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Multiscale analysis of the acoustic scattering by many scatterers of impedance type

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Abstract

We are concerned with the acoustic scattering problem, at a frequency κ, by many small obstacles of arbitrary shapes with impedance boundary condition. These scatterers are assumed to be included in a bounded domain Ω in $\mathbb{R}^3$ which is embedded in an acoustic background characterized by an eventually locally varying index of refraction. The collection of the scatterers $D_m,\ m=1,...,M$ is modeled by four parameters: their number M, their maximum radius a, their minimum distance d and the surface impedances $\lambda_m,\ m=1,...,M$. We consider the parameters M, d and λ_m's having the following scaling properties: $M := M(a) = O(a^{-s})$, $d := d(a) \approx a^t$ and $\lambda_m := \lambda_m(a) = \lambda_{m,0}a^{-\beta}$, as $a \to 0$, with non negative constants s, t and β and complex numbers $\lambda_{m,0}$'s with eventually negative imaginary parts.

We derive the asymptotic expansion of the farfields with explicit error estimate in terms of a, as $a \to 0$. The dominant term is the Foldy-Lax field corresponding to the scattering by the point-like scatterers located at the centers z_m's of the scatterers D_m's with $\lambda_m|\partial D_m|$ as the related scattering coefficients. This asymptotic expansion is justified under the following conditions

$$a \leq a_0,\ |\Re(\lambda_{m,0})| \geq \lambda_-,\ |\lambda_{m,0}| \leq \lambda_+, \quad \beta < 1, \quad 0 \leq s \leq 2-\beta, \quad \frac{s}{3} \leq t$$

and the error of the approximation is $C\ a^{3-2\beta-s}$, as $a \to 0$, where the positive constants $a_0,\ \lambda_-,\ \lambda_+$ and C depend only on the a priori uniform bounds of the Lipschitz characters of the obstacles D_m's and the ones of $M(a)a^s$ and $\frac{d(a)}{a^t}$. We do not assume the periodicity in distributing the small scatterers. In addition, the scatterers can be arbitrary close since t can be arbitrary large, i.e. we can handle the mesoscale regime. Finally, for spherical scatterers, we can also allow the limit case $\beta = 1$ with a slightly better error of the approximation.

Keywords: Acoustic scattering, Small-scatterers, Multiple scattering, Foldy-Lax approximation.

1 Introduction and statement of the results

Let $B_1, B_2, \dots, B_M$ be M open, bounded and simply connected sets in $\mathbb{R}^3$ with Lipschitz boundaries containing the origin. We assume that the Lipschitz constants of $B_j,\ j=1,...,M$ are uniformly bounded. We set $D_m := \epsilon B_m + z_m$ to be the small bodies characterized by the parameter $\epsilon > 0$ and the locations $z_m \in \mathbb{R}^3$, $m=1,\dots,M$. Let U^i be a solution of the Helmholtz equation $(\Delta+\kappa^2)U^i = 0$ in $\mathbb{R}^3$. We denote by U^s the acoustic field scattered by the M small bodies $D_m \subset \mathbb{R}^3$, due to the incident field U^i (mainly the plane incident waves $U^i(x,\theta) := e^{ikx\cdot\theta}$ with the incident direction $\theta \in \mathbb{S}^2$, where $\mathbb{S}^2$ being the unit sphere), with impedance boundary conditions. Hence the total field $U^t := U^i + U^s$ satisfies the following exterior impedance problem of the acoustic waves

$$(\Delta+\kappa^2)U^t = 0 \text{ in } \mathbb{R}^3 \backslash \left(\overset{M}{\underset{m=1}{\cup}} \bar{D}_m \right), \tag{1.1}$$

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$$\frac{\partial U^t}{\partial \nu_m} + \lambda_m U^t \Big|_{\partial D_m} = 0,\ 1 \leq m \leq M, \tag{1.2}$$

$$\frac{\partial U^s}{\partial |x|} - i\kappa U^s = o\left(\frac{1}{|x|}\right), |x| \to \infty,\ \text{(S.R.C)} \tag{1.3}$$

where ν_m is the outward unit normal vector of ∂D_m, $\kappa > 0$ is the wave number and S.R.C stands for the Sommerfeld radiation condition. The scattering problem (1.1-1.3) is well posed in the Hölder or Sobolev spaces, see [18, 19, 27, 29] for instance, in the case $\Im\lambda_m \geq 0$. We will see that this condition can be relaxed to allow $\Im\lambda_m$ to be negative under some conditions. Applying Green's formula to U^s, we can show that the scattered field $U^s(x, \theta)$ has the following asymptotic expansion:

$$U^s(x, \theta) = \frac{e^{i\kappa|x|}}{4\pi|x|} U^\infty(\hat{x}, \theta) + O(|x|^{-2}), \quad |x| \to \infty, \tag{1.4}$$

with $\hat{x} := \frac{x}{|x|}$, where the function $U^\infty(\hat{x}, \theta)$ for $(\hat{x}, \theta) \in \mathbb{S}^2 \times \mathbb{S}^2$ is called the far-field pattern. We recall that the fundamental solution, $\Phi_\kappa(x, y)$, of the Helmholtz equation in $\mathbb{R}^3$ with the fixed wave number κ is given by $\Phi_\kappa(x, y) := \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$, for all $x, y \in \mathbb{R}^3$.

Definition 1.1. *We define*

1. $a := \max\limits_{1\leq m \leq M} diam(D_m)$ $\left[= \epsilon \max\limits_{1\leq m\leq M} diam(B_m)\right]$,
2. $d := \min\limits_{\substack{m\neq j\\ 1\leq m,j\leq M}} d_{mj}$, *where* $d_{mj} := dist(D_m, D_j)$.
3. $\kappa_{\max}$ *as the upper bound of the used wave numbers, i.e.* $\kappa \in [0, \kappa_{\max}]$.

The distribution of the scatterers is modeled as follows:

4. *the number* $M := M(a) := O(a^{-s}) \leq M_{max} a^{-s}$ *with a given positive constant* M_{max}
5. *the minimum distance* $d := d(a) \approx a^t$, *i.e.* $d_{min} a^t \leq d(a) \leq d_{max} a^t$, *with given positive constants* d_{min} *and* d_{max}.
6. *the surface impedance* $\lambda_m := \lambda_{m,0} a^{-\beta}$, *where* $\lambda_{m,0} \neq 0$ *and might be a complex number.*

Here the real numbers s, t and β are assumed to be non negative.

We call the upper bounds of the Lipschitz character of B_m's, $M_{max}, d_{min}, d_{max}$ and κ_{max} the set of the a priori bounds.

The goal of our work is to derive an asymptotic expansion of the scattered field by the collection of the small scatterers $D_m, m = 1, \dots, M$, taking into account these parameters. This is the object of the following theorem.

Theorem 1.2. *There exist positive constants* a_0, λ_- *and* λ_+ *depending only on the set of the a priori bounds such that if*

$$a \leq a_0,\ |\lambda_{m,0}| \leq \lambda_+,\ |\Re(\lambda_{m,0})| \geq \lambda_-, \quad \beta < 1, \quad s \leq 2 - \beta, \quad \frac{s}{3} \leq t \tag{1.5}$$

then the far-field pattern $U^\infty(\hat{x}, \theta)$ *has the following asymptotic expansion*

$$U^\infty(\hat{x}, \theta) = \sum_{m=1}^{M} e^{-i\kappa\hat{x}\cdot z_m} Q_m + O\left(a^{3-s-2\beta}\right), \tag{1.6}$$

uniformly in $\hat{x}$ and θ in $\mathbb{S}^2$. The constant appearing in the estimate $O(.)$ depends only on the set of the a priori bounds, λ_- and λ_+. The coefficients Q_m, $m = 1, ..., M$, are the solutions of the following linear algebraic system

$$Q_m + \sum_{\substack{j=1 \\ j \neq m}}^{M} C_m \Phi_\kappa(z_m, z_j) Q_j \quad = \quad -C_m U^i(z_m, \theta), \tag{1.7}$$

for $m = 1, ..., M$, with $C_m := -\lambda_m |\partial D_m|$.

The algebraic system (1.7) is invertible under the conditins:

$$|\Re(\lambda_{m,0})| \geq \lambda_-, \ s \leq 2 - \beta. \tag{1.8}$$

Let us now assume that the background is not homogeneous but modeled by a locally variable index of refraction n, i.e. there exists a bounded set Ω such that $n(x) = 1, \ x \in \mathbb{R}^3 \setminus \Omega$ and bounded inside Ω, i.e. $|n(x)| \leq n_{max}, \ x \in \Omega$. In this case we model our scattering problem as follows. The total field $U^t_n := U^i + U^s_n$ satisfies the following exterior impedance problem of the acoustic waves

$$(\Delta + \kappa^2 n^2(x)) U^t_n = 0 \text{ in } \mathbb{R}^3 \backslash \left(\overset{M}{\underset{m=1}{\cup}} \bar{D}_m \right), \tag{1.9}$$

$$\frac{\partial U^t_n}{\partial \nu_m} + \lambda_m U^t_n \bigg|_{\partial D_m} = 0, \ 1 \leq m \leq M, \tag{1.10}$$

$$\frac{\partial U^s_n}{\partial |x|} - i\kappa U^s_n = o\left(\frac{1}{|x|}\right), |x| \to \infty, \tag{1.11}$$

Again, the scattering problem (1.9-1.11) is well posed in the Hölder or Sobolev spaces, see [18, 19, 27] in the case $\Im\lambda_m > 0$. As we said for (1.9-1.11), this last condition can be relaxed to allow $\Im\lambda_m$ to be negative. Applying Green's formula to U^s_n, we can show that the scattered field $U^s_n(x, \theta)$ has the following asymptotic expansion:

$$U^s_n(x, \theta) = \frac{e^{i\kappa|x|}}{4\pi|x|} U^\infty_n(\hat{x}, \theta) + O(|x|^{-2}), \quad |x| \to \infty, \tag{1.12}$$

where the function $U^\infty_n(\hat{x}, \theta)$ for $(\hat{x}, \theta) \in \mathbb{S}^2 \times \mathbb{S}^2$ is the corresponding far-field pattern. As a corollary of Theorem 1.2, we derive the following result:

Corollary 1.3. *There exist positive constants a_0, λ_-, λ_+ depending only on the set of the a priori bounds and on n_{max} such that if*

$$a \leq a_0, \ |\lambda_{m,0}| \leq \lambda_+, \ |\Re(\lambda_{m,0})| \geq \lambda_-, \quad \beta < 1, \quad s \leq 2 - \beta, \quad \frac{s}{3} \leq t \tag{1.13}$$

then the far-field pattern $U^\infty_n(\hat{x}, \theta)$ has the following asymptotic expansion [1]

$$U^\infty_n(\hat{x}, \theta) = V^\infty_n(\hat{x}, \theta) + \sum_{m=1}^{M} V^t_n(z_m, -\hat{x}) \mathbb{Q}_m + O\left(a^{3-s-2\beta}\right), \tag{1.14}$$

uniformly in $\hat{x}$ and θ in $\mathbb{S}^2$. The constant appearing in the estimate $O(.)$ depends only on the set of the a priori bounds, λ_-, λ_+ and n_{max}. The quantity [2] *$V^t(z_m, -\hat{x})$ is the total field, evaluated at the point z_m in the direction $-\hat{x}$, corresponding to the scattering problem*

$$(\Delta + \kappa^2 n^2(x)) V^t_n = 0 \ \textit{in} \ \mathbb{R}^3, \tag{1.15}$$

[1] The following remarks apply for both Theorem 1.2 and Corollary 1.3. If $s < 2 - \beta$, then the bounds λ_- and λ_+ can be arbitrary. If $\Re\lambda_m \leq 0$ then, for the invertibility of the systems (1.7) and (1.17), the condition $s \leq 2 - \beta$ on the number of the small bodies can be replaced by the condition $t \leq 2 - \beta$ on the minimum distance between them, see Remark 6.1.

[2] By the mixed reciprocity relations, we have $V^t_n(z_m, -\hat{x}) = G^\infty_\kappa(\hat{x}, z_m)$ where $G^\infty_\kappa(\hat{x}, z_m)$ is the farfield created by a point source $G_\kappa(x, z_m)$, located at the point z_m, in the scattering model (1.15-1.16).

$$\frac{\partial V_n^s}{\partial |x|} - i\kappa V_n^s = o\left(\frac{1}{|x|}\right), |x| \to \infty, \tag{1.16}$$

The coefficients $\mathbb{Q}_m$, $m = 1, ..., M$, *are the solutions of the following linear algebraic system*

$$\mathbb{Q}_m + \sum_{\substack{j=1 \\ j \neq m}}^{M} C_m G_\kappa(z_m, z_j)\mathbb{Q}_j \quad = \quad -C_m V^t(z_m, \theta), \tag{1.17}$$

for $m = 1, ..., M$, *with*

$$C_m := -\lambda_m |\partial D_m|. \tag{1.18}$$

Here $G_\kappa(x, z)$ *is the outgoing Green's function corresponding to the scattering problem (1.15-1.16).*

The algebraic system (1.17) *is invertible under the conditions:*

$$|\Re(\lambda_{m,0})| \geq \lambda_-, \quad s \leq 2 - \beta. \tag{1.19}$$

Before discussing these results comparing them to some of the literature, we add the following remark on the particular case when the scatterers are spherical.

