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A. Alsaedi, B. Ahmed, D. P. Challa, M. Kirane, M. Sini

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A cluster of many small holes with negative imaginary surface impedances may generate a negative refraction index

Ahmed Alsaedi * Bashir Ahmad * Durga Prasad Challa †‡
Mokhtar Kirane*§ Mourad Sini ¶

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Abstract

We deal with the scattering of an acoustic medium modeled by an index of refraction n varying in a bounded region Ω of $\mathbb{R}^3$ and equal to unity outside Ω. This region is perforated with an extremely large number of small holes D_m's of maximum radius a, $a << 1$, modeled by surface impedance functions. Precisely, we are in the regime described by the number of holes of the order $M := O(a^{\beta-2})$, the minimum distance between the holes is $d \sim a^t$ and the surface impedance functions of the form $\lambda_m \sim \lambda_{m,0} a^{-\beta}$ with $\beta > 0$ and $\lambda_{m,0}$ being constants and eventually complex numbers. Under some natural conditions on the parameters β, t and $\lambda_{m,0}$, we characterize the equivalent medium generating, approximately, the same scattered waves as the original perforated acoustic medium. We give an explicit error estimate between the scattered waves generated by the perforated medium and the equivalent one respectively, as $a \to 0$. As applications of these results, we discuss the following findings:

1. If we choose negative valued imaginary surface impedance functions, attached to each surface of the holes, then the equivalent medium behaves as a passive acoustic medium only if it is an acoustic metamaterial with index of refraction $\tilde{n}(x) = -n(x),\ x \in \Omega$ and $\tilde{n}(x) = 1,\ x \in \mathbb{R}^3 \setminus \overline{\Omega}$. This means that, with this process, we can switch the sign of the index of the refraction from positive to negative values.
2. We can choose the surface impedance functions attached to each surface of the holes so that the equivalent index of refraction $\tilde{n}$ is $\tilde{n}(x) = 1,\ x \in \mathbb{R}^3$. This means that the region Ω modeled by the original index of refraction n is approximately cloaked.

*Nonlinear Analysis and Applied Mathematics Research Group (NAAM), Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.

†Corresponding author: Durga Prasad Challa.

‡Department of mathematics, Tallinn University of Technology, Tallinn 19068, Estonia. (Email: durga.challa@ttu.ee).

§Laboratoire de Mathématiques, Pôle Sciences et Technologies, Université de La Rochelle, Avenue Michel Crépeau 17042, La Rochelle Cedex, France, (Email: mokhtar.kirane@univ-lr.fr).

¶RICAM, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040, Linz, Austria. (Email: mourad.sini@oeaw.ac.at).

1 Introduction

The derivation of the macroscopic behavior of a given physical system (or other biological, chemical systems, etc.) as an interaction of a 'dense' microscopic particles is a well known procedure for a long time, see [13, 5]. However, the mathematical modeling as well as the justification of this procedure was understood only in the middle of the last century, see [7, 19, 20]. One of the most known ideas to describe the passage from the microscopic states to the equivalent macroscopic state is the homogenization theory, see [7, 19, 20]. There are two approaches: the deterministic one and the probabilistic one. If one wants to estimate the macroscopic state deterministically, then one needs to assume the periodicity in distributing the small particles. To avoid the periodicity, one can assume the small particles to be randomly distributed and in this case we estimate the equivalent medium in the probabilistic sense. Let us emphasize here that in both approaches, we estimate the limit of the fields created by the microscopic structure to the fields created by the macroscopic one with energy norms stated in the whole domain where the small particles are distributed.

If we are interested only in estimating the limit of the fields away from these small particles (as in the inverse problems and the design theories), then another alternative, avoiding both the periodicity and the randomness, is possible.

A root of this alternative goes back to the seminal work by L. Foldy, see [17], where he gave a close form of the field scattered by M isotropic point-like scatterers, see [17, 21] for details on this model. The justification, or the mathematical foundation, of Foldy's representation was proposed by Berezin and Faddeev in another seminal work, see [6]. The idea is that based on the Krein extension theory of self-adjoint operators, one can model the diffusion by point-like particles by the Schroedinger model with singular potentials of Dirac type supported on those point-like scatterers. This has opened a very fruitful direction of research, see the book [2], on exact models. Following Fadeev's approach, using Krein's extension theory, the diffusion by small particles was studied in several works, see for instance [18, 25].

A different, but still related, approach to describe the diffusion by small particles is based on the integral equations. This is suggested by different authors, as A. Ramm [27] and H. Ammari and H. Kang [4], for instance. In particular, A. Ramm [27, 28] shows that the dominant term in the expansion of the scattered field has the same form as the Foldy's close form where the centers of the small scatterers play the role of the point-like particles. He used the (rough) condition $\frac{a}{d} \ll 1$, where a is the maximum radius of the small particles and d the minimum distance between them, and no error estimate is derived, but he formally characterized the equivalent medium. We also cite the recent works by V. Maz'ya and A. Movchan [22, 23] where they study the Poisson problem and obtain error estimates. In their analysis, they rely on the maximum principle to extend the boundary estimates, which argument does not go smoothly for stationary models as the Helmholtz one.

A rigorous approximation of the scattered fields, with error estimates in terms of the number M of the small scatterers, their minimum distance d and the maximum radius a, was derived in [9, 10] using the integral equations approach. Based on these last estimates, we can characterize the equivalent medium with explicit error estimates in terms of the parameter a, $a << 1$, in appropriate regimes described by the other parameters M and d in terms of a, namely $M := M(a) := O(a^{-s})$ and $d := d(a) \approx a^t$, as $a << 1$, with non negative parameters s and t. This was done in [1, 3] where the small particles are taken to be soft acoustic or rigid elastic particles. The objective of the present work is to extend this study to the case where the small particles have impedance type surfaces with the scaled surface impedences of the form $\lambda_m \approx a^{-\beta}$ with non negative β, in addition to the scaled coefficients M and d described above. Compared to the works in [1], we derive here better error estimates. Precisely, fixing $s = 1$, $\beta = 0$ and $t = \frac{1}{3}$ for simplicity and as an example, we derive here an error of the form $O(a^{\min\{\gamma,\, \frac{2}{3}\}})$ while in [1] it is of the form $O(a^{\min\{\gamma,\, \frac{1}{15}\}})$. Here $\gamma \in (0, 1]$ is the Holder regularity exponent of the coefficients λ_0 and K appearing in the equivalent medium, as discussed below.

The design of materials with desired, and in particular negative, index of refraction is a hot topic in the last years, see [8] for instance. Concerning this topic, our contribution is to have shown mathematically that this is possible with high generality. Indeed, we show that the equivalent medium is modeled by three coefficients K, P_0 and λ_0 modeling respectively, the local distribution of the particles, their geometry and the impedance coefficient attached to each particle. Since we have the freedom in choosing the three functions K, P_0 and λ_0, then we can generate a large family of indices of refraction. In particular

1. if we choose the surface impedance to have negative imaginary parts, which is mathematically possible, see [10], then we show that the equivalent medium will be passive only if the index of refraction is negative, i.e. it behaves as a metamaterial.

2. we can choose the coefficients K, P_0 and λ_0 so that the equivalent index of refraction is educed to the unity. This means that the domain Ω modeled by the original index of refraction n is cloaked.

Let us emphasize that the derived explicit error estimates between the fields generated by the microscopic structure and the one generated by the equivalent macroscopic structure might be useful to quantify the accuracy of the design.

Let us also cite some related works on the derivation of the equivalent media. The first works go at least to Rauch-Taylor, see [29], see also the works by Cioranescu and Murat [14, 15], who characterized the limiting problems for some Poisson type problems and provided convergence results (but with no rates of convergence) of the corresponding resolvent operators. Later, these results were extended and refined in the works of Ozawa and the ones of Figari et al., see respectively [26] and [18] for instance, using point interaction approximations of the Green's kernels. Compared to these results, we do not need the periodicity nor the randomness in distributing the small particles in addition we model them via the scaled parameters M, d and the λ_m's with a high generality as described above.

The rest of the paper is organized as follows. In section 2, we state the main results. Precisely, we describe the mathematical model of the stationary scattering by many small bodies of impedance type in section 2.1, then we state the main mathematical results in section 2.2. We end this section with a discussion on the possible applications of these results in the acoustic metamaterials and the acoustic cloaking in section 2.3. Section 3 is devoted to the proof of the main theorem of the paper.

2 Statement of the results

2.1 The acoustic scattering by many impedance type holes

Let $B_1, B_2, \ldots, B_M$ be M open, bounded and simply connected sets in $\mathbb{R}^3$ with Lipschitz boundaries containing the origin. We assume that the Lipschitz constants of B_j, $j = 1, ..., M$ are uniformly bounded. We set $D_m := \epsilon B_m + z_m$ to be the small bodies characterized by the parameter $\epsilon > 0$ and the locations $z_m \in \mathbb{R}^3$, $m = 1, \ldots, M$. Let U^i be a solution of the Helmholtz equation $(\Delta + \kappa^2)U^i = 0$ in $\mathbb{R}^3$. We denote by U^s the acoustic field scattered by the M small bodies $D_m \subset \mathbb{R}^3$, due to the incident field U^i (mainly the plane incident waves $U^i(x, \theta) := e^{ikx\cdot\theta}$ with the incident direction $\theta \in \mathbb{S}^2$, where $\mathbb{S}^2$ being the unit sphere), with impedance boundary conditions. Hence the total field $U^t := U^i + U^s$ satisfies the following exterior impedance problem of the acoustic waves

$$(\Delta + \kappa^2 n^2(x))U^t = 0 \text{ in } \mathbb{R}^3 \backslash \left(\overset{M}{\underset{m=1}{\cup}} \bar{D}_m \right), \tag{2.1}$$

$$\left.\frac{\partial U^t}{\partial \nu_m} + \lambda_m U^t \right|_{\partial D_m} = 0, \ 1 \leq m \leq M, \tag{2.2}$$

$$\frac{\partial U^s}{\partial |x|} - i\kappa U^s = o\left(\frac{1}{|x|}\right), |x| \to \infty, \tag{2.3}$$

Again, the scattering problem (2.1-2.3) is well posed in the Hölder or Sobolev spaces, see [11, 12, 24] in the case $\Im\lambda_m > 0$. As we said for (2.1-2.3), this last condition can be relaxed to allow $\Im\lambda_m$ to be negative, see [10]. Applying Green's formula to U^s, we can show that the scattered field $U^s(x, \theta)$ has the following asymptotic expansion:

$$U^s(x, \theta) = \frac{e^{i\kappa|x|}}{4\pi|x|} U^\infty(\hat{x}, \theta) + O(|x|^{-2}), \quad |x| \to \infty, \tag{2.4}$$

where the function $U^\infty(\hat{x}, \theta)$ for $(\hat{x}, \theta) \in \mathbb{S}^2 \times \mathbb{S}^2$ is the corresponding far-field pattern.

