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Abstract

We prove a sharp decay rate for the total energy of two classes of systems of weakly coupled hyperbolic equations. We show that we can stabilize the full system through a single damping term, acting on one component only of the system (*indirect stabilization*). The energy estimate is achieved by means of suitable estimates of the resolvent operator norm. We apply this technique to a system of wave-wave equation and to a wave-Petrovsky system.

Keywords: Indirect stabilization, systems of hyperbolic PDEs, resolvent estimates.

AMS subject classifications: 93D15, 35L53, 47D06, 47A10.

1 Introduction

The issue of stabilizing a system of partial differential equations through suitable damping terms (possibly in feedback form) on each component of the system has its origin in the pioneering works of Lagnese and Lions [9] and Russell [11]. In their approach, the multiplier method is the main tool to reach the desired estimates on the energy of each component of the system. This techniques has been further developed in [8] and later in [1] for systems of hyperbolic equations.

In particular, Russell [11] addressed the indirect stabilization problem, that occurs when the damping (or the control) acts on a reduced number of equations of the system. In this situation the (uniformly) exponential decay rate usually cannot be achieved, but weaker decay rates might hold. This is the case in [2] and [3], where (uniform) polynomial stability for the whole system is showed, under a suitable compatibility condition on the operators involved in the system, by means of multipliers properly adapted to the peculiar structure of the system under investigation. Indeed, it turns out that different compatibility condition and multipliers are required to cope with systems with boundary conditions of similar type on each component [2] or mixed [3].

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In [4] a different method has been proposed to prove polynomial stability for the solution of abstract first order Cauchy problems. When applied to the indirect stabilization of a weakly coupled system, this technique requires stronger compatibility conditions among the operators of each component of the system, in order to perform the needed spectral analysis, thus applying, so far, to fewer systems of hyperbolic equations.

An operator-theoretical approach has been recently proposed in [5] and [7], addressing the optimality issue for the established decay rate. Indeed, the decay rate of the system ruled by the dynamics $\mathcal{A}$ is related to the asymptotic behaviour of the norm of the resolvent operator of $\mathcal{A}$ on the imaginary axis (see, for example, [10]). This technique has been successfully applied in [12] to the indirect stabilization of systems of Euler-Bernoulli and wave equations with globally distributed coupling and damping. Let us point out that, however, multipliers cannot be completely avoided, since also this approach requires suitable multipliers (adapted to the operators of the system) in order to deal with elliptic estimates of the resolvent operator.

In this paper we study the indirect stabilization problem for two classes of systems of hyperbolic-type equations, by means of the general criterion given in [7]. Since these systems fall into the general description given in [3], polynomial stabilization is already ensured (see equation (8)). Here we succeed in improving the decay rate of the total energy of the systems, thanks to a sharp analysis of the behaviour of the resolvent operator along the imaginary axis.

More precisely, let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be open and bounded, with sufficiently smooth boundary Γ. We denote by $\Delta = \sum_{i=1}^{d} \partial^2_{x_i x_i}$ the Laplacian operator; moreover, for sake of simplicity, we use the subscript u_t to denote the partial derivative of u with respect of the variable t. Let $\lambda \geq 0$ and $\alpha, \beta > 0$. We will first consider the weakly coupled system of wave equations

$$\begin{cases} u_{tt} - \Delta u + \lambda u + \beta u_t + \alpha v = 0 & \text{in } \Omega \times (0, +\infty) \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in } \Omega \times (0, +\infty) \end{cases} \tag{1}$$

with boundary conditions

$$\frac{\partial u}{\partial \nu}(\cdot, t) + \sigma u(\cdot, t) = 0 = v(\cdot, t) \quad \text{on } \Gamma\,,\ t > 0\,, \quad \sigma \geq 0\,, \tag{2}$$

and initial conditions

$$u(0) = u^0\,,\ u_t(0) = u^1\,,\ v(0) = v^0\,,\ v_t(0) = v^1 \quad \text{in } \Omega\,, \tag{3}$$

for functions u^i, v^i $(i = 0, 1)$ in suitable spaces (see (14)). We remark that $\max(\sigma, \lambda) > 0$, meaning that, under Neumann boundary condition on u, we require $\lambda > 0$, in order to ensure the coercivity of the operator in the first component. Also operators with different boundary conditions on separated portions of the boundary can be treated. Indeed, let Γ_0 and Γ_1 be open subsets of Γ such that

$$\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1\,, \quad \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset\,.$$

In this situation, we can consider system (1) with boundary conditions

$$\begin{matrix} u(\cdot, t) = 0 \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \nu}(\cdot, t) + \sigma u(\cdot, t) = 0 \quad \text{on } \Gamma_1 \\ v(\cdot, t) = 0 \quad \text{on } \Gamma \end{matrix} \qquad t > 0$$

and initial conditions (3). Finally, we will focus on the Petrowsky-wave system

$$\begin{cases} u_{tt} - \Delta u + \beta u_t + \alpha v = 0 & \text{in } \Omega \times (0, +\infty) \\ v_{tt} + \Delta^2 v + \alpha u = 0 & \text{in } \Omega \times (0, +\infty)\,, \end{cases} \tag{4}$$

with Robin boundary conditions

$$\frac{\partial u}{\partial \nu}(\cdot, t) + \sigma u(\cdot, t) = 0 \quad \text{on } \Gamma\,,\ t > 0\,, \quad \sigma > 0\,, \tag{5}$$

on u and either hinged

$$v(\cdot, t) = 0 = \Delta v(\cdot, t) \quad \text{on } \Gamma\,,\ t > 0 \tag{6}$$

or clamped boundary conditions

$$v(\cdot, t) = 0 = \frac{\partial v}{\partial \nu}(\cdot, t) \quad \text{on } \Gamma\,,\ t > 0 \tag{7}$$

on v, with initial conditions (3).

We first point out that the system of two evolution equations associated with the systems above fulfills the compatibility condition introduced in [3], ensuring the estimate on the total energy

$$\mathcal{E}(U(t)) \le \frac{C}{(1+t)^{1/4}} |U_0|^2_{\mathcal{H}} \tag{8}$$

for every initial condition $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathcal{A})$ and for some constant $C > 0$ (see the section below for precise definitions of the spaces $D(\mathcal{A})$, $\mathcal{H}$ and the energy $\mathcal{E}(U)$). The aim of the present paper is to derive the optimal decay rate of the total energy $\mathcal{E}(t)$; indeed, we will show that it decays polynomially in time with a decay rate $1/2$ for initial condition in $D(\mathcal{A})$, that is,

$$\mathcal{E}(U(t)) \le \frac{C}{(1+t)^{1/2}} |U_0|^2_{\mathcal{H}}$$

for every $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathcal{A})$ and for some $C > 0$ (see also equation (31)). In this way, we succeed to improve the decay rate of an exponential factor 2.
Moreover, we observe that in all the systems above the constants α, β, λ and σ can be replaced by bounded functions of the space variable, provided they are strictly positive.