Remark 1.4. *If the scatterers have spherical shapes, then we can slightly improve the error estimates in (1.14) by slightly changing the form of the algebraic system (1.17). Precisely we have* $O\left(a^{3-s-\beta}\right)$ *instead of* $O\left(a^{3-s-2\beta}\right)$. *For these spherical shapes, we can also handle the case* $\beta = 1$. *The result reads as follows. There exist positive constants* a_0, λ_-, λ_+ *depending only on the set of the a priori bounds and on* n_{max} *such that if*

$$a \leq a_0, \; |\lambda_{m,0}| \leq \lambda_+, \; |\Re(\lambda_{m,0})| \geq \lambda_-, \quad \beta \leq 1, \quad s \leq 2 - \beta, \quad \frac{s}{3} \leq t \tag{1.20}$$

then the the far-field pattern $U_n^\infty(\hat{x}, \theta)$ *has the following asymptotic expansion*

$$U_n^\infty(\hat{x}, \theta) = V_n^\infty(\hat{x}, \theta) + \sum_{m=1}^{M} V_n^t(z_m, -\hat{x})\mathbb{Q}_m + O\left(a^{3-s-\beta}\right), \tag{1.21}$$

uniformly in $\hat{x}$ *and* θ *in* $\mathbb{S}^2$. *The coefficients* $\mathbb{Q}_m$, $m = 1, ..., M$, *are the solutions of the following linear algebraic system*

$$\mathbb{Q}_m + \sum_{\substack{j=1 \\ j \neq m}}^{M} C_m G_\kappa(z_m, z_j)\mathbb{Q}_j \quad = \quad -C_m V^t(z_m, \theta), \tag{1.22}$$

for $m = 1, ..., M$, *where now we have a slight change in the constant* C_m *compared to (1.18), i.e.*

$$C_m := \frac{\lambda_m |\partial D_m|}{-1 + \lambda_m I_m}. \tag{1.23}$$

where $I_m := \int_{\partial D_m} \Phi_0(s_m, t) ds_m$, $t \in \partial D_m$, *is a constant if* D_m *is a ball.* [3]

This algebraic system is invertible under the same conditions as (1.17).

These results say that the dominant term in the approximation (1.6), and similarly the one in (1.14), is the Foldy-Lax field corresponding to the scattering by the point-like scatterers located at the 'centers' z_m's

[3] The property that I_m is a constant only for balls is known as Gruber's conjecture and it is solved for the 2D for Lipschitz regular domains and in 3D for convex domains, see [23]. This is the main restriction why we cannot obtain in Theorem 1.2 and Corollary 1.3 the same results as in Remark 1.4 for general shapes. Actually, if D_m is a ball then the value of I_m is nothing but the radius of D_m ($= \epsilon$ radius(B_m)).

of the scatterers D_m's with $\lambda_m|\partial D_m|$ as the related scattering coefficients. This Foldy-Lax field describes the field that results in the multiple scattering between the different scatterers D_m's, see [22] and the references therein for more information on this issue. The accuracy of the approximation of the scattered field by the Foldy-Lax field depends of course on the error estimates. In its generality, this issue is still largely open but there is an increase of interest to understand it, see for instance [11, 12, 14–16, 24–26].

The formal asymptotic expansion (1.6) was already given in [30, 31]. The results in Theorem 1.2 and in Corollary 1.3 provide the rigorous justification of the approximation of the scattered field taking into account all the involved parameters: the maximum diameter a, the number M, the minimum distance d and the surface impedance λ_m's including the regime defined by $M = M(a) := a^{-s},\ d := d(a) := a^t,\ \lambda_m := \lambda_m(a) := \lambda_{m,0}a^{-\beta}$. We characterized the set of parameters s, t and β where the approximation makes sense and we provided the approximation with explicit error estimates in terms of these parameters. These approximation formulas can be used for different purposes:

1. First, they can be used for imaging where the small anomalies are modeled by the small bodies. There is a large literature on the mathematical imaging of small anomalies, see for instance [5, 6] and the references therein. Compared to those results, we can allow the small bodies to be dense and very close, since their number M can be very large and the minimum distance d can be as small as we want. This last property means that we deal with the mesoscale regime, where $\frac{a}{\min_{j\neq m} dist(z_j, z_m)} \sim 1$ and $\min_{j\neq m} dist(z_j, z_m)$ is the minimum distance between the centers of the small scatterers, compare with [24, 25].

2. Second, these approximations can be used for the design of new indices of refraction. When the small scatterers are periodically distributed, this can be justified by homogenization, see [13,17,20,21]. Using the approximations in (1.6) and (1.14), we do not need such periodicity in the distribution of the small scatterers. This observation has been already made in [31] with quite formal computations. Let us observe that in our approximations in Theorem 1.2 and in Corollary 1.3, we allow the surface impedance to have negative imaginary parts. With this freedom of taking the surface impedance, we can generate a large class of indices of refraction with possible applications to the acoustic metamaterials. Details on this issue are reported in [1] and [2]. Let us emphasize that we provide those results with explicit error estimates. This gives more credits to the feasibility of the design process.

3. Third, these approximations can be used to extract the values of the index of refraction $n(x), x \in \Omega$, from the farfields corresponding to few incident directions. The idea of adding small inclusions to deform the medium and then extract the field inside the support of the medium is already described and used in the literature, see for instance [4]. The extraction of the index of refraction (or other coefficients) from the internal measurements, related also to the hybrid imaging methods, has recently attracted the attention of many authors, see for instance [7–10] and the references therein. One of the main difficulties is to handle the possible zeros of the total field. What we propose is to deform the medium using multiple (and close) inclusions instead of only single ones. In this case, what we derive from the asymptotic expansions are the internal values of the Green's function and not only the total fields. Finally, the values of the index of refraction can be extracted from the singularities of these Green's function. Hence, we avoid the problems coming from the zeros of the internal fields. These arguments are reported in [3].

To finish this introduction, we discuss the two 'extreme' cases given by the Neumann and the Dirichlet boundary conditions. In the former case, λ_m's are all zero. In the results stated above, we see that λ_m's are not allowed to be simultaneously zero. Otherwise all the coefficients $C_m,\ m = 1, ..., M$, will be zero and hence all the constants $Q_m,\ m = 1, ..., M$, vanish and the expansion (1.6) will make no sense. The case where $\lambda_m = 0,\ m = 1, ..., M$, is quite tedious but interesting.

- Technically, we see in (3.73), for instance, that in this case we need to go to the higher order in the expansion. Doing that requires quite tedious computations remembering that we are taking into account all the parameters describing the scatterers (M, a and d).

- This particular case is interesting because in this case the dominant coefficients are defined by *matrices* and not vectors (as the vector $(C_1, C_2, ..., C_M)$ in the case where $\lambda_m \neq 0$). But this is not a surprise since the dominant term of the expansion of the far-fields are the Foldy-Lax fields modeling the interaction of the multiple point-like scatterers (given here by the 'centers' of the scatterers). For hard scatterers, we talk about anisotropic interaction between the scatterers, see the book [22] for more information about this issue, contrary to the impedance case where the interaction is isotropic. Due to this 'anisotropic' character of the dominant term, the equivalent medium (when we distribute a cluster of small scatterers with Neumann boundary conditions) is characterized by a divergence form Helmholtz model where the coefficient appearing in the higher order derivative is a matrix (defined by the (scaled) matrix-coefficients appearing in the dominant term of the expansions).

These arguments need to be mathematically justified and quantified. The approximation in the Dirichlet case is discussed in [15] where it is justified under the condition that $\sqrt{M-1}\frac{a}{d}$ is bounded, by a constant depending only some a priori bounds, which means, in the scales we use here, that $0 \leq s \leq 2-2t$, or $t \leq \frac{2-s}{2}$. Since the small obstacles are distributed in a bounded domain Ω, then we have the natural condition on their number $M = O(d^{-3})$, i.e. $\frac{s}{3} \leq t$. In the present work, this last condition is the only one we (naturally) impose on t. Hence, compared to [15], we allow here the small scatterers to be as close as we want since t can be as large as we want, i.e. we cover the mesoscale regime. However, we believe that the conditions used in [15] can be improved to be comparable to the ones we impose here.

The rest of the paper is devoted to prove Theorem 1.2 and Corollary 1.3. In section 2, we describe briefly the main steps of the proof of Theorem 1.2. The detailed proof of Theorem 1.2 is done in section 3 while the one of Corollary 1.3 is given in section 4. The justification of Remark 1.4 is discussed in section 5. In the appendix, we deal with invertibility of the algebraic system (1.7) (and similarly for (1.17)).

2 A brief description of the proof of Theorem 1.2

Let us here describe very briefly the main steps of the proof of Theorem 1.2. Firts of all, the scattering problem has a unique solution and it can be represented via single layer potentials

$$U^t(x) = U^i(x) + \sum_{m=1}^{M} \int_{\partial D_m} \Phi_\kappa(x,s)\sigma_m(s)ds, \; x \in \mathbb{R}^3 \backslash \left(\bigcup_{m=1}^{M} \bar{D}_m\right), \tag{2.1}$$

where $\sigma := (\sigma_1, \dots, \sigma_M)^T$ satisfies the corresponding system of integral equations. We show that this system of integral equation is invertible in the space $\prod_{m=1}^{M} L^2(\partial D_m)$ and that the solution of the scattering problem is unique under some natural conditions on the eventual negative imaginary parts of the surface impedance.

We divide the rest of the analysis into few steps:

1. We have the following a priori estimate of the densities σ_m's. If $s \leq 2-\beta$ then $\|\sigma_m\|_{L^2(\partial D_m)} \leq C a^{1-\beta}$ where C depends only on the Lipschitz characters of $B_m,\ m = 1, ..., M$. The delicate work here is to derive the precise scaling of the corresponding boundary integral operators in the appropriate boundary Sobolev spaces (taking into account the three parameters s, t and β).

2. From the representation $U^s(x) = \sum_{m=1}^{M} \int_{\partial D_m} \Phi_\kappa(x,s)\sigma_m(s)ds,$ for $x \in \mathbb{R}^3 \backslash \left(\bigcup_{m=1}^{M} \bar{D}_m\right)$, we deduce, using the above a priori estimates on σ_m's, that

$$\begin{aligned} U^\infty(\hat{x}) &= \sum_{m=1}^{M} \int_{\partial D_m} e^{-i\kappa\hat{x}\cdot s}\sigma_m(s)ds \\ &= \sum_{m=1}^{M} \left(\int_{\partial D_m} e^{-i\kappa\hat{x}\cdot z_m}\sigma_m(s)ds + \int_{\partial D_m} [e^{-i\kappa\hat{x}\cdot s} - e^{-i\kappa\hat{x}\cdot z_m}]\sigma_m(s)ds\right) \end{aligned}$$

$$= \sum_{m=1}^{M} e^{-i\kappa\hat{x}\cdot z_m}\tilde{Q}_m + O(\kappa M a^{3-\beta}) \tag{2.2}$$

where $\tilde{Q}_m := \int_{\partial D_m} \sigma_m(s)ds$.

3. To estimate the terms $\tilde{Q}_m$, we use the boundary conditions. For $s_m \in \partial D_m$, using the impedance boundary condition (1.2), we have

$$\begin{aligned} 0 &= \frac{\partial U^t}{\partial \nu_m}(s_m) + \lambda_m U^t(s_m) = -\frac{\sigma_m(s_m)}{2} + \int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m,s)\sigma_m(s)ds + \sum_{\substack{j=1\\ j\neq m}}^{M} \int_{\partial D_j} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m,s)\sigma_j(s)ds \\ &+\lambda_m \sum_{j=1}^{M} \int_{\partial D_j} \Phi_\kappa(s_m,s)\sigma_m(s)ds + \frac{\partial U^i}{\partial \nu_m}(s_m) + \lambda_m U^i(s_m) \end{aligned} \tag{2.3}$$

Integrating the above on ∂D_m, we obtain

$$\begin{aligned} &-\frac{1}{2}\int_{\partial D_m} \sigma_m(s_m)ds_m + \int_{\partial D_m}\left(\int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m,s)ds_m\right)\sigma_m(s)ds + \sum_{\substack{j=1\\ j\neq m}}^{M} \int_{\partial D_j}\left(\int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m,s)ds_m\right)\sigma_j(s)ds \\ &\qquad +\lambda_m \sum_{j=1}^{M} \int_{\partial D_j}\left(\int_{\partial D_m} \Phi_\kappa(s_m,s)ds_m\right)\sigma_m(s)ds = -\int_{\partial D_m} \frac{\partial U^i}{\partial \nu_m}(s_m)ds_m - \int_{\partial D_m} \lambda_m U^i(s_m)ds_m \end{aligned}$$

It can be rewritten as

$$\begin{aligned} &-\frac{1}{2}\tilde{Q}_m + \underbrace{\int_{\partial D_m}\left(\int_{\partial D_m} \frac{\partial \Phi_0}{\partial \nu_m}(s_m,s)ds_m\right)\sigma_m(s)ds}_{=:A} + \underbrace{\sum_{j\neq m}^{M} \int_{\partial D_j}\left(\int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m,s)ds_m\right)\sigma_j(s)ds}_{=:B} \\ &\quad + \lambda_m \underbrace{\int_{\partial D_m}\left(\int_{\partial D_m} \Phi_\kappa(s_m,s)ds_m\right)\sigma_m(s)ds}_{=:C} + \lambda_m \underbrace{\sum_{j\neq m}^{M} \int_{\partial D_j}\left(\int_{\partial D_m} \Phi_\kappa(s_m,z_j)ds_m\right)\sigma_j(s)ds}_{=:D} \\ &\quad = -\int_{\partial D_m} \frac{\partial U^i}{\partial \nu_m}(s_m)ds_m - \int_{\partial D_m} \lambda_m U^i(s_m)ds_m + A' + \lambda_m D', \end{aligned} \tag{2.4}$$

with

$$A' := \int_{\partial D_m}\left(\int_{\partial D_m}\left[\frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m,s) - \frac{\partial \Phi_0}{\partial \nu_m}(s_m,s)\right]ds_m\right)\sigma_m(s)ds \tag{2.5}$$

$$D' := \sum_{j\neq m}^{M} \int_{\partial D_j}\left(\int_{\partial D_m} \left[\Phi_\kappa(s_m,s) - \Phi_\kappa(s_m,z_j)\right]ds_m\right)\sigma_j(s)ds. \tag{2.6}$$