Definition 2.1. *We define* $a := \max\limits_{1\leq m\leq M} diam(D_m)$, $d := \min\limits_{\substack{m\neq j \\ 1\leq m,j\leq M}} d_{mj}$ *where* $d_{mj} := dist(D_m, D_j)$ *and set* $\kappa_{\max}$ *as the upper bound of the used wave numbers, i.e.* $\kappa \in [0, \kappa_{\max}]$. *The distribution of the scatterers is modeled as follows.*

1. *The number* $M := M(a) := O(a^{-s}) \leq M_{max} a^{-s}$ *with a given positive constant* M_{max}.
2. *The minimum distance* $d := d(a) \approx a^t$, *i.e.* $d_{min} a^t \leq d(a) \leq d_{max} a^t$, *with given positive constants* d_{min} *and* d_{max}.
3. *The surface impedance* $\lambda_m := \lambda_{m,0} a^{-\beta}$, *where* $\lambda_{m,0} \neq 0$ *and might be a complex number.*

Here the real numbers s, t and β are assumed to be non negative.

We call the upper bounds of the Lipschitz character of B_m's, $M_{max}, d_{min}, d_{max}$ and κ_{max} the set of the apriori bounds. In [10], we have shown that there exist a positive constant a_0, λ_- and λ_+ depending only on the set of the apriori bounds and on n_{max} such that if

$$a \leq a_0, \ |\lambda_{m,0}| \leq \lambda_+, \ |\Re(\lambda_{m,0})| \geq \lambda_-, \quad \beta < 1, \quad s \leq 2 - \beta, \quad \frac{s}{3} \leq t \tag{2.5}$$

then the far-field pattern $U^\infty(\hat{x},\theta)$ has the following asymptotic expansion

$$U^\infty(\hat{x},\theta) = V_n^\infty(\hat{x},\theta) + \sum_{m=1}^{M} V_n^t(-\hat{x}, z_m)\mathbb{Q}_m + O\left(a^{3-s-2\beta}\right), \tag{2.6}$$

uniformly in $\hat{x}$ and θ in $\mathbb{S}^2$. The constant appearing in the estimate $O(.)$ depends only on the set of the apriori bounds, λ_-, λ_+ and on n_{max}. The quantity $V_n^t(z_m, -\hat{x})$ is the total field evaluated at the point z_m in the direction $-\hat{x}$, corresponding to the scattering problem

$$(\Delta + \kappa^2 n^2(x))V_n^t = 0 \text{ in } \mathbb{R}^3, \tag{2.7}$$

$$\frac{\partial V_n^s}{\partial |x|} - i\kappa V_n^s = o\left(\frac{1}{|x|}\right), |x| \to \infty, \tag{2.8}$$

i.e. $V_n^t(z_m, -\hat{x}) := e^{-i\hat{x}\cdot z_m} + V_n^s(z_m, -\hat{x})$, where V_n^s is the scattered field. The coefficients $\mathbb{Q}_m$, $m = 1, ..., M$, are the solutions of the following linear algebraic system

$$\mathbb{Q}_m + \sum_{\substack{j=1 \\ j\neq m}}^{M} C_m G_\kappa(z_m, z_j)\mathbb{Q}_j = -C_m V_n^t(z_m, \theta), \tag{2.9}$$

for $m = 1, ..., M$ where $C_m := -\lambda_m\ |\partial D_m|$. Here $G_\kappa(x, z)$ is the outgoing Green's function corresponding to the scattering problem (2.7-2.8).

The algebraic system (2.9) is invertible under the condition:

$$s \leq 2 - \beta. \tag{2.10}$$

2.2 The equivalent model

As the diameter a tends to zero the error term in (2.6) tends to zero for t and s such that

$$\beta < 1, \quad s \leq 2 - \beta, \quad \frac{s}{3} \leq t, \tag{2.11}$$

and it is at least of the order $O(a^{1-\beta})$. Observe that we have the upper bound

$$|\sum_{m=1}^{M} e^{-i\kappa\hat{x}\cdot z_m} Q_m| \leq M \sup_{m=1,...,M} |Q_m| = O(a^{2-\beta-s}) \tag{2.12}$$

since $Q_m \approx |\lambda_m||D_m| \approx a^{2-\beta}$, see [10]. Hence if the number of holes is $M := M(a) := O(a^{-s})$, $s < 2 - \beta$ and t satisfies (2.11), $a \to 0$, then from (2.6), we deduce that

$$U^\infty(\hat{x},\theta) \to V_n^\infty(\hat{x},\theta), \text{ as } a \to 0, \text{ uniformly in terms of } \theta \text{ and } \hat{x} \text{ in } \mathbb{S}^2. \tag{2.13}$$

This means that this collection of holes has no effect on the homogeneous medium as $a \to 0$. The main concern of this work is to consider the case when $s = 2 - \beta$. Let Ω be a bounded domain, say of unit volume, containing the holes $D_m, m = 1, ..., M$. We divide Ω into $[a^{\beta-2}]$ sub-domains

$\Omega_m,\ m=1,...,[a^{\beta-2}]$ such that each Ω_m contains D_m, with $z_m \in \Omega_m$ as its center, and some of the other D_j's. We assume the number of holes in Ω_m, for $m=1,...,[a^{\beta-2}]$, to be uniformly bounded in terms of m. To be precise, we introduce $K:\mathbb{R}^3 \to \mathbb{R}$ as a positive continuous and bounded function. Let each Ω_m, $m \in \mathbb{N}$, be a cube such that $\Omega_m \cap \Omega$ (which we denote also by Ω_m) is of volume $a^{2-\beta}\frac{[K(z_m)+1]}{K(z_m)+1}$ and contains $[K(z_m)+1]$ holes (where $[a]$ stands for the integral part of $a \in \mathbb{R}$). We set $K_{max} := \sup_{z_m}(K(z_m)+1)$, hence $M = \sum_{j=1}^{[a^{\beta-2}]}[K(z_m)+1] \leq K_{max}[a^{\beta-2}] = O(a^{\beta-2})$.

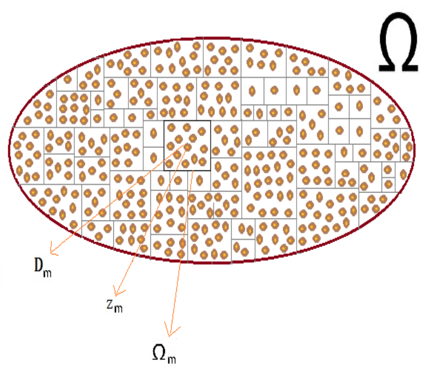

Figure 1: An example on how the holes are distributed in Ω.

We prove the following result.

Theorem 2.2. *Let $\lambda_0 : \Omega \longmapsto \mathbb{C}$ be a continuous function and take $\lambda_m := \lambda_0(z_m)a^{-\beta}$. Consider the small holes to be distributed, as described above, in a bounded domain Ω, say of unit volume, with their number $M := M(a) := O(a^{\beta-2})$ and their minimum distance $d := d(a) := a^t$, $\frac{2-\beta}{3} \leq t \leq 2-\beta$ with $\beta < 1$, as $a \to 0$.*

1. *If the shapes of the holes are different, under the condition that the reference bodies B_m's have uniformly upper bounded perimeters with uniformly lower bounded radiis, then there exist a function $\mathbf{P}_0$ in $\cap_{p\geq 1}L^p(\mathbb{R}^3)$ with support in Ω such that*

$$\lim_{a\to 0} U^\infty(\hat{x},\theta) = U_0^\infty(\hat{x},\theta) \text{ uniformly in terms of } \theta \text{ and } \hat{x} \text{ in } \mathbb{S}^2 \tag{2.14}$$

where $U_0^\infty(\hat{x},\theta)$ is the far-field corresponding to the scattering problem

$$(\Delta + \kappa^2 n^2 + (K+1)\mathbf{P}_0\lambda_0)U_0^t = 0 \text{ in } \mathbb{R}^3, \tag{2.15}$$

$$U_0^t = U_0^s + e^{i\kappa x\cdot\theta}, \tag{2.16}$$

$$\frac{\partial U_0^s}{\partial |x|} - i\kappa U_0^s = o\left(\frac{1}{|x|}\right), |x| \to \infty. \tag{2.17}$$

2. *Assume in addition that λ_0 and $K\mid_\Omega$ are in $C^{0,\gamma}(\Omega)$, $\gamma \in (0,1]$ and the reference bodies B_m's have the same perimeter and diameter, and denote by $P := \frac{|\partial B|}{diam(B)}$. Then*

$$U^\infty(\hat{x},\theta) = U_0^\infty(\hat{x},\theta) + O(a^{\min\{\gamma,\, \frac{2-\beta}{3},\, 1-3\beta,\, 2-\beta-t\}}) \tag{2.18}$$

uniformly in terms of θ and $\hat{x}$ in $\mathbb{S}^2$, where $\mathbf{P}_0 = P$ in Ω and $P_0 = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$.