The paper is organized as follows: in the next section we introduce the abstract setting that fits the previous PDEs systems into a first order Cauchy problem and, in this framework, we state our main result together with general stability and spectral properties. Finally, in Section 3, we prove the required estimates on the two classes of systems of hyperbolic equations introduced above.

2 Abstract setting

Let $(H, \langle\ ,\ \rangle)$ be a Hilbert space on the field $\mathbb{C}$ of complex numbers with associated norm $|\ |_H$. We consider the abstract system of evolution equations

$$\begin{cases} u''(t) + A_1 u(t) + B u'(t) + \alpha v = 0 & \text{in } H \\ v''(t) + A_2 v(t) + \alpha u = 0 & \text{in } H \end{cases} \tag{9}$$

where

(H1) $A_i : D(A_i) \subset H \to H$ $(i = 1, 2)$ are densely defined closed linear selfadjoint operators such that

$$\langle A_i u, u\rangle \geq \omega_i |u|^2 \qquad \forall u \in D(A_i)\,,\ i = 1, 2\,, \quad \text{for some } \ \omega_1, \omega_2 > 0\,. \tag{10}$$

(H2) B is a bounded linear selfadjoint operator on H such that

$$\langle Bu, u\rangle \geq \beta |u|^2 \qquad \forall u \in H\,, \text{ for some } \beta > 0. \tag{11}$$

(H3) α is a real number such that $0 < |\alpha| < \sqrt{\omega_1 \omega_2}\,$.

We associate to the operator A_i, $i = 1, 2$, the energy

$$E_i(u, p) = \frac{1}{2}\left(|A_i^{1/2} u|_H^2 + |p|_H^2\right) \qquad \forall (u, p) \in D(A_i^{1/2}) \times H \quad (i = 1, 2)\,, \tag{12}$$

so that assumption (H1) yields

$$|u|_H^2 \leq \frac{2}{\omega_i} E_i(u, p) \qquad \forall (u, p) \in D(A_i^{1/2}) \times H \quad (i = 1, 2)\,. \tag{13}$$

System (9), with initial conditions

$$\begin{cases} u(0) = u^0 \in D(A_1^{1/2})\,, & u'(0) = u^1 \in H\,, \\ v(0) = v^0 \in D(A_2^{1/2})\,, & v'(0) = v^1 \in H\,, \end{cases} \tag{14}$$

can be formulated as a first order Cauchy problem in the space

$$\mathcal{H} = D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H\,,$$

that becomes a Hilbert space on $\mathbb{C}$ endowed with the scalar product

$$(U, \hat{U}) := \langle A_1^{1/2} u, A_1^{1/2} \hat{u}\rangle + \langle p, \hat{p}\rangle + \langle A_2^{1/2} v, A_2^{1/2} \hat{v}\rangle + \langle q, \hat{q}\rangle + \alpha\langle u, \hat{v}\rangle + \alpha\langle v, \hat{u}\rangle \tag{15}$$

for every $U = (u, p, v, q)$, $\hat{U} = (\hat{u}, \hat{p}, \hat{v}, \hat{q}) \in \mathcal{H}$, that also induces the norm $|U|_{\mathcal{H}} := (U, U)^{1/2}$ on $\mathcal{H}$. Indeed, introducing the operator

$$\begin{cases} D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}) \\ \mathcal{A}U = (p, -A_1 u - Bp - \alpha v, q, -A_2 v - \alpha u) \quad \forall U \in D(\mathcal{A})\,, \end{cases} \tag{16}$$

problem (9) can be recast as

$$\begin{cases} U'(t) = \mathcal{A}U(t) = \begin{pmatrix} 0 & I & 0 & 0 \\ -A_1 & -B & -\alpha I & 0 \\ 0 & 0 & 0 & I \\ -\alpha I & 0 & -A_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} \\ U(0) = U_0 = (u^0, u^1, v^0, v^1) \in \mathcal{H}\,, \end{cases} \tag{17}$$

where I stands for the identity operator on H. Moreover, for every $U \in \mathcal{H}$, we define the total energy of system (17) by

$$\mathcal{E}(U(t)) := E_1(u, p) + E_2(v, q) + 2\alpha \mathtt{Re}\langle u, v\rangle = \frac{|U|_{\mathcal{H}}^2}{2}\,, \tag{18}$$

where $\mathtt{Re}(z)$ denotes the real part of $z \in \mathbb{C}$. By assumption (H3), the total energy $\mathcal{E}$ is equivalent to the sum of the energy of each component, indeed it satisfies

$$\nu_1(\alpha)\,[E_1(u,p) + E_2(v,q)] \leq \mathcal{E}(U(t)) \leq \nu_2(\alpha)\,[E_1(u,p) + E_2(v,q)]\ , \qquad (19)$$

where $\nu_1(\alpha) = 1 - |\alpha|(\omega_1\omega_2)^{-1/2} > 0$ and $\nu_2(\alpha) = 1 + |\alpha|(\omega_1\omega_2)^{-1/2}$.
The operator $\mathcal{A}$ generates a C_0-semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$ (see Lemma 4.2 in [3]), that satisfies $e^{t\mathcal{A}}U_0 = (u(t), p(t), v(t), q(t))$, where (u, v) is the solution of problem (9) with initial conditions (14) and $(p, q) = (u', v')$.

2.1 Stability properties and spectral criteria

In [2] the authors prove that system (9)-(14), or, equivalently, system (17), fails to be exponentially (uniformly) stable, at least when H is infinite dimensional and the operator A_1 has compact resolvent. This feature is a consequence of the compactness of the coupling operator $K\binom{u}{u'} = \binom{0}{\alpha u}$ in the energy space $D(A_i^{1/2}) \times H$ $(i = 1, 2)$. Thus, if any stability property holds for system (9)-(14), then such a property must be weaker than exponential stability. A first step in this direction consists in checking that the total energy of the system fulfills a dissipation relation.

Proposition 1. *For every $U_0 \in D(\mathcal{A})$ we have that*

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}p|_H^2 = \mathtt{Re}(\mathcal{A}U, U)_{\mathcal{H}}\,. \qquad (20)$$

Our approach strongly relies on the spectral properties of the operator $\mathcal{A}$. The next result gives a first insight into the spectrum of $\mathcal{A}$.

