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RICAM-Report 2015-45

On Equivalence of Anchored and ANOVA Spaces; Lower Bounds

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November 18, 2015

Dedicated to the memory of Joseph F. Traub (1932-2015)

Abstract

We provide lower bounds for the norms of embeddings between γ -weighted Anchored and ANOVA spaces of s -variate functions with mixed partial derivatives of order one bounded in L_p norm ($p \in [1, \infty]$). In particular we show that the norms behave polynomially in s for *Finite Order Weights* and *Finite Diameter Weights* if $p > 1$, and increase faster than any polynomial in s for *Product Order-Dependent Weights* and any p .

Keywords: Embeddings, Weighted function spaces, Anchored decomposition, ANOVA decomposition, Equivalence of norms

MSC 2000: 65D30, 65Y20, 41A55, 41A63

1 Introduction

In this short note, we continue research on the equivalence of γ -weighted Anchored and ANOVA spaces of s -variate functions with mixed partial derivatives of order one bounded in L_p norm ($p \in [1, \infty]$). The ANOVA spaces have been investigated in a number of papers and many of their interesting properties have been found. This includes small truncation and superposition dimensions, see, e.g., [10] and papers cited there. However, in general, these results cannot be utilized in practice since ANOVA decomposition involves variances that are impossible to compute numerically. On the other hand, small truncation or superposition dimension for Anchored spaces is easy to exploit, see, e.g., [6, 8, 12]. This is why it is important to know for which weights γ the Anchored and ANOVA spaces are equivalent, or more precisely how fast the norms of the corresponding embeddings increase when the number s of variables increases. If the norms are uniformly bounded then we say that *the spaces are uniformly equivalent*. If the norms increase like a polynomial in s then we say that *the spaces are polynomially equivalent*.

*P. Kritzer is supported by the Austrian Science Fund (FWF): Project F5506-N26, which is a part of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

[†]F. Pillichshammer is partially supported by the Austrian Science Fund (FWF): Project F5509-N26, which is a part of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

This question of equivalence was first addressed in [3] for product weights and in the Hilbert space setting ($p = 2$). More precisely the authors provided lower and upper bounds on the norms of the corresponding embeddings and concluded that there is uniform equivalence if and only if the weights are summable. This result was later slightly strengthened in [6] by showing that the embedding and its inverse have the same norm and by delivering an exact formula for it.

In [4], exact values of the embedding norm were delivered for general weights but only with $p = 1$ and $p = \infty$. The results were then applied to various types of weights to see when there is uniform or polynomial equivalence. In particular, it was shown that for an important class of *Product Order-Dependent* weights there is no polynomial equivalence.

Next in [5], the authors applied the complex interpolation in conjunction with the results mentioned above to get interesting upper bounds on the norms of the embeddings for general weights and for all values of $p \in (1, \infty)$.

With the exception of product weights, the result of [5] does not provide general lower bounds for the norms of the embeddings. This is why, in this short note, we prove that the norms of the embedding and its inverse are equal and provide lower bounds for them. These lower bounds are sharp for $p = 1$ and $p = \infty$ and general weights. They are also sharp for *finite order weights* and *finite diameter weights* for any p , and we believe that they are sharp for any p and general weights.

We next use these lower bounds for the following three cases of weights.

Finite Order Weights: It was shown in [4] that the norms are uniformly bounded for $p = 1$ and are proportional to a polynomial in s for $p = \infty$. Then it was concluded in [5] that the embedding norms are bounded from above by a polynomial in s for every $p \in (1, \infty)$. Using our lower bounds we show that these norms are indeed polynomial in s for $p > 1$. More precisely they are equal to $\Theta(s^{q/p^*})$ where p^* is the conjugate of p and q is the order of the weights.

Finite Diameter Weights: Such kinds of weights have not been investigated in this context so far. We use our lower bound to show that for $p \in (1, \infty)$, the norms of the embeddings increase at least as fast as s^{1/p^*} , where p^* is the conjugate of p . We also deliver matching upper bounds to conclude that the norms are equal to $\Theta(s^{1/p^*})$.

Product Order-Dependent Weights: Since there is no polynomial equivalence for $p = 1$ and $p = \infty$, the techniques employed in [5] could not answer whether there is polynomial equivalence for $p \in (1, \infty)$ or not. Using our lower bounds we conclude that for $p \in (1, \infty)$, the norms of the embeddings increase faster than any polynomial in s .

2 Basic Facts and Notation

Following [4], we recall basic facts and notation pertaining to the *Anchored* and *ANOVA* spaces considered in this paper.

We begin with the notation that is used in this paper: by $s \in \mathbb{N}$ we denote the dimension, and the set of coordinate indices is

$$[s] = \{1, 2, \dots, s\}.$$

By $\mathbf{u}, \mathbf{v}, \mathbf{w}$ we denote subsets of $[s]$. For $\mathbf{u} \subseteq [s]$ its complement is denoted by $\mathbf{u}^c = [s] \setminus \mathbf{u}$.

Moreover, for $\mathbf{t}, \mathbf{x} \in \mathbb{R}^s$, where $\mathbf{t} = (t_1, \dots, t_s)$ and $\mathbf{x} = (x_1, \dots, x_s)$, and $\mathbf{u} \subseteq [s]$, we define

$$[\mathbf{x}_{\mathbf{u}}; \mathbf{t}_{\mathbf{u}^c}] = (y_1, \dots, y_s) \quad \text{with} \quad y_j = \begin{cases} x_j & \text{if } j \in \mathbf{u}, \\ t_j & \text{if } j \notin \mathbf{u}. \end{cases}$$

We also write $\mathbf{x}_{\mathbf{u}}$ to denote the $|\mathbf{u}|$ -dimensional vector $(x_j)_{j \in \mathbf{u}}$ and

$$f^{(\mathbf{u})} = \frac{\partial^{|\mathbf{u}|} f}{\partial \mathbf{x}_{\mathbf{u}}} = \prod_{j \in \mathbf{u}} \frac{\partial}{\partial x_j} f \quad \text{with} \quad f^{(\emptyset)} = f.$$

2.1 Anchored Spaces

For $p \in [1, \infty]$, let $F_p = W_{p,0}^1$ be the space of functions defined on $D = [0, 1]$ that are absolutely continuous, vanish at zero, and with the derivative bounded in the L_p norm. It is a Banach space with respect to the norm $\|f\|_{F_p} = \|f'\|_{L_p}$. The space F_p is the building block for the anchored spaces of s -variate functions.