We see from case 2 of Theorem 2.2 that

$$U^\infty(\hat{x},\theta) = U^\infty_0(\hat{x},\theta) + \left\{ \begin{array}{ll} O(a^{\min\{\gamma,\, \frac{2-\beta}{3},\, 2-\beta-t\}}) & \text{if } 0 \leq \beta \leq \frac{1}{8} \\ O(a^{\min\{\gamma,\, 1-3\beta,\, 2-\beta-t\}}) & \text{if } \frac{1}{8} \leq \beta \leq \frac{1}{3} \end{array} \right. \tag{2.19}$$

and for $\beta \geq \frac{1}{3}$, the remainder is no longer tending to zero as a tends to zero.

We also see that the best error estimate is attained for $\beta = 0$ and it is $O(a^{\min\{\gamma,\, \frac{2}{3},\, 2-t\}})$. In particular if we reasonably take $t \leq 1$, which means that the minimum distance is of the order of the diameters, i.e. $d \approx a$, and $\gamma \geq \frac{2}{3}$, then

$$U^\infty(\hat{x},\theta) = U^\infty_0(\hat{x},\theta) + O(a^{\frac{2}{3}}),\ a \longrightarrow 0. \tag{2.20}$$

However in this case, i.e. $\beta = 0$, the number of small holes attains its maximum, $M = O(a^{\beta-2}) = O(a^{-2})$. Actually, there is a compromise between the number of the small holes and the order of the approximation. In short, the larger is the number of the small holes (or the smaller is β) the better is the approximation.

2.3 Some applications

As a corollary of Theorem 2.2, we deduce the following results.

1. We write $\kappa^2 n^2 + (K+1)P_0\lambda_0 = \kappa^2\tilde{n}^2$, i.e. $\tilde{n}^2 = n^2 + \frac{(K+1)P_0\lambda_0}{\kappa^2}$. This way of representing the equivalent coefficient $\kappa^2 n^2 + (K+1)P_0\lambda_0$ means that the equivalent material behaves as an acoustic material whose index of refraction is $\tilde{n}$ satisfying $\tilde{n}^2 = n^2 + \frac{(K+1)P_0\lambda_0}{\kappa^2}$. In particular, we can choose λ_0, see remark 2.3, as

$$\lambda_0 := \tilde{\lambda}_0\, \kappa^2; \tag{2.21}$$

then

$$\tilde{n}^2 = n^2 + (K+1)P_0\tilde{\lambda}_0. \tag{2.22}$$

We set $\tilde{n} = \tilde{n}_1 + i\tilde{n}_2$. This new acoustic material will be passive only if $\Im\, \tilde{n} = \tilde{n}_2 \geq 0$. From (2.22), we deduce that

$$\tilde{n}_1^2 - \tilde{n}_2^2 = n^2 \ +\ 2\pi(K+1)P_0\Re\, \tilde{\lambda}_0 \quad \text{and} \quad \tilde{n}_1\, \tilde{n}_2 = \pi(K+1)P_0\Im\, \tilde{\lambda}_0. \tag{2.23}$$

Recall that the coefficient λ_0 comes from the surface impedance functions λ_m attached to every small body of the collection of the small bodies generating the coefficient $n^2 + (K+1)P_0\tilde{\lambda}_0$. Hence, if we choose λ_m's so that $\Im\tilde{\lambda}_0 > 0$, then necessarily $\tilde{n}_1 > 0$ and then we can generate acoustic materials having index of refraction as

$$\Re\tilde{n} > 0 \quad \text{and} \quad \Im\tilde{n} > 0. \tag{2.24}$$

Now, if we choose λ_m's so that $\Im(\tilde{\lambda}_0) < 0$, then we deduce from (2.23) and the fact that $\tilde{n}_2 \geq 0$ that necessarily $\tilde{n}_1 < 0$. With this way, we can generate acoustic materials of the form:

$$\Re\, \tilde{n} < 0 \ \text{ and } \ \Im\, \tilde{n} > 0. \tag{2.25}$$

In addition, if we choose the surface impedance function so that $\lambda_0 := \lambda_0(\epsilon)$ with $\Re\, \lambda_0(\epsilon)\ (\, > 0\,) \to 0$ and $\Im\, \lambda_0(\epsilon)(\, < 0\,) \to 0$, as $\epsilon \to 0$, so that the condition (3.9) is satisfied i.e. $\frac{\Re\, \lambda_0(\epsilon)}{|\lambda_0(\epsilon)|^2} > \frac{\sqrt{26 M_{max}}}{\pi}$, then the two equations in (2.23) imply that

$$\Re\, \tilde{n}(\epsilon) \to -n \;\text{ and }\; \Im\, \tilde{n}(\epsilon) \to 0 \quad \text{as } \epsilon \to 0.$$

As a conclusion, if we perforate a given acoustic material, modeled by the index of refraction $n(x)$, with appropriately distributed small holes with negative imaginary part surface impedance functions, attached to each surface, the equivalent medium behaves as a passive acoustic medium only if it is an acoustic metamaterial, i.e. with index of refraction

$$\tilde{n}(x) = -n(x). \tag{2.26}$$

2. We can choose the surface impedance functions so that $n^2 + \frac{(K+1)P_0\lambda_0}{\kappa^2} = 1,\ x \in \Omega$. For instance we take $\lambda_0 := \tilde{\lambda}_0\, \kappa^2$ and $\tilde{\lambda}_0 := \frac{1-n^2}{(K+1)P_0}$. Hence the equivalent index of refraction $\tilde{n}$ is $\tilde{n}(x) = 1,\ x \in \mathbb{R}^3$. This means that the region Ω modeled by the index of refraction n is approximately cloaked. Observe that since we can take the surface impedances complexe valued with any sign then we can cloak the region Ω defined by complex valued index of refraction with any sign of the real and imaginary parts.

Remark 2.3. *The surface impedance in (2.21), i.e.* $\lambda_0 := \tilde{\lambda}_0\, \kappa^2$, *will be achieved if we choose the surface impedances of the small holes as*

$$\lambda_m := \lambda_m(\kappa) := \tilde{\lambda}_{m,0}\, \kappa^2 a^{-\beta}, \tag{2.27}$$

with $a << 1$, *for instance. Since these surface impedance functions appear in the boundary conditions (2.2), i.e.*

$$\frac{\partial U_m}{\partial \nu_m} + \lambda_m(\kappa) U_m = 0,\ 1 \le m \le M, \quad \text{on } \partial D_m,$$

which we can rewrite for convenience, to link the acoustic pressure U_m *to the velocity on the boundary* $\frac{\partial U_m}{\partial \nu_m}$, *as*

$$U_m + \sigma_m(\kappa)\, \frac{\partial U_m}{\partial \nu_m} = 0,\ 1 \le m \le M, \quad \text{on } \partial D_m \tag{2.28}$$

where $\sigma_m(\kappa) := \lambda_m^{-1}(\kappa)$. *In the time domain these impedance boundary conditions are translated as*

$$\tilde{U}_m(t,x) + \int_{-\infty}^{\infty} \tilde{\sigma}_m(t-t') \frac{\partial \tilde{U}_m}{\partial \nu_m}(t',x) dt' = 0,\ x \text{ on } \partial D_m. \tag{2.29}$$

where $\tilde{\sigma}_m(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma(\kappa) e^{-i\kappa t} d\kappa$ *and* $\tilde{U}_m(t,x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{U}_m(\kappa, x) e^{-i\kappa t} d\kappa$.

The wave propagation with the type of time domain impedance boundary conditions in (2.29) were recently object of studies see for instance [16]. Sufficient conditions on the admissibility of the impedance boundary conditions (2.29), as the reality, passivity and causality conditions, are given in the book [30] and also discussed in [16]. These conditions are given in the frequency domain and, if $\Re\, \tilde{\lambda}_0 \geq 0$ *and due to the decay in terms of* κ, *our surface impedances* $\sigma_m(\kappa) = \lambda_m^{-1}$ *with* λ_m *given in (2.27) satisfy those conditions. As a consequence, we do hope that the choice of the surface impedances used in the applications described above might make sense in practice.*

3 Proof of Theorem 2.2

3.1 The relative distribution of the small bodies

We start with the following observation from [3] on the relative distribution of the small bodies. For $m = 1, \dots, M$ fixed, we distinguish between the obstacles D_j, $j \neq m$, by keeping them into different layers based on their distance from D_m. Let us first assume that $K(z_m) = 0$ for every z_m. Hence each Ω_m has the (same) volume $a^{2-\beta}$ and contains only one obstacle D_m. We arrange these cubes in cuboids, in different layers such that the total cubes upto the n^{th} layer consists of $(2n+1)^3$ cubes for $n = 0, \dots, [\left(a^{\frac{2-\beta}{3}} - \frac{a}{2}\right)^{-1}]$, and Ω_m is located at the center, see Fig 2.2. Hence the number of obstacles, we denote by D_j^n and located in the n^{th}, $n \neq 0$, layer will be $[(2n+1)^3 - (2n-1)^3] = 24n^2 + 2$ and their distance from D_m is greater than $n\left(a^{\frac{2-\beta}{3}} - \frac{a}{2}\right)$. Observe that, $\frac{2-\beta}{2} a^{\frac{2-\beta}{3}} \leq \left(a^{\frac{2-\beta}{3}} - \frac{a}{2}\right) \leq a^{\frac{2-\beta}{3}}$. Hence we deduce the needed estimate

$$d(D_j^n, D_m) \geq \frac{n a^{\frac{2-\beta}{3}}}{2}. \tag{3.1}$$

Now, we come back to the case where $K(z_m) \neq 0$. As $\frac{1}{2} \leq \frac{[K(z_m)+1]}{K(z_m)+1} \leq 1$, then with such Ω_m's, the total cubes located in the n^{th} layer consists of at most the double of $[(2n+1)^3 - (2n-1)^3]$, i.e. $48n^2 + 4$ and the inequality (3.1) is also verified.