Proposition 2. *Assume hypotheses (H1)-(H2)-(H3) hold. Then $i\mathbb{R} \subset \rho(\mathcal{A})$, that is, the resolvent set of $\mathcal{A}$ contains the imaginary axis.*

Proof. Let $b \in \mathbb{R}$ and $U = (u, p, v, q) \in D(\mathcal{A})$ such that

$$\mathcal{A}U = ibU\,. \qquad (21)$$

We claim that $U = 0$. We first address the case $b \neq 0$. Then, equation (21) can be recast as

$$\begin{cases} ibu = p \\ ibp = -A_1u - Bp - \alpha v \\ ibv = q \\ ibq = -A_2v - \alpha u\,. \end{cases} \qquad (22)$$

We multiply both sides of (21) for U and take the real part. Thus, owing to equation (20), we deduce that $|B^{1/2}p|_H = 0$. Then, hypothesis (H2) implies $p = 0$; by the first equation in (22) we have $u = 0$ (since $b \neq 0$); from the second equation in (22) we find $v = 0$ (since $\alpha \neq 0$); finally, the third equation in (22) ensures that $q = 0$, so we conclude that $U = 0$. On the other hand, if $b = 0$, system (22) reduces to $p = q = 0$ and

$$\begin{cases} A_1u + \alpha v = 0 \\ A_2v + \alpha u = 0\,. \end{cases} \qquad (23)$$

We now argue by contradiction. Suppose there exist $u \neq 0 \neq v$ satisfying system (23). Thus, multiplying the first equation therein by u and the second by v, thanks to hypothesis (H1), we have

$$\begin{aligned} \omega_1 |u|_H^2 &\leq \langle A_1 u, u\rangle = -\alpha \langle v, u\rangle \,, \\ \omega_2 |v|_H^2 &\leq \langle A_2 v, v\rangle = -\alpha \langle u, v\rangle \,. \end{aligned} \tag{24}$$

Since the left upper terms in (24) are positive, we multiply both sides together and we use the Cauchy-Schwarz inequality,

$$\omega_1 \omega_2 |u|_H^2 |v|_H^2 \leq \alpha^2 |\langle u, v\rangle|^2 \leq \alpha^2 |u|_H^2 |v|_H^2 \,, \tag{25}$$

infering $\omega_1 \omega_2 \leq \alpha^2$, that negates hypothesis (H3). Therefore $u = v = 0$ and we conclude again that $U = 0$. □

As a consequence, thanks to a characterization due to Benchimol [6], we deduce that system (17) is strongly stable.

Corollary 1. *Under hypotheses (H1)-(H2)-(H3), the semigroup $e^{t\mathcal{A}}$ is strongly stable, that is,*

$$\lim_{t \to +\infty} |e^{t\mathcal{A}} U_0|_{\mathcal{H}} = 0 \qquad \forall U_0 \in \mathcal{H} \,. \tag{26}$$

We now further analyze the asymptotic behaviour of the contraction semigroup $e^{t\mathcal{A}}$ with generator $(\mathcal{A}, D(\mathcal{A}))$. For this purpose, we recall a result due to Borichev and Tomilov [7, Theorem 2.4], which gives a necessary and sufficient condition for the polynomial decay of the semigroup norm. Given two functions f and g, we use the notations $f(t) = O(g(t))$ as $t \to \infty$ when the function $|f(t)/g(t)|$ is bounded for large t; and $f(t) = o(g(t))$ as $t \to \infty$ if the function $f(t)/g(t)$ tends to 0 as $t \to \infty$.

Proposition 3. *Let $T(t)$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that the imaginary axis $i\mathbb{R}$ lies in the resolvent set $\rho(A)$ of A. For every fixed $\gamma > 0$, the following conditions are equivalent:*

$$\|R(ib, A)\|_{\mathcal{L}(H)} = O(|b|^\gamma) \qquad \text{as } b \to +\infty \,; \tag{27}$$

$$\|T(t) A^{-1}\|_{\mathcal{L}(H)} = O(|t|^{-1/\gamma}) \qquad \text{as } t \to +\infty \,; \tag{28}$$

$$\|T(t) A^{-1} x\|_H = o(|t|^{-1/\gamma}) \qquad \text{as } t \to +\infty \quad \forall x \in H \,. \tag{29}$$

In particular, we are interested in the case $\gamma = 4$, that is the decay rate we will show for all the systems addressed in Section 1. Indeed, in the following sections we will prove for each of those systems the next stabilization result.

Theorem 1. *Suppose hypotheses (H1)-(H2)-(H3) hold, and let $(\mathcal{A}, D(\mathcal{A}))$ the operator defined in*(16)-(17)*. Assume moreover that*

$$\|R(ib, \mathcal{A})\|_{\mathcal{L}(\mathcal{H})} = O(|b|^4) \quad \text{as } b \to +\infty \,. \tag{30}$$

Then, for every integer $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$|U(t)|_{\mathcal{H}} = |e^{t\mathcal{A}} U_0|_{\mathcal{H}} \leq \frac{C_m}{(1+t)^{m/4}} |U_0|_{D(\mathcal{A}^m)} \quad \forall t \geq 0 \,,\; U_0 \in D(\mathcal{A}^m) \,. \tag{31}$$

Once condition (30) has been proved for each system under consideration in Section 1, we can conclude that the total energy decays polynomially at infinity, with respect to the regularity of the initial condition U_0. In particular, for every $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathcal{A})$,

$$\mathcal{E}(U(t)) \leq \frac{C}{(1+t)^{1/2}} |U_0|^2_{\mathcal{H}} \quad \forall t \geq 0 \,. \tag{32}$$

Remark 1. i) Whether the decay rate in (32) is sharp or not, is related to the optimality of the estimate we give in the relation (30). Since, along the computations of the proof below, we try to get the finest estimate on the exponent γ (see equations (42)-(75)), we conjecture that the estimate (32) is optimal for the systems at hand.

ii) We can consider systems with more general coupling operators such as

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 P v = 0 & \text{in } H \\ v''(t) + A_2 v(t) + \alpha_2 P^* u = 0 & \text{in } H \end{cases} \tag{33}$$

where P is a bounded linear coercive operator on H, P^* is its adjoint operator, and hypothesis $(H3)$ is replaced by

(H3)' α_1 and α_2 are two real numbers such that $0 < \alpha_1\alpha_2 < \dfrac{\omega_1\omega_2}{\|P\|^2_{\mathcal{L}(H)}}$.

In the case $\alpha_i > 0$, the total energy of system (33) is defined by

$$\mathcal{E}(U(t)) := \alpha_2 E_1(u, p) + \alpha_1 E_2(v, q) + \alpha_1\alpha_2 \langle Pu, v\rangle + \alpha_1\alpha_2 \langle P^* v, u\rangle$$

and still verifies the estimate of Theorem 1.

3 Indirect stabilization by resolvent estimates

In the following, we will denote by $H^k(\Omega), H^k_0(\Omega)$ the usual Sobolev spaces with norm

$$|u|_{H^k} = \left(\int_\Omega \sum_{|p|\leq k} |D^p u|^2 dx \right)^{1/2}$$

where we have set $D^p = \partial^{p_1}_{x_1} \dots \partial^{p_d}_{x_d}$ for any multi-index $p = (p_1, \dots, p_d)$. Moreover, we will refer to $C_\Omega > 0$ as the largest constant such that the Poincars inequality

$$C_\Omega |u|^2_{L^2(\Omega)} \leq |\nabla u|^2_{L^2(\Omega)} \tag{34}$$

holds true for any $u \in H^1_0(\Omega)$. In the following sections we set $H := L^2(\Omega)$ and we define the bounded operator $B : H \to H$ by $Bu = \beta u$ for all $u \in H$, for some positive constant $\beta > 0$.