For non-empty \mathbf{u} , let $F_{\mathbf{u},p}$ be a Banach space that is the completion of the space spanned by $f(\mathbf{x}) = \prod_{j \in \mathbf{u}} f_j(x_j)$ with $f_j \in F_p$ with respect to the norm

$$\|f\|_{F_{\mathbf{u},p}} = \|f^{(\mathbf{u})}\|_{L_p}.$$

For $\mathbf{u} = \emptyset$, $F_{\emptyset,p}$ is the space of constant functions with the norm given by the absolute value.

Consider next a family $\boldsymbol{\gamma} = (\gamma_{\mathbf{u}})_{\mathbf{u} \subseteq [s]}$ of non-negative numbers, called *weights*. Let

$$\mathfrak{U} = \{\mathbf{u} \subseteq [s] : \gamma_{\mathbf{u}} > 0\}.$$

The corresponding $\boldsymbol{\gamma}$ -weighted *anchored space* $F_{s,p,\boldsymbol{\gamma}}$ is the completion of $\bigoplus_{\mathbf{u} \in \mathfrak{U}} F_{\mathbf{u},p}$ with respect to the norm given by

$$\|f\|_{F_{s,p,\boldsymbol{\gamma}}} = \left(\sum_{\mathbf{u} \in \mathfrak{U}} \gamma_{\mathbf{u}}^{-p} \|f^{(\mathbf{u})}([\cdot; \mathbf{0}_{\mathbf{u}^c}])\|_{L_p}^p \right)^{1/p}.$$

It is known, see, e.g., [4, Section 3], that for non-empty \mathbf{u} ,

$$F_{\mathbf{u},p} = T_{\mathbf{u},p}(L_p(D^{\mathbf{u}})),$$

where

$$T_{\mathbf{u},p}(h)(\mathbf{x}) = \int_{D^{\mathbf{u}}} h(\mathbf{t}_{\mathbf{u}}) \prod_{j \in \mathbf{u}} \mathbf{1}_{[0,x_j]}(t_j) d\mathbf{t}_{\mathbf{u}}$$

and $\mathbf{1}_I$ is the characteristic function of the set I , i.e., $\mathbf{1}_I(t) = 1$ if $t \in I$ and 0 otherwise. Then any $f \in F_{s,p,\boldsymbol{\gamma}}$ has a unique decomposition, called *anchored decomposition*,

$$f = \sum_{\mathbf{u} \in \mathfrak{U}} f_{\mathbf{u}} \quad \text{with} \quad f_{\mathbf{u}} = T_{\mathbf{u},p}(h_{\mathbf{u}}) \quad \text{for} \quad h_{\mathbf{u}} \in L_p(D^{\mathbf{u}}),$$

where here and in the following we write $D^{\mathbf{u}}$ for $D^{|\mathbf{u}|}$. Moreover, $f_{\mathbf{u}} \in F_{\mathbf{u},p}$ and

$$f_{\mathbf{u}}^{(\mathbf{u})} = h_{\mathbf{u}} = f^{(\mathbf{u})}([\cdot; \mathbf{0}_{\mathbf{u}^c}]).$$

2.2 ANOVA Spaces

The definition of the corresponding γ -weighted ANOVA spaces is very similar to that of Anchored spaces with the only difference that instead of the space $F_p = W_{p,0}^1$ we use the space $W_{p,\text{int}}^1$ of absolutely continuous functions on $[0, 1]$ with $\|f'\|_{L_p} < \infty$ and such that

$$\int_0^1 f(t) dt = 0.$$

Then the corresponding γ -weighted ANOVA space $H_{s,p,\gamma}$ has the norm given by

$$\|f\|_{H_{s,p,\gamma}} = \left(\sum_{\mathbf{u} \in \mathfrak{U}} \gamma_{\mathbf{u}}^{-p} \left\| \int_{D^{\mathbf{u}^c}} f^{(\mathbf{u})}([\cdot_{\mathbf{u}}; \mathbf{t}_{\mathbf{u}^c}]) d\mathbf{t}_{-\mathbf{u}} \right\|_{L_p}^p \right)^{1/p}.$$

Any function from $H_{s,p,\gamma}$ has a unique ANOVA decomposition

$$f = \sum_{\mathbf{u} \in \mathfrak{U}} f_{\mathbf{u}},$$

where

$$f_{\mathbf{u}}(\mathbf{x}) = \int_{D^{\mathbf{u}}} g_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) \prod_{j \in \mathbf{u}} (\mathbf{1}_{[0,x_j]}(t_j) - (1 - t_j)) d\mathbf{t}_{\mathbf{u}}$$

for some $g_{\mathbf{u}} \in L_p(D^{\mathbf{u}})$ and

$$f_{\mathbf{u}}^{(\mathbf{u})} = g_{\mathbf{u}} = \int_{D^{\mathbf{u}^c}} f^{(\mathbf{u})}([\cdot_{\mathbf{u}}; \mathbf{t}_{\mathbf{u}^c}]) d\mathbf{t}_{\mathbf{u}^c}.$$

3 Equivalence of Anchored and ANOVA Spaces

It was shown in [4] that the Anchored and ANOVA spaces are equal (as sets of functions) if and only if the following holds:

$$\gamma_{\mathfrak{w}} > 0 \quad \text{implies that} \quad \gamma_{\mathbf{u}} > 0 \quad \text{for all } \mathbf{u} \subset \mathfrak{w}. \quad (1)$$

This is why from now on we assume that (1) is satisfied.