Using the a priori estimate of σ_m's, the singularities of the fundamental solutions Φ_κ and the harmonicity of Φ_0, we derive the estimates:

$$A = -\frac{1}{2}\tilde{Q}_m, \quad A' = O\left(\kappa^2 a^{4-\beta}\right), \; B = O\left(2\kappa^2 \frac{a^{5-\beta}}{d^{2\alpha}}[\frac{6}{d^\alpha}+7]\right), \quad C = O\left(a^{3-\beta}\right), \tag{2.7}$$

$$D=\sum_{j\neq m}^{M}\Phi_\kappa(z_m,z_j)\tilde{Q}_j|\partial D_m|+O\left(2\frac{a^{5-\beta}}{d^{2\alpha}}\left[7\kappa+\frac{6\kappa+13}{d^\alpha}\right]\right) \tag{2.8}$$

and

$$D'=O\left(2\frac{a^{5-\beta}}{d^{2\alpha}}\left[7\kappa+\frac{6\kappa+13}{d^\alpha}\right]\right). \tag{2.9}$$

Here the parameter α is introduced to count the number of small scatterers surrounding a given and fixed one, see Fig 1 and the discussion before. From (2.4), we obtain the approximation below

$$\begin{aligned}-\tilde{Q}_m \quad &+ \quad \sum_{j\neq m}^{M}\Phi_\kappa(z_m,z_j)\lambda_m|\partial D_m|\tilde{Q}_j\\ &= \quad -\lambda_m|\partial D_m|e^{i\kappa\theta\cdot z_m}+O\left((|\lambda_m|+\kappa)\kappa a^3\right)+\lambda_m O\left(a^{3-\beta}\right),\\ &\quad +O\left(\kappa^2a^{4-\beta}\right)+O\left(2\kappa^2\frac{a^{5-\beta}}{d^{2\alpha}}[\frac{6}{d^\alpha}+7]\right)+\lambda_m O\left(2\frac{a^{5-\beta}}{d^{2\alpha}}\left[7\kappa+\frac{6\kappa+13}{d^\alpha}\right]\right).\end{aligned} \tag{2.10}$$

Dividing by $C_m:=-\lambda_m|\partial D_m|$ and since $\lambda_m=O(a^{-\beta})$, and then $C_m=O(a^{2-\beta})$, we can rewrite the above system as

$$\frac{\tilde{Q}_m}{C_m} \quad = \quad -e^{i\kappa\theta\cdot z_m}-\sum_{j\neq m}^{M}C_j\Phi_\kappa(z_m,z_j)\frac{\tilde{Q}_j}{C_j}+O\left(a^{1-\beta}+\frac{a^{3-\beta}}{d^{3\alpha}}\right). \tag{2.11}$$

4. Let now the vector $Y:=(Y_1,Y_2,...,Y_M)$ be the solution of the Foldy-Lax algebraic system

$$Y_m=-e^{i\kappa\cdot z_m}-\sum_{\substack{j=1\\ j\neq m}}^{M}C_j\Phi_\kappa(z_m,z_j)Y_j,\quad m=1,...,\ M. \tag{2.12}$$

We show that the algebraic system (2.12) is invertible under general condition on s and β and derive an error estimate. Based on this error estimate, we deduce from (2.11) and (2.12) that

$$\sum_{m=1}^{M}|\tilde{Q}_m-C_mY_m| \quad = \quad O\left(Ma^{2-\beta}\left(a^{1-\beta}+\frac{a^{3-\beta}}{d^{3\alpha}}\right)\right). \tag{2.13}$$

5. The proof ends by plugging (2.13) and (2.12) in (2.2) and setting $Q_m:=C_mY_m,\quad m=1,...,\ M$.

3 The detailed proof of Theorem 1.2

3.1 The representation via layer potential

We start with the following proposition on the solution of the problem (1.1-1.3) via the layer potential representation.

Proposition 3.1. *Assume that the negative part of the imaginary part of λ_m are small enough. In addition, suppose that κ^2 is not a Dirichlet* [4] *eigenvalue of the Laplacian in D_m, for $m = 1, ..., M$. Then* [5] *for $m = 1, 2, \ldots, M$, there exists $\sigma_m \in L^2(\partial D_m)$ such that the problem (1.1-1.3) has one and a unique solution and it is of the form*

$$U^t(x) = U^i(x) + \sum_{m=1}^{M} \int_{\partial D_m} \Phi_\kappa(x, s)\sigma_m(s)ds, \;\; x \in \mathbb{R}^3 \backslash \left(\overset{M}{\underset{m=1}{\cup}} \bar{D}_m \right), \tag{3.1}$$

Proof of Proposition 3.1. We look for the solution of the problem (1.1-1.3) of the form (3.1), then from the impedance boundary condition (1.2), we obtain

$$\begin{aligned} -\frac{\sigma_j(s_j)}{2} + \int_{\partial D_j} \frac{\partial \Phi_\kappa(s_j, s)}{\partial \nu_j(s_j)} \sigma_j(s)ds + \sum_{\substack{m=1 \\ m\neq j}}^{M} \int_{\partial D_m} \frac{\partial \Phi_\kappa(s_j, s)}{\partial \nu_j(s_j)} \sigma_m(s)ds & \\ + \lambda_j \sum_{m=1}^{M} \int_{\partial D_m} \Phi_\kappa(s_j, s)\sigma_m(s)ds &= -\frac{\partial U^i(s_j)}{\partial \nu_j(s_j)} - \lambda U^i(s_j), \; \forall s_j \in \partial D_j, \; j = 1, \ldots, M. \end{aligned} \tag{3.2}$$

One can write it in a compact form as $(-\frac{1}{2}\mathbf{I} + DL^* + DK^* + \lambda(L + K))\sigma = -(\partial_\nu + \lambda)U^{In}$ with $\partial_\nu := (\partial_{\nu mj})_{m,j=1}^M$, $\lambda := (\lambda_{mj})_{m,j=1}^M$, $DL^* := (DL^*_{mj})_{m,j=1}^M$, $DK^* := (DK^*_{mj})_{m,j=1}^M$, $L := (L_{mj})_{m,j=1}^M$ and $K := (K_{mj})_{m,j=1}^M$, where

$$\mathbf{I}_{mj} = \begin{cases} I, \text{Identity operator} & m = j \\ 0, \text{zero operator} & else \end{cases}, \quad DL^*_{mj} = \begin{cases} \mathcal{D}^*_{mj} & m = j \\ 0 & else \end{cases}, \quad DK^*_{mj} = \begin{cases} \mathcal{D}^*_{mj} & m \neq j \\ 0 & else \end{cases}, \tag{3.3}$$

$$\partial_{\nu mj} = \begin{cases} \partial_{\nu_m} & m = j \\ 0 & else \end{cases}, \quad \lambda_{mj} = \begin{cases} \lambda_m & m = j \\ 0 & else \end{cases}, \quad L_{mj} = \begin{cases} \mathcal{S}_{mj} & m = j \\ 0 & else \end{cases}, \quad K_{mj} = \begin{cases} \mathcal{S}_{mj} & m \neq j \\ 0 & else \end{cases} \tag{3.4}$$

$U^I = U^I(s_1, \ldots, s_M) := \left(U^i(s_1), \ldots, U^i(s_M)\right)^T$ and $\sigma = \sigma(s_1, \ldots, s_M) := (\sigma_1(s_1), \ldots, \sigma_M(s_M))^T$. Here, for the indices m and j fixed, $\mathcal{S}_{mj}$ is the integral operator acting as

$$\mathcal{S}_{mj}(\sigma_j)(t) := \int_{\partial D_j} \Phi_\kappa(t, s)\sigma_j(s)ds, \quad t \in \partial D_m, \tag{3.5}$$

and $\mathcal{D}^*_{mj}$ is the adjoint of the integral operator defined by,

$$\mathcal{D}_{mj}(\sigma_j)(t) := \int_{\partial D_j} \frac{\partial \Phi_\kappa(t, s)}{\partial \nu_m(s)} \sigma_j(s)ds, \quad t \in \partial D_m. \tag{3.6}$$

Then the operator $\mathcal{D}^*_{mm} : L^2(\partial D_m) \to L^2(\partial D_m)$ is adjoint of the double layer operator $\mathcal{D}_{mm} : L^2(\partial D_m) \to L^2(\partial D_m)$, defined by,

$$\mathcal{D}_{mm}(\sigma_m)(t) := \int_{\partial D_m} \frac{\partial \Phi_\kappa(t, s)}{\partial \nu_m(s)} \sigma_j(s)ds, \quad t \in \partial D_m, \tag{3.7}$$

[4] This last condition is satisfied for every κ such that $\kappa \leq \kappa_{\max}$ and $a < \frac{1}{\kappa_{\max}} \sqrt[3]{\frac{4\pi}{3}} \mathrm{j}_{1/2,1}$. Here $\mathrm{j}_{1/2,1}$ is the 1st positive zero of the Bessel function $\mathrm{J}_{1/2}$.

[5] The result of this proposition is valid regardless of the smallness of the obstacles D_m's nor the conditions on M, d and λ_m's. The emphasize is on the possibility to deal with surface impedance eventually having negative imaginary parts.

and the operator $-\frac{1}{2}I+\mathcal{D}^*_{mm}: L^2(\partial D_m)\rightarrow L^2(\partial D_m)$ is isomorphic and hence Fredholm with zero index. For $m\neq j$, $\mathcal{D}^*_{mj}: L^2(\partial D_j)\rightarrow L^2(\partial D_m)$ is compact, see [28, Theorem 4.1].[6]

Also notice that $\mathcal{S}_{mj}: L^2(\partial D_j)\rightarrow L^2(\partial D_m)$ is compact. So, $(-\frac{1}{2}\mathbf{I}+DL^*+DK^*+\lambda(L+K)):\prod_{m=1}^{M}L^2(\partial D_m)\rightarrow\prod_{m=1}^{M}L^2(\partial D_m)$ is Fredholm with zero index. We induce the product of spaces by the maximum of the norms of the space. To show that $(-\frac{1}{2}\mathbf{I}+DL^*+DK^*+\lambda(L+K))$ is invertible it is enough to show that it is injective. i.e. $(-\frac{1}{2}\mathbf{I}+DL^*+DK^*+\lambda(L+K))\sigma=0$ implies $\sigma=0$. We write

$$\tilde{U}(x)=\sum_{m=1}^{M}\int_{\partial D_m}\Phi_\kappa(x,s)\sigma_m(s)ds,\ \text{ in }\mathbb{R}^3\backslash\left(\overset{M}{\underset{m=1}{\cup}}\bar{D}_m\right)$$

and

$$\tilde{\tilde{U}}(x)=\sum_{m=1}^{M}\int_{\partial D_m}\Phi_\kappa(x,s)\sigma_m(s)ds,\ \text{ in }\overset{M}{\underset{m=1}{\cup}}D_m.$$

Then $\tilde{U}$ satisfies $\Delta\tilde{U}+\kappa^2\tilde{U}=0$ for $x\in\mathbb{R}^3\backslash\left(\overset{M}{\underset{m=1}{\cup}}\bar{D}_m\right)$, with S.R.C and $(\partial_{\nu_m}+\lambda_m)\tilde{U}(x)=0$ on $\overset{M}{\underset{m=1}{\cup}}\partial D_m$.

Let us assume for the moment that the exterior problem has a unique solution, then we deduce that $\tilde{U}=0$ in $\mathbb{R}^3\backslash\left(\overset{M}{\underset{m=1}{\cup}}\bar{D}_m\right)$. Due to the continuity of the single layer potentials, we deduce that $\tilde{\tilde{U}}=0$ on $\overset{M}{\underset{m=1}{\cup}}\partial D_m$. In addition, we know that $\Delta\tilde{\tilde{U}}+\kappa^2\tilde{\tilde{U}}=0$ for $x\in\overset{M}{\underset{m=1}{\cup}}D_m$. From the condition on κ^2, we deduce that $\tilde{\tilde{U}}=0$ in $\left(\overset{M}{\underset{m=1}{\cup}}D_m\right)$.

By the jump relations, we have

$$\frac{\partial\tilde{U}}{\partial\nu}(x)+\lambda_m\tilde{U}(x)=0\implies(\mathbf{K}^*\sigma_m)(x)-\frac{\sigma_m(x)}{2}+\sum_{\substack{j=1\\j\neq m}}^{M}(\mathcal{D}^*_{mj}+\lambda_m\mathcal{S}_{mj})(\sigma_j)(x)=0 \tag{3.8}$$

and

$$\frac{\partial\tilde{\tilde{U}}}{\partial\nu}(x)+\lambda_m\tilde{\tilde{U}}(x)=0\implies(\mathbf{K}^*\sigma_m)(x)+\frac{\sigma_m(x)}{2}+\sum_{\substack{j=1\\j\neq m}}^{M}(\mathcal{D}^*_{mj}+\lambda_m\mathcal{S}_{mj})(\sigma_j)(x)=0 \tag{3.9}$$

for $x\in\partial D_m$ and for $m=1,\dots,M$. Here, $\mathbf{K}^*$ is the adjoint of the double layer operator $\mathbf{K}$,

$$(\mathbf{K}\sigma_m)(x):=\int_{\partial D_m}\frac{\partial}{\partial\nu_s}\Phi_\kappa(x,s)\sigma_m(s)ds,\ \text{ for }m=1,\dots,M. \tag{3.10}$$

Difference between (3.8) and (3.9) provides us, $\sigma_m=0$ for all m.
We conclude then that $-\frac{1}{2}\mathbf{I}+DL^*+DK^*+\lambda(L+K)=:-\frac{1}{2}\mathbf{I}+\mathcal{D}^*+\lambda\mathcal{S}:\prod_{m=1}^{M}L^2(\partial D_m)\rightarrow\prod_{m=1}^{M}L^2(\partial D_m)$ is invertible.