3.1 Indirect stabilization of a wave-wave system

Consider the weakly coupled system of wave equations

$$\begin{cases} u_{tt} - \Delta u + \lambda u + \beta u_t + \alpha v = 0 & \text{in } \Omega \times (0, +\infty) \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in } \Omega \times (0, +\infty) \\ \frac{\partial u}{\partial \nu} + \sigma u = 0 = v & \text{on } \Gamma \times (0, +\infty) \\ u(0) = u^0\,,\ u_t(0) = u^1\,,\ v(0) = v^0\,,\ v_t(0) = v^1 & \text{in } \Omega\,, \end{cases} \tag{35}$$

where $\alpha \in \mathbb{R}$ and $\lambda, \sigma \geq 0$ with $\max(\lambda, \sigma) > 0$.
We can rewrite system (35) as (9)-(14) (or (17), equivalently) introducing the operators

$$D(A_1) := \{u \in H^2(\Omega) : \left(\tfrac{\partial u}{\partial \nu} + \sigma u\right)(\cdot, t) = 0 \text{ on } \Gamma,\ t > 0\},\ A_1 u = -\Delta u + \lambda u\,,$$
$$D(A_2) := H^2(\Omega) \cap H^1_0(\Omega)\,, \quad A_2 v = -\Delta v\,.$$

Let μ_0^2 be the least eigenvalue of $-\Delta + \lambda I$ with Neumann/Robin boundary condition, and λ_0^2 be the least eigenvalue of $-\Delta$ on $H^2(\Omega) \cap H^1_0(\Omega)$. So the assumption (H3) on α requires

$$0 < |\alpha| < \lambda_0 \mu_0\,. \tag{36}$$

For every $U \in \mathcal{H} = H^1(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)$, the energy associated to system (35) is

$$\mathcal{E}(U(t)) := \frac{1}{2} \int_\Omega \left[u_t^2 + |\nabla u|^2 + \lambda u^2 + v_t^2 + |\nabla v|^2 + 2\alpha \texttt{Re}(u\bar{v})\right] dx + \frac{1}{2} \int_\Gamma \sigma u^2 d\Sigma = \frac{|U|^2_{\mathcal{H}}}{2}\,, \tag{37}$$

which, for every $U \in D(\mathcal{A})$, satisfies

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_\Omega \beta u_t^2 dx = \texttt{Re}(\mathcal{A}U(t), U(t))_{\mathcal{H}}\,. \tag{38}$$

Corollary 1 ensures that system (35) is strongly stable. In the sequel we want to show that condition (30) holds for system (35). In this way, we prove that the total energy of the system decays polynomially at infinity, with respect to the regularity of the initial condition U_0. In particular, for every $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathcal{A})$,

$$\mathcal{E}(U(t)) \leq \frac{C}{(1+t)^{1/2}} \left(|u^0|^2_{H^2} + |u^1|^2_{H^1} + |v^0|^2_{H^2} + |v^1|^2_{H^1_0}\right)\,. \tag{39}$$

Proof of Theorem 1 for system (35). Thanks to Proposition 2 we know that $i\mathbb{R} \subset \rho(\mathcal{A})$. Thus, we need to show that

$$\|(ibI - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} = O(|b|^4) \quad \text{as } b \to +\infty\,. \tag{40}$$

Let $U \in \mathcal{H}$ and $b \in \mathbb{R}$ such that $|b| \geq \max(1, \beta, \lambda)$. Since $Rank(ibI - \mathcal{A}) = \mathcal{H}$, there exists $Z \in D(\mathcal{A})$ such that

$$ibZ - \mathcal{A}Z = U \qquad \text{in } \mathcal{H}\,. \tag{41}$$

Thus, the estimate (40) will hold once provided that there exists $C_\alpha > 0$ (depending on Ω and α but not on b) such that

$$|Z| \leq C_\alpha |b|^4 |U| \,, \tag{42}$$

where C_α blows up as $|\alpha|$ goes to 0 or to $\lambda_0\mu_0$, and, for simplicity of notation, we set $|Z| = |Z|_{\mathcal{H}}$. Denoting $Z = (u,p,v,q) \in D(\mathcal{A})$ and $U = (f,g,h,k) \in \mathcal{H}$, equation (41) reads as

$$\begin{cases} ibu - p = f & \text{in } H^1(\Omega) \\ ibp - \Delta u + \lambda u + \beta p + \alpha v = g & \text{in } L^2(\Omega) \\ ibv - q = h & \text{in } H^1_0(\Omega) \\ ibq - \Delta v + \alpha u = k & \text{in } L^2(\Omega) \,. \end{cases} \tag{43}$$

We will proceed in several steps, evaluating each term of the norm $|Z|$.

Step 1: Estimate of $|p|_H$ and $|bu|_H$.
We first multiply by Z both sides of equation (41) and then take the real part of it. Thanks to the right identity in (38), we deduce that $\beta|p|_H^2 = \mathtt{Re}(U,Z)$, so

$$\beta|p|_H^2 \leq |U|\,|Z| \,. \tag{44}$$

Then, from the first equation of system (43), and using equation (19), we deduce that

$$\begin{aligned} |bu|_H^2 \leq 2|p|_H^2 + 2|f|_H^2 \leq \frac{2}{\beta}|U|\,|Z| + \frac{2}{\mu_0^2}|f|^2_{H^1(\Omega)} \\ \leq \frac{2}{\beta}|U|\,|Z| + \frac{2\mu_0^{-2}}{1-|\alpha|(\lambda_0\mu_0)^{-1}}|U|^2 \,, \end{aligned}$$

so

$$|bu|_H^2 \leq \frac{2}{\beta}|U|\,|Z| + K_\alpha|U|^2 \,, \tag{45}$$

where here and in the following K_α denotes a generic constant depending on α and Ω, which blows up as $|\alpha| \nearrow \lambda_0\mu_0$.