Let

$$\iota = \iota_{s,p,\gamma} : F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}$$

be the *embedding operator*, $\iota(f) = f$. As mentioned in the introduction, [5] provides interesting upper bounds on the norms of ι and its inverse ι^{-1} .

We now prove that ι and ι^{-1} have the same norm. Moreover we provide a lower bound for that norm which, as will be illustrated, is sharp for a number of special cases.

Theorem 1 *The norms $\|\iota\|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}}$ and $\|\iota^{-1}\|_{H_{s,p,\gamma} \hookrightarrow F_{s,p,\gamma}}$ are the same and are equal to*

$$\sup_{\{c_{\mathbf{u}}, h_{\mathbf{u}}\}_{\mathbf{u} \in \mathfrak{U}}} \frac{\left(\sum_{\mathbf{v} \in \mathfrak{U}} \gamma_{\mathbf{v}}^{-p} \int_{D^{\mathbf{v}}} \left| \sum_{\mathfrak{w} \subseteq [s] \setminus \mathbf{v}} c_{\mathbf{v} \cup \mathfrak{w}} \int_{D^{\mathfrak{w}}} h_{\mathbf{v} \cup \mathfrak{w}}([\mathbf{x}_{\mathbf{v}}; \mathbf{t}_{\mathfrak{w}}]) \prod_{j \in \mathfrak{w}} (1 - t_j) d\mathbf{t}_{\mathfrak{w}} \right|^p d\mathbf{x}_{\mathbf{v}} \right)^{1/p}}{\left(\sum_{\mathbf{u} \in \mathfrak{U}} c_{\mathbf{u}}^p \right)^{1/p}}, \quad (2)$$

where the supremum is with respect to non-negative numbers c_u and functions $h_u \in L_p(D^u)$ such that $\|h_u\|_{L_p(D^u)} = \gamma_u$.

The norms are bounded from below by

$$\sup_{\{c_u \geq 0\}_{u \in \mathfrak{U}}} \frac{\left(\sum_{\mathfrak{v} \in \mathfrak{U}} \gamma_{\mathfrak{v}}^{-p} \left(\sum_{\mathfrak{w} \subseteq [s] \setminus \mathfrak{v}} c_{\mathfrak{v} \cup \mathfrak{w}} \gamma_{\mathfrak{v} \cup \mathfrak{w}} m_{p^*}^{|\mathfrak{w}|} \right)^p \right)^{1/p}}{\left(\sum_{u \in \mathfrak{U}} c_u^p \right)^{1/p}}, \quad (3)$$

where

$$m_{p^*} = \frac{1}{(p^* + 1)^{1/p^*}} \quad \text{and} \quad \frac{1}{p} + \frac{1}{p^*} = 1$$

with $m_\infty = 1$.

Proof. As already mentioned, any $f \in F_{s,p,\gamma}$ can be written as

$$f(\mathbf{x}) = c_\emptyset \gamma_\emptyset + \sum_{\emptyset \neq u \in \mathfrak{U}} f_{\mathfrak{u},u}(\mathbf{x}), \quad \text{where} \quad f_{\mathfrak{u},u}(\mathbf{x}) = c_u \int_{D^u} h_u(\mathbf{t}_u) \prod_{j \in u} \mathbf{1}_{[0,x_j]}(t_j) d\mathbf{t}_u,$$

for non-negative numbers c_u and functions h_u such that

$$\|h_u\|_{L_p(D^u)} = \gamma_u.$$

Of course, the terms $f_{\mathfrak{u},u}$ are from the anchored decomposition of f , and

$$f_{\mathfrak{u},u}^{(\mathfrak{u})} = c_u h_u.$$

Therefore,

$$\|f\|_{F_{s,p,\gamma}} = \left(\sum_{u \in \mathfrak{U}} c_u^p \right)^{1/p}.$$

For $\mathfrak{v} \subseteq [s]$, $f_{\mathfrak{u},u}^{(\mathfrak{v})} = 0$ if $\mathfrak{v} \not\subseteq u$. Consider therefore u that contains \mathfrak{v} . Then

$$\int_{D^{\mathfrak{v}^c}} f_{\mathfrak{u},u}^{(\mathfrak{v})}([\mathbf{x}_{\mathfrak{v}}; \mathbf{x}_{\mathfrak{v}^c}]) d\mathbf{x}_{\mathfrak{v}^c} = c_u \int_{D^{u \setminus \mathfrak{v}}} h_u(\mathbf{x}_{\mathfrak{v}}; \mathbf{t}_{u \setminus \mathfrak{v}}) \prod_{j \in u \setminus \mathfrak{v}} (1 - t_j) d\mathbf{t}_{u \setminus \mathfrak{v}}$$

and the ANOVA term $f_{A,\mathfrak{v}}$ is given by

$$f_{A,\mathfrak{v}}(\mathbf{x}) = \sum_{\mathfrak{w} \subseteq [s] \setminus \mathfrak{v}} c_{\mathfrak{v} \cup \mathfrak{w}} \int_{D^{\mathfrak{w}}} h_{\mathfrak{v} \cup \mathfrak{w}}([\mathbf{x}_{\mathfrak{v}}; \mathbf{t}_{\mathfrak{w}}]) \prod_{j \in \mathfrak{w}} (1 - t_j) d\mathbf{t}_{\mathfrak{w}}.$$

