 14 formal solutions of the linear differential operator $G_{14}^{D_{fcc}}$, it is easy to show that its exterior square is of order 90, instead of the generically expected order 91. The differential Galois group of the operator $G_{14}^{D_{fcc}}$ is included in $Sp(14, \mathbb{C})$.

The exterior square of the adjoint of $G_{14}^{D_{fcc}}$ either has the order 90, or annihilates a rational solution. We find that the exterior square of the adjoint of $G_{14}^{D_{fcc}}$ annihilates the following rational function

$$\text{sol}_R(\text{ext}^2(\text{adjoint}(G_{14}^{D_{fcc}}))) = \frac{x^{21} \cdot P_{84}(x) \cdot S_8(x)^2}{P_{14}(x)},$$

(14)

where $P_{14}(x)$ is the degree-95 apparent polynomial of $G_{14}^{D_{fcc}}$, $S_8(x)$ is a degree-8 polynomial corresponding to the finite singularities given in Appendix A, and $P_{84}$ is a degree-84 polynomial.

From these results, we should expect, in the “canonical decomposition” of $G_{14}^{D_{fcc}}$, the factor equivalent to $A_1$ in (13) to be an order-two self-adjoint operator with (14) as the Wronskian. We should expect also the equivalent to $E_7$ in (13) to be self-adjoint with even order greater than two. It is tempting to find the “canonical decomposition” of $G_{14}^{D_{fcc}}$, and see whether the order of the ”last” factor (i.e. the equivalent of $E_7$ in (13)) is equal to the dimension $d = 8$ as we conjectured in [9].

Indeed, the “canonical decomposition” [11] of $G_{14}^{D_{fcc}}$ is

$$G_{14}^{D_{fcc}} = (A_2 \cdot B_2 \cdot C_2 \cdot D_8 + A_2 \cdot B_2 + C_2 \cdot D_8 + A_2 \cdot D_8 + 1) \cdot r(x),$$

(15)

where $r(x)$ is a rational function, and where all the factors are self-adjoint, with the indices indicating the order.

The starting relation to obtain this decomposition is the homomorphism that maps the solutions of $G_{14}^{D_{fcc}}$ to the solutions of the adjoint:

$$\text{adjoint}(L_{12}) \cdot G_{14}^{D_{fcc}} = \text{adjoint}(c_{14}^{D_{fcc}}) \cdot L_{12}$$

(16)

The sequence of Euclidean right divisions (the indices indicate the orders)

$$G_{14}^{D_{fcc}} = A_2 \cdot L_{12} + L_{10}, \quad L_{12} = B_2 \cdot L_{10} + L_8, \quad L_{10} = C_2 \cdot L_8 + r(x),$$

(17)

and substitutions, give the decomposition (15). We have shown in [11] that the factors $A_2$, $B_2$, $C_2$ are automatically self-adjoint. The sequence ends when the rest of the last Euclidean right division is a rational function: one, then, obtains the order-8 self-adjoint operator $D_8 = L_8/r(x)$. If the exterior square of $G_{14}^{D_{fcc}}$ was of the generic order 91, and annihilated a rational function, the sequence of right divisions would continue, and the last Euclidean right division would be $L_4 = F_2 \cdot L_2 + r(x)$. 

4.2. The differential Galois group $G_{18}^{D_{f_{cc}}}$

Similar calculations performed on the operator $G_{18}^{D_{f_{cc}}}$ show that the symmetric square is of order 170, instead of the generically expected order 171. The differential Galois group of the operator $G_{18}^{D_{f_{cc}}}$ is included in $SO(18, \mathbb{C})$.

The symmetric square of the adjoint of $G_{18}^{D_{f_{cc}}}$ annihilates the rational function
\[
\text{sol}_R(\text{sym}^2(\text{adjoint}(G_{18}^{D_{f_{cc}}}))) = \frac{x^{28} \cdot P_{260}(x) \cdot S_9(x)^2}{P_{18}(x)^2},
\]
where $P_{18}(x)$ is the apparent polynomial of the operator $G_{18}^{D_{f_{cc}}}$, $S_9(x)$ is a degree-9 polynomial corresponding to the finite singularities given in Appendix A, and $P_{260}$ is a degree-260 polynomial.

Heavy calculations give the canonical decomposition [11] of $G_{18}^{D_{f_{cc}}}$ as (again, the indices denote the order):
\[
G_{18}^{D_{f_{cc}}} = (A_1 \cdot B_1 \cdot C_1 \cdot D_1 \cdot E_1 \cdot F_1 \cdot G_1 \cdot H_1 \cdot I_1 \cdot J_9 + \cdots) \cdot r(x).
\]
All the factors are self-adjoint. The decomposition contains 89 terms and is obtained through a sequence of nine Euclidean right divisions starting from the homomorphism that maps the solutions of $G_{18}^{D_{f_{cc}}}$ to the solutions of the adjoint:
\[
\text{adjoint}(L_{17}) \cdot G_{18}^{D_{f_{cc}}} = \text{adjoint} \left( G_{18}^{D_{f_{cc}}} \right) \cdot L_{17}.
\]
Here also, we see that the conjecture of [9] is verified. The order of the last self-adjoint factor (i.e. $J_9$) is equal to the dimension of the lattice, $d = 9$.

4.3. The differential Galois groups of $G_{22}^{D_{f_{cc}}}$, $G_{27}^{D_{f_{cc}}}$ and $G_{32}^{D_{f_{cc}}}$

The detailed calculations done for the decompositions of $G_{14}^{D_{f_{cc}}}$ and $G_{18}^{D_{f_{cc}}}$ are too huge to be performed on $G_{22}^{D_{f_{cc}}}$ and $G_{27}^{D_{f_{cc}}}$. However, it is straightforward to obtain that the exterior square of $G_{22}^{D_{f_{cc}}}$ is of order 230, instead of the generic order 231. The differential Galois group of the linear differential operator $G_{22}^{D_{f_{cc}}}$ is included in $Sp(22, \mathbb{C})$. The symmetric square of $G_{27}^{D_{f_{cc}}}$ is of order 377, instead of the generic order 378. The differential Galois group of the operator $G_{27}^{D_{f_{cc}}}$ is included in $SO(27, \mathbb{C})$. Also, the exterior square of the known modulo a prime $G_{32}^{D_{f_{cc}}}$, is of order 495 instead of the generic order 496. The differential Galois group of the operator $G_{32}^{D_{f_{cc}}}$ is included in $Sp(32, \mathbb{C})$.

5. Coupling of lattices

In Section 2.2, we mentioned that some recurrences on the coefficients $T_d(n, p)$ have been used to improve the efficiency of the computations. Recall that to obtain the recursion relation giving the coefficients $T_d(n, p)$, we introduced [9]
\[
\zeta_d = \sum_{i=1}^{d} \sum_{j=i+1}^{d} \cos(k_i) \cdot \cos(k_j), \quad \sigma_d = \sum_{i=1}^{d} \cos(k_i),
\]
in terms of which the coefficients $T_d(n, p)$ are given by
\[
T_d(n, p) = A^{n+p} \cdot \langle \zeta_d^n \cdot \sigma_d^{2p} \rangle
\]
where the symbol $\langle \rangle$ means that the integration on the variables $k_j$, occurring in the integrand, has been performed (with the normalization $\pi^d$).
It is straightforward to see that the coefficients $T_d(n,p)$ correspond to the coefficients in the expansion around $(0,0)$ of the $d$-dimensional integral
\[
T_d(z, y) = \frac{1}{\pi^d} \int_0^\pi \cdots \int_0^\pi \frac{dk_1 \cdots dk_d}{(1-4z\zeta_d) \cdot (1-4y\sigma_d^2)}
= \sum_{n=0} \sum_{p=0} z^n \cdot y^p \cdot T_d(n,p),
\]
which gives the $LGF_d(z)$ of the $d$-dimensional fcc lattice for $y = 0$, and the $LGF_d(y)$ of the $d$-dimensional simple cubic lattice for $z = 0$. Note that for the simple cubic lattice, $\sigma_d^2$ should be $\sigma_d$. The expansion of the LGF of the simple cubic lattice corresponds to the expansion of (23) with $z = 0$ and $y = (x/d)^2/4$, where $x$ is the expansion parameter.

From the computation of $T_d(n,p)$ for some values of $d$, we infer the first terms of the expansion around $(0,0)$ of $T_d(z,y)$
\[
T_d(z,y) = 1 + 2d \cdot y + 2d \cdot (d-1) \cdot z^2 + 4d \cdot (d-1) \cdot zy + 6d \cdot (2d-1) \cdot y^2
+ 8d \cdot (d-1) \cdot (d-2) \cdot z^3 + 4d \cdot (d-1) \cdot (5d-7) \cdot z^2y + 48d \cdot (d-1)^2 \cdot zy^2
+ 20d \cdot (4 + 6d^2 - 9d) \cdot y^3 + \cdots
\]
(24)

Even if there is no obvious lattice corresponding to the Green function (23), we found that it might be worthy to analyze the bivariate series $T_d(z,y)$, per se.

In the sequel, we address the system of linear partial differential equations (PDE) that annihilates the $D$-finite bivariate series $T_d(z,y)$ for $d = 2$.

6. Partial differential equations for $T_2(z,y)$

To find the PDEs that annihilate $T_2(z,y)$ we can either proceed as for the linear ODEs, i.e. by the guessing method, or apply the creative telescoping technique [16, 17, 18]; the latter is computationally more costly, but provides a certificate of correctness of the obtained differential equations. For example [4], it was powerful enough to find and prove the ODEs satisfied by the LGF for $d = 4, 5, 6$, but failed for $d \geq 7$. All the differential equations mentioned in this and the following sections are also available in electronic form [15].