Step 2: Estimate of $|\nabla u|_H^2 + \lambda|u|_H^2 + \sigma|u|^2_{L^2(\Gamma)}$.
Consider the scalar product in H of the second identity in system (43) with u

$$\int_\Omega (ibp - \Delta u + \lambda u + \beta p + \alpha v)\overline{u}dx = \int_\Omega g\overline{u}dx \,.$$

Integration by parts leads to

$$\int_\Omega (|\nabla u|^2 + \lambda u^2)dx + \int_\Gamma \sigma u^2 d\Sigma = \mathtt{Re}\int_\Omega (g - (ib+\beta)p - \alpha v)\overline{u}dx \,. \tag{46}$$

We now evaluate each terms in the right-hand side integral. First,

$$\begin{aligned} \left|\int_\Omega g\overline{u}dx\right| &\leq |g|_H|u|_H \leq \frac{1}{\mu_0}|g|_H\left[\int_\Omega (|\nabla u|^2 + \lambda u^2)dx + \int_\Gamma \sigma u^2 d\Sigma\right]^{1/2} \\ &\leq \frac{1}{2\mu_0^2}|g|_H^2 + \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)dx + \frac{1}{2}\int_\Gamma \sigma u^2 d\Sigma \\ &\leq K_\alpha|U|^2 + \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)dx + \frac{1}{2}\int_\Gamma \sigma u^2 d\Sigma \,. \end{aligned}$$

Moreover, thanks to (44)-(45), and keeping in mind that $|b| \geq \beta$,

$$\left|\int_\Omega (ib+\beta)p\overline{u}dx\right| \leq \frac{b^2+\beta^2}{4}|u|_H^2 + |p|_H^2 \leq \frac{1}{2}|bu|_H^2 + |p|_H^2 \leq \frac{2}{\beta}|U|\,|Z| + K_\alpha |U|^2 .$$

Finally, thanks to (45),

$$\left|\int_\Omega \alpha v\overline{u}dx\right| \leq |\alpha||u|_H|v|_H \leq \frac{|\alpha|}{|b|}K_\alpha |Z|\,|bu|_H$$
$$\leq \frac{|\alpha|K_\alpha}{|b|}|Z|\left(|U|^{1/2}|Z|^{1/2} + K_\alpha|U|\right) \leq \frac{|\alpha|K_\alpha}{|b|}\left(|U|^{1/2}|Z|^{3/2} + K_\alpha|U|\,|Z|\right) .$$

Plugging the last three inequalities in equation (46), we derive that

$$\int_\Omega (|\nabla u|^2 + \lambda u^2)dx + \int_\Gamma \sigma u^2 d\Sigma \leq K_\alpha \left(|U|\,|Z| + |U|^2 + \frac{|\alpha|}{|b|}|U|^{1/2}|Z|^{3/2}\right) , \quad (47)$$

or, for a partial estimate,

$$\int_\Omega (|\nabla u|^2 + \lambda u^2)dx + \int_\Gamma \sigma u^2 d\Sigma \leq \frac{4}{\beta}|U|\,|Z| + K_\alpha|U|^2 + 2|\alpha||u|_H|v|_H . \quad (48)$$

Step 3: Partial estimate of $|\nabla v|_H$.
From the third identity in system (43) we derive that $q = ibv - h$, and using this relation in the fourth equation of (43) we obtain $-b^2 v - \Delta v + \alpha u = ibh + k$. The scalar product by v of both sides of this relation gives, after integration by parts,

$$\int_\Omega |\nabla v|^2 dx = \int_\Omega b^2 v^2 dx + \mathtt{Re}\int_\Omega (ibh + k - \alpha u)\overline{v}dx . \quad (49)$$

Since

$$\left|\int_\Omega (ibh + k - \alpha u)\overline{v}dx\right| \leq |h|_H|bv|_H + \frac{|k|_H}{|b|}|bv|_H + |\alpha|\frac{|u|_H}{|b|}|bv|_H$$
$$\leq |bv|_H^2 + \frac{1}{4}(|h|_H^2 + |k|_H^2) + \frac{|\alpha|^2}{2b^4}|bu|_H^2 \leq |bv|_H^2 + \frac{|\alpha|^2}{\beta b^4}|U|\,|Z| + K_\alpha|U|^2 ,$$

from equation (49) we deduce that

$$\int_\Omega |\nabla v|^2 dx \leq 2|bv|_H^2 + \frac{|\alpha|^2}{\beta b^4}|U|\,|Z| + K_\alpha|U|^2 . \quad (50)$$

Step 4: Estimate of $|bv|_H$.
Taking the inner product between the second equation in (43) and $b^2 v$, we have

$$\int_\Omega \{[(ib+\beta)p - \Delta u + \lambda u + \alpha v]b^2\overline{v}\}dx = \int_\Omega b^2 g\overline{v}dx ,$$

so we have that

$$\int_\Omega b^2 v^2 dx = \frac{b^2}{\alpha}\mathtt{Re}\int_\Omega [(g - (ib+\beta)p - \lambda u)\overline{v} - \nabla u . \nabla\overline{v}]dx . \quad (51)$$

First, note that, thanks to (44), (45) and $|b| \geq \max(1, \beta, \lambda)$,

$$\left|\frac{b^2}{\alpha}\int_\Omega (g-(ib+\beta)p-\lambda u)\overline{v}dx\right| \leq \int_\Omega \left[\frac{3b^2v^2}{4}+\frac{b^2g^2}{\alpha^2}+\frac{b^4p^2}{\alpha^2}+\frac{\lambda^2b^2u^2}{\alpha^2}\right]dx$$
$$\leq \frac{3}{4}|bv|_H^2 + \frac{K_\alpha}{\alpha^2}b^2|U|^2 + \frac{C}{\alpha^2}b^4|U|\,|Z|\,, \quad (52)$$

where here and in the following C denotes a generic positive constant, independent from α and b. Second, observe that, thanks to (48) and (50), we find out that

$$\left|\frac{b^2}{\alpha}\int_\Omega \nabla u.\,\nabla\overline{v}dx\right| \leq \frac{b^2}{|\alpha|}|\nabla u|_H|\nabla v|_H$$
$$\leq \frac{b^2}{|\alpha|}\left(\sqrt{2}|\alpha|^{1/2}|u|_H^{1/2}|v|_H^{1/2}+\frac{2}{\sqrt{\beta}}|U|^{1/2}|Z|^{1/2}+K_\alpha|U|\right)$$
$$\cdot\left(\sqrt{2}|bv|_H+\frac{|\alpha|}{\sqrt{\beta}b^2}|U|^{1/2}|Z|^{1/2}+K_\alpha|U|\right)$$
$$\leq C\left(\frac{b}{|\alpha|^{1/2}}|bv|_H^{3/2}|bu|_H^{1/2}+\frac{b^2}{|\alpha|}|U|^{1/2}|Z|^{1/2}|bv|_H+\frac{K_\alpha}{|\alpha|}b^2|U|\,|bv|_H\right.$$
$$+|\alpha|^{1/2}|u|_H^{1/2}|v|_H^{1/2}|U|^{1/2}|Z|^{1/2}+|U|\,|Z|+K_\alpha|U|^{3/2}|Z|^{1/2}$$
$$\left.+\frac{K_\alpha}{|\alpha|^{1/2}}b^2|u|_H^{1/2}|v|_H^{1/2}|U|+\frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}+\frac{K_\alpha}{|\alpha|}b^2|U|^2\right)\,. \quad (53)$$