Therefore

$$\begin{aligned} \|f\|_{H_{s,p,\gamma}} &= \left(\gamma_\emptyset^{-p} \left| c_\emptyset \gamma_\emptyset + \sum_{\emptyset \neq u \in \mathfrak{U}} c_u \int_{D^u} h_u(\mathbf{t}_u) \prod_{j \in u} (1 - t_j) d\mathbf{t}_u \right|^p \right. \\ &\quad \left. + \sum_{\emptyset \neq \mathfrak{v} \in \mathfrak{U}} \gamma_{\mathfrak{v}}^{-p} \int_{D^{\mathfrak{v}}} \left| \sum_{\mathfrak{w} \subseteq [s] \setminus \mathfrak{v}} c_{\mathfrak{v} \cup \mathfrak{w}} \int_{D^{\mathfrak{w}}} h_{\mathfrak{v} \cup \mathfrak{w}}([\mathbf{x}_{\mathfrak{v}}; \mathbf{t}_{\mathfrak{w}}]) \prod_{j \in \mathfrak{w}} (1 - t_j) d\mathbf{t}_{\mathfrak{w}} \right|^p d\mathbf{x}_{\mathfrak{v}} \right)^{1/p} \end{aligned}$$

and the norm of the embedding ι is given by the supremum of the right hand side of the above equation divided by $\left(\sum_{\mathbf{u} \subseteq [s]} c_{\mathbf{u}}^p\right)^{1/p}$. The supremum is with respect to non-negative numbers $c_{\mathbf{u}}$ and functions $h_{\mathbf{u}} \in L_p(D^{\mathbf{u}})$ such that $\|h_{\mathbf{u}}\|_{L_p(D^{\mathbf{u}})} = \gamma_{\mathbf{u}}$.

Consider now $f \in H_{s,p,\gamma}$. It can be written as

$$f(\mathbf{x}) = c_{\emptyset} \gamma_{\emptyset} + \sum_{\emptyset \neq \mathbf{u} \in \mathfrak{U}} f_{A,\mathbf{u}}(\mathbf{x}),$$

where

$$f_{A,\mathbf{u}}(\mathbf{x}) = (-1)^{|\mathbf{u}|} c_{\mathbf{u}} \int_{D^{\mathbf{u}}} h_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) \prod_{j \in \mathbf{u}} (\mathbf{1}_{[0,x_j]}(t_j) - (1 - t_j)) \, d\mathbf{t}_{\mathbf{u}}.$$

The terms $f_{A,\mathbf{u}}$ are from the ANOVA decomposition of f . Again, we choose $\|h_{\mathbf{u}}\|_{L_p} = \gamma_{\mathbf{u}}$ and, therefore,

$$\|f\|_{H_{s,p,\gamma}} = \left(\sum_{\mathbf{u} \in \mathfrak{U}} c_{\mathbf{u}}^p\right)^{1/p}.$$

For $\mathbf{v} \subseteq \mathbf{u}$,

$$f_{A,\mathbf{u}}^{(\mathbf{v})}([\mathbf{x}_{\mathbf{v}}; \mathbf{0}_{\mathbf{v}^c}]) = (-1)^{|\mathbf{v}|} c_{\mathbf{u}} \int_{D^{\mathbf{u} \setminus \mathbf{v}}} h_{\mathbf{u}}([\mathbf{x}_{\mathbf{v}}; \mathbf{t}_{\mathbf{u} \setminus \mathbf{v}}]) \prod_{j \in \mathbf{u} \setminus \mathbf{v}} (1 - t_j) \, d\mathbf{t}_{\mathbf{u} \setminus \mathbf{v}}.$$

Therefore the anchored norm of f is given by

$$\begin{aligned} \|f\|_{F_{s,p,\gamma}} &= \left(\gamma_{\emptyset}^{-p} \left| c_{\emptyset} \gamma_{\emptyset} + \sum_{\emptyset \neq \mathbf{u} \in \mathfrak{U}} c_{\mathbf{u}} \int_{D^{\mathbf{u}}} h_{\mathbf{u}}(\mathbf{t}_{\mathbf{u}}) \prod_{j \in \mathbf{u}} (1 - t_j) \, d\mathbf{t}_{\mathbf{u}} \right|^p \right. \\ &\quad \left. + \sum_{\emptyset \neq \mathbf{v} \in \mathfrak{U}} \gamma_{\mathbf{v}}^{-p} \int_{D^{\mathbf{v}}} \left| \sum_{\mathbf{w} \subseteq [s] \setminus \mathbf{v}} c_{\mathbf{v} \cup \mathbf{w}} \int_{D^{\mathbf{w}}} h_{\mathbf{v} \cup \mathbf{w}}([\mathbf{x}_{\mathbf{v}}; \mathbf{t}_{\mathbf{w}}]) \prod_{j \in \mathbf{w}} (1 - t_j) \, d\mathbf{t}_{\mathbf{w}} \right|^p \, d\mathbf{x}_{\mathbf{v}} \right)^{1/p}. \end{aligned}$$

This proves (2).

We now prove the lower bound (3). Consider

$$h_{\mathbf{u}}(\mathbf{x}) = \gamma_{\mathbf{u}} \prod_{j \in \mathbf{u}} h(x_j),$$

where the univariate function $h \in L_p(D)$ is such that

$$\|h\|_{L_p(D)} = 1 \quad \text{and} \quad \int_D h(t) (1 - t) \, dt = \|h\|_{L_p(D)} \|(1 - \cdot)\|_{L_{p^*}(D)} = m_{p^*}.$$