Figure 2: Rubik's cube consisting of two layers

3.2 Solvability of the linear-algebraic system (2.9)

The algebraic system (2.9) can be written in compact form as

$$\mathbf{B} Q = \mathrm{U}^I, \tag{3.2}$$

where $Q, \mathrm{U}^I \in \mathbb{C}^{M\times 1}$ and $\mathbf{B} \in \mathbb{C}^{M\times M}$ are defined as;

$$\mathbf{B} := \begin{pmatrix} -\frac{1}{C_1} & -G_\kappa(z_1, z_2) & -G_\kappa(z_1, z_3) & \cdots & -G_\kappa(z_1, z_M) \\ -G_\kappa(z_2, z_1) & -\frac{1}{C_2} & -G_\kappa(z_2, z_3) & \cdots & -G_\kappa(z_2, z_M) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -G_\kappa(z_M, z_1) & -G_\kappa(z_M, z_2) & \cdots & -G_\kappa(z_M, z_{M-1}) & -\frac{1}{C_M} \end{pmatrix}, \tag{3.3}$$

$$Q := \begin{pmatrix} Q_1 & Q_2 & \dots & Q_M \end{pmatrix}^\top \text{ and } \mathrm{U}^I := \begin{pmatrix} V_n^t(z_1) & V_n^t(z_2) & \dots & V_n^t(z_M) \end{pmatrix}^\top. \quad (3.4)$$

The following lemma provides us with the needed estimate on the invertibility of (3.2).

Lemma 3.1. *We distinguish the following two cases:*

- *Let* $\Re(\lambda_{m,0}) < 0$ *and assume that* $\min\limits_{1\leq m\leq M} \frac{\Re C_m}{|C_m|^2} > \frac{\sqrt{2M_{max}}}{\pi\, a^{2-\beta}}$ *then the matrix* $\mathbf{B}$ *is invertible and the solution vector* Q *of* (3.2) *satisfies the estimate*

$$\sum_{m=1}^{M} |Q_m|^2 \leq 4 \left(\frac{\min\limits_{1\leq m\leq M} \Re C_m}{\max\limits_{1\leq m\leq M} |C_m|^2} - \frac{\sqrt{26 M_{max}}}{\pi\, a^{2-\beta}} \right)^{-2} \sum_{m=1}^{M} \left| V_n^t(z_m) \right|^2 \quad (3.5)$$

and then

$$\sum_{m=1}^{M} |Q_m| \leq 2 \left(\frac{\min\limits_{1\leq m\leq M} \Re C_m}{\max\limits_{1\leq m\leq M} |C_m|} - \frac{\max\limits_{1\leq m\leq M} |C_m| \sqrt{26 M_{max}}}{\pi\, a^{2-\beta}} \right)^{-1} M \max_{1\leq m\leq M} |C_m| \sum_{m=1}^{M} \left| V_n^t(z_m) \right|, \quad (3.6)$$

- *Let* $\Re(\lambda_{m,0}) > 0$ *and assume that* $\frac{\min\limits_{1\leq m\leq M} \Re(-C_m)}{(\max\limits_{1\leq m\leq M} |C_m|)^2} > \frac{\sqrt{2M_{max}}}{\pi a^{2-\beta}}$ *then the matrix* $\mathbf{B}$ *is invertible and the solution vector* Q *of* (3.2) *satisfies the estimate*

$$\sum_{m=1}^{M} |Q_m|^2 \leq 4 \left(\frac{\min\limits_{1\leq m\leq M} \Re(-C_m)}{\max\limits_{1\leq m\leq M} |C_m|^2} - \frac{\sqrt{2 M_{max}}}{\pi\, a^{2-\beta}} \right)^{-2} \sum_{m=1}^{M} \left| V_n^t(z_m) \right|^2. \quad (3.7)$$

and then

$$\sum_{m=1}^{M} |Q_m| \leq 2 \left(\frac{\min\limits_{1\leq m\leq M} \Re{-C_m}}{\max\limits_{1\leq m\leq M} |C_m|} - \frac{\max\limits_{1\leq m\leq M} |C_m| \sqrt{26 M_{max}}}{\pi\, a^{2-\beta}} \right)^{-1} M \max_{1\leq m\leq M} |C_m| \sum_{m=1}^{M} \left| V_n^t(z_m) \right|, \quad (3.8)$$

The proof of this lemma is given in [10] for the case where $n = 1$, i.e. G_κ is the fundamental solution $\Phi_\kappa(x, y) := \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$ of the free Helmholtz model. But that proof goes smoothly for G_κ as well.

Since $C_m := -\lambda_m |\partial D_m|$, the condition $\frac{\min\limits_{1\leq m\leq M} |\Re C_m|}{\max\limits_{1\leq m\leq M} |C_m|^2} > \frac{\sqrt{26 M_{max}}}{\pi\, a^{2-\beta}}$ is satisfied if λ_- and λ_+ satisfy

$$\frac{\lambda_-}{\lambda_+^2} > \frac{\sqrt{26 M_{max}}}{\pi}. \quad (3.9)$$

3.3 The limiting model

From the function K, we define a bounded function $K_a : \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$K_a(x) := K_a(z_m) := \begin{cases} K(z_m)+1 & \text{if} \quad x \in \Omega_m, \\ 0 & \text{if} \quad x \notin \Omega_m \text{ for any } m = 1, \dots, [a^{\beta-2}]. \end{cases} \tag{3.10}$$

Hence each Ω_m contains $[K_a(z_m)]$ obstacles and $K_{max} := \sup_{z_m} K_a(z_m)$.

Let $\mathbf{C}_a$ be the piecewise constant functions such that $\mathbf{C}_a|_{\Omega_m} = \bar{C}_m := \lambda_{m,0} \frac{|\partial B_m|}{\left(\max\limits_m diam(B_m)\right)^2}$ for all $m = 1, \dots, M$ and vanishes outside Ω. The constants $\bar{C}_m$ are independent of a. We set

$$\mathcal{C} := \max_{1 \leq m \leq M} |\bar{C}_m|_\infty. \tag{3.11}$$

Consider the Lippmann-Schwinger equation

$$U_a(z) + \int_\Omega G_\kappa(z,y) K_a(y) \mathbf{C}_a(y) U_a(y) dy \;=\; -V_n^t(z,\theta), z \in \Omega \tag{3.12}$$

and set the Poisson potential

$$V(Y)(x) := \int_\Omega G_\kappa(x,y) K_a(y) \mathbf{C}_a(y) Y(y) dy, \qquad x \in \mathbb{R}^3. \tag{3.13}$$

The coefficients K_a and $\mathbf{C}_a$ are uniformly bounded. The next lemma concerns the mapping properties of the Poisson potential.

Lemma 3.2. *The operator $V : L^2(\Omega) \to H^2(\Omega)$ is well defined and it is a bounded operator for any bounded domain Ω in $\mathbb{R}^3$, i.e. there exists a positive constant c_0 such that*

$$\|V(Y)\|_{H^2(\Omega)} \leq c_0 \|Y\|_{L^2(\Omega)}. \tag{3.14}$$

Its proof is given in[12], for instance, in the case when $G_\kappa(x,y) = \Phi_\kappa(x,y) := \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$. However $G_\kappa - \Phi_\kappa$ satisfies $(\Delta + \kappa^2)(G_\kappa - \Phi_\kappa) = \kappa^2(1-n^2)\Phi_\kappa$ in $\mathbb{R}^3$. By interior estimates, we deduce that $\|G_\kappa(\cdot,z) - \Phi_\kappa(\cdot,z)\|_{H^2(\Omega)}$ is uniformly bounded in terms of z. By this last property, we show that Lemma 3.2 is also valid for $n \neq 1$ and $n = 1$ in $\mathbb{R}^3 \setminus \Omega$.

Using Lemma 3.2, the fact that the operator $I + V : L^2(\Omega) \longmapsto L^2(\Omega)$ is Fredholm with zero index and the uniqueness of the scattering problem corresponding to the model

$$(\Delta + \kappa^2 n^2 - K_a \mathbf{C}_a) Y = 0, \;\text{ in } \mathbb{R}^3 \tag{3.15}$$

(where $Y := Y^i + Y^s$ and Y^s satisfies the Sommerfeld radiation conditions and Y^i is an incident field), we have the following lemma, see [1] the details.

Lemma 3.3. *There exists one and only one solution Y of the Lippmann-Schwinger equation (3.12) and it satisfies the estimate*

$$\|Y\|_{L^\infty(\Omega)} \leq C \|V_n^t\|_{H^2(\Omega)} \quad \textit{and} \quad \|\nabla Y\|_{L^\infty(\Omega)} \leq C' \|V_n^t\|_{H^2(\tilde{\Omega})}, \tag{3.16}$$

where $\tilde{\Omega}$ being a large bounded domain which contains $\bar{\Omega}$.

3.3.1 Case when the obstacles are arbitrarily distributed

From the definition of $\mathbf{C}_a$, we have $\mathbf{C}_a\ |_{\Omega_m}=\ \lambda_0(z_m)\frac{|B_m|}{\max\{diam(B_m)^2\}}$. We have assumed the reference bodies B_m's to have different but uniformly bounded perimeters with uniformly lower bounded radiis. In addition, since λ_0 is a continuous function, the function $\lambda_a : \Omega \longrightarrow \mathbb{C}$ defined $\lambda_a\ |_{\Omega_m}:= \lambda(z_m)$, converges to λ_0 uniformly. Then there exists a function $\mathbf{P}_0$ in $L^2(\Omega)$ such that $\mathbf{C}_a$ converges weakly to $\mathbf{C}_0 := \lambda_0\mathbf{P_0}$ in $L^2(\Omega)$. Now, since K is continuous hence K_a converges to $(K+1)$ in $L^\infty(\Omega)$ and hence in $L^2(\Omega)$. Then we can show that $K_a\mathbf{C}_a$ converges to $(K+1)\mathbf{C}_0$ in $L^2(\Omega)$.