We need now to show that the exterior problem $\Delta\tilde{U}+\kappa^2\tilde{U}=0$ for $x\in\mathbb{R}^3\backslash\left(\overset{M}{\underset{m=1}{\cup}}\bar{D}_m\right)$, with S.R.C and $(\partial_{\nu_m}+\lambda_m)\tilde{U}(x)=0$ on $\overset{M}{\underset{m=1}{\cup}}\partial D_m$, has a unique solution. This result is known under the condition

[6]Observe that $-\frac{1}{2}I+\mathcal{D}^*_{mm}$ is the adjoint of $-\frac{1}{2}I+\mathcal{D}_{mm}$. Hence $-\frac{1}{2}I+\mathcal{D}^*_{mm}$ is Fredholm if we show that $-\frac{1}{2}I+\mathcal{D}_{mm}$ is. In [28], this last property is proved for the case $\kappa=0$ and by a perturbation argument, we can obtain the same results for every κ. Using the condition on κ^2, we deduce that $-\frac{1}{2}I+\mathcal{D}^*_{mm}$ is an isomorphism.

$\Im \lambda_m \geq 0$ on $\overset{M}{\underset{m=1}{\cup}} \partial D_m$, see [19] for instance. To relax this positivity condition and consider $\Im \lambda_m < 0$ for some m's, we proceed as follows. Set $f := (f_1, ..., f_M)$ with $f_m := i(\Im\lambda)_- \tilde{U}$ on ∂D_m where $(\Im\lambda)_- := \max_{\overset{M}{\underset{m=1}{\cup}} \partial D_m} \{-\Im\lambda_m, 0\}$. Hence $\tilde{U}$ satisfies $\Delta\tilde{U} + \kappa^2\tilde{U} = 0$ for $x \in \mathbb{R}^3 \setminus \left(\overset{M}{\underset{m=1}{\cup}} \bar{D}_m\right)$, with S.R.C and $(\partial_{\nu_m} + \lambda_m + i(\Im\lambda)_-)\tilde{U}(x) = f_m$ on $\overset{M}{\underset{m=1}{\cup}} \partial D_m$. Since now $\Im(\lambda_m + i(\Im\lambda)_-) > 0$, then this last problem has a unique solution and it can be represented via single layer potentials $\sum_{m=1}^M S(\psi_j)$. Taking the normal trace on ∂D_m's, we deduce that $\left(-\frac{1}{2}\mathbf{I} + DL^* + DK^* + (\lambda + i(\Im\lambda)_-\mathbf{I})(L+K)\right)\psi = f$, where $\psi := (\psi_1, ...\psi_M)$. But from the form of f, we obviously have $f = i(\Im\lambda)_-(L+K)\psi$. Hence

$$(-\frac{1}{2}\mathbf{I} + DL^* + DK^* + \lambda(L+K))\psi = 0. \tag{3.11}$$

Let us first consider, for simplicity, only one scatterer and assume that the surface impedance is a constant $\lambda := \lambda^r + i\,\lambda^i$. Then (3.11) reduces to

$$(-\frac{1}{2}I + \mathbf{K}^* + \lambda S)\psi = 0. \tag{3.12}$$

1. If $\lambda^i \geq 0$, then, as usual in the scattering theory [19], we derive that $S\psi = 0$ and then $\psi = 0$.

2. How about $\lambda^i < 0$? Since the operator $-\frac{1}{2}I + \mathbf{K}^* + \lambda^r S$ is invertible in the $L^2(\partial D_m)$ spaces, then the equation (3.12) can be reduced to

$$(-\frac{1}{2}I + \mathbf{K}^* + \lambda^r S)^{-1} S\psi = -(i\lambda^i)^{-1}\psi = i(\lambda^i)^{-1}\psi \tag{3.13}$$

i.e. $(\psi, i(\lambda^i)^{-1})$ is an eigenelement of the operator $(-\frac{1}{2}I + \mathbf{K}^* + \lambda^r S)^{-1}S$. But $(-\frac{1}{2}I + \mathbf{K}^* + \lambda^r S)^{-1}S$ is compact hence it has only a discrete set of eigenvalues. Then if we take the surface impedance λ such that $i(\lambda^i)^{-1}$ is different from these discrete values, then $\psi = 0$ and hence $U = 0$.

We conclude that the scattering by an obstacle with an impedance type boundary condition modeled by a constant $\lambda^r + i\lambda^i$ is well posed as soon as the imaginary part λ^i is such that $i(\lambda^i)^{-1}$ is not an eigenvalue of the corresponding compact operator $(-\frac{1}{2}I + \mathbf{K}^* + \lambda^r S)^{-1}S$.

In particular, the operator in (3.12), i.e. $-\frac{1}{2}I + \mathbf{K}^* + \lambda S$, can be inverted using the Neumann series if λ satisfies $|\lambda^i| \|(-\frac{1}{2}I + \mathbf{K}^* + \lambda^r S)^{-1}\| \|S\| < 1$. This is of course a stronger condition on λ but it is enough for our purposes. In addition this Neumann series argument applies smoothly to the case where we have variable surface impedance's and multiple scatterers. Indeed, we know that the operator $-\frac{1}{2}\mathbf{I} + DL^* + DK^* + (\lambda + i(\Im\lambda)_-\mathbf{I})(L+K) : \prod_{m=1}^{M} L^2(\partial D_m) \to \prod_{m=1}^{M} L^2(\partial D_m)$ is invertible. Hence if the negative part of the imaginary part of λ is small so that [7]

$$(\Im\lambda)_- \|L+K\| \|(-\frac{1}{2}\mathbf{I} + DL^* + DK^* + (\lambda + i(\Im\lambda)_-\mathbf{I})(L+K))^{-1}\| < 1, \tag{3.14}$$

then by the Neumann series expansion $-\frac{1}{2}\mathbf{I} + DL^* + DK^* + \lambda(L+K)$ is also invertible and hence $\psi = 0$. □

[7]A general condition to inverte (3.11) is to assume that -1 is not an eigenvalue of the compact operator $(-\frac{1}{2}\mathbf{I} + DL^* + DK^* + (\lambda + i(\Im\lambda)_-\mathbf{I})(L+K))^{-1}(i(\Im\lambda)_-(L+K))$.

3.2 An appropriate estimate of the densities σ_m, $m=1,\dots,M$

From the above theorem, we have the following representation of σ:

$$
\begin{aligned}
\sigma &= -(-\frac{1}{2}\mathbf{I}+DL^*+DK^*+\lambda(L+K))^{-1}(\partial_\nu+\lambda)U^{In} \\
&= -(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}(\mathbf{I}+(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}(DK^*+\lambda K))^{-1}(\partial_\nu+\lambda)U^{In} \\
&= -(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}\sum_{l=0}^{\infty}\left((-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}(DK^*+\lambda K)\right)^l(\partial_\nu+\lambda)U^{In},
\end{aligned} \tag{3.15}
$$

if $\left\|(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}(DK^*+\lambda K)\right\|<1$. By the assumption $a<\frac{1}{\kappa_{\max}}\sqrt[3]{\frac{4\pi}{3}}\mathrm{j}_{1/2,1}$, the operator $-\frac{1}{2}\mathbf{I}+DL^*$ is invertible. We write $-\frac{1}{2}\mathbf{I}+DL^*+\lambda L=(I+\lambda L(-\frac{1}{2}\mathbf{I}+DL^*)^{-1})(-\frac{1}{2}\mathbf{I}+DL^*)$. From the scaling of the operators L and $(-\frac{1}{2}\mathbf{I}+DL^*)$, we show that there exists a constant $\tilde{c}$, depending only on the Lipschitz character of the reference bodies B_m's, such that if

$$
\|\lambda\|a=\max_m|\lambda_{m,0}|a^{1-\beta}<\tilde{c}, \tag{3.16}
$$

we have $\|\lambda L(-\frac{1}{2}\mathbf{I}+DL^*)^{-1}\|<1$ and hence the operator $-\frac{1}{2}\mathbf{I}+DL^*+\lambda L$ is invertible. The condition (3.16) is verified if $\beta<1$ and a small enough. For the convinience, we denote $\|\lambda\|$ by $|\lambda|$.

This implies that

$$
\|\sigma\| \leq \frac{\left\|(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}\right\|}{1-\left\|(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}\right\|\|DK^*+\lambda K)\|}\left\|(\partial_\nu+\lambda)U^{In}\right\|. \tag{3.17}
$$

Here we use the following notations:

$$
\begin{aligned}
\|DK^*+\lambda K\| &:= \|DK^*+\lambda K\|_{\mathcal{L}\left(\prod_{m=1}^{M}L^2(\partial D_m),\prod_{m=1}^{M}L^2(\partial D_m)\right)} \\
&\equiv \max_{1\leq m\leq M}\sum_{j=1}^{M}\left\|DK^*_{mj}+\lambda_m K_{mj}\right\|_{\mathcal{L}(L^2(\partial D_j),L^2(\partial D_m))} \\
&= \max_{1\leq m\leq M}\sum_{\substack{j=1\\ j\neq m}}^{M}\left\|\mathcal{D}^*_{mj}+\lambda_m\mathcal{S}_{mj}\right\|_{\mathcal{L}(L^2(\partial D_j),L^2(\partial D_m))},
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
\left\|(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}\right\| &:= \left\|(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}\right\|_{\mathcal{L}\left(\prod_{m=1}^{M}L^2(\partial D_m),\prod_{m=1}^{M}L^2(\partial D_m)\right)} \\
&\equiv \max_{1\leq m\leq M}\sum_{j=1}^{M}\left\|\left((-\frac{1}{2}\mathbf{I}+DL^*+\lambda_m L)^{-1}\right)_{mj}\right\|_{\mathcal{L}(L^2(\partial D_m),L^2(\partial D_j))} \\
&= \max_{1\leq m\leq M}\left\|(-\frac{1}{2}I+\mathcal{D}^*_{mm}+\lambda_m\mathcal{S}_{mm})^{-1}\right\|_{\mathcal{L}(L^2(\partial D_m),L^2(\partial D_m))},
\end{aligned} \tag{3.19}
$$

$$
\|\sigma\| := \|\sigma\|_{\prod_{m=1}^{M}L^2(\partial D_m)} \equiv \max_{1\leq m\leq M}\|\sigma_m\|_{L^2(\partial D_m)}, \tag{3.20}
$$

$$
\left\|U^{In}\right\| := \left\|U^{In}\right\|_{\prod_{m=1}^{M}L^2(\partial D_m)} \equiv \max_{1\leq m\leq M}\left\|U^i\right\|_{L^2(\partial D_m)} \tag{3.21}
$$

and

$$
\left\|\partial_\nu U^{In}\right\| := \left\|\partial_\nu U^{In}\right\|_{\prod_{m=1}^{M}L^2(\partial D_m)} \equiv \max_{1\leq m\leq M}\left\|\partial_{\nu_m}U^i\right\|_{L^2(\partial D_m)}. \tag{3.22}
$$

In the following proposition, we provide conditions under which $\left\|(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}(DK^*+\lambda K)\right\|<1$ and then estimate $\|\sigma\|$ via (3.17).

Proposition 3.2. *There exists a_0 depending only on the set of the a priori bounds such that if $a \leq a_0$ and $s \leq 2-\beta$, then we have the following estimate*

$$\|\sigma_m\|_{L^2(\partial D_m)} \leq c\epsilon^{1-\beta}$$

where c is a positive constant depending only on the set of the a priori bounds.

Proof of Proposition 3.2.
Suppose $0 < \epsilon \leq 1$ and $D_\epsilon := \epsilon B + z \subset \mathbb{R}^n$. For any functions f, g defined on ∂D_ϵ and ∂B respectively, we use the notations;

$$(f)^\wedge(\xi) := \hat{f}(\xi) := f(\epsilon\xi + z) \quad \text{and} \quad (g)^\vee(x) := \check{g}(x) := g\left(\frac{x-z}{\epsilon}\right). \tag{3.23}$$

Then for each $\psi \in L^2(\partial D_\epsilon)$, we have

$$\|\psi\|_{L^2(\partial D_\epsilon)} = \epsilon^{\frac{n-1}{2}}\|\hat{\psi}\|_{L^2(\partial B)} \tag{3.24}$$

We divide the rest of the proof of Proposition 3.2 into two steps. In the first step, we assume we have a single obstacle and then in the second step we deal with the multiple obstacles case.

3.2.1 The case of a single obstacle

Let us consider a single obstacle $D_\epsilon := \epsilon B + z$ with unit outword normal ν to its boundary. Then define the operator $\mathcal{D}_{D_\epsilon} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon)$ by

$$\left(\mathcal{D}_{D_\epsilon}\psi\right)(s) = \int_{\partial D_\epsilon} \frac{\Phi_\kappa(s,t)}{\partial \nu(t)}\psi(t)dt. \tag{3.25}$$

Following the arguments in the proof of Proposition 3.1, the integral operator $-\frac{1}{2}I + \mathcal{D}^*_{D_\epsilon} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon)$ is invertible. If we consider the problem (1.1-1.3) in $\mathbb{R}^3 \setminus \bar{D}_\epsilon$, we obtain

$$\sigma = (-\frac{1}{2}I + \mathcal{D}^*_{D_\epsilon} + \lambda \mathcal{S}_{D_\epsilon})^{-1}(\partial_\nu + \lambda)U^i,$$

and then

$$\|\sigma\|_{L^2(\partial D_\epsilon)} \leq \|(-\frac{1}{2}I + \mathcal{D}^*_{D_\epsilon} + \lambda \mathcal{S}_{D_\epsilon})^{-1}\|_{\mathcal{L}(L^2(\partial D_\epsilon), L^2(\partial D_\epsilon))}\|(\partial_\nu + \lambda)U^i\|_{L^2(\partial D_\epsilon)}. \tag{3.26}$$

We have the following lemma, see [15, Lemma 2.4 and Lemma 2.15].