Then

$$\int_{D^{\mathbf{v}}} \left(\sum_{\mathbf{w} \subseteq [s] \setminus \mathbf{v}} c_{\mathbf{v} \cup \mathbf{w}} \int_{D^{\mathbf{w}}} h_{\mathbf{v} \cup \mathbf{w}}([\mathbf{x}_{\mathbf{v}}; \mathbf{t}_{\mathbf{w}}]) \prod_{j \in \mathbf{w}} (1 - t_j) \, d\mathbf{t}_{\mathbf{w}} \right)^p \, d\mathbf{x}_{\mathbf{v}}$$

$$\begin{aligned}
&= \int_{D^{\mathfrak{v}}} \prod_{j \in \mathfrak{v}} h^p(x_j) \, d\mathbf{x}_{\mathfrak{v}} \left(\sum_{\mathfrak{w} \subseteq [s] \setminus \mathfrak{v}} c_{\mathfrak{v} \cup \mathfrak{w}} \gamma_{\mathfrak{v} \cup \mathfrak{w}} m_{p^*}^{|\mathfrak{w}|} \right)^p \\
&= \left(\sum_{\mathfrak{w} \subseteq [s] \setminus \mathfrak{v}} c_{\mathfrak{v} \cup \mathfrak{w}} \gamma_{\mathfrak{v} \cup \mathfrak{w}} m_{p^*}^{|\mathfrak{w}|} \right)^p.
\end{aligned}$$

This completes the proof. \square

We have the following proposition.

Proposition 2 *The lower bound (3) is sharp for $p = 1$ and $p = \infty$ and arbitrary weights.*

Proof. To simplify the notation let us use

$$\bar{\gamma}_{\mathfrak{u}} = \gamma_{\mathfrak{u}} m_{p^*}^{|\mathfrak{u}|} = \frac{\gamma_{\mathfrak{u}}}{(p^* + 1)^{|\mathfrak{u}|/p^*}.$$

Then the numerator in the lower bound (3) can be rewritten as

$$N_p((c_{\mathfrak{u}})_{\mathfrak{u}}) = \left(\sum_{\mathfrak{v} \in \mathfrak{U}} \bar{\gamma}_{\mathfrak{v}}^{-p} \left(\sum_{\substack{\mathfrak{u} \subseteq [s] \\ \mathfrak{v} \subseteq \mathfrak{u}}} c_{\mathfrak{u}} \bar{\gamma}_{\mathfrak{u}} \right)^p \right)^{1/p}.$$

For $p = 1$ we have $p^* = \infty$. Hence $(p^* + 1)^{1/p^*} = 1$ and

$$\frac{N_p((c_{\mathfrak{u}})_{\mathfrak{u}})}{\sum_{\mathfrak{u} \in \mathfrak{U}} c_{\mathfrak{u}}} \leq \max_{\mathfrak{u} \subseteq [s]} \sum_{\mathfrak{v} \subseteq \mathfrak{u}} \frac{\gamma_{\mathfrak{u}}}{\gamma_{\mathfrak{v}}}$$

and the inequality above is sharp. From [4] we know that for $p = 1$,

$$\|z\|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}} = \max_{\mathfrak{u} \in \mathfrak{U}} \sum_{\mathfrak{v} \subseteq \mathfrak{u}} \frac{\gamma_{\mathfrak{u}}}{\gamma_{\mathfrak{v}}}.$$

This proves the claim for $p = 1$.

Consider next $p = \infty$. Then, of course, $(p^* + 1)^{1/p^*} = 2$. The lower bound (3) can be rewritten as

$$\max_{(c_{\mathfrak{u}} \geq 0)_{\mathfrak{u} \in \mathfrak{U}}} \frac{\max_{\mathfrak{v} \in \mathfrak{U}} \gamma_{\mathfrak{v}}^{-1} \sum_{\mathfrak{w} \subseteq [s] \setminus \mathfrak{v}} c_{\mathfrak{v} \cup \mathfrak{w}} \frac{\gamma_{\mathfrak{v} \cup \mathfrak{w}}}{2^{|\mathfrak{w}|}}}{\max_{\mathfrak{u} \subseteq [s]} c_{\mathfrak{u}}} = \max_{\mathfrak{v} \in \mathfrak{U}} \sum_{\mathfrak{w} \subseteq [s] \setminus \mathfrak{v}} \frac{\gamma_{\mathfrak{v} \cup \mathfrak{w}}}{2^{|\mathfrak{w}|} \gamma_{\mathfrak{v}}}.$$

This lower bound for $p = \infty$ is also sharp since it is equal to the norm of the corresponding embedding, as shown in [4]. \square

As already mentioned, [5] provides an upper bound on the norm of the embedding for product weights and any $p \in (1, \infty)$. It also provides a matching lower bound. The purpose of the proposition below is to show that a sharp lower bound can also be obtained from (3). Recall that *product weights*, introduced in [11], are of the form

$$\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j \quad \text{for a sequence } (\gamma_j)_{j \geq 1} \text{ in } \mathbb{R}^+.$$

In particular, for product weights we have $\mathfrak{U} = \mathcal{P}([s])$, the set of all subsets of $[s]$.

Proposition 3 For $p \in (1, \infty)$, the lower bound (3) is (modulo a multiplicative constant) sharp for product weights and

$$\| \iota \|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}} \geq \prod_{j=1}^s \left(1 + \gamma_j \left(\frac{p-1}{p^*+1} \right)^{1/p^*} \right)^{1/p}. \quad (4)$$

In particular it shows that for the uniform equivalence it is necessary that

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

Proof. Consider $c_u = \prod_{j \in u} c_j$ with $c_j \geq 0$ for $j \geq 1$. Then the numerator in the expression in the lower bound (3) is equal to

$$\begin{aligned} \left(\sum_{\mathfrak{v} \subseteq [s]} c_{\mathfrak{v}}^p \left(\sum_{\mathfrak{w} \subseteq [s] \setminus \mathfrak{v}} c_{\mathfrak{w}} \gamma_{\mathfrak{w}} m_{p^*}^{|\mathfrak{w}|} \right)^p \right)^{1/p} &= \left(\sum_{\mathfrak{v} \subseteq [s]} c_{\mathfrak{v}}^p \left(\sum_{\mathfrak{w} \subseteq [s] \setminus \mathfrak{v}} \prod_{j \in \mathfrak{w}} (c_j \gamma_j m_{p^*}) \right)^p \right)^{1/p} \\ &= \left(\sum_{\mathfrak{v} \subseteq [s]} \prod_{j \in \mathfrak{v}} c_j^p \prod_{j \in [s] \setminus \mathfrak{v}} (1 + c_j \gamma_j m_{p^*})^p \right)^{1/p} \\ &= \left(\prod_{j=1}^s (c_j^p + (1 + c_j \gamma_j m_{p^*})^p) \right)^{1/p} \\ &= \prod_{j=1}^s (1 + c_j^p + (1 + c_j \gamma_j m_{p^*})^p - 1)^{1/p}. \end{aligned}$$