Since $K\mathbf{C}_a$ is bounded in $L^\infty(\Omega)$, then from the invertibility of the Lippmann-Schwinger equation and the mapping properties of the Poisson potential, see Lemma 3.3, we deduce that $\|U^t_a\|_{H^2(\Omega)}$ is bounded and in particular, up to a sub-sequence, U^t_a tends to U^t_0 in $L^2(\Omega)$. From the convergence of $K_a\mathbf{C}_a$ to $(K+1)\mathbf{C}_0$ and the one of U^t_a to U^t_0 and (3.12), we derive the following equation satisfied by $U^t_0(x)$

$$U^t_0(x) + \int_\Omega G_\kappa(x,y)(K_a)(y)\mathbf{C}_0(y)U^t_0(y)dy = -V^t_n(x,\theta) \text{ in } \Omega.$$

This is the Lippmann-Schwinger equation corresponding to the scattering problem $(\Delta+\kappa^2-(K+1)\mathbf{C}_0)U^t_0 = 0$ in $\mathbb{R}^3$, $U^t_0 = U^s_0 + U^i$, and U^s satisfies the Sommerfeld radiation conditions. As the corresponding far-fields are of the form

$$U^\infty_0(\hat{x},\theta) = \int_\Omega G^\infty_\kappa(\hat{x},y)(K+1)(y)\mathbf{C}_0(y)U^t_0(y)dy$$

which can be written, by the mixed reciprocity relation $G^\infty_\kappa(\hat{x},y) = V_n(y,-\hat{x})$, as

$$U^\infty_0(\hat{x},\theta) = \int_\Omega V_n(y,-\hat{x})(K+1)(y)\mathbf{C}_0(y)U^t_0(y)dy$$

and similarly the ones of U^t_a are of the form

$$U^\infty_a(\hat{x},\theta) = \int_\Omega V_n(y,-\hat{x})K_a(y)\mathbf{C}_a(y)U^t_a(y)dy$$

we deduce that

$$U^\infty_a(\hat{x},\theta) - U^\infty_0(\hat{x},\theta) = o(1),\ a\to 0,\ \text{uniformly in terms of } \hat{x},\theta\ \in \mathbb{S}^2.$$

3.3.2 Case when K is Hölder continuous

If we assume that $K \in C^{0,\gamma}(\Omega)$, $\gamma \in (0,1]$, then we have the estimate $\|(K+1)-K_a\|_{L^\infty(\Omega)} \leq Ca^\gamma$, $a << 1$. Similarly, since we assume the shapes to have the same perimeter and diameter and λ_0 to be in $C^{0,\gamma}(\Omega)$, $\gamma \in (0,1]$, then we have also $\|\mathbf{C}_0 - \mathbf{C}_a\|_{L^\infty(\Omega)} \leq Ca^\gamma$ since $\mathbf{C}_0 = \lambda_0\frac{|\partial B|}{diam(B)}$ where B is the common reference body. Since the obstacles have the same perimeter, we set $\mathbf{C}_0$ to be a constant in Ω and $\mathbf{C}_0 = 0$ in $\mathbb{R}^3 \setminus \Omega$. Recall that U_0 and U_a are solutions of the Lippmann-Schwinger equations

$$U_0 + \int_\Omega G_\kappa(x,y)(K+1)(y)\mathbf{C}_0(y)U^t_0(y)dy = V^t_n$$

and

$$U_a + \int_\Omega G_\kappa(x,y) K_a(y) \mathbf{C}_0(y) U_a^t(y) dy = V_n^t.$$

From the estimate $\|(K+1) - K_a\|_{L^\infty(\Omega)} \leq Ca^\gamma$, $a << 1$, we derive the estimate

$$U_0^\infty(\hat{x},\theta) - U_a^\infty(\hat{x},\theta) = O(a^\gamma),\ a << 1,\ \text{uniformly in terms of } \hat{x},\theta\ \in \mathbb{S}^2. \tag{3.17}$$

3.4 The approximation by the algebraic system

For each $m = 1, \dots, M$, we rewrite the equation (3.12) as follows

$$\begin{aligned} U_a(z_m) &+ \sum_{\substack{j=1 \\ j\neq m}}^{M} G_\kappa(z_m, z_j) \bar{C}_j U_a(z_j) a^{2-\beta} \\ &= -V_n^t(z_m,\theta) + \sum_{\substack{j=1 \\ j\neq m}}^{M} G_\kappa(z_m, z_j) \bar{C}_j U_a(z_j) a^{2-\beta} - \sum_{\substack{j=1 \\ j\neq m}}^{[a^{\beta-2}]} G_\kappa(z_m, z_j) K_a(z_j) \bar{C}_j U_a(z_j) Vol(\Omega_j) \\ &\quad + \sum_{\substack{j=1 \\ j\neq m}}^{[a^{\beta-2}]} G_\kappa(z_m, z_j) K_a(z_j) \bar{C}_j U_a(z_j) Vol(\Omega_j) - \int_\Omega G_\kappa(z_m, y) K_a(y) \mathbf{C}_a(y) U_a(y) dy. \end{aligned} \tag{3.18}$$

Let us estimate the following quantities:

$$A := \sum_{\substack{j=1 \\ j\neq m}}^{[a^{\beta-2}]} G_\kappa(z_m, z_j) K_a(z_j) \bar{C}_j U_a(z_j) Vol(\Omega_j) - \int_\Omega G_\kappa(z_m, y) K_a(y) \mathbf{C}_a(y) U_a(y) dy$$

and

$$B := \sum_{\substack{j=1 \\ j\neq m}}^{M} G_\kappa(z_m, z_j) \bar{C}_j U_a(z_j) a^{2-\beta} - \sum_{\substack{j=1 \\ j\neq m}}^{[a^{\beta-2}]} G_\kappa(z_m, z_j) K_a(z_j) \bar{C}_j U_a(z_j) Vol(\Omega_j).$$

3.4.1 Estimate of A

By the decomposition of Ω, $\Omega := \cup_{l=1}^{[a^{\beta-2}]} \Omega_l$, we have

$$\int_\Omega G_\kappa(z_m, y) K_a(y) \mathbf{C}_a(y) U_a(y) dy = \sum_{l=1}^{[a^{\beta-2}]} \int_{\Omega_l} G_\kappa(z_m, y) K_a(y) \mathbf{C}_a(y) U_a(y) dy. \tag{3.19}$$

Hence,

$$\begin{aligned} A := &\int_{\Omega_m} G_\kappa(z_m, y) K_a(y) \mathbf{C}_a(y) U_a(y) dy \\ &+ \sum_{\substack{j=1 \\ j\neq m}}^{[a^{\beta-2}]} \left[G_\kappa(z_m, z_j) K_a(z_j) \bar{C}_j U_a(z_j) Vol(\Omega_j) - \int_{\Omega_j} G_\kappa(z_m, y) K_a(y) \mathbf{C}_a(y) U_a(y) dy \right]. \end{aligned} \tag{3.20}$$

For $l \neq m$, we have

$$\int_{\Omega_l} G_\kappa(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy - G_\kappa(z_m,z_l)K_a(z_l)\bar{C}_l U_a(z_l)Vol(\Omega_l)$$
$$= K_a(z_l)\bar{C}_l \int_{\Omega_l} \left[G_\kappa(z_m,y)U_a(y) - G_\kappa(z_m,z_l)U_a(z_l)\right] dy. \tag{3.21}$$

We set $f(z_m,y) = G_\kappa(z_m,y)U_a(y)$ then $f(z_m,y)$ satisfies

$$f(z_m,y) - f(z_m,z_l) = (y-z_l)R^i(z_m,y)$$

where

$$\begin{aligned} R^i(z_m,y) &= \int_0^1 \nabla_y f(z_m, y-\beta(y-z_l))\, d\beta \\ &= \int_0^1 \nabla_y \left[G_\kappa(z_m, y-\beta(y-z_l))U_a(y-\beta(y-z_l))\right]\, d\beta \\ &= \int_0^1 \left[\nabla_y G_\kappa(z_m, y-\beta(y-z_l))\right] U_a(y-\beta(y-z_l))\, d\beta \\ &\quad + \int_0^1 G_\kappa(z_m, y-\beta(y-z_l)) \left[\nabla_y U_a(y-\beta(y-z_l))\right]\, d\beta. \end{aligned} \tag{3.22}$$

We set $\Phi_{\kappa,j}(x,y) := \frac{e^{i\kappa n(z_j)|x-y|}}{4\pi|x-y|}$. We have the following lemma:

Lemma 3.4. *We have the asymptotic expansion:*

$$\partial^\alpha_{x_i} G_\kappa(z_j,y) = \partial^\alpha_{x_i}\Phi_{\kappa,j}(z_j,y) + O(1), \quad \text{for } y \text{ in } \Omega, \ \text{with } \alpha = 0,1. \tag{3.23}$$