In order to apply the guessing method we assume a partial differential equation of order $Q$ in the homogeneous partial derivatives $z \cdot \partial / \partial z$ and $y \cdot \partial / \partial y$, with polynomials in $z$ and $y$ of degree $D$ that annihilates $T_2(z,y)$:
\[
\sum_{q=0}^{Q} \sum_{n=0}^{D} \sum_{p=0}^{D} a_{n,p}^{(q)} \cdot z^n \cdot y^p \cdot \left(z \cdot \frac{\partial}{\partial z}\right)^q \cdot \left(y \cdot \frac{\partial}{\partial y}\right)^{Q-q} \cdot T_2(z,y) = 0. \tag{25}
\]
This linear system fixes the coefficients $a_{n,p}^{(q)}$, and leaves some of them free. The number of non-fixed coefficients is the number of PDEs with order $Q$ and degree $D$.

If all the coefficients are such that $a_{n,p}^{(q)} = 0$, we increase the order $Q$ and/or the degree $D$.

For $Q = 1$, and various increasing values of $D$, all the coefficients are such that $a_{n,p}^{(q)} = 0$. For $Q = 2$ and $D = 2$, there is only one PDE, that we denote $PDE_2$.

For $Q = 3$ and $D = 1$, there are only two PDEs, called $PDE_3^{(1)}$ and $PDE_3^{(2)}$. Note that there is no concept of “minimal order” for PDEs, while there is one for ODEs.
Instead, one can consider a Gröbner basis (see Appendix B) in the ring of partial differential operators, as is demonstrated in Section 6.4. It is obvious that there are as many PDEs that annihilate $T_2(z, y)$ as we wish (namely, all elements of the left ideal $\text{ann}(T_2)$, see Appendix B). For instance, for $Q = 4$ and $D = 1$, we obtain five PDEs, three of them are of order four and two of them are combinations of $PDE_3^{(1)}$ and $PDE_3^{(2)}$.

6.1. Two PDEs for $T_2(z, y)$

With the notation

$$D_{xy}^{(n, p)} = \frac{\partial^{n+p}}{\partial x^n \partial y^p},$$

(26)

the system of two partial differential operators for $T_2(z, y)$ reads

$$PDE_3^{(1)} = 2y^3 \cdot (16y - 1) \cdot D_{xy}^{(0,3)} + 3y^2z \cdot (16zy + 12y - 1) \cdot D_{xy}^{(1,2)}$$
$$+ yz^2 \cdot (48y + 12y - 1 - 2z) \cdot D_{zy}^{(2,1)} + 12yz^2 \cdot (4z + 1) \cdot D_{zy}^{(2,0)}$$
$$+ 4yz \cdot (60zy + 24y - 1 - z) \cdot D_{zy}^{(1,1)} + 2y^2 \cdot (24zy + 80y - 3) \cdot D_{zy}^{(0,2)}$$
$$+ 24yz \cdot (6z + 1) \cdot D_{y}^{(1,0)} + 2y \cdot (68y - 1 + 72zy) \cdot D_{y}^{(0,1)}$$
$$+ 8y \cdot (6z + 1) \cdot D_{y}^{(0,0)},$$

(27)

and:

$$PDE_3^{(2)} = 2y^3 \cdot (16y - 1) \cdot D_{xy}^{(0,3)} + y^2z \cdot (24y + 32zy - 3 + 4z) \cdot D_{xy}^{(1,2)}$$
$$+ yz^2 \cdot (32zy - 1 + 8y) \cdot D_{zy}^{(2,1)} + 8yz^2 \cdot (4z + 1) \cdot D_{zy}^{(2,0)}$$
$$+ 4yz \cdot (40zy + 16y - 1 + z) \cdot D_{zy}^{(1,1)} + 2y^2 \cdot (16zy + 80y - 3 + 2z) \cdot D_{zy}^{(0,2)}$$
$$+ 16yz \cdot (6z + 1) \cdot D_{y}^{(1,0)} + 2y \cdot (68y + 2z + 48zy - 1) \cdot D_{y}^{(0,1)}$$
$$+ 8y \cdot (4z + 1) \cdot D_{y}^{(0,0)}.$$

(28)

These two operators annihilate (by construction) a finite truncation of the power series $T_2(z, y)$, where the truncation index has been chosen such that it is very likely that they also annihilate the infinite series $T_2(z, y)$. That they indeed annihilate $T_2(z, y)$ will be made rigorous in Section 6.4. In particular, it follows that $PDE_3^{(1)}$ and $PDE_3^{(2)}$ are compatible. By assuming a common solution of the form

$$\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} b_{n,p} \cdot z^n \cdot y^p,$$

(29)

we find a unique solution to the system of PDEs that identifies with $T_2(z, y)$, up to an overall constant. Only one constant means that if we switch to recurrences on the coefficients we will need only one initial condition.

The recursions for $T_2(n, p) = U(n, p)$ are

$$- 4 \cdot (8p^2 + 17p + 9np + 9n + 8 + 3n^2) \cdot U(n + 1, p)$$
$$+ 2 \cdot n(n + 1) \cdot U(n, p + 1) + (n + p + 2)(2p + n + 3) \cdot U(n + 1, p + 1)$$
$$- 48 \cdot (n + 1)(n + p + 1) \cdot U(n, p) = 0,$$

(30)

and

$$8 \cdot (3 + 4p^2 + 7p + 3np + 3n + n^2) \cdot U(n + 1, p)$$
$$+ 4 \cdot (n + 1)(p + 1) \cdot U(n, p + 1) + (p - n + 2)(2p + n + 3) \cdot U(n + 1, p + 1)$$
$$+ 32 \cdot (n + 1)(n + p + 1) \cdot U(n, p) = 0,$$

(31)
with the auxiliary recurrence:

\[(p + 1)^2 \cdot U(0, p + 1) - 4 \cdot (2p + 1)^2 \cdot U(0, p) = 0.\]  

(32)

With the coefficient \(U(0, 0)\) given, these three recurrences generate all the \(U(n, p)\).

\[6.2. \text{One PDE for } T_2(z, y)\]

There is only one partial differential operator of order \(Q = 2\) and degree \(D = 2\) that annihilates \(T_2(z, y)\):

\[PDE_2 = z^2 \cdot (4z - 1)(4z + 1)(4y - 1) \cdot D_{z,y}^{(2,0)} \]

\[+ zy \cdot (64z^2y - 16z^2 - 4z - 20y + 3) \cdot D_{z,y}^{(1,1)} \]

\[- 2y^2 \cdot (16y - 1) \cdot D_{z,y}^{(0,2)} + z(192z^2y - 48z^2 - 32zy - 12y + 1) \cdot D_{z,y}^{(1,0)} \]

\[+ 2y \cdot (32z^2y - 8z^2 - 32y - 24zy - 2z + 1) \cdot D_{z,y}^{(0,1)} \]

\[- (16z^2 + 32zy + 8y - 64z^2y) \cdot D_{z,y}^{(0,0)},\]  

(33)

which acting on the bivariate form (29) generates, remarkably, a unique solution that identifies with \(T_2(z, y)\), up to an overall constant.

The coefficients \(T_2(n, p) = U(n, p)\) are given by the recursion

\[- (20np + 72p + 40 + 24n + 32p^2 + 4n^2) \cdot U(n + 2, p) \]

\[+ 64(n + 1)(n + p + 1) \cdot U(n, p) - 4(p + 1)(n + 2) \cdot U(n + 1, p + 1) \]

\[+ (p + 3 + n)(2p + 4 + n) \cdot U(n + 2, p + 1) \]

\[- (64 + 48p + 32n) \cdot U(n + 1, p) - 16 \cdot (n + 1)(n + 2 + p) \cdot U(n, p + 1) = 0,\]  

(34)

with the auxiliary recursions:

\[(n + 2)^2 \cdot U(n + 2, 0) - 16 \cdot (n + 1)^2 \cdot U(n, 0) = 0, \quad U(1, 0) = 0, \quad U(1, 0) = 0,\]  

\[(p + 1)^2 \cdot U(0, p + 1) - 4 \cdot (2p + 1)^2 \cdot U(0, p) = 0, \quad U(0, 1) = 0, \quad U(0, 1) = 0,\]  

\[(p + 2)(2p + 3) \cdot U(1, p + 1) - 4 \cdot (p + 1)(5 + 8p) \cdot U(1, p) = 16 \cdot (2 + 3p) \cdot U(0, p) + 4 \cdot (p + 1) \cdot U(0, p + 1).\]

(35)

Here also, these recurrences generate all the coefficients starting with \(U(0, 0)\).

\[6.3. \text{On the logarithmic solutions}\]

The system of PDEs given in (27, 28) has no logarithmic solution of the form

\[\sum_{n=0}^{\infty} \sum_{p=0}^{n} F_{n,p}(z, y) \cdot \ln(z)^n \cdot \ln(y)^{n-p},\]  

(36)

where \(F_{n,p}(z, y)\) are analytic at \((0, 0)\) bivariate series.

In contrast, the PDE given in (33) seems to have no bound in the summation on \(n\) (we obtained logarithmic solutions up to \(n = 17\)). Furthermore, we find that the logarithms \(\ln(z)\), and \(\ln(y)\), appear in the solutions as:

\[\sum_{n=0}^{\infty} \sum_{p=0}^{n} F_{n,p}(z, y) \cdot (\ln(z) - \mu \cdot \ln(y))^p.\]

(37)

The number of logarithmic solutions depends now on the value of \(\mu\). For \(\mu = 1\) and \(\mu = 1/2\), we find no bound to \(n\) (in our calculations, we reached \(n = 17\)). For
any other value of $\mu \neq 1, 1/2$, there is only one logarithmic solution i.e. $n = 1$. For generic $\mu$ the solutions are $T_2(z, y)$ and:

\[
T_2(z, y) \cdot (\ln(z) - \mu \cdot \ln(y)) \\
+ \left(\frac{1}{2} + 2\mu - 4\mu \cdot z + 2 \cdot (3 + 5\mu) \cdot z^2 + 4 \cdot (1 - \mu) \cdot zy - (13 + 12\mu) \cdot y^2 - \frac{304}{9} \mu \cdot z^3 + \frac{2}{3} \cdot (33 - 16\mu) \cdot z^2 y + \frac{8}{15} \cdot (31 - 90\mu) \cdot zy^2 - \frac{4}{9} \cdot (559 + 420\mu) \cdot y^3 + \cdots \right). \tag{38}
\]

Let us address the details of the computations on how the values $\mu = 1$, and $\mu = 1/2$ appear. Acting by $PDE_2$ on the form (37) rewritten as

\[
F_{n,n}(z,y) \cdot (\ln(z) - \mu \cdot \ln(y))^n + \cdots, \tag{39}
\]
gives the choice of zeroing

\[
a_{0,0} \cdot (\mu - 1/2)(\mu - 1) = 0, \tag{40}
\]

where $a_{0,0}$ is the leading coefficient of the bivariate series $F_{n,n}(z,y)$. The choice $\mu = 1$ (or $\mu = 1/2$) allows $a_{0,0} \neq 0$, which permits $n$ to be higher. The choice $a_{0,0} = 0$ will decrease the degree $n$, and the process continues with $n - 1$.