Lemma 3.3. *Let $\phi, \psi \in L^2(\partial D_\epsilon)$. Then,*

$$\mathcal{S}_{D_\epsilon}\psi = \epsilon\ (\mathcal{S}^\epsilon_B \hat{\psi})^\vee, \tag{3.27}$$

$$\|\mathcal{S}_{D_\epsilon}\|_{\mathcal{L}(L^2(\partial D_\epsilon), L^2(\partial D_\epsilon))} = \epsilon\, \|\mathcal{S}^\epsilon_B\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))}\,, \tag{3.28}$$

$$\mathcal{D}^*_{D_\epsilon}\psi = (\mathcal{D}^{\epsilon^*}_B \hat{\psi})^\vee, \tag{3.29}$$

$$\left(-\frac{1}{2}I + \mathcal{D}^*_{D_\epsilon}\right)\psi = \left(\left(-\frac{1}{2}I + \mathcal{D}^{\epsilon^*}_B\right)\hat{\psi}\right)^\vee, \tag{3.30}$$

$$\left(-\frac{1}{2}I + \mathcal{D}^*_{D_\epsilon}\right)^{-1}\phi = \left(\left(-\frac{1}{2}I + \mathcal{D}^{\epsilon^*}_B\right)^{-1}\hat{\phi}\right)^\vee, \tag{3.31}$$

$$\left\| \left(-\frac{1}{2}I + \mathcal{D}^*_{D_\epsilon} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_\epsilon), L^2(\partial D_\epsilon))} = \left\| \left(-\frac{1}{2}I + \mathcal{D}^{\epsilon^*}_{B} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))} \tag{3.32}$$

and hence

$$\left(-\frac{1}{2}I + \mathcal{D}^*_{D_\epsilon} + \lambda \mathcal{S}_{D_\epsilon} \right) \psi = \left(\left(-\frac{1}{2}I + \mathcal{D}^{\epsilon^*}_{B} + \lambda \epsilon \mathcal{S}^{\epsilon}_{B} \right) \hat{\psi} \right)^{\vee}, \tag{3.33}$$

$$\left(-\frac{1}{2}I + \mathcal{D}^*_{D_\epsilon} + \lambda \mathcal{S}_{D_\epsilon} \right)^{-1} \phi = \left(\left(-\frac{1}{2}I + \mathcal{D}^{\epsilon^*}_{B} + \lambda \epsilon \mathcal{S}^{\epsilon}_{B} \right)^{-1} \hat{\phi} \right)^{\vee}, \tag{3.34}$$

$$\left\| \left(-\frac{1}{2}I + \mathcal{D}^*_{D_\epsilon} + \lambda \mathcal{S}_{D_\epsilon} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_\epsilon), L^2(\partial D_\epsilon))} = \left\| \left(-\frac{1}{2}I + \mathcal{D}^{\epsilon^*}_{B} + \lambda \epsilon \mathcal{S}^{\epsilon}_{B} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))}, \tag{3.35}$$

with $\mathcal{S}^{\epsilon}_{B}\hat{\psi}(\xi) := \int_{\partial B} \Phi^{\epsilon}(\xi,\eta)\hat{\psi}(\eta)d\eta$, $\mathcal{D}^{\epsilon^*}_{B}\hat{\psi}(\xi) := \int_{\partial B} \frac{\partial \Phi^{\epsilon}(\xi,\eta)}{\partial \nu(\xi)}\hat{\psi}(\eta)d\eta$ *and* $\Phi^{\epsilon}(\xi,\eta) := \frac{e^{i\kappa\epsilon|\xi-\eta|}}{4\pi|\xi-\eta|}$.

Let us estimate the norm of $\|\mathcal{S}^{\epsilon}_{B}\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))}$.

Lemma 3.4. *The operator norm of the compact operator* $\mathcal{S}_{D_\epsilon} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon)$*, defined in* (3.25)*, is estimated by* ϵ*, i.e.*

$$\|\mathcal{S}_{D_\epsilon}\|_{\mathcal{L}(L^2(\partial D_\epsilon), L^2(\partial D_\epsilon))} \leq \epsilon \left(\left\|\mathcal{S}^{0}_{B}\right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))} + \frac{1}{2\pi}\kappa\epsilon^2|\partial B| \right), \tag{3.36}$$

Proof of Lemma 3.4. To estimate the operator norm of $\mathcal{S}_{D_\epsilon}$, we decompose $\mathcal{S}_{D_\epsilon} =: \mathcal{S}^{\kappa}_{D_\epsilon} = \mathcal{S}^{i_\kappa}_{D_\epsilon} + \mathcal{S}^{d_\kappa}_{D_\epsilon}$ into two parts $\mathcal{S}^{i_\kappa}_{D_\epsilon}$ (independent of κ) and $\mathcal{S}^{d_\kappa}_{D_\epsilon}$ (dependent of κ) given by

$$\mathcal{S}^{i_\kappa}_{D_\epsilon}\psi(x) := \int_{\partial D_\epsilon} \frac{1}{4\pi|x-y|}\psi(y)dy, \tag{3.37}$$

$$\mathcal{S}^{d_\kappa}_{D_\epsilon}\psi(x) := \int_{\partial D_\epsilon} \frac{e^{i\kappa|x-y|}-1}{4\pi|x-y|}\psi(y)dy. \tag{3.38}$$

With this definition, $\mathcal{S}^{i_\kappa}_{D_\epsilon} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon)$ and $\mathcal{S}^{d_\kappa}_{D_\epsilon} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon)$ are compact. From (3.28), it can be observed that

$$\left\|\mathcal{S}^{i_\kappa}_{D_\epsilon}\right\|_{\mathcal{L}(L^2(\partial D_\epsilon), L^2(\partial D_\epsilon))} = \epsilon \left\|\mathcal{S}^{i_\kappa}_{B}\right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))} = \epsilon \left\|\mathcal{S}^{0}_{B}\right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))} \tag{3.39}$$

On the other hand, as mentioned in (2.30) of [15, Lemma 2.5], the following estimate can be obtained,

$$\left\|\mathcal{S}^{d_\kappa}_{D_\epsilon}\right\|_{\mathcal{L}(L^2(\partial D_\epsilon), L^2(\partial D_\epsilon))} \leq \frac{1}{2\pi}\kappa\epsilon^3|\partial B| \text{ for } \kappa_{\max} diam(D_\epsilon) \leq 1. \tag{3.40}$$

Hence the result follows.

□

3.2.2 The multiple obstacle case

A way of counting the small scatterers

Before proceeding further we make the following observation. For $m = 1, \dots, M$ fixed, we distinguish between the obstacles D_j, $j \neq m$ by keeping them into different layers based on their distance from D_m. Let Ω_m,

$1 \leq m \leq M$ be the cubes of center z_m such that each side is of size $(\frac{a}{2} + d^\alpha)$ with $0 \leq \alpha \leq 1$ and it contains only D_m. Let us suppose that these cubes are arranged in a cuboid, for example unit rubics cube, see Fig 1, in different layers such that the total cubes upto the n^{th} layer consists $(2n+1)^3$ cubes for $n = 0, \dots, [d^{-\alpha}]$, and Ω_m is located on the center. Hence the number of obstacles located in the n^{th}, $n \neq 0$ layer will be $[(2n+1)^3 - (2n-1)^3]$ and their distance from D_m is more than nd^α.

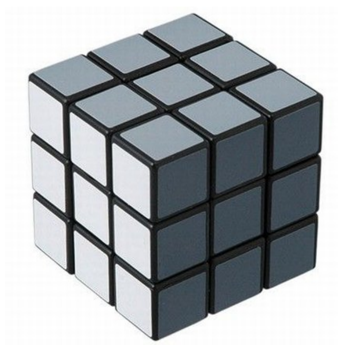

Figure 1: Rubik's cube consisting of two layers

Lemma 3.5. *For $m, j = 1, 2, \dots, M$, the operator $\mathcal{S}_{mj} : L^2(\partial D_j) \to L^2(\partial D_m)$ defined in Proposition 3.1, see* (3.5)*, satisfies the following estimates,*

- *For $j = m$,*

$$\|\mathcal{S}_{mm}\|_{\mathcal{L}(L^2(\partial D_m), L^2(\partial D_m))} \leq \epsilon \left(\left\| \mathcal{S}^{i_\kappa}_{\underset{\sim}{B}} \right\| + \frac{1}{2\pi} \kappa \epsilon^2 |\partial \underset{\sim}{B}| \right), \tag{3.41}$$

for $\kappa_{\max} a \leq 1$.

- *For $j \neq m$, such that $D_j \in N^n_m, n = 1, \dots, [d^{-\alpha}]$*

$$\|\mathcal{S}_{mj}\|_{\mathcal{L}(L^2(\partial D_j), L^2(\partial D_m))} \leq \frac{1}{4\pi} \frac{1}{nd^\alpha} |\partial \underset{\sim}{B}| \, \epsilon^2, \tag{3.42}$$

where $|\partial \underset{\sim}{B}| := \max\limits_m |\partial B_m|$ and $\left\| \mathcal{S}^{i_\kappa}_{\underset{\sim}{B}} \right\| := \max\limits_m \left\| \mathcal{S}^{i_\kappa}_{B_m} \right\|_{\mathcal{L}(L^2(\partial B_m), L^2(\partial B_m))}$.

Proof of Lemma 3.5. The estimate (3.41) is nothing else but (3.36) of Lemma 3.4, replacing B by B_m, z by z_m and D_ϵ by D_m respectively. The proof of the estimate (3.42) is a straightforward consequences of (2.37) in [15, Lemma 2.6].

□

Proposition 3.6. *For $m, j = 1, 2, \dots, M$, the operator $\mathcal{D}^*_{mj} : L^2(\partial D_j) \to L^2(\partial D_m)$ defined in Proposition 3.1, see* (3.6)*, satisfies the following estimates,*

- *For $j = m$,*

$$\left\| \left(-\frac{1}{2} I + \mathcal{D}^*_{mm} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_m), L^2(\partial D_m))} \leq \grave{C}_{6m}, \tag{3.43}$$

where $\grave{C}_{6m} := \dfrac{2\pi \left\| \left(-\frac{1}{2} I + \mathcal{D}^{i_\kappa *}_{B_m} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B_m), L^2(\partial B_m))}}{2\pi - \kappa^2 \epsilon^2 |\partial B_m| \left\| \left(-\frac{1}{2} I + \mathcal{D}^{i_\kappa *}_{B_m} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B_m), L^2(\partial B_m))}}.$

- *For $j \neq m$, such that $D_j \in N_m^n, n = 1, \dots, [d^{-\alpha}]$*

$$\left\|\mathcal{D}_{mj}^*\right\|_{\mathcal{L}(L^2(\partial D_j),L^2(\partial D_m))} \leq \frac{1}{4\pi}\left(\frac{\kappa}{nd^\alpha}+\frac{1}{n^2d^{2\alpha}}\right)|\partial \c{B}|\,\epsilon^2, \tag{3.44}$$

where $|\partial \c{B}| := \max_m \partial B_m$.

In addition, as a consequence of (3.34)*, we can also prove that*

- *For $j = m$,*

$$\left\|\left(-\frac{1}{2}I+\mathcal{D}_{mm}^*+\lambda_m\mathcal{S}_{mm}\right)^{-1}\right\|_{\mathcal{L}(L^2(\partial D_m),L^2(\partial D_m))} \leq C_{6m}, \tag{3.45}$$

where $C_{6m} := \frac{2\pi\left\|\left(-\frac{1}{2}I+\mathcal{D}_{B_m}^{i_\kappa}{}^*+\lambda_m\epsilon\mathcal{S}_{B_m}^{i_\kappa}\right)^{-1}\right\|_{\mathcal{L}(L^2(\partial B_m),L^2(\partial B_m))}}{2\pi-(|\lambda_m|+\kappa)\kappa\epsilon^2|\partial B_m|\left\|\left(-\frac{1}{2}I+\mathcal{D}_{B_m}^{i_\kappa}{}^*+\lambda_m\epsilon\mathcal{S}_{B_m}^{i_\kappa}\right)^{-1}\right\|_{\mathcal{L}(L^2(\partial B_m),L^2(\partial B_m))}}$.

Proof of Proposition 3.6. This result can be proved in the similar lines of the proof of [15, Proposition 2.17].