Proof. We know that $(\Delta + \kappa^2 n^2(x))G_\kappa = -\delta(\cdot - y)$, in $\mathbb{R}^3$ and $\Phi_{\kappa,j}(x,y) := \frac{e^{i\kappa n(z_j)|x-y|}}{4\pi|x-y|}$ satisfies $(\Delta + \kappa^2 n^2(z_j))\Phi_\kappa = -\delta(\cdot - y)$, in $\mathbb{R}^3$, where both G_κ and $\Phi_{\kappa,j}$ satisfy the Sommerfeld radiation condition. Then $H_{\kappa,j}(x,z) := (G_\kappa - \Phi_{\kappa,j})(x,z)$ satisfies the same radiation condition and

$$(\Delta + \kappa^2 n^2(z_j))H_{\kappa,j} = \kappa^2(n^2(z_j) - n^2(x))G_\kappa, \ \text{in } \mathbb{R}^3. \tag{3.24}$$

Multiplying both sides of (3.24) by $\Phi_{\kappa,j}$ and integrating over $B \supset\supset \Omega$ we obtain:

$$\begin{aligned} H_{\kappa,j}(x,y) &= \kappa^2 \int_B (n^2(z_j) - n^2(t))\Phi_{\kappa,j}(t,x)G_{\kappa,j}(t,y)dt \\ &\quad + \int_{\partial B} H_{\kappa,j}(t,y)\partial_{\nu(t)}\Phi_{\kappa,j}(t,x)ds(t) + \int_{\partial\Omega} \partial_{\nu(t)}H_{\kappa,j}(t,x)\Phi_{\kappa,j}(t,x)ds(t), \end{aligned} \tag{3.25}$$

and then

$$\begin{aligned} \nabla_x H_{\kappa,j}(x,y) &= \kappa^2 \int_B (n^2(z_j) - n^2(t))G_\kappa(t,y)\nabla_x\Phi_{\kappa,j}(t,x)dt \\ &\quad + \int_{\partial B} H_{\kappa,j}(t,y)\nabla_x\partial_{\nu(t)}\Phi_{\kappa,j}(t,x)ds(t) + \int_{\partial B} \partial_{\nu(t)}H_{\kappa,j}(t,y)\nabla_x\Phi_{\kappa,j}(t,x)ds(t). \end{aligned} \tag{3.26}$$

Since $(n^2(z_j) - n^2(t))\nabla_x\Phi_\kappa(t,z_j) = O(|t-z_j|), t \in B$, as the singularity of $\nabla_x\Phi_{\kappa,j}(t,x)$ is of the order $|t-x|^{-2}$, then both integrals appearing in (3.25) and (3.26) are of the order $O(1)$ for $x = z_j$ and $y \in \Omega \subset\subset B$. □

From the explicit form of $\Phi_{\kappa,m}$, we have $\nabla_y \Phi_{\kappa,m}(x,y) = \Phi_{\kappa,m}(x,y)\left[\frac{1}{|x-y|} - i\kappa n(z_m)\right]\frac{x-y}{|x-y|}, x \neq y$. Now from Section 3.1, precisely the inequality (3.1), we see that for $l \neq m$

$$|\Phi_{\kappa,m}(z_m, y-\beta(y-z_l))| \leq \frac{\mathring{c}}{4\pi\; n\frac{a^{\frac{2-\beta}{3}}}{2}}, \quad \text{and} \quad |\nabla_y \Phi_{\kappa,m}(z_m, y-\beta(y-z_l))| \leq \frac{\mathring{c}}{4\pi\; n^2\left(\frac{a^{\frac{2-\beta}{3}}}{2}\right)^2}$$

where $\mathring{c}$ depends only on κ and $n(z_m)$. Combining these estimates with (3.23) of Lemma 3.4, we derive the inequalities

$$|G_\kappa(z_m, y-\beta(y-z_l))| \leq \frac{\mathring{c}}{4\pi\; n\frac{a^{\frac{2-\beta}{3}}}{2}}, \text{ and } |\nabla_y G_\kappa(z_m, y-\beta(y-z_l))| \leq \frac{\mathring{c}}{4\pi\; n^2\left(\frac{a^{\frac{2-\beta}{3}}}{2}\right)^2}. \tag{3.27}$$

Then,

$$|R_l(z_m,y)| \leq \frac{\mathring{c}}{2\pi\; na^{\frac{2-\beta}{3}}}\left(\frac{1}{na^{\frac{2-\beta}{3}}}\int_0^1 |U_a(y-\beta(y-z_l))|d\beta + \int_0^1 |\nabla_y U_a(y-\beta(y-z_l))|d\beta\right). \tag{3.28}$$

Then, for $l \neq m$, (3.21) and (3.28) and observing that $\bar{C}_l$ is a constant in Ω_l, imply the estimate

$$\begin{aligned}
&\Big|\int_{\Omega_l} G_\kappa(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy - G_\kappa(z_m,z_l)K_a(z_l)\bar{C}_l U_a(z_l)Vol(\Omega_l)\Big| \\
&\qquad \leq \frac{\mathring{c}\bar{C}_l K_a(z_l)}{\pi\; n^2 a^{\frac{2(2-\beta)}{3}}}\int_{\Omega_l}\Big[\int_0^1 |U_a(y-\beta(y-z_l))|\;d\beta\Big]|y-z_l|dy \\
&\qquad\qquad + \frac{\mathring{c}\bar{C}_l K_a(z_l)}{2\pi\; na^{\frac{2-\beta}{3}}}\int_{\Omega_l}\left[\int_0^1 |\nabla_y U_a(y-\beta(y-z_l))|d\beta\right]|y-z_l|dy \\
&\underset{(3.16)}{\leq} c_1\frac{[K_a(z_l)]\bar{C}_l}{n^2 a^{\frac{2(2-\beta)}{3}}}\, a^{\frac{4(2-\beta)}{3}} \underset{(3.11)}{\leq} c_1\frac{K_{\max}\mathcal{C}}{n^2}\, a^{\frac{2(2-\beta)}{3}},
\end{aligned} \tag{3.29}$$

for a suitable constant c_1.

Regarding the integral $\int_{\Omega_m} G_\kappa(z_m,y)\mathbf{C}_a(y)U_a(y)dy$ we do the following estimates:

$$\left|\int_{\Omega_m} G_\kappa(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy\right| \leq \frac{3}{8\pi}c_1 K_{max}\mathcal{C}\left(\frac{4}{3\pi}\right)^{\frac{1}{3}} a^{\frac{2(2-\beta)}{3}}. \tag{3.30}$$

From (3.20), we can have

$$|A| \leq |\int_{\Omega_m} G_\kappa(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy|$$

$$+\sum_{\substack{j=1\\j\neq m}}^{[a^{\beta-2}]}\left[|G_\kappa(z_m,z_j)K_a(z_j)\bar{C}_jU_a(z_j)Vol(\Omega_j)-\int_{\Omega_j}G_\kappa(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy|\right]$$

which we can estimate by

$$|A|\leq\sum_{n=1}^{[2a^{\frac{\beta-2}{3}}]}2[(2n+1)^3-(2n-1)^3]\Big[|G_\kappa(z_m,z_j)K_a(z_j)\bar{C}_jU_a(z_j)Vol(\Omega_j)|$$
$$-\int_{\Omega_j}|G_\kappa(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)|dy\Big]+|\int_{\Omega_m}G_\kappa(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy|.$$

and then

$$|A|\leq CK_{max}[a^{\frac{2(2-\beta)}{3}}+a^{\frac{2-\beta}{3}}].$$

Finally

$$|A|\leq CK_{max}a^{\frac{2-\beta}{3}}.$$

3.4.2 Estimate of B

$$\sum_{\substack{j=1\\j\neq m}}^{M}G_\kappa(z_m,z_j)\bar{C}_jU_a(z_j)a^{2-\beta}-\sum_{\substack{j=1\\j\neq m}}^{[a^{\beta-2}]}G_\kappa(z_m,z_j)K_a(z_j)\bar{C}_jU_a(z_j)Vol(\Omega_j)$$
$$=\sum_{\substack{l=1\\l\neq m\\z_l\in\Omega_m}}^{[K_a(z_m)]}G_\kappa(z_m,z_l)\bar{C}_lU_a(z_l)a^{2-\beta}+\sum_{\substack{j=1\\j\neq m}}^{[a^{\beta-2}]}\sum_{\substack{l=1\\z_l\in\Omega_j}}^{[K_a(z_j)]}G_\kappa(z_m,z_l)\bar{C}_lU_a(z_l)a^{2-\beta}$$
$$-\sum_{\substack{j=1\\j\neq m}}^{[a^{\beta-2}]}G_\kappa(z_m,z_j)K_a(z_j)\bar{C}_jU_a(z_j)Vol(\Omega_j)$$
$$=\bar{C}_ma^{2-\beta}\sum_{\substack{l=1\\l\neq m\\z_l\in\Omega_m}}^{[K_a(z_m)]}G_\kappa(z_m,z_l)U_a(z_l)$$
$$+\sum_{\substack{j=1\\j\neq m}}^{[a^{\beta-2}]}\bar{C}_ja^{2-\beta}\big[\big(\sum_{\substack{l=1\\z_l\in\Omega_j}}^{[K_a(z_j)]}G_\kappa(z_m,z_l)U_a(z_l)\big)-G_\kappa(z_m,z_j)[K_a(z_j)]U_a(z_j)\big],$$

since $Vol(\Omega_j)=a^{2-\beta}\frac{[K_a(z_j)]}{K_a(z_j)}$ and $\bar{C}_l=\bar{C}_j,$ for $l=1,...,\ K_a(z_j).$ We write,

$$E_1^j\ :=\ \sum_{\substack{l=1\\l\neq m\\z_l\in\Omega_m}}^{[K_a(z_m)]}G_\kappa(z_m,z_l)U_a(z_l)\tag{3.31}$$

and
$$E_2^j \;:=\; \Big[\Big(\sum_{\substack{l=1\\ z_l\in\Omega_j}}^{[K_a(z_j)]} G_\kappa(z_m,z_l)U_a(z_l)\Big) - G_\kappa(z_m,z_j)[K_a(z_j)]U_a(z_j)\Big]$$
$$= \sum_{\substack{l=1\\ z_l\in\Omega_j}}^{[K_a(z_j)]} \Big(G_\kappa(z_m,z_l)U_a(z_l) - G_\kappa(z_m,z_j)U_a(z_j)\Big). \tag{3.32}$$

We need to estimate $\bar{C}_m a^{2-\beta}E_1^j$ and $\sum_{\substack{j=1\\ j\neq m}}^{[a^{\beta-2}]}\bar{C}_j a^{2-\beta}E_2^j$.