The choice (40) comes from the action of $PDE_2$ on (39), and the leading coefficient of the expansion to be cancelled is:

\[
n \cdot (n - 1) \cdot a_{0,0} \cdot (\mu - 1/2)(\mu - 1) \cdot (\ln(z) - \mu \cdot \ln(y))^{n-2} + O(z^3, y^1). \tag{41}
\]

6.4. Gröbner basis of PDEs for $T_2(z, y)$

A Gröbner basis for $\text{ann}(T_2)$, the annihilating ideal of PDEs for $T_2(z, y)$, can be obtained by applying Buchberger’s algorithm to the input \{PDE$_1^{(1)}$, PDE$_3^{(2)}$, PDE$_2$\} (some basics about Gröbner bases are given in Appendix B). Alternatively, we can compute the annihilating ideal from scratch, i.e. from the integral representation (23) of $T_2(z, y)$, by the method of creative telescoping. Both tasks can be performed with the HolonomicFunctions package [19] and yield the same result. The second approach, however, gives an independent proof that the guessed PDEs presented in the sections are correct.

The Gröbner basis of $\text{ann}(T_2)$ (with respect to degree-lexicographic order and $D_y < D_z$) consists of 3 operators, whose supports are given as follows:

\[
\{D_{zy}^{(2,0)}, D_{zy}^{(1,1)}, D_{zy}^{(0,2)}, D_{zy}^{(1,0)}, D_{zy}^{(0,1)}, D_{zy}^{(0,0)} \}, \\
\{D_{zy}^{(0,3)}, D_{zy}^{(1,2)}, D_{zy}^{(2,1)}, D_{zy}^{(0,0)}, D_{zy}^{(0,1)}, D_{zy}^{(0,0)} \}, \\
\{D_{zy}^{(1,2)}, D_{zy}^{(1,1)}, D_{zy}^{(0,2)}, D_{zy}^{(1,0)}, D_{zy}^{(0,1)}, D_{zy}^{(0,0)} \}.
\]

Note that the first basis element is exactly $PDE_2$. By investigating the leading monomials $D_{zy}^{(2,0)}, D_{zy}^{(0,3)}, D_{zy}^{(1,2)}$ one immediately finds that there are 5 monomials under the stairs, namely the monomials $D_{zy}^{(1,1)}, D_{zy}^{(0,2)}, D_{zy}^{(1,0)}, D_{zy}^{(0,1)}, D_{zy}^{(0,0)}$, which cannot be reduced by either of the leading monomials. We say that $\text{ann}(T_2)$ has holonomic rank 5. Hence one could expect that 5 initial conditions have to be given to identify the particular solution $T_2(z, y)$. As discussed before, we remarkably need only one initial condition.

‡ Note that the derivative with respect to $\mu$ of the logarithmic solution is also a solution.
7. Ordinary differential equations for $T_2(z,y)$

The bivariate series $T_2(z,y)$ may be seen as depending on the variable $z$ (or $y$) where $y$ (or $z$) is a parameter. By derivation of the PDE system, and elimination of the unwanted derivatives, one obtains a linear ODE on the variable $z$ (or $y$) that annihilates $T_2(z,y)$. Such elimination can be conveniently performed by using the Gröbner basis presented in Section 6.4.

7.1. ODE with the derivative on $z$ for $T_2(z,y)$

The linear ODE with the variable $z$, that annihilates $T_2(z,y)$, is of order five, and we call the corresponding operator $L_5^{(z)}$, (with the derivative $D_z = \frac{\partial}{\partial z}$):

$$L_5^{(z)} = \sum_{n=0}^{5} P_n(z,y) \cdot D_z^n.$$  

(42)

The polynomial in front of the highest derivative is

$$z^2 \cdot (4z-1) \cdot (4z+1) \cdot (z-4y) \cdot (16z^2y+y+8zy-4z^2) \cdot P_{app},$$

(43)

where $P_{app}$ carries apparent singularities:

$$P_{app} = 192y \cdot (4y-1) \cdot z^5 - (128y^2 + 32y - 12) \cdot z^4 + 4y(80y - 19) \cdot z^3 - (40y^2 + 4y - 1) \cdot z^2 + y \cdot (y - 1) \cdot z + y^2.$$  

(44)

The factorization of the order-five linear differential operator $L_5^{(z)}$ reads (the indices are the orders)

$$L_5^{(z)} = \left(L_1^{(2)} \cdot L_1^{(1)}\right) \oplus \left(L_3^{(1)} \cdot L_1^{(1)}\right) \oplus \left(L_2 \cdot L_1^{(1)}\right),$$

(45)

where the four factors are given in Appendix C.

The solution of $L_1^{(1)}$ reads:

$$\text{sol}\left(L_1^{(1)}\right) = \sqrt{\frac{z}{(z-4y) \cdot (4 \cdot (4y-1) \cdot z^2 + 8zy + y)}},$$

(46)

The second solution of $L_1^{(2)} \cdot L_1^{(1)}$ reads:

$$\text{sol}(L_1^{(1)}) \cdot \int\frac{z^{-3/2}}{\sqrt{(z-4y) \cdot (4 \cdot (4y-1) \cdot z^2 + 8zy + y)}} \cdot dz.$$  

(47)

The integral can be evaluated in terms of the incomplete elliptic integrals, so that the second solution of $L_1^{(2)} \cdot L_1^{(1)}$ reads

$$\text{sol}(L_1^{(1)}) \cdot \left(\frac{4y - \sqrt{y}}{2y^2} \cdot E(z_1, z_2) - \frac{4 \cdot (8y - 1 - 2\sqrt{y})}{y \cdot (16y - 1)} \cdot F(z_1, z_2)\right) - \frac{2}{(z-4y)y},$$

with

$$z_1 = \sqrt{\frac{(4\sqrt{y} - 1)^2z}{z-4y}}, \quad z_2 = \sqrt{\frac{(4\sqrt{y} + 1)^2}{(4\sqrt{y} - 1)^2}}$$

(48)

and where $E$ and $F$ are the incomplete elliptic integrals:

$$E(z,k) = \int_0^z \frac{1}{\sqrt{1-k^2t^2}} \cdot dt, \quad F(z,k) = \int_0^z \frac{1}{\sqrt{1-k^2t^2} \cdot \sqrt{1-t^2}} \cdot dt.$$  

(49)
The second solution of \( L_1^{(3)} \cdot L_1^{(1)} \) can be written as the general Heun function
\[
f(z) \cdot \text{Heun}(a, q, \frac{1}{2}, \frac{3}{2}, z, g(z)),
\]
with
\[
f(z) = \frac{z \cdot \sqrt{4 \cdot z} \sqrt{y} + \sqrt{y} - 2z}{(16 \cdot z^2 y + y + 8 \cdot z y - 4 \cdot z^2) \cdot \sqrt{z} - 4 \cdot y},
\]
\[
g(z) = \frac{z y \cdot (64 \cdot z y + 16 \cdot y - 12 \cdot z + 1) - 2 \cdot z \cdot \sqrt{y} \cdot (z - 4 \cdot y)}{4 \cdot y \cdot (16 \cdot z^2 y + y + 8 \cdot z y - 4 \cdot z^2)},
\]
and:
\[
a = \frac{1}{2} + \frac{16 \cdot y + 1}{16 \sqrt{y}}, \quad q = \frac{1 + a}{4}.
\]

The solution of \( L_2 \cdot L_1^{(1)} \) which is not solution of \( L_1^{(1)} \) can be written as
\[
sol(L_1^{(1)}) \cdot \int \frac{sol(L_2)}{sol(L_1^{(1)})} \cdot dz,
\]
where one of the solutions of \( L_2 \) reads
\[
sol(L_2) = \frac{z \cdot (16 \cdot z^2 - 1) (64 \cdot z^3 y - 80 \cdot z^2 y + 16 \cdot z^2 - 36 \cdot z y - 3 \cdot y) \cdot dH(z)}{3 \cdot y (z - 4 \cdot y) (y + 8 \cdot z y + 16 \cdot z^2 y - 4 \cdot z^2) \cdot dz} + 4 \cdot z \cdot \frac{256 \cdot z^4 y + 112 \cdot z^3 - 512 \cdot z^2 y - 224 \cdot z^2 y + z - 32 \cdot z y - 3 \cdot y}{y \cdot (z - 4 \cdot y) (y + 8 \cdot z y + 16 \cdot z^2 y - 4 \cdot z^2)} \cdot H(z),
\]
where \( H(z) \) is the hypergeometric function
\[
H(z) = _2F_1 \left( \frac{3}{2}, \frac{5}{2}, \frac{1}{1}, 16 \cdot z^2 \right).
\]

7.2. ODE with the derivative on \( y \) for \( T_2(z,y) \)

The bivariate series \( T_2(z,y) \), where \( z \) is a parameter, is annihilated by an order-five linear differential operator \( N_5^{(y)} \) (with the derivative \( D_y = \frac{\partial}{\partial y} \)):
\[
N_5^{(y)} = \sum_{n=0}^{5} Q_n(z,y) \cdot D_y^n.
\]