Let ε be a positive real number. We recall Young's inequality in the form

$$|xy| \leq \frac{|x|^p}{pq^{1/q}\varepsilon^{p/q}}+\varepsilon|y|^q \quad (54)$$

that holds for every real numbers x, y and for a suitable pair of conjugate exponents (p,q). For $p=4$ and $q=4/3$, we deduce that

$$\frac{b}{|\alpha|^{1/2}}|bv|_H^{3/2}|bu|_H^{1/2} \leq \varepsilon|bv|_H^2+C_\varepsilon\frac{b^4}{\alpha^2}|bu|_H^2$$
$$\leq \varepsilon|bv|_H^2+C_\varepsilon\frac{b^4}{\alpha^2}|U|\,|Z|+\frac{K_{\alpha,\varepsilon}}{\alpha^2}b^4|U|^2\,, \quad (55)$$

where C_ε and $K_{\alpha,\varepsilon}$ are positive constants that diverge as ε goes to 0^+. Similarly, thanks to standard Young's inequality with parameter ε, we deduce that

$$\frac{b^2}{|\alpha|}|U|^{1/2}|Z|^{1/2}|bv|_H \leq \varepsilon|bv|_H^2+C_\varepsilon\frac{b^4}{\alpha^2}|U|\,|Z|\,, \quad (56)$$

$$\frac{K_\alpha}{|\alpha|}b^2|U|\,|bv|_H \leq \varepsilon|bv|_H^2+\frac{K_{\alpha,\varepsilon}}{\alpha^2}b^4|U|^2\,, \quad (57)$$

$$|\alpha|^{1/2}|u|_H^{1/2}|v|_H^{1/2}|U|^{1/2}|Z|^{1/2}$$
$$=(|bv|_H^{1/2}|U|^{1/4}|Z|^{1/4})\left(\frac{|\alpha|^{1/2}}{|b|}|bu|_H^{1/2}|U|^{1/4}|Z|^{1/4}\right)$$
$$\leq |bv|_H|U|^{1/2}|Z|^{1/2}+\frac{|\alpha|}{4b^2}|bu|_H|U|^{1/2}|Z|^{1/2}$$
$$\leq \varepsilon|bv|_H^2+C_\varepsilon|U|\,|Z|+\frac{1}{b^4}|bu|_H^2 \leq \varepsilon|bv|_H^2+C_\varepsilon|U|\,|Z|+\frac{K_\alpha}{b^4}|U|^2\,, \quad (58)$$

$$\frac{K_\alpha}{|\alpha|^{1/2}} b^2 |u|_H^{1/2} |v|_H^{1/2} |U| \le \frac{K_\alpha}{|\alpha|^{1/2}} |b||U| \left(\frac{|bu|_H}{2} + \frac{|bv|_H}{2} \right)$$
$$\le \varepsilon |bv|_H^2 + \frac{K_{\alpha,\varepsilon}}{|\alpha|} b^2 |U|^2 + |bu|_H^2 + \frac{K_\alpha}{|\alpha|} b^2 |U|^2$$
$$\le \varepsilon |bv|_H^2 + \frac{K_{\alpha,\varepsilon}}{|\alpha|} b^2 |U|^2 + \frac{2}{\beta} |U|\,|Z| \,. \quad (59)$$

Gathering estimates (55)-...-(59) in relation (53), we end up with

$$\left| \frac{b^2}{\alpha} \int_\Omega \nabla u . \nabla \overline{v} dx \right| \le 5C\varepsilon |bv|_H^2 + C_\varepsilon \frac{b^4}{\alpha^2} |U|\,|Z| + \frac{K_{\alpha,\varepsilon}}{\alpha^2} b^4 |U|^2 + \frac{K_\alpha}{|\alpha|} b^2 |U|^{3/2} |Z|^{1/2} \,. \quad (60)$$

Back to relation (51), owing to (52) and (60), we have

$$(1 - 20C\varepsilon)|bv|_H^2 \le \frac{K_{\alpha,\varepsilon}}{\alpha^2} b^4 |U|^2 + \frac{C_\varepsilon}{\alpha^2} b^4 |U|\,|Z| + \frac{K_\alpha}{|\alpha|} b^2 |U|^{3/2} |Z|^{1/2} \,,$$

that, for a sufficiently small $\varepsilon > 0$, ensures that

$$|bv|_H^2 \le \frac{K_\alpha}{\alpha^2} b^4 |U|^2 + \frac{C}{\alpha^2} b^4 |U|\,|Z| + \frac{K_\alpha}{|\alpha|} b^2 |U|^{3/2} |Z|^{1/2} \,. \quad (61)$$

Step 5: Estimate of $|\nabla v|_H$ and $|q|_H$.
Using estimate (61), relation (50) yields

$$|\nabla v|_H^2 \le \frac{K_\alpha}{\alpha^2} b^4 |U|^2 + \frac{C}{\alpha^2} b^4 |U|\,|Z| + \frac{K_\alpha}{|\alpha|} b^2 |U|^{3/2} |Z|^{1/2} \,. \quad (62)$$

On the other hand, by the third equation in system (43), we conclude that

$$|q|_H^2 \le 2|bv|_H^2 + 2|h|_H^2 \le \frac{K_\alpha}{\alpha^2} b^4 |U|^2 + \frac{C}{\alpha^2} b^4 |U|\,|Z| + \frac{K_\alpha}{|\alpha|} b^2 |U|^{3/2} |Z|^{1/2} \,. \quad (63)$$

Final Step: Resolvent estimate.
Thanks to equations (44)-(47)-(62)-(63), we deduce that

$$|Z|^2 \le \nu_2(\alpha) \left[\int_\Omega \left(p^2 + |\nabla u|^2 + \lambda u^2 + q^2 + |\nabla v|^2 \right) dx \right]$$
$$\le C_\alpha \left[b^4 |U|^2 + b^4 |U|\,|Z| + b^2 |U|^{3/2} |Z|^{1/2} + |U|^{1/2} |Z|^{3/2} \right] \,,$$

where C_α is a positive constant depending only on Ω and α (but not on b) that blows up as $|\alpha|$ goes to 0 or to $\lambda_0 \mu_0$. Applying again Young's inequality with suitable choices of conjugate exponents (p, q), we conclude that

$$|Z|^2 \le C_\alpha |b|^8 |U|^2 \,,$$

that completes the proof of relation (42). □

Observe that similar calculations, with suitably different space $D(A_1)$ and total energy, ensure the same indirect stabilization result also for a system with different boundary conditions on different portion of the boundary, as in the case of system (1) with boundary conditions

$$\begin{matrix} u(\cdot,t) = 0 \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \nu}(\cdot,t) + \sigma u(\cdot,t) = 0 \quad \text{on } \Gamma_1 \\ v(\cdot,t) = 0 \quad \text{on } \Gamma \end{matrix} \qquad t > 0\,,$$

with $\max(\lambda, \sigma) > 0$, and where Γ_0, Γ_1 are open subsets of Γ such that

$$\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1\,, \quad \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset\,.$$