Now by writing $f'(z_m,y) := G_\kappa(z_m,y)U_a(y)$. For $z_l\in\Omega_j$, $j\neq m$, using Taylor series, we can write
$$f'(z_m,z_j) - f'(z_m,z_l) = (z_j - z_l)R'(z_m;z_j,z_l),$$
with
$$R'(z_m;z_j,z_l) \;=\; \int_0^1 \nabla_y f'(z_m, z_j - \beta(z_j - z_l))\,d\beta. \tag{3.33}$$
By doing the computations similar to the ones we have performed in (3.22-3.28) and by using Lemma 3.3 and Lemma 3.4, we obtain
$$|\sum_{\substack{j=1\\ j\neq m}}^{[a^{\beta-2}]}\bar{C}_j a^{2-\beta}E_2^j| \leq c_2\mathcal{C}K_{max}a^{\frac{2-\beta}{3}} \tag{3.34}$$
One can easily see that,
$$|\bar{C}_m a^{2-\beta}E_1^j| \leq \frac{c_1(K_{max}-1)\mathcal{C}}{4\pi}\frac{a^{2-\beta}}{d} = \frac{c_1(K_{max}-1)\mathcal{C}}{4\pi}a^{2-\beta-t}. \tag{3.35}$$

Substitution of (3.19) in (3.18) and using the estimates (3.29) and (3.30) associated to A and the estimates (3.34) and (3.35) associated to B gives us

$$U_a(z_m) + \sum_{\substack{j=1\\ j\neq m}}^{M} G_\kappa(z_m,z_j)\bar{C}_j U_a(z_j)a^{2-\beta} \;=\; -V_n^t(z_m,\theta) \tag{3.36}$$
$$+O\left(c_2\mathcal{C}K_{max}a^{\frac{2-\beta}{3}}\right) + O\left(\frac{c_1(K_{max}-1)\mathcal{C}}{4\pi}a^{2-\beta-t}\right).$$

We rewrite the algebraic system (2.9) as
$$U_{a,m} + \sum_{\substack{j=1\\ j\neq m}}^{M} G_\kappa(z_m,z_j)\bar{C}_j U_{a,j}a^{2-\beta} = -V_n^t(z_m) \tag{3.37}$$

where we set $U_{a,m} := C_m^{-1} Q_m$, recalling that $C_m = \bar{C}_m\ a^{2-\beta}$.

Taking the difference between (3.37) and (3.37) produces the algebraic system

$$(U_{a,m} - U_a(z_m)) + \sum_{\substack{j=1 \\ j\neq m}}^{M} G_\kappa(z_m, z_j)\bar{C}_j(U_{a,j} - U_a(z_j))a^{2-\beta} \ = \ O\left(\mathcal{C}K_{max}(a^{\frac{2-\beta}{3}} + a^{2-\beta-t})\right).$$

Comparing this system with (2.9) and by using Lemma 3.1, we obtain the estimate

$$\sum_{m=1}^{M}(U_{a,m} - U_a(z_m)) \ = \ O\left(\mathcal{C}K_{max}M(a^{\frac{2-\beta}{3}} + a^{2-\beta-t})\right). \tag{3.38}$$

For the special case $d = a^t$, $M = O(a^{\beta-2})$ with $t > 0$, we have the following approximation of the far-field from the Foldy-Lax asymptotic expansion (2.6) and from the definitions $U_{a,m} := C_m^{-1}Q_m$ and $C_m := -\lambda_m|\partial D_m| = \bar{C}_m a^{2-\beta}$, for $m = 1, \ldots, M$:

$$\begin{aligned} 4\pi\ U^\infty(\hat{x}, \theta) \ &= \ V_n^\infty + \sum_{j=1}^{M} V_n(z_j, -\hat{x})\bar{C}_j U_{a,j}\ a^{2-\beta} + O\left(a^{3-s-2\beta}\right) \\ &= \ V_n^\infty + \sum_{j=1}^{M} V_n(z_j, -\hat{x})\bar{C}_j U_{a,j}\ a^{2-\beta} + O\left(a^{1-3\beta}\right). \end{aligned} \tag{3.39}$$

Consider the far-field of type:

$$U^\infty_{\mathbf{C}_a}(\hat{x}, \theta) \ = \ V_n^\infty + \frac{1}{4\pi}\int_\Omega V_n(y, -\hat{x})K_a(y)\mathbf{C}_a(y)U_a(y)dy.$$

corresponding to the scattering problem (3.15). Taking the difference between (3.40) and (3.39) we have:

$$\begin{aligned} &4\pi(U^\infty_{\mathbf{C}_a}(\hat{x}, \theta) - U^\infty(\hat{x}, \theta)) \\ &= \int_\Omega V_n(y, -\hat{x})K_a(y)\mathbf{C}_a(y)U_a(y)dy - \sum_{j=1}^{M} V_n(z_j, -\hat{x})\bar{C}_j U_{a,j} a^{2-\beta} + O\left(a^{1-3\beta}\right) \\ &= \sum_{j=1}^{[a^{\beta-2}]}\int_{\Omega_j} V_n(y, -\hat{x})K_a(y)\mathbf{C}_a(y)U_a(y)dy - \sum_{j=1}^{[a^{\beta-2}]}\sum_{\substack{l=1 \\ z_l\in\Omega_j}}^{[K_a(z_j)]} V_n(z_l, -\hat{x})\bar{C}_l U_{a,l} a^{2-\beta} + O\left(a^{1-3\beta}\right) \\ &= \sum_{j=1}^{[a^{\beta-2}]} K_a(z_j)\bar{C}_j\int_{\Omega_j}\left[V_n(y, -\hat{x})U_a(y) - V_n(z_j, -\hat{x})U_a(z_j)\right]dy \\ &\quad + \sum_{j=1}^{[a^{\beta-2}]}\bar{C}_j a^{2-\beta}\left[\sum_{\substack{l=1 \\ z_l\in\Omega_j}}^{[K_a(z_j)]}(V_n(z_j, -\hat{x})U_a(z_j) - V_n(z_l, -\hat{x})U_a(z_l)) + \sum_{\substack{l=1 \\ z_l\in\Omega_j}}^{[K_a(z_j)]} V_n(z_l, -\hat{x})\left(U_a(z_l) - U_{a,l}\right)\right] \end{aligned}$$

$$+ O\left(a^{1-3\beta}\right)$$

$$= \sum_{j=1}^{[a^{\beta-2}]} \int_{\Omega_j} K_a(z_j)\bar{C}_j \left[V_n(y,-\hat{x})U_a(y) - V_n(z_j,-\hat{x})U_a(z_j)\right] dy$$

$$+ \sum_{j=1}^{[a^{\beta-2}]} \bar{C}_j a^{2-\beta} \sum_{\substack{l=1 \\ z_l\in\Omega_j}}^{[K_a(z_j)]} \left(V_n(z_j,-\hat{x})U_a(z_j) - V_n(z_l,-\hat{x})U_a(z_l)\right) + \sum_{j=1}^{M} V_n(z_j,-\hat{x})\bar{C}_j a^{2-\beta}\left[U_a(z_j) - U_{a,j}\right]$$

$$+ O\left(a^{1-3\beta}\right)$$

$$\overset{(3.38)}{=} \sum_{j=1}^{[a^{\beta-2}]} K_a(z_j)\bar{C}_j \int_{\Omega_j} \left[V_n(y,-\hat{x})U_a(y) - V_n(z_j,-\hat{x})U_a(z_j)\right] dy$$

$$+ \sum_{j=1}^{[a^{\beta-2}]} \bar{C}_j a^{2-\beta} \sum_{\substack{l=1 \\ z_l\in\Omega_j}}^{[K_a(z_j)]} \left(V_n(z_j,-\hat{x})U_a(z_j) - V_n(z_l,-\hat{x})U_a(z_l)\right) + O\left(\mathcal{C}^2 K_{max}(a^{\frac{2-\beta}{3}} + a^{2-\beta-t})\right)$$

$$+ O\left(a^{1-3\beta}\right). \tag{3.40}$$

Now, let us estimate the difference $\sum_{j=1}^{[a^{\beta-2}]} K_a(z_j)\bar{C}_j \int_{\Omega_j} \left[V_n(y,-\hat{x})U_a(y) - V_n(z_j,-\hat{x})U_a(z_j)\right] dy$. Write, $f_1(y) = V_n(y,-\hat{x})U_a(y)$. Using Taylor series, we can write

$$f_1(y) - f_1(z_j) = (y - z_j)\cdot R_j(y),$$

with

$$\begin{aligned} R_j(y) &= \int_0^1 \nabla_y(f_1)(y-\beta(y-z_j))\, d\beta \\ &= \int_0^1 \left[\nabla_y\left[V_n(-\hat{x}, y-\beta(y-z_j))U_a(y-\beta(y-z_j))\right]\right] d\beta \\ &= \int_0^1 \left[\nabla_y V_n(-\hat{x}, y-\beta(y-z_j))\right] U_a(y-\beta(y-z_j))\, d\beta \\ &\quad + \int_0^1 V_n(-\hat{x}, y-\beta(y-z_j))\left[\nabla_y U_a(y-\beta(y-z_j))\right] d\beta. \end{aligned} \tag{3.41}$$