The polynomial, in front of the highest derivative, reads
\[
y^2 \cdot (16 \cdot y - 1) \cdot (z - 4 \cdot y) \cdot (16 \cdot z^2 y + y + 8 \cdot z y - 4 \cdot z^2) \cdot P_{app},
\]
where \( P_{app} \) carries apparent singularities:
\[
P_{app} = -12 \cdot (4 \cdot z + 1) \cdot (32 \cdot z^2 - 12 \cdot z - 1) \cdot y^3
+ (1344 \cdot z^3 - 8 + 20 \cdot z^2 - 114 \cdot z + 64 \cdot z^4) \cdot y^2
- (1 + 12 \cdot z - 20 \cdot z^2 - 264 \cdot z^3 + 48 \cdot z^4) \cdot y + (32 \cdot z^2 - 1 - 6 \cdot z) \cdot z^2.
\]

The order-five operator \( N_5^{(y)} \) has the following direct sum factorization
\[
N_5^{(y)} = \left( N_1^{(2)} \cdot N_1^{(1)} \right) \oplus \left( N_1^{(3)} \cdot N_1^{(1)} \right) \oplus \left( N_2 \cdot N_1^{(1)} \right),
\]
where the four factors are given in Appendix C.
The solution of $N_1^{(1)}$ reads:

$$sol(N_1^{(1)}) = \sqrt{\frac{y}{(z-4y)(4 \cdot (4y-1) \cdot \frac{z^2}{2} + 8zy + y)}}. \quad (60)$$

The second solution of $N_1^{(2)} \cdot N_1^{(1)}$ reads:

$$sol(N_1^{(1)}) \cdot \int \frac{y^{-3/2} \cdot (16y+1)}{\sqrt{(z-4y) \cdot (4 \cdot \frac{z^2}{2} + 8zy + y)}} \cdot dy. \quad (61)$$

The integral can be evaluated in terms of the incomplete elliptic integrals and the second solution of $N_1^{(2)} \cdot N_1^{(1)}$ reads

$$sol(N_1^{(1)}) \cdot \left(\frac{4z-1}{2} \cdot \frac{\sqrt{(z-4y) \cdot (4 \cdot \frac{z^2}{2} + 8zy + y)}}{2} \cdot \frac{384}{z} \cdot (4 \cdot \frac{z^3}{2} + 8zy + y)\right) - \frac{1}{2}z^3, \quad (62)$$

with:

$$z_1 = \sqrt{-\frac{16z^2y + y + 8zy - 4z^2}{4z^2}}, \quad z_2 = \sqrt{-\frac{16z}{(4z-1)^2}}. \quad (63)$$

The second solution of $N_1^{(3)} \cdot N_1^{(1)}$ is a general Heun function

$$sol(N_1^{(1)}) \cdot \sqrt{y} \cdot Heun\left(a, q, \frac{1}{2}, 1, \frac{1}{2}, \frac{2}{z}\right), \quad (64)$$

with:

$$a = \frac{16z}{(4z+1)^2}, \quad q = \frac{1+a}{4}. \quad (65)$$

The solution of $N_2 \cdot N_1^{(1)}$ which is not solution of $N_1^{(1)}$ can be written as

$$sol(N_1^{(1)}) \cdot \int \frac{sol(N_2)}{sol(N_1^{(1)})} \cdot dy, \quad (66)$$

where one of the solutions of $N_2$ reads

$$sol(N_2) = y \cdot (16y-1) \cdot \frac{3 \cdot z \cdot (z-4y) \cdot (4z+1) \cdot \frac{z^2}{2} + 8zy + 6y - 20z^2 - 3z}{3 \cdot z^5 \cdot (4z-1) \cdot (4z+1) \cdot \frac{z^2}{2} + 8zy + 6y - 20z^2 - 3z} \cdot dy, \quad (67)$$

where $H_y(y)$ is the hypergeometric function

$$H_y(y) = 2F_{1}\left(\frac{3}{2}, \frac{5}{2}, 1, 16y\right). \quad (68)$$

7.3. The linear ODE on $z$ and $y$ as PDE for $T_2(z, y)$

The linear differential equations corresponding to the operators $L_5^{(z)}$ and $N_5^{(y)}$ act on $T_2(z, y)$ as a system of decoupled PDEs. Both ODEs annihilate (as it should) the bivariate series $T_2(z, y)$, and they generate a unique common bivariate series solution, analytic at (0, 0), that identifies with $T_2(z, y)$.

As for the solutions of the three PDEs of the previous sections, one remarks that $sol(L_1^{(1)})$ and $sol(N_1^{(1)})$ have simple structures, and we can check that

$$\sqrt{\frac{yz}{(z-4y) \cdot (y + 8zy + 4 \cdot (4y-1) \cdot \frac{z^2}{2})}}. \quad (69)$$
is actually a solution of the three PDEs, \(PDE_3^{(1)}\), \(PDE_3^{(2)}\), and \(PDE_2\). Unfortunately, the other solutions are too complicated to be used to fabricate more general common solutions of the three PDEs.

However, the bivariate series \(T_2(z, y)\) can be written as a combination of the solutions of \(L_5^{(z)}\). Let us call \(S_1^{(z)}\), \(S_2^{(z)}\) and \(S_3^{(z)}\) the formal solutions analytic at \(z = 0\) of (respectively) the operators \(L_1^{(2)} \cdot L_1^{(1)}\), \(L_1^{(3)} \cdot L_1^{(1)}\) and \(L_2 \cdot L_1^{(1)}\). The first terms of these solutions are given in Appendix D.

The bivariate series \(T_2(z, y)\) reads

\[
T_2(z, y) = C_1^{(z)}(y) \cdot S_1^{(z)} + C_2^{(z)}(y) \cdot S_2^{(z)} + C_3^{(z)}(y) \cdot S_3^{(z)},
\]

(70)

where the combination coefficients \(C_j^{(z)}(y)\) are given in Appendix D.

Similarly, one may consider the bivariate series \(T_2(z, y)\) as a combination of the solutions of \(N_5^{(y)}\). With \(S_1^{(y)}\), \(S_2^{(y)}\) and \(S_3^{(y)}\) the formal solutions analytic at \(y = 0\) of (respectively) the operators \(N_1^{(2)} \cdot N_1^{(3)}\), \(N_1^{(3)} \cdot N_1^{(1)}\) and \(N_2 \cdot N_1^{(1)}\) (see Appendix E), the bivariate series \(T_2(z, y)\) reads

\[
T_2(z, y) = C_1^{(y)}(z) \cdot S_1^{(y)} + C_2^{(y)}(z) \cdot S_2^{(y)} + C_3^{(y)}(z) \cdot S_3^{(y)},
\]

(71)

where the combination coefficient \(C_j^{(y)}(z)\) are given in Appendix E.

Note that we have used for the solutions \(S_j^{(z)}\) (resp. \(S_j^{(y)}\)) the formal solutions of the corresponding operators since this is easier. Otherwise a full closed expression for \(T_2(z, y)\) is given in Appendix F, which is obtained by integration of the double integral. One should note that the expression is a "partition" that does not reflect the factorization of \((\operatorname{e.g.}) L_5^{(z)}\).

8. Partial differential equations for \(T_3(z, y)\)

Similar calculations can be performed for the bivariate series \(T_3(z, y)\) corresponding to the expansion around \((0, 0)\) of the integral (23) with \(d = 3\). In this instance, however, the creative telescoping method turned out to be too costly, and hence, all the PDEs presented below have been obtained by the guessing method.

We find that, for \(Q = 3\) and \(D = 3\), there is only one PDE (denoted \(PDE_3\)) and for \(Q = 4, D = 2\) there are two PDEs (called \(PDE_4^{(1)}, PDE_4^{(2)}\)). Here also, and similarly to \(T_2(z, y)\), both \(PDE_4^{(1)}\), \(PDE_4^{(2)}\) acting on the generic bivariate series (29) generate the unique \(T_3(z, y)\), while \(PDE_3\) is sufficient to generate a unique solution that identifies with \(T_3(z, y)\).

As for the logarithmic solutions, there is no solution of the form (36) for the system \((PDE_4^{(1)}, PDE_4^{(2)})\). However, and similarly to what happened for \(T_2(z, y)\), the number of logarithmic solutions for \(PDE_3\) depends on the value of \(\mu\) in the combination (37). For \(\mu = 1\) and \(\mu = 1/2\), there is non finite number of such solutions (we reached \(n = 17\) in our calculations).

For generic values of \(\mu \neq 1, 1/2\), one obtains three solutions, the bivariate series \(T_3(z, y)\) and the logarithmic solutions

\[
T_3(z, y) \cdot (\ln(z) - \mu \cdot \ln(y))^2 + T_3^{(1)} \cdot (\ln(z) - \mu \cdot \ln(y)) + T_3^{(0)},
\]

\[
T_3(z, y) \cdot (\ln(z) - \mu \cdot \ln(y)) + \frac{1}{2} T_3^{(1)},
\]

(72)
where:
\[
T_3^{(1)} = 4 \cdot (1 - 4\mu) \cdot z - 4 \cdot (7\mu - 1) \cdot y - 2 \cdot (24\mu - 17) \cdot z^2 + 4 \cdot (13 - 40\mu) \cdot yz - 6 \cdot (87\mu - 7) \cdot y^2 + \cdots
\]
(73)
\[
T_3^{(0)} = 8 \cdot \mu \cdot (1 - 5\mu) \cdot z + 2 \cdot (8\mu - 11) \cdot y - 2 \cdot (3\mu - 2 + 31\mu^2) \cdot z^2 - \frac{4}{3} \cdot (57\mu^2 + 41 - 9\mu) \cdot yz + \frac{1}{2} \cdot (700\mu + 392\mu^2 - 831) \cdot y^2 + \cdots
\]
(74)

8.1. Decoupled linear differential equations for \( T_3(z,y) \)

For the PDE system \( \{PDE_3, PDE_4^{(1)}, PDE_4^{(2)}\} \) annihilating \( T_3(z,y) \), we used Buchberger’s algorithm as implemented in the HolonomicFunctions program [19] and obtained immediately a Gröbner basis (given in electronic form in [15]). It allows us to derive two (order-nine) ordinary differential equations\‡ for \( T_3(z,y) \), one involving only \( D_z \), the other one only \( D_y \):
\[
L_9^{(z)} = \sum_{n=0}^{9} P_n(z,y) \cdot D_z^n, \quad N_9^{(y)} = \sum_{n=0}^{9} Q_n(z,y) \cdot D_y^n.
\]
(75)