□

End of the proof of Proposition 3.2. By substituting (3.42) and (3.44) in (3.18), (3.45) in (3.19), and using our discusion related to Fig 1 on how we count the number of small scatterers, we obtain

$$\begin{aligned}
\|DK^*+\lambda K\| &\equiv \max_{1\leq m\leq M}\sum_{\substack{j=1\\ j\neq m}}^{M}\left\|\mathcal{D}_{mj}^*+\lambda_m\mathcal{S}_{mj}\right\|_{\mathcal{L}(L^2(\partial D_j),L^2(\partial D_m))}\\
&\leq \sum_{n=1}^{[d^{-\alpha}]}[(2n+1)^3-(2n-1)^3]\frac{1}{4\pi}\left(\frac{|\lambda|+\kappa}{nd^\alpha}+\frac{1}{n^2d^{2\alpha}}\right)|\partial \c{B}|\,\epsilon^2\\
&= \sum_{n=1}^{[d^{-\alpha}]}[24n^2+2]\frac{1}{4\pi}\left(\frac{|\lambda|+\kappa}{nd^\alpha}+\frac{1}{n^2d^{2\alpha}}\right)|\partial \c{B}|\,\epsilon^2\\
&= \sum_{n=1}^{[d^{-\alpha}]}\frac{1}{2\pi}[12n^2+1]\left(\frac{|\lambda|+\kappa}{nd^\alpha}+\frac{1}{n^2d^{2\alpha}}\right)|\partial \c{B}|\,\epsilon^2\\
&= \frac{1}{2\pi}\left[(|\lambda|+\kappa)d^{-\alpha}\sum_{n=1}^{[d^{-\alpha}]}[12n+\frac{1}{n}]+d^{-2\alpha}\sum_{n=1}^{[d^{-\alpha}]}[12+\frac{1}{n^2}]\right]|\partial \c{B}|\,\epsilon^2\\
&\leq \frac{1}{2\pi}\left[(|\lambda|+\kappa)(6d^{-3\alpha}+7d^{-2\alpha})+13d^{-3\alpha}\right]|\partial \c{B}|\,\epsilon^2
\end{aligned} \tag{3.46}$$

and

$$\begin{aligned}
\left\|\left(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L\right)^{-1}\right\| &\equiv \max_{1\leq m\leq M}\left\|\left(-\frac{1}{2}I+\mathcal{D}_{mm}^*+\lambda_m\mathcal{S}_{mm}\right)^{-1}\right\|_{\mathcal{L}(L^2(\partial D_m),L^2(\partial D_m))}\\
&\equiv \max_{1\leq m\leq M} C_{6m}.
\end{aligned} \tag{3.47}$$

Hence, (3.46-3.47) provides

$$\left\|(-\frac{1}{2}\mathbf{I}+DL^*+\lambda L)^{-1}\right\|\|DK^*+\lambda K\|$$

$$\leq \underbrace{\left(\max_{1\leq m\leq M} C_{6m}\right)|\partial Ḅ|\,\frac{1}{2\pi}\,[(|\lambda|+\kappa)(6+7d^{\alpha})+13]\,d^{-3\alpha}\epsilon^{2}}_{=:C_s}. \quad (3.48)$$

By imposing the condition $\left\|(-\frac{1}{2}\mathbf{I}+DL^{*}+\lambda L)^{-1}\right\|\|DK^{*}+\lambda K\|<1$, we have from (3.17) and (3.20-3.22);

$$\begin{aligned}
\|\sigma_m\|_{L^2(\partial D_m)}\leq\|\sigma\| &\leq \frac{\left\|(-\frac{1}{2}\mathbf{I}+DL^{*}+\lambda L)^{-1}\right\|}{1-\left\|(-\frac{1}{2}\mathbf{I}+DL^{*}+\lambda L)^{-1}\right\|\|DK^{*}+\lambda K\|}\left\|(\partial_\nu+\lambda)U^{In}\right\| \\
&\leq C_p\left\|(-\frac{1}{2}\mathbf{I}+DL^{*}+\lambda L)^{-1}\right\|\max_{1\leq m\leq M}\left\|(\partial_\nu+\lambda)U^{i}\right\|_{L^2(\partial D_m)}\quad\left(C_p\geq\frac{1}{1-C_s}\right) \\
&\underset{(3.47)}{\leq} \mathrm{C}\max_{1\leq m\leq M}\left\|(\partial_\nu+\lambda)U^{i}\right\|_{L^2(\partial D_m)}\quad\left(\mathrm{C}:=C_p\max_{1\leq m\leq M}C_{6m}\right), \qquad (3.49)
\end{aligned}$$

for all $m\in\{1,2,\dots,M\}$. But,

$$\begin{aligned}
\left\|(\partial_\nu+\lambda)U^{i}\right\|_{L^2(\partial D_m)} &\leq |\lambda|\left\|U^{i}\right\|_{L^2(\partial D_m)}+\left\|\partial_\nu U^{i}\right\|_{L^2(\partial D_m)} \\
&= |\lambda|\epsilon\;|\partial B_m|^{\frac{1}{2}}+k\epsilon\,|\partial B_m|^{\frac{1}{2}}\quad\left(\text{Since },U^{i}(x,\theta)=e^{i\kappa x\cdot\theta}\right) \\
&\leq (|\lambda|+\kappa)\epsilon\,|\partial Ḅ|^{\frac{1}{2}},\forall m=1,2,\dots,M. \qquad (3.50)
\end{aligned}$$

Now by substituting (3.50) in (3.49), for each $m=1,\dots,M$, we obtain

$$\|\sigma_m\|_{L^2(\partial D_m)} \leq \mathcal{C}(\kappa)\epsilon, \qquad (3.51)$$

where $\mathcal{C}(\kappa):=\mathrm{C}\,|\partial Ḅ|^{\frac{1}{2}}\,(|\lambda|+\kappa)$.

The condition $\left\|(-\frac{1}{2}\mathbf{I}+DL^{*}+\lambda L)^{-1}\right\|\|DK^{*}+\lambda K)\|<1$ is satisfied if

$$C_s = \left(\max_{1\leq m\leq M}C_{6m}\right)|\partial Ḅ|\,\frac{1}{2\pi}\,[(|\lambda|+\kappa)(6+7d^{\alpha})+13]\,d^{-3\alpha}\epsilon^{2}\;<\;1. \qquad (3.52)$$

Since $\lambda_m=\lambda_{m0}\,a^{-\beta}$ and $d\approx a^{t}$, in particular $d\geq d_{min}a^{t}$, then (3.52) reads as $a^{-3\alpha t+2-\beta}<\grave{c}$, where we set

$$\grave{c}:=\left(\left[(\lambda_{+}+\kappa_{\max}a^{\beta})(6+7[d_{\max}]^{\alpha})+13a^{\beta}\right]\frac{1}{2\pi}\frac{|\partial Ḅ|}{[\max\limits_{1\leq m\leq M}diam(B_m)]^{2}}\max_{1\leq m\leq M}C_{6m}d_{min}^{-\alpha}\right)^{-1}>1 \qquad (3.53)$$

with $\lambda_{+}:=\max\limits_{1\leq m\leq M}|\lambda_{m0}|$ and with the rewritten form of C_{6m} mentioned in Proposition 3.6 as

$$C_{6m}:=\frac{2\pi\left\|\left(-\frac{1}{2}I+\mathcal{D}_{B_m}^{i_\kappa\,*}+\frac{\lambda_{m0}\epsilon^{1-\beta}}{[\max\limits_{1\leq m\leq M}diam(B_m)]^{\beta}}\mathcal{S}_{B_m}^{i_\kappa}\right)^{-1}\right\|_{\mathcal{L}(L^2(\partial B_m),L^2(\partial B_m))}}{2\pi-(\frac{\lambda_{m0}\kappa\epsilon^{2-\beta}}{[\max\limits_{1\leq m\leq M}diam(B_m)]^{\beta}}+\kappa^{2}\epsilon^{2})|\partial B_m|\left\|\left(-\frac{1}{2}I+\mathcal{D}_{B_m}^{i_\kappa\,*}+\frac{\lambda_{m0}\epsilon^{1-\beta}}{[\max\limits_{1\leq m\leq M}diam(B_m)]^{\beta}}\mathcal{S}_{B_m}^{i_\kappa}\right)^{-1}\right\|_{\mathcal{L}(L^2(\partial B_m),L^2(\partial B_m))}}.$$

Observe that $s=3\alpha t$. Hence (3.52) makes sense if $s\leq 2-\beta$ and λ_{+} satisfies (3.53).

Again, since $\lambda_m=\lambda_{m0}a^{-\beta}$, (3.51) can be rewritten as

$$\|\sigma_m\|_{L^2(\partial D_m)} \leq \grave{\mathcal{C}}(\kappa)\epsilon^{1-\beta}, \qquad (3.54)$$

with[8]

$$\grave{\mathcal{C}}(\kappa) := \mathrm{C}\, |\partial \mathrm{B}|^{\frac{1}{2}} \, (\frac{\lambda_+}{[\max\limits_{1\leq m\leq M} diam(B_m)]^\beta} + \kappa\epsilon^\beta) \, \left(< \mathrm{C}\, |\partial \mathrm{B}|^{\frac{1}{2}} \, (\frac{\lambda_+}{[\max\limits_{1\leq m\leq M} diam(B_m)]^\beta} + \kappa_{\max})\right).$$

□

3.3 Approximation of the far-fields. I. Approximation by the total charges

We start with the definition of the total charges Q_m, $m = 1, \dots M$.

Definition 3.7. *We call the σ_m's used in* (3.1)*, the solution of the problem (1.1-1.3), the surface charge distributions. Using these surface charge distributions, we define the total charge on each surface ∂D_m denoted by Q_m as*

$$Q_m := \int_{\partial D_m} \sigma_m(s) ds. \tag{3.55}$$

In the following proposition, we provide an approximate of the far-fields in terms of the total charges Q_m.

Proposition 3.8. *The far-field pattern U^∞ of the scattered solution of the problem (1.1-1.3) has the following asymptotic expansion*

$$U^\infty(\hat{x}) = \sum_{m=1}^{M} [e^{-i\kappa\hat{x}\cdot z_m} Q_m + O(\kappa\, a^{3-\beta})], \tag{3.56}$$

with Q_m given by (3.55), if $\kappa_{\max}\, a < 1$ where $O(\kappa\, a^{3-\beta}) \leq C\kappa\, a^{3-\beta}$ and $C := \frac{|\partial B| \mathrm{C}(\lambda_+ + \kappa_{\max})}{\left(\max\limits_{1\leq m\leq M} diam(B_m)\right)^{2-\beta}}$.

Proof of Proposition 3.8. From (3.1), we have

$$U^s(x) \;=\; \sum_{m=1}^{M} \int_{\partial D_m} \Phi_\kappa(x,s)\sigma_m(s) ds, \text{ for } x \in \mathbb{R}^3 \backslash \left(\overset{M}{\underset{m=1}{\cup}} \bar{D}_m\right).$$

Hence

$$\begin{aligned} U^\infty(\hat{x}) &= \sum_{m=1}^{M} \int_{\partial D_m} e^{-i\kappa\hat{x}\cdot s} \sigma_m(s) ds \\ &= \sum_{m=1}^{M} \left(\int_{\partial D_m} e^{-i\kappa\hat{x}\cdot\, z_m} \sigma_m(s) ds + \int_{\partial D_m} [e^{-i\kappa\hat{x}\cdot\, s} - e^{-i\kappa\hat{x}\cdot\, z_m}] \sigma_m(s) ds \right) \\ &= \sum_{m=1}^{M} \left(e^{-i\kappa\hat{x}\cdot\, z_m} Q_m + \int_{\partial D_m} [e^{-i\kappa\hat{x}\cdot\, s} - e^{-i\kappa\hat{x}\cdot\, z_m}] \sigma_m(s) ds \right). \end{aligned} \tag{3.57}$$

[8]It is important to remark that, if we do not distinguish the near by and far obstacles, as it is discussed in the beginning of section 3.2.2, and by following the way it was done in [15,16] we can get the estimate $\|DK^* + \lambda K\| \leq \frac{M-1}{4\pi}\left(\frac{|\lambda|+\kappa}{d} + \frac{1}{d^2}\right) |\partial B|\, \epsilon^2$ in place of (3.46) and hence the condition (3.52) will be replaced by $(M-1)\frac{a^{2-\beta}}{d^2} < c_0$, for some suitable constant c_0. However, this condition is too strong to enable us to apply our asymptotic expansion to the effective medium theory where we need to choose $M \sim a^{-s}$ with $s = 2-\beta$ and $d \sim a^t$with $t \geq \frac{s}{3}$.

For every $m = 1, 2, \ldots, M$, we have from Proposition 3.2;

$$
\begin{aligned}
\left| \int_{\partial D_m} |\sigma_m(s)| ds \right| &\leq \|1\|_{L^2(\partial D_m)} \|\sigma_m\|_{L^2(\partial D_m)} \\
&\underset{(3.54)}{\leq} \|1\|_{L^2(\partial D_m)} \mathrm{C}\, |\partial \mathrm{B}|^{\frac{1}{2}} \, (\lambda_+ + \kappa_{\max})\, \epsilon^{1-\beta} \\
&\leq |\partial \mathrm{B}| \mathrm{C} (\frac{\lambda_+}{[\max\limits_{1\leq m\leq M} diam(B_m)]^{\beta}} + \kappa_{\max})\, \epsilon^{2-\beta} \qquad (3.58)
\end{aligned}
$$

$$
\text{with} \quad C :== \frac{|\partial \mathrm{B}| \mathrm{C} (\lambda_+ + \kappa_{\max} [\max\limits_{1\leq m\leq M} diam(B_m)]^{\beta})}{\left(\max\limits_{1\leq m\leq M} diam(B_m) \right)^2}. \qquad (3.59)
$$

It gives us the following estimate;

$$
\begin{aligned}
\left| \int_{\partial D_m} [e^{-i\kappa \hat{x}\cdot\, s} - e^{-i\kappa \hat{x}\cdot\, z_m}] \sigma_m(s) ds \right| &\leq \int_{\partial D_m} \left| e^{-i\kappa \hat{x}\cdot\, s} - e^{-i\kappa \hat{x}\cdot\, z_m} \right| |\sigma_m(s)| ds \\
&\leq \int_{\partial D_m} \sum_{l=1}^{\infty} \kappa^l |s - z_m|^l |\sigma_m(s)| ds \\
&\leq \int_{\partial D_m} \sum_{l=1}^{\infty} \kappa^l \left(\frac{a}{2} \right)^l |\sigma_m(s)| ds \\
&\underset{(3.58)}{\leq} C a^{2-\beta} \sum_{l=1}^{\infty} \kappa^l \left(\frac{a}{2} \right)^l \\
&= \frac{1}{2} C \kappa\, a^{3-\beta} \frac{1}{1 - \kappa \frac{a}{2}}, \text{ if } a < \frac{2}{\kappa_{\max}} \left(\leq \frac{2}{\kappa} \right) \qquad (3.60)
\end{aligned}
$$

which means

$$
\int_{\partial D_m} [e^{-i\kappa \hat{x}\cdot\, s} - e^{-i\kappa \hat{x}\cdot\, z_m}] \sigma_m(s) ds \quad \leq \quad C\kappa\, a^{3-\beta}, \text{ for } a \leq \frac{1}{\kappa_{\max}}. \qquad (3.61)
$$

Now substitution of (3.61) in (3.57) gives the required result (3.56).