Since the denominator is equal to $\prod_{j=1}^s (1 + c_j^p)$, we get that the norm of the embedding is bounded from below by

$$\sup_{\{c_j \geq 0\}_{j=1}^s} \prod_{j=1}^s \left(1 + \frac{(1 + c_j \gamma_j m_{p^*})^p - 1}{1 + c_j^p} \right)^{1/p}. \quad (5)$$

Clearly, for $p \in (1, \infty)$ and $c_j = 1/(p-1)^{1/p}$

$$\begin{aligned} \left(1 + \frac{(1 + c_j \gamma_j m_{p^*})^p - 1}{1 + c_j^p} \right)^{1/p} &\geq \left(1 + \frac{p c_j \gamma_j m_{p^*}}{1 + c_j^p} \right)^{1/p} \\ &= \left(1 + \gamma_j \left(\frac{p-1}{p^*+1} \right)^{1/p^*} \right)^{1/p} \\ &= 1 + \gamma_j p^{-1} \left(\frac{p-1}{p^*+1} \right)^{1/p^*} + O(\gamma_j^2). \end{aligned}$$

This completes the proof. \square

Remark 4 For $p = 1$ we have $m_{p^*} = m_\infty = 1$ and hence we get from (5)

$$\sup_{\{c_j \geq 0\}_{j=1}^s} \prod_{j=1}^s \left(1 + \frac{(1 + c_j \gamma_j m_{p^*})^p - 1}{1 + c_j^p} \right)^{1/p} = \sup_{\{c_j \geq 0\}_{j=1}^s} \prod_{j=1}^s \left(1 + \frac{c_j}{1 + c_j} \gamma_j \right) = \prod_{j=1}^s (1 + \gamma_j).$$

This matches exactly the result from [4, Proposition 17 for $p = 1$].

For $p = 2$ we know from [6] that

$$\|i\|_{F_{s,2,\gamma} \hookrightarrow H_{s,p,\gamma}} = \prod_{j=1}^s \left(1 + \frac{\gamma_j}{\sqrt{3}} \left(\sqrt{1 + \frac{\gamma_j^2}{12}} + \frac{\gamma_j^3}{\sqrt{12}} \right) \right)^{1/2}. \quad (6)$$

Note that the lower bound (4) for $p = 2$ takes the form

$$\prod_{j=1}^s \left(1 + \frac{\gamma_j}{\sqrt{3}} \right)^{1/2}$$

and is very close to the true value in (6).

The following proposition provides a lower bound that is sometimes easier to use.

Proposition 5 *The lower bound (3) is bounded from below by*

$$\max_{\mathbf{u} \in \mathfrak{U}} \left(\sum_{\mathbf{v} \subseteq \mathbf{u}} \frac{\gamma_{\mathbf{u}}^p}{(p^* + 1)^{p|\mathbf{u} \setminus \mathbf{v}|/p^*} \gamma_{\mathbf{v}}^p} \right)^{1/p}. \quad (7)$$

Proof. Clearly the numerator in (3) is not smaller than

$$\left(\sum_{\mathbf{v} \in \mathfrak{U}} \gamma_{\mathbf{v}}^{-p} \sum_{\mathbf{w} \subseteq [s] \setminus \mathbf{v}} c_{\mathbf{v} \cup \mathbf{w}}^p \frac{\gamma_{\mathbf{v} \cup \mathbf{w}}^p}{(p^* + 1)^{p|\mathbf{w}|/p^*}} \right)^{1/p} = \left(\sum_{\mathbf{u} \in \mathfrak{U}} c_{\mathbf{u}}^p \gamma_{\mathbf{u}}^p \sum_{\mathbf{v} \subseteq \mathbf{u}} \gamma_{\mathbf{v}}^{-p} (p^* + 1)^{-p|\mathbf{u} \setminus \mathbf{v}|/p^*} \right)^{1/p}.$$

Let \mathbf{u}^* be such that

$$\max_{\mathbf{u} \in \mathfrak{U}} \sum_{\mathbf{v} \subseteq \mathbf{u}} \frac{\gamma_{\mathbf{u}}^p}{(p^* + 1)^{p|\mathbf{u} \setminus \mathbf{v}|/p^*} \gamma_{\mathbf{v}}^p}$$

is attained at \mathbf{u}^* . Then taking $c_{\mathbf{u}^*} = 1$ and $c_{\mathbf{u}} = 0$ for all $\mathbf{u} \neq \mathbf{u}^*$ completes the proof. \square

We now apply (3) to get lower bounds for special classes of weights.

3.1 Finite Order Weights

Consider *finite order weights*, for the first time dealt with in [2], of the form

$$\gamma_{\mathbf{u}} = \begin{cases} \omega^{|\mathbf{u}|} & \text{if } |\mathbf{u}| \leq q, \\ 0 & \text{if } |\mathbf{u}| > q. \end{cases} \quad (8)$$

It was shown in [4] that the norm of the embedding is uniformly bounded for $p = 1$, and grows like a polynomial in s for $p = \infty$. The authors of [5] proved using the above results and complex interpolation theory that for $p \in (1, \infty)$ the norm of the embedding is bounded from above by a polynomial in s . It is therefore of interest to see whether the embedding norm is uniformly bounded or indeed behaves polynomially for $p \in (1, \infty)$.