One factorization of \( L_9^{(z)} \) reads\† (the indices denote orders):
\[
L_9^{(z)} = L_3 \cdot L_4^{(4)} \cdot L_1^{(3)} \cdot L_1^{(2)} \cdot L_1^{(1)} \cdot L_2.
\]
(76)
The similar factorization of \( N_9^{(y)} \) reads:
\[
N_9^{(y)} = N_3 \cdot N_1^{(4)} \cdot N_1^{(3)} \cdot N_1^{(2)} \cdot N_1^{(1)} \cdot N_2.
\]
(77)

The two order-two operators \( L_2 \) and \( N_2 \) are given in Appendix G. They are self-adjoint, up to a conjugation by their Wronskians \( W(L_2) \) and \( W(N_2) \) (see Appendix G):
\[
L_2 \cdot W(L_2) = W(L_2) \cdot \text{adjoint}(L_2), \quad N_2 \cdot W(N_2) = W(N_2) \cdot \text{adjoint}(N_2).
\]
(78)

We have been able to find one solution for \( L_9^{(z)} \) (and \( N_9^{(y)} \)). Defining the hypergeometric function
\[
S_{zy} = \frac{\sqrt{yz}}{P_{zy}} \cdot \frac{\sqrt{yz}}{P_{zy}} \cdot \text{F}_1 \left( \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, [1], \frac{64 \cdot yz \cdot (z - 3y)^3 \cdot (1 + 4z)^2}{P_{zy}^2} \right),
\]
(79)
where
\[
P_{zy} = 1728 y^2 z^3 - 432 y z^3 + 16 z^3 + 864 z^2 y^2 - 72 z^2 y + 108 z y^2 + z y - 4 y^2,
\]
(80)
one checks that \( S_{zy} \) is solution of the two most right order-two operators \( L_2 \) and \( N_2 \)
\[
L_2(S_{zy}) = 0, \quad N_2(S_{zy}) = 0.
\]
(81)
As was seen for the operators \( L_5^{(z)} \) and \( N_5^{(y)} \) corresponding to \( T_2(z,y) \), with the solution (69), one can check that the solution (79) of \( L_9^{(z)} \) (and \( N_9^{(y)} \)), is one solution to the whole PDE system:
\[
PDE_3(S_{zy}) = 0, \quad PDE_4^{(1)}(S_{zy}) = 0, \quad PDE_4^{(2)}(S_{zy}) = 0.
\]
(82)

\‡ Do not confuse the labels of some factors with those occurring for \( T_2(z,y) \).

\† The full factorization of \( L_9^{(z)} \) as a direct sum is \( L_9^{(z)} = L_3 \oplus L_2 \oplus L_1^{(4)} \oplus L_1^{(3)} \oplus L_2 \oplus L_1^{(2)} \oplus L_1^{(1)} \oplus L_2 \).

\† The full factorization of \( L_9^{(z)} \) as a direct sum is \( L_9^{(z)} = L_3 \oplus L_2 \oplus L_1^{(4)} \oplus L_2 \oplus L_1^{(3)} \oplus L_2 \oplus L_1^{(2)} \oplus L_1^{(1)} \oplus L_2 \).
This solution (79) of the whole PDE system is, in fact, quite remarkable. It corresponds to a modular form [20, 21, 22]. In order to see this modular form structure, let us recall various (non trivial) identities on hypergeometric functions.

The use of the identity

$$2F_1\left(\frac{1}{4}, \frac{3}{4}, [1], X\right) = \frac{1}{(1 + 3X)^{1/4}} \cdot 2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], \frac{27X \cdot (1 - X)^2}{(1 + 3X)^3}\right),$$  

(83)

together with the identity

$$2F_1\left(\frac{1}{4}, \frac{3}{4}, [1], X\right) = \left(\frac{4}{4 - 3X}\right)^{1/4} \cdot 2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], \frac{27X^2 \cdot (X - 1)}{(3X - 4)^3}\right),$$  

(84)

implies the following identity on the same hypergeometric function

$$2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], A_1\right) = \sqrt{2} \cdot \left(\frac{1 + 3X}{4 - 3X}\right)^{1/4} \cdot 2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], A_2\right),$$  

(85)

with the two different arguments†

$$A_1(X) = \frac{27 \cdot X \cdot (1 - X)^2}{(1 + 3X)^3}, \quad A_2(X) = \frac{27 \cdot X^2 \cdot (X - 1)}{(3X - 4)^3} = A_1(1 - X).$$  

(86)

This enables to rewrite the solution $S_{zy}$ of (82), where $S_{zy}$ is given in (79), as a $2F_1$ hypergeometric function with two different pullbacks, namely $A_1$ and $A_2$ given by (86) where $X$ is given by (with $P_{zy}$ given by (80)):

$$X = \frac{64 \cdot yz \cdot (z - 3y)^3 \cdot (1 + 4z)^2}{P_{zy}^2},$$  

(87)

and where $1 - X$ reads:

$$\frac{(144y^2z^2 + 96yz - 40y^2 + 16y^2 - 8yz + z^2)(144yz^2 + 24yz - 16z^2 + y^2)}{P_{zy}^2}. $$  

(88)

This shows that the solution $S_{zy}$, given in (79), corresponds to a modular form, seen as a function of $z$, or seen as a function of $y$.

The two pullbacks $A_1$ and $A_2$ are lying on the algebraic genus zero modular curve

$$1953125 \cdot A_1^3 A_2^3 - 187500 \cdot A_1^4 A_2^2 \cdot (A_1 + A_2)$$

$$+ 375 \cdot A_1 A_2 \cdot (16 A_1^2 - 4027 A_1 A_2 + 16 A_2^2)$$

$$- 64 \cdot (A_1 + A_2) \cdot (A_1^2 + 1487 A_1 A_2 + A_2^2) + 110592 A_1 A_2 = 0.$$  

(89)

If one introduces $Z$ such that $X = Z/(Z + 64)$, one can see clearly that this modular equation‡ (89) is the same as the one corresponding to the fundamental modular curve $X_0$, associated with the Landen transformation, and its well-known Hauptmodul rational parametrization [20, 21, 22]:

$$A_1 = \frac{1728 \cdot Z}{(Z + 16)^3}, \quad A_2 = \frac{1728 \cdot Z^2}{(Z + 256)^3}.$$  

(90)

**Remark:** We have a quite remarkable result for the $X = \text{const.}$ foliation. When $X$ is a constant, one finds that the curves $X = \text{const.}$, are genus zero curves.

† Such non-trivial identity (85) is characteristic of modular forms [20, 21, 22].

‡ G.S. Joyce already noticed the emergence of modular equations on lattice Green functions [5].
9. Remarks and comments

We give here some miscellaneous remarks on the calculations presented in the previous sections.

**Remark 1:** The system of recurrence equations for $T_2(n, j)$ can be obtained by the creative telescoping method [16, 18]. For higher $d$, the computations get too heavy. But with the guessing method, the recurrences for $d = 5$ can be reached and this may yield an efficient implementation, since one could take $T_5(n, j)$ as initial values instead of $T_2(n, j)$ in (4).

**Remark 2:** In our calculations, (sections 3 and 2.2), we have experienced that we do not gain anything by doing the computation modulo primes, and then using Chinese remaindering to construct the true result in $\mathbb{Z}$. The reason is that here we do not encounter an intermediate expression swell, but rather the fact that the largest integers that occur during the computation are basically those that are given as the final result (when this is considered to be the list of Taylor coefficients). Of course, we are mostly interested in the linear differential operator, which, itself, has much smaller integer coefficients. Therefore the natural strategy would be to compute the Taylor coefficients modulo prime, and guess the operator modulo prime, and, only after this is done for sufficiently many primes, use Chinese remaindering and rational reconstruction to get the true operator. Unfortunately, the operator also has quite large integers in its coefficients so that this strategy is unfavorable. However, we can still use homomorphic images for the purpose of prediction, e.g. how many terms are required for guessing the linear differential operator.

As an example, in the case ($d = 11$, 20 processes, 2464 terms), the timing is 18 days in exact arithmetic calculations. When we compute modulo the prime $2^{31} - 1$, our implementation needs 58 hours, but the rational reconstruction of the linear differential operator requires 185 primes of this size.

**Remark 3:** With the emergence of the algebraic solution (69) for the system of $d = 2$ PDEs, and the emergence of the modular form solution (79) for the system of $d = 3$ PDEs, it is tempting to conjecture that similar solutions of two variables exist for all the system of PDEs for arbitrary value of $d$.

**Remark 4:** For one complex variable, the holonomic (or D-finite [6]) functions are solutions of linear ODEs with polynomial coefficients in the complex variable. The singularities (and apparent singularities) can be seen immediately as solutions of the head polynomial coefficient of the linear ODE. For partial differential equations system annihilating holonomic functions of several complex variables, the singular manifolds would be too complex or simply could not be well defined. By considering several Picard-Fuchs systems of two-variables “associated” to Calabi-Yau ODEs†, we showed [23] that D-finite (holonomic) functions are actually a good framework for actually finding properly the singular manifolds. The singular algebraic varieties for some $T_d(z, y)$ are given in Appendix H.

† Along the line of the relation between lattice Green functions and Calabi-Yau ODEs see for instance [8].
10. Conclusion

A recursive method has been introduced in [9] to generate the expansion of the lattice Green function of the \(d\)-dimensional face-centred cubic lattice. The method has been used to generate many coefficients for \(d = 7\) and the corresponding linear differential equation has been obtained [9].

We have shown, here, the strength and the limit of this recursive method. Some observations on the recursive method allow us to improve the computations and produce the series up to \(d = 12\). The corresponding linear differential equations have been obtained and show that the pattern (order, singularities, differential Galois group) seen for the lower \(d\)'s continues.