3.2 Stabilization for a wave-Petrowsky system

We now focus on the stabilization problem for the weakly coupled system

$$\begin{cases} u_{tt} - \Delta u + \beta u_t + \alpha v = 0 & \text{in } \Omega \times (0,+\infty) \\ v_{tt} + \Delta^2 v + \alpha u = 0 & \text{in } \Omega \times (0,+\infty) \\ u(0) = u^0\,,\ u_t(0) = u^1\,,\ v(0) = v^0\,,\ v_t(0) = v^1 & \text{in } \Omega\,, \end{cases} \tag{64}$$

for some $\beta > 0$ and $\alpha \in \mathbb{R}$, with Robin boundary conditions

$$\frac{\partial u}{\partial \nu}(\cdot,t) + \sigma u(\cdot,t) = 0 \quad \text{on } \Gamma\,,\ t > 0 \tag{65}$$

on u (for some $\sigma > 0$) and either clamped

$$v(\cdot,t) = 0 = \frac{\partial v}{\partial \nu}(\cdot,t) \quad \text{on } \Gamma\,,\ t > 0 \tag{66}$$

or hinged boundary conditions

$$v(\cdot,t) = 0 = \Delta v(\cdot,t) \quad \text{on } \Gamma\,,\ t > 0 \tag{67}$$

on v. We can rewrite systems (64)-(65)-(66) and (64)-(65)-(67) as (9) (or (17), equivalently) introducing the operators

$$D(A_1) := \{u \in H^2(\Omega) : \left(\frac{\partial u}{\partial \nu} + \sigma u\right)(\cdot,t) = 0 \text{ on } \Gamma\,,\ t > 0\}\,,\ A_1 u = -\Delta u \tag{68}$$

and

$$D(A_2) := \{v \in H^4(\Omega) : v(\cdot,t) = 0 = \frac{\partial v}{\partial \nu}(\cdot,t) \text{ on } \Gamma\,,\ t > 0\}\,,\ A_2 v = \Delta^2 v \tag{69}$$

or

$$D(A_2) := \{v \in H^4(\Omega) : v(\cdot,t) = 0 = \Delta v(\cdot,t) \text{ on } \Gamma\,,\ t > 0\}\,,\ A_2 v = \Delta^2 v\,, \tag{70}$$

and defining $(\mathcal{A}, D(\mathcal{A}))$ as in (16). Let λ_0^2 be the least eigenvalue of $-\Delta$ with Robin boundary conditions, and μ_0^2 be the least eigenvalue of Δ^2 with either clamped or hinged boundary conditions. So the assumption (H3) on α yields $0 < |\alpha| < \lambda_0\mu_0$. For every $U \in \mathcal{H} = H^1(\Omega) \times L^2(\Omega) \times H_0^2(\Omega) \times L^2(\Omega)$, the energy associated to systems (64)-(65)-(66) or (64)-(65)-(67) is

$$\mathcal{E}(U(t)) = \frac{1}{2}\int_\Omega \left[u_t^2 + |\nabla u|^2 + v_t^2 + |\Delta v|^2 + 2\alpha \mathtt{Re}(u\bar{v})\right] dx + \frac{1}{2}\int_\Gamma \sigma u^2 d\Sigma \tag{71}$$

which, for every $U \in D(\mathcal{A})$, satisfies

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_\Omega \beta u_t^2 dx = \mathtt{Re}(\mathcal{A}U(t), U(t))_{\mathcal{H}}\,. \tag{72}$$

We now analyze the asymptotic behaviour of the contraction semigroup $e^{t\mathcal{A}}$ with generator $(\mathcal{A}, D(\mathcal{A}))$.

Proof of Theorem 1 for systems (64)-(65)-(66) *and* (64)-(65)-(67).
Proposition 2 ensures that $i\mathbb{R} \subset \rho(\mathcal{A})$. Thus, systems (64)-(65)-(66) and (64)-(65)-(67) are strongly stable and we are left to show that

$$\|(ibI - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} = O(|b|^4) \quad \text{as } s \to +\infty\,. \tag{73}$$

Let $U \in \mathcal{H}$ and $b \in \mathbb{R}$ such that $|b| \geq \max(1, \beta)$. Since $Rank(ibI - \mathcal{A}) = \mathcal{H}$, there exists $Z \in D(\mathcal{A})$ such that

$$ibZ - \mathcal{A}Z = U \qquad \text{in } \mathcal{H}\,. \tag{74}$$

Thus, the estimate (73) will hold once provided that there exists $C_\alpha > 0$ (depending on Ω and α but not on b) such that

$$|Z| \leq C_\alpha |b|^4 |U|\,, \tag{75}$$

where C blows up as $|\alpha|$ goes to 0 or to $\lambda_0\mu_0$. Denoting $Z = (u, p, v, q) \in D(\mathcal{A})$ and $U = (f, g, h, k) \in \mathcal{H}$, equation (74) reads as

$$\begin{cases} ibu - p = f & \text{in } H^1(\Omega) \\ ibp - \Delta u + \beta p + \alpha v = g & \text{in } L^2(\Omega) \\ ibv - q = h & \text{in } H_0^2(\Omega) \\ ibq + \Delta^2 v + \alpha u = k & \text{in } L^2(\Omega)\,. \end{cases} \tag{76}$$

<u>Step 1:</u> Estimate of $|p|_H$ and $|bu|_H$.
We first multiply by Z both sides of equation (74) and then take the real part of it. Thanks to the right identity in (72), we deduce that $|\beta p|_H^2 = \mathtt{Re}(U, Z)$, so

$$|p|_H^2 \leq \frac{1}{\beta}|U|\,|Z|\,, \tag{77}$$

and consequently, from the first equation of system (76), we deduce that

$$|bu|_H^2 \leq \frac{2}{\beta}|U|\,|Z| + K_\alpha |U|^2\,, \tag{78}$$

where the constant K_α blows up as $|\alpha| \nearrow \lambda_0\mu_0$.
<u>Step 2:</u> Estimate of $\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma$.
We can estimate this term of the energy as in the previous example, proving that, for $|b| \geq \max(1, \beta)$,

$$\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma \leq K_\alpha \left(|U|\,|Z| + |U|^2 + \frac{|\alpha|}{|b|}|U|^{1/2}|Z|^{3/2} \right)\,, \tag{79}$$

or

$$\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma \le \frac{4}{\beta}|U|\,|Z| + K_\alpha |U|^2 + |\alpha||u|_H |v|_H \,. \tag{80}$$

Step 3: Partial estimate of $|\Delta v|_H$.