Recall that V_n satisfies the scattering problem (2.7)- (2.8), hence it is also solution of the corresponding Lippmann-Schwinger equation $V_n(x) + \int_\Omega \Phi(x,y)(n^2(y)-1)V_n(y)dy = -e^{\kappa x\cdot\theta}$. This is the same integral equation as (3.12) at the expense of replacing K_aC_a by $n-1$ and V_n by $e^{\kappa x\cdot\theta}$. Then replacing in Lemma 3.3 V_n by $e^{\kappa x\cdot\theta}$ and then Y by V_n, we have the following estimates

$$\|V_n\|_{L^\infty(\Omega)}, \|\nabla V_n\|_{L^\infty(\Omega)} \leq \tilde{C} \tag{3.42}$$

where $\tilde{C}$ depends only on $\|n\|_{L^\infty(\Omega)}$ and κ. Then

$$|R_j(y)| \;\leq\; \tilde{C}\left(\int_0^1 |U_a(y-\beta(y-z_j))|\, d\beta + \int_0^1 |\nabla_y U_a(y-\beta(y-z_j))|\, d\beta\right). \tag{3.43}$$

Using (3.43) we get the estimate

$$
\begin{aligned}
&\Big|\sum_{j=1}^{[a^{\beta-2}]} K_a(z_j)\bar{C}_j \int_{\Omega_j} \big[V_n(y,-\hat{x})(y)U_a(y) - V_n(z_j,-\hat{x})U_a(z_j)\big]\,dy\Big| \\
&\leq \tilde{C}\sum_{j=1}^{[a^{\beta-2}]} K_a(z_j)\bar{C}_j\Big(\kappa \int_{\Omega_j} |y-z_j| \int_0^1 |U_a(y-\beta(y-z_j))|\,d\beta\,dy\Big) \\
&\qquad + \tilde{C}\sum_{j=1}^{[a^{\beta-2}]} K_a(z_j)\bar{C}_j\left(\int_{\Omega_j} |y-z_j| \int_0^1 |\nabla_y U_a(y-\beta(y-z_j))|\,d\beta\,dy\right) \\
&\leq \tilde{C}\sum_{j=1}^{[a^{\beta-2}]} K_a(z_j)\bar{C}_j c_1\, a^{2-\beta}\, a^{\frac{2-\beta}{3}}\ (\kappa + c_5) \\
&\leq \tilde{C} K_{max}\mathcal{C} c_1\big(\kappa + c_5\big)\, a^{\frac{2-\beta}{3}}.
\end{aligned} \tag{3.44}
$$

In the similar way, using (3.38), we have,

$$
\left|\sum_{j=1}^{[a^{\beta-2}]} \bar{C}_j a^{2-\beta} \sum_{\substack{l=1 \\ z_l\in\Omega_j}}^{[K_a(z_j)]} \left(V_n(z_j,-\hat{x})U_a(z_j) - V_n(z_l,-\hat{x})U_a(z_l)\right)\right| \leq O\left(K_{max}\mathcal{C}(a^{\frac{2-\beta}{3}} + a^{2-\beta-t})\right). \tag{3.45}
$$

Using the estimates (3.44) and (3.45) in (3.40), we obtain

$$
\begin{aligned}
&\frac{1}{4\pi}U^\infty_{\mathbf{C}_a}(\hat{x},\theta) - U^\infty(\hat{x},\theta) \\
&= O\left(K_{max}a^{\frac{2-\beta}{3}}\mathcal{C}c_1\left(\kappa + c_5\right)\right) + O(\mathcal{C}(\mathcal{C}+1)K_{max}M\ a^{2-\beta}(a^{\frac{2-\beta}{3}} + a^{2-\beta-t})) + O\left(a^{1-3\beta}\right) \\
&= O\left(a^{\frac{2-\beta}{3}} + a^{2-\beta-t} + a^{1-3\beta}\right).
\end{aligned} \tag{3.46}
$$

Since $Vol(\Omega)$ is of order $a^{\beta-2}(\frac{a^{2-\beta}}{2} + \frac{d}{2})^3$, and d is of the order a^t, we should have $t \geq \frac{2-\beta}{3}$. Hence, we need to impose the following conditions

$$
t \geq \frac{2-\beta}{3},\ 2-\beta-t \geq 0 \ \text{ and }\ 1-3\beta > 0.
$$

3.5 End of the proof of Theorem 2.2

Combining the estimates (3.46) and (3.17), we deduce that

$$
\frac{1}{4\pi}\left[U^\infty(\hat{x},\theta) - U^\infty_0(\hat{x},\theta)\right]\cdot\hat{x} = O(a^{\min\{\gamma,\,\frac{1}{3},\,\frac{2-\beta}{3},\,2-\beta-t,\,1-3\beta\}}),\ a << 1, \quad \frac{2-\beta}{3} \leq t \leq 2-\beta \tag{3.47}
$$

uniformly in terms of $\hat{x},\theta \ \in \mathbb{S}^2$.

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References

[1] B. Ahmad, D. P. Challa, M. Kirane, and M. Sini. The equivalent refraction index for the acoustic scattering by many small obstacles: with error estimates. *J. Math. Anal. Appl.*, 424(1):563–583, 2015.

[2] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden. *Solvable models in quantum mechanics*. AMS Chelsea Publishing, Providence, RI, second edition, 2005. With an appendix by Pavel Exner.

[3] F. Al-Musallam, D. P. Challa and M. Sini. The equivalent mass density for the elastic scattering by many small rigid bodies and applications. `arXiv:1504.06947`

[4] H. Ammari and H. Kang. *Polarization and moment tensors*, volume 162 of *Applied Mathematical Sciences*. Springer, New York, 2007. With applications to inverse problems and effective medium theory.

[5] A. S. Barnard. *Modelling of nanoparticles: approaches to morphology and evolution*. Rep. Prog. Phys, 73 (2010) 086502.

[6] F. A. Berezin and L. D. Faddeev. *A remark on Scroedinger's equation with a singular potential*. Soviet Math. Dokl. 2 (1961), 372-375.

[7] A. Bensoussan; J. L. Lions and G. Papanicolaou. Asymptotic analysis for periodic structures. Studies in Mathematics and its Applications, 5. North-Holland Publishing Co., Amsterdam-New York, 1978.

[8] W. Cai and V. Shalaev. *Optical Metamaterials: Fundamental and Applications*. Hardcover ISBN: 978-1-4419-1150-6, Springer-Verlag New York, 2010.

[9] D. P. Challa and M. Sini. On the justification of the Foldy-Lax approximation for the acoustic scattering by small rigid bodies of arbitrary shapes. *Multiscale Model. Simul.*, 12(1):55–108, 2014.

[10] D. P. Challa; M. Sini. Multiscale analysis of the acoustic scattering by many scatterers of impedance type. *Preprint , arXiv: 1504. 02665*.

[11] D. L. Colton and R. Kress. *Integral equation methods in scattering theory*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1983. A Wiley-Interscience Publication.

[12] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 1998.

[13] T. Choy. *Effective Medium Theory: principles and applications*, International series of monographs on Physics: 102, Clarendon Press-Oxford, 1999.

[14] D. Cioranescu and F. Murat. *Un terme étrange venu d'ailleurs. (French) [A strange term brought from somewhere else] Nonlinear partial differential equations and their applications.* Collège de France Seminar, Vol. II (Paris, 1979/1980), pp. 9838, 38990, Res. Notes in Math., 60, Pitman, Boston, Mass.-London, 1982.

[15] D. Cioranescu and F. Murat. A strange term coming from nowhere Topics in the Mathematical Modelling of Composite Materials. Progress in Nonlinear Differential Equations and Their Applications Volume 31, 1997, pp 45-93

[16] D. Dragna and P. Blanc-Benon. Physically admissible impedance models for time-domain computations of outdoor sound propagation Acta Acustica united with Acustica. Vol. 100 (2014), 401-410. DOI 10.3813/AAA.918719

[17] L. L. Foldy. The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers. *Phys. Rev. (2)*, 67:107–119, 1945.

[18] R. Figari, E. Orlandi and S. Teta. The Laplacian in regions with many small obstacles: fluctuations around the limit operator. J. Statist. Phys. 41 (1985), no. 3-4, 465177.

[19] V. Jikov, S. Kozlov and O. Oleinik. *Homogenization of differential operators and integral functionals.* Springer-Verlag, 1994.

[20] V. Marchenko and E. Khruslov *Homogenization of partial differential equations.* Birkhauser-Boston, 2006.

[21] P. A. Martin. *Multiple scattering*, volume 107 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2006. Interaction of time-harmonic waves with N obstacles.

[22] V. Maz'ya and A. Movchan. Asymptotic treatment of perforated domains without homogenization. *Math. Nachr.*, 283(1):104–125, 2010.

[23] V. Maz'ya, A. Movchan, and M. Nieves. *Green's Kernels and Meso-Scale Approximations in Perforated Domains*, volume 2077 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2013.

[24] W. McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[25] S. A. Nazarov, and J. Sokolowski. Self-adjoint extensions for the Neumann Laplacian and applications. Acta Math. Sin. (Engl. Ser.) 22 (2006), no. 3, 879-906.

[26] S. Ozawa, Point interaction potential approximation for $(-\Delta + U)^{-1}$ and eigenvalues of the Laplacian on wildly perturbed domain. Osaka J. Math. 20 (1983), no. 4, 923177.

[27] A. G. Ramm. *Inverse problems.* Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005.

[28] A. G. Ramm. Many-body wave scattering by small bodies and applications. *J. Math. Phys.*, 48(10):103511, 29, 2007.

[29] J. Rauch and M. Taylor, Potential and scattering theory on wildly perturbed domains. J. Funct. Anal. 18 (1975), 2717.

[30] S. W. Rienstra and A. Hirschberg An Introduction to Acoustics. Eindhoven University of Technology, 2015. Revised edition of IWDE 92-06.