In the recursive method, a two-dimensional array \(T_d(n,j)\) is computed where only the coefficients \(T_d(n,0)\) correspond to the expansion of the lattice Green function. The two-dimensional array \(T_d(n,j)\) gives the expansion of a "lattice" Green function that depends on two variables. These \(D\)-finite bivariate series are studied and the differential equations they are solution of, are addressed.

We have been able to produce some solutions of the partial differential equations annihilating the bivariate series \(T_d(z,y)\). A remarkable modular form solution emerged for \(d = 3\). The corresponding Hauptmodul pullback is a simple rational function of \(y\) and \(z\). In terms of this Hauptmodul, the \((y, z)\) plane is a foliation of rational curves. Such kind of results are clearly a strong incentive to generalize the search of solutions of the \(D\)-finite systems corresponding to higher dimensions \(d\).

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Appendix A. Singularities of the ODEs for \(d = 8, 9, 10, 11, 12\)

The singularities, occurring at the head derivative of \(G^{8D_{fcc}}\), are \(x^{11} S_8(x) \cdot P_{14}(x)\). The roots of the degree-95 polynomial \(P_{14}(x)\) are apparent singularities. The polynomial \(S_8(x)\) corresponding to the finite singularities reads:

\[
S_8(x) = (14 + x)^2 (x - 1) (x - 7) (x - 2) (6 + x) (7 + x) (20 + x) (28 + x) (48 + x) (21 + 2 x) (4 + 3 x) (28 + 3 x) (32 + 3 x) (16 + 5 x) (28 + 5 x) (28 + 11 x) (112 + 11 x) (224 + 13 x) (112 + 19 x).
\]

(A.1)

The singularities of \(G^{9D_{fcc}}\) are \(x^{14} \cdot S_9(x) P_{18}(x)\). The roots of the degree-133 polynomial \(P_{18}(x)\) are apparent singularities. The polynomial \(S_9(x)\) corresponding to the finite singularities reads:

\[
S_9(x) = (x + 9)^4 (7 x + 9) (4 x + 9) (x + 3) (2 x + 9) (7 x + 36) (5 x + 27) (x + 12) (2 x + 27) (5 x + 72) (x + 15) (x + 18) (2 x + 45) (x + 27) (x + 36) (x + 63) (2 x - 9) (5 x - 9) (x - 1).
\]

(A.2)
Lattice Green Functions

The singularities of $G_{22}^{10D/fcc}$ are $x^{18} \cdot S_{10}(x) \cdot P_{22}(x)$. The roots of the degree-252 polynomial $P_{22}(x)$ are apparent singularities. The polynomial $S_{10}(x)$, corresponding to the finite singularities, reads:

$$S_{10}(x) = (15 + x)^2 (x - 1) (4 x + 75) (19 x + 540) (x - 15) (4 x + 15) (x + 80)
\quad (2 x + 45) (22 x + 45) (2 x + 25) (7 x + 20) (7 x + 120) (2 x + 15) (23 x + 180)
\quad (13 x + 60) (29 x + 360) (17 x + 135) (x + 45) (4 x + 45) (x + 35) (3 x - 5)
\quad (4 x + 5) (11 x + 135) (x + 20) (13 x - 45) (x + 12) (x + 9) (x + 8) (x + 5).$$

(A.3)

The singularities of $G_{27}^{11D/fcc}$ are $x^{22} S_{11}(x) P_{27}(x)$. The roots of the degree-352 polynomial $P_{27}(x)$ are apparent singularities. The polynomial $S_{11}(x)$, corresponding to the finite singularities, reads:

$$S_{11}(x) = (x + 11)^6 (55 + x)^2 (x - 1) (8 x + 55) (29 x + 55) (4 x + 55) (2 x + 55)
\quad (4 x + 11) (7 x + 165) (7 x - 55) (2 x + 33) (17 x + 55) (x + 44) (13 x + 275)
\quad (3 x + 55) (7 x - 11) (13 x + 55) (7 x + 110) (x + 35) (3 x + 22) (x + 99)
\quad (19 x - 55) (7 x + 33) (9 x + 11) (x + 15) (9 x + 55) (17 x + 275) (3 x + 77)
\quad (23 x + 165).$$

(A.4)

The singularities of $G_{32}^{12D/fcc}$ are $x^{27} S_{12}(x) P_{32}(x)$. The roots of the degree-580 polynomial $P_{32}(x)$ are apparent singularities. The polynomial $S_{12}(x)$, corresponding to the finite singularities, reads:

$$S_{12}(x) = (2 x + 33)^2 (43 x + 264) (5 x + 6) (37 x + 66) (3 x + 8) (23 x + 66)
\quad (67 x + 264) (2 x + 9) (13 x + 66) (5 x + 33) (7 x + 48) (7 x + 66) (9 x + 88)
\quad (10 x + 99) (53 x + 528) (x + 10) (x + 11) (5 x + 66) (19 x + 264) (7 x + 99)
\quad (37 x + 528) (23 x + 330) (5 x + 72) (29 x + 528) (17 x + 330) (13 x + 264)
\quad (x + 21) (x + 22) (31 x + 792) (8 x + 231) (x + 32) (x + 33) (25 x + 1056)
\quad (x + 54) (x + 66) (x + 120) (x - 33) (2 x - 11)
\quad (13 x - 33) (2 x - 3) (x - 1).$$

(A.5)

Appendix B. Gröbner basis basics

The theory of Gröbner bases has been initiated by Bruno Buchberger in his Ph.D. thesis [24] in 1965. While originally it was formulated for commutative multivariate polynomial rings, we are interested in its generalization to noncommutative rings. Here we can only mention a few key facts that are important for the kind of applications that we have in mind.

Let $D_z$ denote the operator $\partial / \partial z$. The motivation for using operator notation is that it turns ODEs and PDEs into (univariate resp. multivariate) polynomials. For example the PDEs appearing in Sections 6-8 can be represented by polynomials in the ring $\mathbb{C}(z,y)[D_z, D_y]$, i.e., the ring of partial differential operators in $z$ and $y$ (it is an instance of an Ore algebra). Note that this ring is not commutative, because of the Leibniz rule $D_z a = a D_z + \partial a / \partial z$ for all $a \in \mathbb{C}(z,y)$.

Let $f$ be a power series (or some other kind of “function”); we define

$$\text{ann}_\mathcal{O}(f) = \{ P \in \mathcal{O} \mid P(f) = 0 \},$$

called the annihilating ideal of $f$. It can be easily seen that this set is indeed a left ideal in $\mathcal{O}$, as for example, the left-multiplication of $P \in \mathcal{O}$ by $D_z$ corresponds to
differentiating the differential equation represented by $P$ with respect to $z$. Since univariate polynomial rings are principal ideal domains, $\text{ann}(f)$ is generated by a single element if we consider only one derivation; this unique generator corresponds to the minimal-order ODE. Of course, in the case of PDEs, an annihilating ideal in general is generated by several operators.

We need some notion of leading term for PDEs (in the case of ODEs it is clear). For this purpose one imposes a total order on the monomials in the ring under consideration, that is compatible with multiplication and that has 1 as the smallest monomial; such an ordering is called a monomial order. For example, the degree-lexicographic order on the ring $\mathbb{C}(z,y)[D_z, D_y]$ with $D_y < D_z$ is defined by

$$D^i_z D^j_y < D^k_z D^\ell_y \iff i + j < k + \ell \lor (i + j = k + \ell \land i < k).$$

Using this notion of leading term, it is straightforward to define a multivariate polynomial division (called reduction) of $P \in \mathcal{O}$ by some $Q_1, \ldots, Q_r \in \mathcal{O}$. It works by subtracting, in each step, a suitable multiple of some $Q_i$ such that the leading term of the dividend vanishes. In some steps of this process one may have the choice between several of the $Q_i$, and this has the consequence that the remainder of the multivariate polynomial division is not unique in general. Now, if $I$ is an ideal, then a set $G$ of generators of $I$ is called a Gröbner basis if for each polynomial the remainder of the division by $G$ is unique. In particular, we have that the division of $P$ by $G$ has remainder 0 if and only if $P \in I$; this property allows to decide the ideal membership problem. Gröbner bases are also a powerful tool for elimination purposes, i.e., for finding elements in an ideal that do not depend on some of the variables of the polynomial ring. There are several algorithms to compute, from an arbitrary set of generators of an ideal, a Gröbner basis, the most classic one being Buchberger’s algorithm [24].

### Appendix C. The factorization of $L_5^{(z)}$ and $L_5^{(y)}$ for $T_2(z,y)$

The factors in the decomposition (45) of $L_5^{(z)}$ read

$$L_1^{(1)} = D_z + 2 \cdot \frac{8 z^3 y - 16 z^2 y^2 + 6 z^2 y - 2 \cdot z^3 + y^2}{(16 z^2 y + y + 8 z y - 4 z^2) (z - 4 y) \cdot z},$$

$$L_1^{(2)} = D_z + 2 \cdot \frac{32 z^3 y - 8 z^3 y - 96 z^2 y^2 + 36 z^2 y + z y - 32 y^2 z - 2 y^2}{(16 z^2 y + y + 8 z y - 4 z^2) (z - 4 y) \cdot z},$$

$$L_1^{(3)} = L_1^{(2)} - \frac{1}{z},$$

and

$$L_2 = D_z^2 + \frac{p_1(z,y)}{p_2(z,y)} \cdot D_z + \frac{p_0(z,y)}{p_2(z,y)},$$

where:

$$p_2(z,y) = z \cdot (4 z - 1) \cdot (4 z + 1) \cdot (y + 8 y z + 16 y z^2 - 4 z^2) \cdot (z - 4 y)$$

$$(256 z^4 y^2 - 64 z^4 y - 128 z^3 y^2 + 4 z^3 + 64 z^2 y^2 - 20 y z^2 + 40 y^2 z - 2 y z + 3 y^2).$$
\[ p_1(z, y) = 2z \cdot \left( y^3 + 16y^4 - 3200z^4y^2 - 208z^4y + 816z^3y^2 + 84z^2y^2 + y^2z \right. \]
\[ - 18z^3y + 576yz^5 - 40704z^5y^2 - 46080z^6y^2 + 8704yz^6 + 3584z^7y \]
\[ - 98304z^8y^2 + 12288z^8y + 196608z^9y^3 - 217088z^9y^4 + 181248z^{10}y^3 \]
\[ + 212992z^{11}y^3 - 24576z^6y^3 + 196608z^7y^4 - 524288z^7y^4 - 98304z^{11}y^4 \]
\[ - 63488z^{12}y^4 + 89088z^{12}y^3 - 20480z^{13}y^3 - 1440z^7y^3 + 6080z^3y^3 \]
\[ - 116z^3y^3 - 3840z^3y^4 + 384zy^4 + 24z^5 - 896z^7 \].