From the third identity in system (76) we derive that $q = ibv - h$, and using this relation in the fourth equation of (76) we obtain $\Delta^2 v - b^2 v + \alpha u = ibh + k$. The scalar product by v of both sides of this relation gives, after integrations by parts,

$$\int_\Omega |\Delta v|^2 dx = \int_\Omega b^2 v^2 dx + \mathtt{Re}\int_\Omega (ibh + k - \alpha u)\overline{v} dx \,. \tag{81}$$

Thanks to (78) and the assumption $|b| \ge 1$,

$$\begin{aligned}\left|\int_\Omega (ibh + k - \alpha u)\overline{v} dx\right| &\le |h|_H |bv|_H + \frac{|k|_H}{|b|}|bv|_H + |\alpha|\frac{|bu|_H}{b^2}|bv|_H \\ &\le |bv|_H^2 + |h|_H^2 + \frac{|k|_H^2}{b^2} + \frac{\alpha^2}{2b^4}|bu|_H^2 \le |bv|_H^2 + \frac{\alpha^2}{\beta b^4}|U|\,|Z| + K_\alpha |U|^2 \,,\end{aligned}$$

so from equation (81) we deduce that

$$\int_\Omega |\Delta v|^2 dx \le 2|bv|_H^2 + \frac{\alpha^2}{\beta b^4}|U|\,|Z| + K_\alpha |U|^2 \,. \tag{82}$$

Step 4: Estimate of $|bv|_H$.

The inner product between the second equation in (76) and $b^2 v$ gives

$$\int_\Omega b^2 v^2 dx = \frac{b^2}{\alpha}\mathtt{Re}\int_\Omega [(g - (ib+\beta)p)\overline{v} - \nabla u.\,\nabla\overline{v}] dx \,. \tag{83}$$

First, note that

$$\begin{aligned}\left|\frac{b^2}{\alpha}\int_\Omega (g - (ib+\beta)p)\overline{v} dx\right| &\le \int_\Omega \left[\frac{3b^2v^2}{4} + \frac{b^2 g^2}{\alpha^2} + \frac{b^4 p^2}{\alpha^2}\right] dx \\ &\le \frac{3}{4}|bv|_H^2 + \frac{K_\alpha}{\alpha^2} b^2 |U|^2 + \frac{C}{\alpha^2} b^4 |U|\,|Z| \,. \end{aligned} \tag{84}$$

Afterwards, thanks to the well-known inequality

$$\int_\Omega (v^2 + |\nabla v|^2) dx \le \int_\Omega |\Delta v|^2 dx \qquad \forall v \in H^2(\Omega)\cap H_0^1(\Omega) \,, \tag{85}$$

and owing to (80) and (82), we find out that

$$\begin{aligned}\left|\frac{b^2}{\alpha}\int_\Omega \nabla u.\,\nabla\overline{v} dx\right| &\le \frac{b^2}{|\alpha|}|\nabla u|_H |\nabla v|_H \le \frac{b^2}{|\alpha|}|\nabla u|_H |\Delta v|_H \\ &\le \frac{Cb^2}{|\alpha|}\left(|\alpha|^{1/2}|u|_H^{1/2}|v|_H^{1/2} + \frac{2}{\sqrt{\beta}}|U|^{1/2}|Z|^{1/2} + K_\alpha |U|\right) \\ &\quad \cdot \left(\sqrt{2}|bv|_H + \frac{|\alpha|}{\sqrt{\beta} b^2}|U|^{1/2}|Z|^{1/2} + K_\alpha |U|\right) \\ &\le C\left(\frac{b}{|\alpha|^{1/2}}|bv|_H^{3/2}|bu|_H^{1/2} + \frac{b^2}{|\alpha|}|U|^{1/2}|Z|^{1/2}|bv|_H + \frac{K_\alpha}{|\alpha|}b^2|U|\,|bv|_H \right. \\ &\quad + |\alpha|^{1/2}|u|_H^{1/2}|v|_H^{1/2}|U|^{1/2}|Z|^{1/2} + |U|\,|Z| + K_\alpha |U|^{3/2}|Z|^{1/2} \\ &\quad \left. + \frac{K_\alpha}{|\alpha|^{1/2}} b^2 |u|_H^{1/2}|v|_H^{1/2}|U| + \frac{K_\alpha}{|\alpha|} b^2 |U|^{3/2}|Z|^{1/2} + \frac{K_\alpha}{|\alpha|} b^2 |U|^2\right) \,. \end{aligned} \tag{86}$$

Let ε be a positive real number. Applying Young's inequality together with estimates (55)-...-(59), from (86) we conclude that

$$\left|\frac{b^2}{\alpha}\int_\Omega \nabla u.\,\nabla\overline{v}dx\right| \leq 5C\varepsilon|bv|_H^2 + C_\varepsilon\frac{b^4}{\alpha^2}|U|\,|Z| + \frac{K_{\alpha,\varepsilon}}{\alpha^2}b^4|U|^2 + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,. \quad (87)$$

Back to relation (83), owing to (84) and (87), we have

$$(1-20C\varepsilon)|bv|_H^2 \leq \frac{K_{\alpha,\varepsilon}}{\alpha^2}b^4|U|^2 + \frac{C_\varepsilon}{\alpha^2}b^4|U|\,|Z| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,.$$

So, for a sufficiently small $\varepsilon > 0$, we deduce that

$$|bv|_H^2 \leq \frac{K_\alpha}{\alpha^2}b^4|U|^2 + \frac{C}{\alpha^2}b^4|U|\,|Z| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,. \quad (88)$$

Step 5: Estimate of $|\nabla v|_H$ and $|q|_H$.
Using estimate (88) in relation (82) ensures that

$$|\nabla v|_H^2 \leq \frac{K_\alpha}{\alpha^2}b^4|U|^2 + \frac{C}{\alpha^2}b^4|U|\,|Z| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,. \quad (89)$$

On the other hand, by the third equation in system (76), we conclude that

$$|q|_H^2 \leq 2|bv|_H^2 + 2|h|_H^2 \leq \frac{K_\alpha}{\alpha^2}b^4|U|^2 + \frac{C}{\alpha^2}b^4|U|\,|Z| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,. \quad (90)$$

Thanks to equations (77)-(79)-(89)-(90), we deduce that

$$\begin{aligned}|Z|^2 &\leq \nu_2(\alpha)\left[\int_\Omega \left(p^2 + |\nabla u|^2 + q^2 + |\nabla v|^2\right)dx + \int_\Gamma \sigma u^2 d\Sigma\right]\\ &\leq C_\alpha\left[b^4|U|^2 + b^4|U|\,|Z| + b^2|U|^{3/2}|Z|^{1/2} + |U|^{1/2}|Z|^{3/2}\right]\,,\end{aligned}$$

where C_α is a positive constant depending only on Ω and α (but not on b) that blows up as $|\alpha|$ goes to 0 or to $\lambda_0\mu_0$. Applying again Young's inequality with suitable choices of conjugate exponents (p,q), we conclude that

$$|Z|^2 \leq C_\alpha|b|^8|U|^2\,,$$

completing the proof of relation (75). □

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