Proposition 6 For finite order weights (8) and $p \in (1, \infty)$ we have

$$\|\iota\|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}} = \Theta(s^{q/p^*}).$$

Proof. It was shown in [4] that the norm of the embedding for $p = 1$ is uniformly bounded and it is proportional to s^q for $p = \infty$. Hence the results of [5] imply the following upper bound

$$\|\iota\|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}} = O(s^{q(1-1/p)}).$$

Hence it is enough to show a matching lower bound. For that purpose we will use (3) for a special choice of c_u . Namely, consider $c_u = 1$ if $|\mathbf{u}| = q$ and $c_u = 0$ otherwise. Note that then the denominator in (3) equals

$$\binom{s}{q}^{1/p}.$$

Consider next the numerator of (3) where instead of the whole summation with respect to \mathbf{v} we consider only one term with $\mathbf{v} = \emptyset$. Then we have

$$\sum_{|\mathbf{w}|=q} c_{\mathbf{w}} \gamma_{\mathbf{w}} m_{p^*}^{|\mathbf{w}|} = m_{p^*}^q \omega^q \binom{s}{q}.$$

Therefore

$$\|\iota\|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}} \geq m_{p^*}^q \omega^q \binom{s}{q}^{1-1/p} \geq \frac{m_{p^*}^q \omega^q}{(q!)^{1-1/p}} (s-q)^{q(1-1/p)}.$$

This completes the proof. \square

3.2 Finite Diameter Weights

Consider *finite diameter weights*, which were first introduced by Creutzig (see [1], and also [9]) of the form

$$\gamma_{\mathbf{u}} = \begin{cases} \omega^{|\mathbf{u}|} & \text{if } \text{diam}(\mathbf{u}) \leq q, \\ 0 & \text{if } \text{diam}(\mathbf{u}) > q, \end{cases} \quad (9)$$

where $\text{diam}(\mathbf{u}) = \max_{i,j \in \mathbf{u}} |i-j|$ and $\text{diam}(\emptyset) = 0$ by convention.

Proposition 7 For finite diameter weights (9) and $p \in [1, \infty]$ we have

$$\|\iota\|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}} = \Theta((s-q)^{1/p^*}). \quad (10)$$

Proof. We will use the following fact in a number of places. For $\ell \in \{2, \dots, q\}$,

$$\sum_{\substack{\mathbf{u} \subseteq [s] \\ \text{diam}(\mathbf{u}) = \ell}} x^{|\mathbf{u}|} = \sum_{k=2}^{\ell+1} x^k \sum_{\substack{\mathbf{u} \subseteq [s], |\mathbf{u}|=k \\ \text{diam}(\mathbf{u}) = \ell}} 1 = \sum_{k=2}^{\ell+1} x^k (s-\ell) \binom{\ell-1}{k-2} = (s-\ell)x^2(1+x)^{\ell-1}. \quad (11)$$

We start by showing the result for the case $p = 1$. In this case, the embedding norm is equal to

$$\max_{\mathbf{u} \in \mathfrak{U}} \sum_{\mathbf{v} \subseteq \mathbf{u}} \frac{\gamma_{\mathbf{u}}}{\gamma_{\mathbf{v}}} = \max_{\substack{\mathbf{u} \subseteq [s] \\ \text{diam}(\mathbf{u}) \leq q}} \omega^{|\mathbf{u}|} \sum_{\mathbf{v} \subseteq \mathbf{u}} \omega^{-|\mathbf{v}|} = \max_{\substack{\mathbf{u} \subseteq [s] \\ \text{diam}(\mathbf{u}) \leq q}} (1 + \omega)^{|\mathbf{u}|} = (1 + \omega)^{q+1},$$

which yields the desired result for $p = 1$. For $p = \infty$, the embedding norm is equal to

$$\begin{aligned} \max_{\mathbf{v} \in \mathfrak{U}} \sum_{\substack{\mathfrak{w} \subseteq [s] \setminus \mathbf{v} \\ \text{diam}(\mathfrak{w} \cup \mathbf{v}) \leq q}} \frac{\omega^{|\mathfrak{w}|}}{2^{|\mathfrak{w}|}} &= \sum_{\substack{\mathfrak{w} \in \mathfrak{U} \\ \text{diam}(\mathfrak{w}) \leq q}} \left(\frac{\omega}{2}\right)^{|\mathfrak{w}|} \\ &= \sum_{\ell=0}^q \left(\frac{\omega}{2}\right)^{\ell} \sum_{\substack{\mathfrak{w} \in \mathfrak{U} \\ \text{diam}(\mathfrak{w}) = \ell}} 1 \\ &= \sum_{\substack{\mathfrak{w} \in \mathfrak{U} \\ \text{diam}(\mathfrak{w}) = 0}} 1 + \sum_{\substack{\mathfrak{w} \in \mathfrak{U} \\ \text{diam}(\mathfrak{w}) = 1}} 1 + \sum_{\ell=2}^q \left(\frac{\omega}{2}\right)^{\ell} \sum_{\substack{\mathfrak{w} \in \mathfrak{U} \\ \text{diam}(\mathfrak{w}) = \ell}} 1 \\ &= s + 1 + \frac{\omega}{2}(s-1) + \sum_{\ell=2}^q \left(\frac{\omega}{2}\right)^{\ell} (s-q)2^{q-1} \\ &= \Theta(s-q), \end{aligned}$$

where we used (11) in the fourth equality. Therefore, using the result of [5] we get that for $p \in (1, \infty)$ the corresponding norm is bounded by

$$O((s-q)^{1-1/p}).$$

Hence to complete the proof, we need to show a matching lower bound. Similar to the case of finite order weights we consider $c_{\mathbf{u}} = 1$ if $\text{diam}(\mathbf{u}) = q$ and $c_{\mathbf{u}} = 0$ otherwise. Then, using (11) again, the denominator in (3) equals

$$\left(\sum_{\text{diam}(\mathbf{u})=q} 1 \right)^{1/p} = (s-q)^{1/p} 2^{(q-1)/p}.$$