\[ p_0(z, y) = 144y^4 - 7296z^4y^2 - 112z^4y^2 + 124z^2y^2 - 2y^2z \]
\[ - 28z^3y + 1600yz^5 - 80896z^5y^2 - 107520z^6y^2 + 18176y^6z + 8192z^7y \]
\[ - 196608z^8y^2 + 24576z^8y + 393216z^8y^3 - 315392z^4y^4 + 347136z^5y^3 \]
\[ + 49152z^7y^3 - 812z^6y^3 + 458752z^6y^4 - 524288z^7y^4 - 229376z^5y^4 \]
\[ - 88064z^3y^4 + 150016z^3y^4 + 20480z^3y^2 - 736z^2y^3 + 11840z^3y^3 \]
\[ - 188yz^3 - 8960z^2y^4 + 384zy^4 + 16z^5 - 2048y^7. \]

The factors in the decomposition (59) of \( N_0^{(y)} \) read

\[ N_1^{(1)} = D_y + 2 \cdot \frac{16z^2y^2 - z^3 + y^2 + 8y^2z}{(16z^2y + y + 8zy - 4z^2) \cdot (4y - z) \cdot y}, \]

\[ N_1^{(2)} = D_y + 2 \cdot \frac{q_0(z, y)}{y \cdot (16z^2y + y + 8zy - 4z^2) \cdot (16y + 1) \cdot (z - 4y)}, \]

where

\[ q_0(z, y) = -2z^3 + 3y + 24z^2y + 16z^3y - 6y^2 - 40y^2z + 96z^2y^2 \]
\[ + 128z^3y^2 - 64y^3 - 512zy^3 - 1024z^2y^3, \]

\[ N_1^{(3)} = N_1^{(2)} - \frac{1}{(16y + 1) \cdot y}, \]

and

\[ N_2 = D_y - \frac{\tilde{p}_1(z, y)}{\tilde{p}_2(z, y)} \cdot D_y + \frac{\tilde{p}_0(z, y)}{\tilde{p}_2(z, y)}, \]

where:

\[ \tilde{p}_2(z, y) = y \cdot (16y - 1) \cdot (z - 4y) \cdot (y + 8zy + 16z^2y^2 - 4z^2) \cdot (32z^4y \]
\[ - 8z^4 + 256z^3y^2 - 32z^2y^2 + 10z^2y - 32y^2y + 2z^3y + 10z^3y + 2y^3), \]

\[ \tilde{p}_1(z, y) = -24y^4 + 15360y^5z + 102400y^5z^2 - 1310720y^5z^3 - 491520y^5z^4 \]
\[ + 163840y^5z^3 - 672z^4y^2 - 60z^4y^2 - 2z^2y^2 + 24yz^5 - 3456z^5y^2 \]
\[ - 20480z^6y^2 + 576yz^6 + 1920z^7y - 104448z^4y^4 + 41472z^5y^3 \]
\[ + 32768z^7y^3 + 94208z^6y^3 + 221184z^5y^4 - 82944z^3y^4 + 26112z^4y^3 \]
\[ - 15360z^7y^2 + 3102z^3y^2 + 4672z^3y^2 + 18zy^3 - 14592z^2y^4 \]
\[ - 1056zy^4 - 32z^7 + 640y^5, \]

\[ \tilde{p}_0(z, y) = -8y^3 + 480y^4 + 96z^6 + 10240z^4y^2 - 104z^4y + 1320z^3y^2 \]
\[ + 152z^2y^2 + 8y^2z - 4z^3y - 672yz^5 + 34048z^5y^2 + 104448z^6y^2 \]
\[ - 11648yz^6 - 8704z^7y - 368640z^4y^4 + 2048z^5y^3 - 155648z^6y^3 \]
\[ - 98304z^5y^4 + 122880z^3y^4 - 64512z^4y^3 + 18432z^7y^2 - 8544z^2y^3 \]
\[ - 44288z^3y^3 - 568zy^3 + 76800z^2y^4 + 11520zy^4 - 8z^5 + 640z^7. \]
Appendix D. The matching of $T_2(z, y)$ with the solutions of $L_5^{(z)}$

$T_2(z, y)$ as a linear combination on the formal solutions of $L_5^{(z)}$ reads

$$T_2(z, y) = C_1^{(z)}(y) \cdot S_1^{(z)} + C_2^{(z)}(y) \cdot S_2^{(z)} + C_3^{(z)}(y) \cdot S_3^{(z)},$$

(D.1)

where

$$S_1^{(z)} = 1 + \left( -\frac{16}{3} + 2y^{-1} \right) \cdot z^2 + \left( \frac{512}{15} - \frac{184}{15} y^{-1} \right) \cdot z^3$$
$$+ \left( -\frac{18176}{105} + \frac{320}{21} y^{-1} + \frac{208}{35} y^{-2} \right) \cdot z^4$$
$$+ \left( \frac{253952}{315} - \frac{2304}{35} y^{-1} - \frac{2816}{35} y^{-2} - \frac{4}{315} y^{-3} \right) \cdot z^5 + \ldots$$

$$S_2^{(z)} = z + \left( -\frac{16}{3} + \frac{1}{6} y^{-1} \right) \cdot z^2 + \left( \frac{368}{15} + \frac{22}{15} y^{-1} + \frac{1}{30} y^{-2} \right) \cdot z^3$$
$$+ \left( -\frac{11264}{105} + \frac{2864}{105} y^{-1} + \frac{8}{35} y^{-2} + \frac{1}{140} y^{-3} \right) \cdot z^4$$
$$+ \left( \frac{144128}{315} - \frac{79424}{315} y^{-1} + \frac{96}{35} y^{-2} + \frac{2}{45} y^{-3} + \frac{1}{630} y^{-4} \right) \cdot z^5 + \ldots$$

$$S_3^{(z)} = z^2 - 6 z^3 + \left( 36 + 3 y^{-1} \right) \cdot z^4 - \left( 180 + 40 y^{-1} \right) \cdot z^5$$
$$+ \left( 900 + 370 y^{-1} + 10 y^{-2} \right) \cdot z^6 + \ldots$$

$$C_1^{(z)}(y) = \frac{1}{\sqrt{1 - 16 y}} \cdot 2F_1 \left( \frac{1}{2}, \frac{1}{2}, [1], 16 \cdot \frac{y}{16 y - 1} \right),$$

(D.2)

$$C_2^{(z)}(y) = -\frac{8 y}{(8 y - 1)^{3/2}} \cdot 2F_1 \left( \frac{3}{4}, \frac{5}{4}, [2], 64 \cdot \frac{y^2}{(1 - 8 y)^2} \right),$$

(D.3)

$$C_3^{(z)}(y) = -2 y^{-1}.$$  

(D.4)

Appendix E. The matching of $T_2(z, y)$ with the solutions of $N_5^{(y)}$

$T_2(z, y)$ as linear combination on the formal solutions of $N_5^{(y)}$ reads

$$T_2(z, y) = C_1^{(y)}(z) \cdot S_1^{(y)} + C_2^{(y)}(z) \cdot S_2^{(y)} + C_3^{(y)}(z) \cdot S_3^{(y)},$$

(E.1)

where

$$S_1^{(y)} = 1 - 16 y - \left( \frac{128}{3} + \frac{208}{3} z^{-1} + \frac{16}{3} z^{-2} + \frac{1}{3} z^{-3} \right) \cdot y^2$$
$$- \left( \frac{2048}{15} + \frac{4096}{15} z^{-1} + \frac{1408}{5} z^{-2} + \frac{464}{15} z^{-3} + \frac{8}{3} z^{-4} + \frac{1}{15} z^{-5} \right) \cdot y^3 + \ldots$$

$$S_2^{(y)} = y + \left( \frac{8}{3} + z^{-1} + \frac{1}{6} z^{-2} \right) \cdot y^2$$
$$+ \left( \frac{128}{15} + 16 z^{-1} + \frac{232}{15} z^{-2} + \frac{1}{30} z^{-3} + \frac{1}{30} z^{-4} \right) \cdot y^3 + \ldots$$
$\sigma_3^{(y)} = y + (8 + 3z^{-1}) \cdot y^2 + (76 + 30z^{-1} + 10z^{-2}) \cdot y^3 + \cdots$

$C_1^{(y)}(z) = \frac{1}{\sqrt{1 - 16z^2}} \cdot 2F_1 \left( \left[ \frac{1}{2}, \frac{1}{2} \right], [1], \frac{16z^2}{16z^2-1} \right), \quad \text{(E.2)}$

$C_2^{(y)}(z) = \frac{2 \cdot (12z + 1)}{z} \cdot H_z(z) + \frac{(4z + 1)(4z - 1)}{2z} \cdot \frac{d}{dz} H_z(z), \quad \text{(E.3)}$

with: $H_z(z) = 2F_1 \left( \left[ \frac{1}{2}, \frac{1}{2} \right], [1], 16z^2 \right), \quad \text{(E.4)}$