□

3.4 Approximation of the far-fields. II. Estimates of the total charges

3.4.1 Derivation of the linear algebraic system

We start with following a priori estimate on the total charges $Q_m,\ m = 1, ..., M$.

Lemma 3.9. *For $m = 1, 2, \ldots, M$, we have*

$$
|Q_m| \quad \leq \quad \tilde{c}\, \epsilon^{2-\beta}, \qquad (3.62)
$$

where $\tilde{c} := |\partial B| \mathrm{C} (\frac{\lambda_+}{[\max\limits_{1\leq m\leq M} diam(B_m)]^{\beta}} + \kappa_{\max})$ *with* ∂B *and* C *are defined in* (3.42) *and* (3.49) *respectively.*

Proof of Lemma 3.9. The proof follows as below;

$$
|Q_m| \quad = \quad \left| \int_{\partial D_m} \sigma_m(s) ds \right|
$$

$$
\begin{aligned}
&\leq && \|1\|_{L^2(\partial D_m)}\|\sigma_m\|_{L^2(\partial D_m)} \\
&\underset{(3.54)}{\leq} && \|1\|_{L^2(\partial D_m)} C\,|\partial \c{B}|^{\frac{1}{2}}\,(\frac{\lambda_+}{[\max\limits_{1\leq m\leq M} diam(B_m)]^{\beta}} + \kappa_{\max})\,\epsilon^{1-\beta} \\
&\leq && |\partial \c{B}| C(\frac{\lambda_+}{[\max\limits_{1\leq m\leq M} diam(B_m)]^{\beta}} + \kappa_{\max})\,\epsilon^{2-\beta}.
\end{aligned}
$$

□

The following proposition gives an approximate characterization of the total charges $Q_m,\ m = 1, ..., M$.

Proposition 3.10. *For $m = 1, 2, \dots, M$, the total charge Q_m on each surface ∂D_m of the small scatterer D_m can be calculated from the algebraic system*

$$
\frac{Q_m}{\bar{C}_m} = -U^i(z_m) - \sum_{\substack{j=1 \\ j\neq m}}^{M} \bar{C}_j \Phi_\kappa(z_m, z_j)\frac{Q_j}{\bar{C}_j} + Err \tag{3.63}
$$

where $Err := O\left(a^{1-\beta} + \frac{a^{3-\beta}}{d^{3\alpha}}\right)$ and $\bar{C}_m := -\lambda_m|\partial D_m|$.

Proof of Proposition 3.10. For $s_m \in \partial D_m$, using the impedance boundary condition (1.2), we have

$$
\begin{aligned}
0 \;=\; & \frac{\partial U^t}{\partial \nu_m}(s_m) + \lambda_m U^t(s_m) = -\frac{\sigma_m(s_m)}{2} + \int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m, s)\sigma_m(s)ds + \sum_{\substack{j=1 \\ j\neq m}}^{M}\int_{\partial D_j} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m, s)\sigma_j(s)ds \\
& + \lambda_m \sum_{j=1}^{M}\int_{\partial D_j} \Phi_\kappa(s_m, s)\sigma_m(s)ds + \frac{\partial U^i}{\partial \nu_m}(s_m) + \lambda_m U^i(s_m)
\end{aligned}
\tag{3.64}
$$

Integrating the above on ∂D_m, we can write it as

$$
\begin{aligned}
& -\frac{1}{2}\int_{\partial D_m} \sigma_m(s_m)ds_m + \int_{\partial D_m}\left(\int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m, s)ds_m\right)\sigma_m(s)ds + \sum_{\substack{j=1 \\ j\neq m}}^{M}\int_{\partial D_j}\left(\int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m, s)ds_m\right)\sigma_j(s)ds \\
& \quad + \lambda_m \sum_{j=1}^{M}\int_{\partial D_j}\left(\int_{\partial D_m} \Phi_\kappa(s_m, s)ds_m\right)\sigma_m(s)ds = -\int_{\partial D_m} \frac{\partial U^i}{\partial \nu_m}(s_m)ds_m - \int_{\partial D_m} \lambda_m U^i(s_m)ds_m
\end{aligned}
$$

It can be rewritten as

$$
\begin{aligned}
& -\frac{1}{2}Q_m + \underbrace{\int_{\partial D_m}\left(\int_{\partial D_m} \frac{\partial \Phi_0}{\partial \nu_m}(s_m, s)ds_m\right)\sigma_m(s)ds}_{=:A} + \underbrace{\sum_{j\neq m}^{M}\int_{\partial D_j}\left(\int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m, s)ds_m\right)\sigma_j(s)ds}_{=:B} \\
& + \lambda_m \underbrace{\int_{\partial D_m}\left(\int_{\partial D_m} \Phi_\kappa(s_m, s)ds_m\right)\sigma_m(s)ds}_{=:C} + \lambda_m \underbrace{\sum_{j\neq m}^{M}\int_{\partial D_j}\left(\int_{\partial D_m} \Phi_\kappa(s_m, z_j)ds_m\right)\sigma_j(s)ds}_{=:D} \\
& = -\int_{\partial D_m} \frac{\partial U^i}{\partial \nu_m}(s_m)ds_m - \int_{\partial D_m} \lambda_m U^i(s_m)ds_m + A' + \lambda_m D',
\end{aligned}
$$

with

$$
A' := \int_{\partial D_m}\left(\int_{\partial D_m}\left[\frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m, s) - \frac{\partial \Phi_0}{\partial \nu_m}(s_m, s)\right]ds_m\right)\sigma_m(s)ds \tag{3.65}
$$

$$D' \quad := \quad \sum_{j\neq m}^{M} \int_{\partial D_j} \left(\int_{\partial D_m} \left[\Phi_\kappa(s_m,s) - \Phi_\kappa(s_m,z_j)\right] ds_m \right) \sigma_j(s) ds. \tag{3.66}$$

- We can approximate A and A' as follows;

$$\begin{aligned} A &= \int_{\partial D_m} \left(\int_{\partial D_m} \frac{\partial \Phi_0}{\partial \nu_m}(s_m,s) ds_m \right) \sigma_m(s) ds \\ &= \int_{\partial D_m} [\mathbf{K}^{i_\kappa}_{D_m}(1)](s)\sigma_m(s) ds \\ &= -\frac{1}{2}\int_{\partial D_m} \sigma_m(s) ds \\ &= -\frac{1}{2} Q_m. \end{aligned} \tag{3.67}$$

Here $\mathbf{K}^{i_\kappa}_{D_m}$ is the double layer operator defined as in (3.10) but with zero frequency and on the boundary of D_m. Observe that,

$$\begin{aligned} |A'| &= \left| \int_{\partial D_m} \left(\int_{\partial D_m} \left[\frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m,s) - \frac{\partial \Phi_0}{\partial \nu_m}(s_m,s) \right] ds_m \right) \sigma_m(s) ds \right| \\ &= \left| \int_{\partial D_m} \left(\int_{\partial D_m} \left[\frac{i\kappa(s_m - s)\cdot \nu_m(s_m)}{4\pi |s_m - s|^2} \sum_{l=1}^{\infty} (i\kappa|s_m - s|)^l \left(\frac{1}{l!} - \frac{1}{(l+1)!} \right) \right] ds_m \right) \sigma_m(s) ds \right| \\ &\leq \left| \int_{\partial D_m} \left(\int_{\partial D_m} \left[\frac{\kappa^2}{4\pi} \sum_{l=0}^{\infty} \frac{(\kappa a)^l}{2^l} \right] ds_m \right) |\sigma_m(s)| ds \right| \\ &= O\left(\kappa^2 a^{4-\beta}\right). \end{aligned} \tag{3.68}$$

- Now, we can approximate B

$$\begin{aligned} B &= \sum_{j\neq m}^{M} \int_{\partial D_j} \left(\int_{\partial D_m} \frac{\partial \Phi_\kappa}{\partial \nu_m}(s_m,s) ds_m \right) \sigma_j(s) ds \\ &= \sum_{j\neq m}^{M} \int_{\partial D_j} \left(\int_{D_m} \Delta\Phi_\kappa(y_m,s) dy_m \right) \sigma_j(s) ds \\ &= \sum_{j\neq m}^{M} \int_{\partial D_j} \left(\int_{D_m} [-\kappa^2 \Phi_\kappa(y_m,s)] dy_m \right) \sigma_j(s) ds \\ &= \sum_{j\neq m}^{M} \int_{\partial D_j} \left[O\left(\kappa^2 \frac{a^3}{d_{mj}} \right) \right] \sigma_j(s) ds \\ &= \sum_{n=1}^{[d^{-\alpha}]} [(2n+1)^3 - (2n-1)^3] \left[O\left(\kappa^2 \frac{a^{5-\beta}}{n d^\alpha} \right) \right] \\ &= O\left(2\kappa^2 \frac{a^{5-\beta}}{d^\alpha} \sum_{n=1}^{[d^{-\alpha}]} [12n + \frac{1}{n}] \right) \\ &= O\left(2\kappa^2 \frac{a^{5-\beta}}{d^{2\alpha}} [\frac{6}{d^\alpha} + 7] \right). \end{aligned} \tag{3.69}$$

- Since $\Phi_\kappa(s_m,s)-\Phi_0(s_m,z_j)=O\left(\frac{\kappa a}{d_{mj}}+\frac{a}{d_{mj}^2}\right)$ for $s\in\partial D_j, j\neq m$, we can approximate D and D' as follows;

$$\begin{aligned}
D &= \sum_{j\neq m}^{M}\int_{\partial D_j}\left(\int_{\partial D_m}\Phi_\kappa(s_m,z_j)ds_m\right)\sigma_j(s)ds\\
&= \sum_{j\neq m}^{M}\left[\int_{\partial D_j}\left(\int_{\partial D_m}\Phi_\kappa(z_m,z_j)ds_m\right)\sigma_j(s)ds\right.\\
&\qquad\left.+\int_{\partial D_j}\left(\int_{\partial D_m}[\Phi_\kappa(s_m,z_j)-\Phi_\kappa(z_m,z_j)]ds_m\right)\sigma_j(s)ds\right]\\
&= \sum_{j\neq m}^{M}\left[\Phi_\kappa(z_m,z_j)Q_j|\partial D_m|+O\left(\frac{a}{d_{mj}}\left(\kappa+\frac{1}{d_{mj}}\right)|\partial D_m|\left|\int_{\partial D_j}\sigma_j(s)ds\right|\right)\right]\\
&= \sum_{j\neq m}^{M}\left[\Phi_\kappa(z_m,z_j)Q_j|\partial D_m|+O\left(\frac{a^{5-\beta}}{d_{mj}}\left(\kappa+\frac{1}{d_{mj}}\right)\right)\right]\\
&= \sum_{j\neq m}^{M}\Phi_\kappa(z_m,z_j)Q_j|\partial D_m|+\sum_{n=1}^{[d^{-\alpha}]}[(2n+1)^3-(2n-1)^3]O\left(\frac{a^{5-\beta}}{nd^\alpha}\left(\kappa+\frac{1}{nd^\alpha}\right)\right)\\
&= \sum_{j\neq m}^{M}\Phi_\kappa(z_m,z_j)Q_j|\partial D_m|+O\left(2\frac{a^{5-\beta}}{d^\alpha}\sum_{n=1}^{[d^{-\alpha}]}\frac{12n^2+1}{n}\left(\kappa+\frac{1}{nd^\alpha}\right)\right)\\
&= \sum_{j\neq m}^{M}\Phi_\kappa(z_m,z_j)Q_j|\partial D_m|+O\left(2\frac{a^{5-\beta}}{d^{2\alpha}}\left[7\kappa+\frac{6\kappa+13}{d^\alpha}\right]\right) \qquad (3.70)
\end{aligned}$$

and

$$\begin{aligned}
|D'| &= \left|\sum_{j\neq m}^{M}\int_{\partial D_j}\left(\int_{\partial D_m}\left[\Phi_\kappa(s_m,s)-\Phi_\kappa(s_m,z_j)\right]ds_m\right)\sigma_j(s)ds\right|\\
&= \sum_{j\neq m}^{M}O\left(\frac{a^{5-\beta}}{d_{mj}}\left(\kappa+\frac{1}{d_{mj}}\right)\right)\\
&= \sum_{n=1}^{[d^{-\alpha}]}[(2n+1)^3-(2n-1)^3]O\left(\frac{a^{5-\beta}}{nd^\alpha}\left(\kappa+\frac{1}{nd^\alpha}\right)\right)\\
&= O\left(2\frac{a^{5-\beta}}{d^\alpha}\sum_{n=1}^{[d^{-\alpha}]}\frac{12n^2+1}{n}\left(\kappa+\frac{1}{nd^\alpha}\right)\right)\\
&= O\left(2\frac{a^{5-\beta}}{d^{2\alpha}}\left[7\kappa+\frac{6\kappa+13}{d^\alpha}\right]\right). \qquad (3.71)
\end{aligned}$$

- Now, let us approximate C. Since $|\int_{\partial D_m}\Phi_\kappa(s_m,s)ds_m|\leq\frac{1}{4\pi}\int_{\partial D_m}\frac{1}{|s_m-s|}ds_m=O(a)$, then

$$\begin{aligned}
|C| &= \left|\int_{\partial D_m}\left(\int_{\partial D_m}\Phi_\kappa(s_m,s)ds_m\right)\sigma_m(s)ds\right|\\
&= O\left(a^{3-\beta}\right). \qquad (3.72)
\end{aligned}$$

Hence, from (3.65-3.72), we obtain the approximation below

$$
\begin{aligned}
-Q_m &+ \sum_{j\neq m}^{M} \Phi_\kappa(z_m,z_j)\lambda_m|\partial D_m|Q_j \\
&= -\lambda_m|\partial D_m|e^{i\kappa\theta\cdot z_m} + O\left((|\lambda_m|+\kappa)\kappa a^3\right) + \lambda_m O\left(a^{3-\beta}\right), \\
&\quad +O\left(\kappa^2 a^{4-\beta}\right) + O\left(2\kappa^2\frac{a^{5-\beta}}{d^{2\alpha}}[\frac{6}{d^\alpha}+7]\right) + \lambda_m O\left(2\frac{a^{5-\beta}}{d^{2\alpha}}\left[7\kappa+\frac{6\kappa+13}{d^\alpha}\right]\right). \qquad (3.73)
\end{aligned}
$$

Indeed, $\int_{\partial D_m}\lambda_m((U^i(s_m)-U^i(z_m))ds_m = O(|\lambda|\kappa a^3)$ and $\int_{\partial D_m}\frac{\partial U^i}{\partial \nu_m}(s_m)ds_m = O(\kappa^2 a^3)$. In addition, since $\beta<1$, then $\lambda_m O\left(a^{3-\beta}\right) = O(a^{3-2\beta}) = o(a^{2-\beta}) = o(\lambda|\partial D_m|e^{i\kappa\theta\cdot z_m})$.