Consider next the numerator of (3) where instead of the whole summation with respect to \mathbf{v} we consider only one term with $\mathbf{v} = \emptyset$. Then we have, once more by (11),

$$\sum_{\text{diam}(\mathfrak{w})=q} c_{\mathfrak{w}} \gamma_{\mathfrak{w}} m_{p^*}^{|\mathfrak{w}|} = \omega^q \sum_{\text{diam}(\mathfrak{w})=q} m_{p^*}^{|\mathfrak{w}|} = \omega^q (s-q) m_{p^*}^2 (1 + m_{p^*})^{q-1}.$$

Therefore

$$\|\iota\|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}} \geq \frac{\omega^q (s-q) m_{p^*}^2 (1 + m_{p^*})^{q-1}}{(s-q)^{1/p} 2^{(q-1)/p}} = \omega^q m_{p^*}^2 \left(\frac{1 + m_{p^*}}{2^{1/p}} \right)^{q-1} (s-q)^{1/p^*}.$$

This completes the proof. \square

3.3 Product Order-Dependent Weights

Consider *product order-dependent weights*, as introduced in [7], of the form

$$\gamma_{\mathbf{u}} = (|\mathbf{u}|!)^{\beta_1} \prod_{j \in \mathbf{u}} \frac{c}{j^{\beta_2}} \quad \text{for } c > 0 \text{ and } 0 < \beta_1 < \beta_2.$$

It was shown in [4] that for $p = 1$ or $p = \infty$, the norm of the embedding converges to infinity faster than any polynomial in s .

Proposition 8 *For product order-dependent weights and $p \in [1, \infty]$,*

$$\|\iota\|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}} = \Omega(s^\tau) \quad \text{for all } \tau > 0.$$

Proof. As already mentioned, the result is known for $p \in \{1, \infty\}$ and, therefore we consider now only $p \in (1, \infty)$. We use the lower bound from Proposition 5 for $\mathbf{u} = [s]$ with the summation restricted to $\mathbf{v} = \{k, k+1, \dots, s\}$ for $k = 1, 2, \dots, s$. Then

$$\begin{aligned} \|\iota\|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}}^p &\geq \sum_{k=1}^s \frac{\gamma_{[s]}^p}{(p^* + 1)^{p(k-1)/p^*} \gamma_{\{k,\dots,s\}}^p} \\ &= \sum_{k=1}^s \left(\frac{s!}{(s-k+1)!} \right)^{p\beta_1} \left(\frac{c}{(p^* + 1)^{1/p^*}} \right)^{p(k-1)} \frac{1}{((k-1)!)^p \beta_2} \\ &\geq \sum_{k=1}^s (s-k+2)^{p(k-1)\beta_1} \left(\frac{c}{(p^* + 1)^{1/p^*}} \right)^{p(k-1)} \frac{1}{((k-1)!)^p \beta_2}. \end{aligned}$$

For given τ consider only one term from the sum above with k such that $k-1 = \lceil \tau/\beta_1 \rceil$. Then for $s \geq k+2$ we have

$$\|\iota\|_{F_{s,p,\gamma} \hookrightarrow H_{s,p,\gamma}} > a_\tau (s+1 - \lceil \tau/\beta_1 \rceil)^\tau$$

for $a_\tau = (c/(p^* + 1)^{1/p^*})^{\lceil \tau/\beta_1 \rceil} / (\lceil \tau/\beta_1 \rceil!)^{\beta_2}$. This completes the proof. \square

References

- [1] J. Creutzig: Finite-diameter weights. Unpublished, 2007.
- [2] J. Dick, I.H. Sloan, X. Wang, and H. Woźniakowski: Good lattice rules in weighted Korobov spaces with general weights, *Numer. Math.* 103: 63–97, 2006.
- [3] M. Hefter and K. Ritter: On embeddings of weighted tensor product Hilbert spaces, *J. Complexity* 31: 405–423, 2015.
- [4] M. Hefter, K. Ritter, and G. W. Wasilkowski: On equivalence of weighted anchored and ANOVA spaces of functions with mixed smoothness of order one in L_1 and L_∞ norms, *J. Complexity*, electronic version. DOI: 10.1016/j.jco.2015.07.001

- [5] A. Hinrichs and J. Schneider: Equivalence of anchored and ANOVA spaces via interpolation, *J. Complexity*, to appear.
- [6] P. Kritzer, F. Pillichshammer, and G. W. Wasilkowski: Very low truncation dimension for high dimensional integration under modest error demand, submitted.
- [7] F.Y. Kuo, C. Schwab, and I.H. Sloan: Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. *SIAM J. Numer. Anal.* 6, 3351–3374, 2012.
- [8] F.Y. Kuo, I.H. Sloan, G. W. Wasilkowski, and H. Woźniakowski: Liberating the dimension, *J. Complexity* 26: 422–454, 2010.
- [9] E. Novak and H. Woźniakowski: *Tractability of multivariate Problems. Volume I: Linear Information*. European Mathematical Society, Zürich, 2008.
- [10] A. Owen: Effective dimension for weighted function spaces, Technical Report, 2014. Available under:
<http://statweb.stanford.edu/owen/reports/effdim-periodic.pdf>
- [11] I. H. Sloan and H. Woźniakowski: When are quasi-Monte Carlo algorithms efficient for high dimensional integrals? *J. Complexity* 14: 1–33, 1998.
- [12] G. W. Wasilkowski: Tractability of approximation of ∞ -variate functions with bounded mixed partial derivatives, *J. Complexity* 30: 325–346, 2014.

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