$C_3^{(y)}(z) = -2z^{-1}. \quad \text{(E.5)}$

**Appendix F. Closed form expression of** $V_2(z, y)$

The bivariate series

$$V_2(z, y) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{d \kappa_1 d \kappa_2}{(1 - z \zeta_2) \cdot (1 - \frac{y}{2} \sigma_2)}, \quad \text{(F.1)}$$

with

$$\zeta_2 = \cos(k_1) \cdot \cos(k_2), \quad \sigma_2 = \cos(k_1) + \cos(k_2), \quad \text{(F.2)}$$

is related to the integral (23) with $d = 2$ by $T_3(z, y) = V_2(4z, \pm \sqrt{y})$. The integral $V_2(z, y)$ can be written in the closed form expression

$$V_2^{\text{closed}}(z, y) = \frac{y}{zy + y - 2z} \cdot K(y) + \frac{z}{\delta} \cdot \left( \Pi \left( \frac{(z - \delta)^2}{y^2}, z \right) - \Pi \left( \frac{(z + \delta)^2}{y^2}, z \right) \right)$$

$$+ \frac{y \cdot (1 - y) \cdot \delta}{(z - y^2)(zy + y - 2z)} \cdot \left( \Pi(\Delta_-, y) - \Pi(\Delta_+, y) \right), \quad \text{(F.3)}$$

with

$$\delta = \sqrt{z^2 - y^2}, \quad \Delta_\pm = \frac{2zy + y^2 - zy^2 - 2z \pm 2(1 - y) \cdot \delta}{zy + y - 2z}, \quad \text{(F.4)}$$

and where $\Pi$ and $K$ are the complete elliptic integrals of the third and first kind:

$$\Pi(\nu, k) = \int_0^\pi \frac{1}{(1 - \nu \cos(\phi))^2} \cdot \frac{1}{\sqrt{1 - k^2 \cos(\phi)^2}} \cdot d\phi, \quad K(k) = \Pi(0, k). \quad \text{(F.5)}$$

One has

$$V_2(z, y) = \text{Re} \left( V_2^{\text{closed}}(z, y) \right), \quad \text{for} \quad |z| < 1, \quad |y| < 1, \quad \text{(F.6)}$$

and

$$V_2(z, y) = V_2^{\text{closed}}(z, y), \quad \text{for} \quad y \cdot (zy + y - 2z) > 0. \quad \text{(F.7)}$$

**Appendix G. The two right-most order-two operators** $L_2$ and $N_2$ for $T_3(z, y)$

- The right-most order-two linear differential operator $L_2$ (see (76)) in the factorization of the order-nine operators $L_0^{(z)}$ reads:

$$L_2 = D_z^2 + z^2 \cdot \frac{p_1(z, y)}{p_2(z, y)} \cdot D_z + \frac{p_0(z, y)}{p_2(z, y)}, \quad \text{(G.1)}$$
where:

\[ p_2(z, y) = z^2 \cdot (432 y^2 z^2 + 180 zy^2 - 36 z^2 y + 12 y^2 - 13 yz + 2 z^2) \]
\[ \times (4 z + 1) \cdot (3 y - z) \cdot (144 z^2 y + 24 yz - 16 z^2 + y) \]
\[ \times (144 y^2 z^2 + 96 zy^2 - 40 z^2 y + 16 y^2 - 8 yz + z^2), \quad (G.2) \]

\[ p_1(z, y) = 512 \cdot (4 y - 1) \cdot (36 y - 1) \cdot (9 y - 1) \cdot (216 y^2 - 18 y + 1) \cdot z^7 \]
\[ - 96 \cdot (10177696 y^6 - 10730880 y^5 + 3050784 y^4 - 361080 y^3 + 20176 y^2 - 481 y + 3) \cdot z^6 \]
\[ - 16 y (25754112 y^5 - 16422912 y^4 + 3259008 y^3 - 270672 y^2 + 9518 y - 103) \cdot z^5 \]
\[ - 2 y (103762944 y^5 - 46033920 y^4 + 6379200 y^3 - 334704 y^2 + 5464 y - 1) \cdot z^4 \]
\[ - 2 y^2 \cdot (26065152 y^4 - 8142336 y^3 + 717600 y^2 - 17920 y + 13) \cdot z^3 \]
\[ - y^3 (6816960 y^3 - 1407888 y^2 + 60196 y - 99) \cdot z^2 \]
\[ - 24 y^4 (18288 y^2 - 1960 y + 3) \cdot z - 10944 y^6 - 144 y^5, \quad (G.3) \]

\[ p_0(z, y) = 256 (4 y - 1) (36 y - 1) (9 y - 1) (216 y^2 - 18 y + 1) \cdot z^8 \]
\[ - (107495424 y^6 - 134369280 y^5 + 41720832 y^4 - 5163264 y^3 + 284928 y^2 - 6240 y + 32) \cdot z^7 \]
\[ - 8 y (14556672 y^5 - 10917504 y^4 + 2372544 y^3 - 202944 y^2 + 6788 y - 63) \cdot z^6 \]
\[ - 16 (3032640 y^4 - 1574640 y^3 + 235404 y^2 - 12305 y + 178) y^2 \cdot z^5 \]
\[ - 4 y^2 (2519424 y^4 - 883872 y^3 + 81840 y^2 - 1956 y + 1) \cdot z^4 \]
\[ - 16 y^3 (75168 y^3 - 15786 y^2 + 746 y - 3) \cdot z^3 \]
\[ - 2 (49032 y^2 - 5550 y + 113) \cdot y^4 \cdot z^2 - 12 \cdot (456 y - 35) y^5 \cdot z - 144 y^6, \quad (G.4) \]

This order-two operator \( L_2 \) is self-adjoint up to a conjugation by its Wronskian \( W(L_2) \)

\[ L_2 \cdot W(L_2) = W(L_2) \cdot \text{adj}(L_2), \quad (G.5) \]

where this Wronskian \( W(L_2) \) reads:

\[ W(L_2) = \frac{432 y^2 z^2 + 180 y^2 z - 36 yz^2 + 12 y^2 - 13 yz + 2 z^2}{(4 z + 1) \cdot (z - 3 y) \cdot d_1 \cdot d_2} \]

where:
\[ d_1 = 144 yz^2 + 24 yz - 16 z^2 + y, \]
and:
\[ d_2 = 144 y^2 z^2 + 96 y^2 z - 40 yz^2 + 16 y^2 - 8 yz + z^2. \]

- The right-most order-two linear differential operator \( N_2 \) (see (77)) in the factorization of the order-nine operators \( N_9^y \) reads:

\[ N_2 = D_y^2 + y^2 \cdot \frac{q_1(z, y)}{q_2(z, y)} \cdot D_y + \frac{q_0(z, y)}{q_2(z, y)}, \quad (G.7) \]

where:

\[ q_2(z, y) = y^2 \cdot (36 yz - 6 y + z) \cdot (3 y - z) \]
\[ \times (144 y^2 z^2 + 96 y^2 z - 40 yz^2 + 16 y^2 - 8 yz + z^2) \]
\[ \times (144 yz^2 + 24 yz - 16 z^2 + y), \quad (G.8) \]

\[ q_1(z, y) = 864 \cdot (6 z - 1) \cdot (3 z + 1)^2 \cdot (12 z + 1)^2 \cdot y^4 \]
\[ - 96 z \cdot (15552 z^5 + 25920 z^4 + 4428 z^3 - 1098 z^2 - 273 z - 7) \cdot y^3 \]
\[ + 6 z^2 \cdot (38016 z^4 + 7200 z^3 - 8184 z^2 - 1850 z - 31) \cdot y^2 \]
\[ + 2 z^3 \cdot (8064 z^3 + 6000 z^2 + 944 z + 11) \cdot y - 1360 z^6 - 104 z^5 - 4, \quad (G.9) \]
of the dimension $z$ series have singularities bearing on the variable or Landau conditions methods [10], we have the following results. The bivariate

$$q_0(z, y) = 4 z^6 + 16 \cdot (25 z - 4) \cdot z^3 \cdot y$$

$$- 2 z^3 \cdot (2448 z^3 + 24 z^2 - 163 z - 1) \cdot y^2$$

$$+ 12 z^2 \cdot (432 z^4 - 792 z^3 - 669 z^2 - 113 z - 2) \cdot y^3$$

$$- 12 z \cdot (15552 z^5 + 22032 z^4 + 1188 z^3 - 1989 z^2 - 363 z - 10) \cdot y^4$$

$$+ 216 \cdot (6 z - 1)(3 z + 1)^2 (12 z + 1)^2 \cdot y^5. \quad (G.10)$$

This order-two operator $N_2$ is self-adjoint up to a conjugation by its Wronskian $W(N_2)$

$$N_2 \cdot W(N_2) = W(N_2) \cdot \text{adjoint}(N_2), \quad (G.11)$$

where this Wronskian $W(N_2)$ reads:

$$W(N_2) = \frac{36 y z - 6 y + z}{z \cdot (z - 3 y)} \cdot d_1 \cdot d_2$$

where: $d_1 = 144 y z^2 + 24 y z - 16 z^2 + y,$

and: $d_2 = 144 y^2 z^2 + 96 y^2 z - 40 y z^2 + 16 y^2 - 8 y z + z^2. \quad (G.12)$

**Appendix H. Singularities of the $T_d(z, y)$**

Based on the singularities of the ODEs in one variable (the other being a parameter) or Landau conditions methods [10], we have the following results. The bivariate

$$d = 2, \quad 4 y - z = 0, \quad 4 z^2 \cdot (4 y - 1) + 8 y z + y = 0,$$

$$d = 3, \quad 3 y - z = 0, \quad 16 z^2 \cdot (9 y - 1) + 24 y z + y = 0,$$

$$16 y^2 \cdot (3 z + 1)^2 - z \cdot (40 y z + 8 y - z) = 0, \quad (H.1)$$

$$d = 4, \quad 8 y - 3 z = 0, \quad 36 z^2 \cdot (16 y - 1) + 48 y z + y = 0,$$

$$4 z^2 \cdot (16 y - 1) + 16 y z + y = 0, \quad 16 y z + 4 y - z = 0.$$

One may infer from the first two varieties of each $d$, the expressions in function of the dimension

$$y - \frac{d - 1}{2d} \cdot z = 0, \quad y - \frac{4 \cdot (d - 1) \cdot z^2}{(1 + 2 \cdot d \cdot (d - 1) \cdot z)^2} = 0. \quad (H.2)$$

**References**


Lattice Green Functions