We can rewrite the above algebraic system as

$$
-\frac{1}{\lambda_m|\partial D_m|}Q_m = -e^{i\kappa\theta\cdot z_m} - \sum_{j\neq m}^{M}\Phi_\kappa(z_m,z_j)Q_j + Err \qquad (3.74)
$$

with

$$
\begin{aligned}
Err &:= \frac{1}{\lambda_m|\partial D_m|}\Bigg[O\left((\lambda_+ + \kappa\epsilon^\beta)\kappa a^{3-\beta}\right) + O\left(\kappa^2 a^{4-\beta}\right) + O\left(2\kappa^2\frac{a^{5-\beta}}{d^{2\alpha}}[\frac{6}{d^\alpha}+7]\right) \\
&\quad +O\left(2\frac{a^{5-2\beta}}{d^{2\alpha}}\left[7\kappa+\frac{6\kappa+13}{d^\alpha}\right]\right) + O\left(a^{3-2\beta}\right)\Bigg] \\
&= \frac{1}{\lambda_m|\partial D_m|}O\left(a^{3-2\beta}+\frac{a^{5-2\beta}}{d^{3\alpha}}\right) \\
&= O\left(a^{1-\beta}+\frac{a^{3-\beta}}{d^{3\alpha}}\right). \qquad (3.75)
\end{aligned}
$$

In the last two lines in the above approximation, we used the fact that $\kappa \le \kappa_{\max}$ and $d \le d_{\max}$.

□

3.4.2 Invertibility of the algebraic system

We define the following algebraic system

$$
\frac{\bar{Q}_m}{\bar{C}_m} := -U^i(z_m) - \sum_{j\neq m}^{M}\bar{C}_j\Phi_\kappa(z_m,z_j)\frac{\bar{Q}_j}{\bar{C}_j} \qquad (3.76)
$$

for all $m=1,2,\dots,M$. It can be written in a compact form as

$$
\mathbf{B}\bar{Q} = \mathrm{U}^I, \qquad (3.77)
$$

where $\bar{Q}, \mathrm{U}^I \in \mathbb{C}^{M\times 1}$ and $\mathbf{B}\in\mathbb{C}^{M\times M}$ are defined as

$$
\mathbf{B} := \begin{pmatrix}
-\frac{1}{\bar{C}_1} & -\Phi_\kappa(z_1,z_2) & -\Phi_\kappa(z_1,z_3) & \cdots & -\Phi_\kappa(z_1,z_M) \\
-\Phi_\kappa(z_2,z_1) & -\frac{1}{\bar{C}_2} & -\Phi_\kappa(z_2,z_3) & \cdots & -\Phi_\kappa(z_2,z_M) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\Phi_\kappa(z_M,z_1) & -\Phi_\kappa(z_M,z_2) & \cdots & -\Phi_\kappa(z_M,z_{M-1}) & -\frac{1}{\bar{C}_M}
\end{pmatrix}, \qquad (3.78)
$$

$$
\bar{Q} := \begin{pmatrix}\bar{Q}_1 & \bar{Q}_2 & \dots & \bar{Q}_M\end{pmatrix}^\top \text{ and } \mathrm{U}^I := \begin{pmatrix}U^i(z_1) & U^i(z_2) & \dots & U^i(z_M)\end{pmatrix}^\top. \qquad (3.79)
$$

The above linear algebraic system is solvable for $\bar{Q}_j,\ 1\le j\le M$, when the matrix $\mathbf{B}$ is invertible. Now we give sufficient conditions for the invertibility of system (3.77). We recall that $\lambda_m := \lambda_{m,0}a^{-\beta},\ \beta\ge 0$.

Lemma 3.11. *We distinguish the following two cases:*

- *Let* $\Re(\lambda_{m,0}) < 0$ [9] *and assume that* $\min\limits_{1\leq m\leq M} \frac{\Re \bar{C}_m}{|\bar{C}_m|^2} > \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}}$ *then the matrix* $\mathbf{B}$ *is invertible and the solution vector* $\bar{Q}$ *of* (3.77) *satisfies the estimate*

$$\sum_{m=1}^{M} |\bar{Q}_m|^2 \leq 4 \left(\frac{\min\limits_{1\leq m\leq M} \Re \bar{C}_m}{\max\limits_{1\leq m\leq M} |\bar{C}_m|^2} - \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}} \right)^{-2} \sum_{m=1}^{M} \left| U^i(z_m) \right|^2. \tag{3.80}$$

- *Let* $\Re(\lambda_{m,0}) > 0$ *and assume that* $\frac{\min\limits_{1\leq m\leq M} \Re(-\bar{C}_m)}{(\max\limits_{1\leq m\leq M} |\bar{C}_m|)^2} > \frac{\sqrt{2M_{max}}}{\pi d^{\frac{s}{t}}}$ *then the matrix* $\mathbf{B}$ *is invertible and the solution vector* $\bar{Q}$ *of* (3.77) *satisfies the estimate*

$$\sum_{m=1}^{M} |\bar{Q}_m|^2 \leq 4 \left(\frac{\min\limits_{1\leq m\leq M} \Re(-\bar{C}_m)}{\max\limits_{1\leq m\leq M} |\bar{C}_m|^2} - \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}} \right)^{-2} \sum_{m=1}^{M} \left| U^i(z_m) \right|^2. \tag{3.81}$$

where $M_{max} := M\ a^s$*, recalling that* $M := O(a^{-s})$*, as* $a \to 0$*.*

Since $\bar{C}_m := -\lambda_m |\partial D_m|$ and $d \geq d_{min} a^t$, the condition $\frac{\min\limits_{1\leq m\leq M} |\Re \bar{C}_m|}{\max\limits_{1\leq m\leq M} |\bar{C}_m|^2} > \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}}$ is satisfied if $\frac{4\lambda_- a^{s-(2-\beta)}}{\pi \lambda_+^2} > \frac{\sqrt{2M_{max}}}{\pi d^{\frac{s}{t}}}$. This is possible if $s < 2-\beta$ and a is small enough or $s = 2-\beta$ and λ_- and λ_+ satisfy $\frac{4\lambda_-}{\pi\lambda_+^2} > \frac{\sqrt{2M_{max}}}{\pi d_{min}^{\frac{s}{t}}}$. Recall that $t \geq \frac{s}{3}$, hence $\frac{s}{t} \leq 3$. If we assume $d_{min} \leq 1$, then the last condition is satisfied if $\frac{4\lambda_-}{\pi\lambda_+^2} > \frac{\sqrt{2M_{max}}}{\pi d_{min}^3}$ and if $d_{min} \geq 1$ then we take $\frac{4\lambda_-}{\pi\lambda_+^2} > \frac{\sqrt{2M_{max}}}{\pi}$. The proof of this lemma will be given in the appendix.

We can rewrite the inequatilies (3.80) and (3.81) using norm inequalities as

$$\sum_{m=1}^{M} |\bar{Q}_m| \leq 2 \left(\frac{\min\limits_{1\leq m\leq M} \Re \bar{C}_m}{\max\limits_{1\leq m\leq M} |\bar{C}_m|} - \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}} \max_{1\leq m\leq M} |\bar{C}_m| \right)^{-1} M \max_{1\leq m\leq M} |\bar{C}_m| \max_{1\leq m\leq M} \left| U^i(z_m) \right| \tag{3.82}$$

$$\sum_{m=1}^{M} |\bar{Q}_m| \leq \left(\frac{\min\limits_{1\leq m\leq M} \Re(-\bar{C}_m)}{\max\limits_{1\leq m\leq M} |\bar{C}_m|} - \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}} \max_{1\leq m\leq M} |\bar{C}_m| \right)^{-1} M \max_{1\leq m\leq M} |\bar{C}_m| \max_{1\leq m\leq M} \left| U^i(z_m) \right| \tag{3.83}$$

which holds for the cases $\Re\lambda_m \leq 0$ and $\Re\lambda_m \geq 0$ respectively.

3.4.3 The dominant part of the total charges

The difference between (3.76) and (3.63) produce the following

$$\frac{Q_m - \bar{Q}_m}{\bar{C}_m} = -\sum_{\substack{j=1 \\ j\neq m}}^{M} \Phi_\kappa(z_m, z_j) \left(Q_j - \bar{Q}_j \right) + Err, \tag{3.84}$$

for $m = 1, \ldots, M$. Comparing the above system of equations (3.84) with (3.76) and by making use of the estimates (3.82) and (3.83), we obtain

$$\sum_{m=1}^{M} (Q_m - \bar{Q}_m) = O\left(M a^{2-\beta} Err \right). \tag{3.85}$$

[9] In this case, we can actually relax the condition $s \leq 2-\beta$ on the number of small scatterers s, see Remark 6.1

We can evaluate the $\bar{Q}_m$'s from the algebraic system (3.76). This means that $\bar{Q}_m$'s are the dominant parts of the total charges Q_m's.

3.5 Approximation of the far-fields. III. End of the proof of Theorem 1.2

Using (3.85) in (3.56) we can represent the far-field pattern in terms of $\bar{Q}_m$ as follows

$$\begin{aligned} U^\infty(\hat{x}) &= \sum_{m=1}^{M}[e^{-i\kappa\hat{x}\cdot z_m}Q_m + O(\kappa\, a^{3-\beta})] \\ &= \sum_{m=1}^{M}\left[e^{-i\kappa\hat{x}\cdot z_m}[\bar{Q}_m + (Q_m - \bar{Q}_m)] + O(\kappa\, a^{3-\beta})\right] \\ &= \sum_{m=1}^{M} e^{-i\kappa\hat{x}\cdot z_m}\bar{Q}_m + O(Ma^{2-\beta}[Err + (\kappa\, a)]) \\ &\underset{(3.75)}{=} \sum_{m=1}^{M} e^{-i\kappa\hat{x}\cdot z_m}\bar{Q}_m + O\left(Ma^{2-\beta}\left(a^{1-\beta} + \frac{a^{3-\beta}}{d^{3\alpha}}\right)\right) \\ &= \sum_{m=1}^{M} e^{-i\kappa\hat{x}\cdot z_m}\bar{Q}_m + O\left(a^{3-s-2\beta}\right) \end{aligned} \tag{3.86}$$

since, as $s = 3t\alpha \leq 2-\beta$, we have $\frac{a^{3-\beta}}{d^{3\alpha}} \sim a^{3-\beta-3t\alpha} = O(a) = \circ(a^{1-\beta})$ if $\beta < 1$. Hence Theorem 1.2 is proved with the replacement of C_m in the statement by $\bar{C}_m$.

□

4 Proof of Corollary 1.3

The steps of the proof of Corollary 1.3 are the same as for the proof of Theorem 1.2. Here we only explain the main changes that are needed to derive it.

We recall that, for an obstacle D_ϵ of radius ϵ, $\mathcal{S}(\phi)(s) := \int_{\partial D_\epsilon}\Phi_\kappa(s,t)\phi(t)dt$ and $\mathcal{D}(\phi)(s) := \int_{\partial D_\epsilon}\frac{\partial\Phi_\kappa(s,t)}{\partial\nu(t)}\phi(t)dt$. Similarly, we set $\mathcal{S}_G(\phi)(s) := \int_{\partial D_\epsilon}G_\kappa(s,t)\phi(t)dt$ and $\mathcal{D}_G(\phi)(s) := \int_{\partial D_\epsilon}\frac{\partial G_\kappa(s,t)}{\partial\nu(t)}\phi(t)dt$.

We see that $W_\kappa(x,z) := G_\kappa(x,z) - \Phi_\kappa(x,z)$ satisfies

$$(\Delta + \kappa^2 n^2)W_\kappa = \kappa^2(1-n^2)\Phi_\kappa, \quad \text{in } \mathbb{R}^3 \tag{4.1}$$

with the Sommerfeld radiation conditions. Since $\Phi_\kappa(\cdot,z)$, $z \in \mathbb{R}^3$ is bounded in $L^p(\Omega)$, for $p<3$, by interior estimates, we deduce that $W(\cdot,z)$, $z\in\mathbb{R}^3$ is bounded in $W^{2,p}(\Omega)$, for $p<3$, and hence, in particular, the normal traces are bounded in $L^2(\partial D_\epsilon)$. Then we can show that the norms of the operators

$$\mathcal{S}_G - \mathcal{S} : L^2(\partial D_\epsilon) \to H^1(\partial D_\epsilon) \tag{4.2}$$

and

$$\mathcal{D}_G - \mathcal{D} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon) \tag{4.3}$$

are of the order $O(\epsilon)$ at least. The representation (3.1), in Proposition 3.1, needs to be replaced by

$$U_n^t(x) = V_n^t(x) + \sum_{m=1}^{M}\int_{\partial D_m}G_\kappa(x,s)\sigma_m(s)ds, \;\; x \in \mathbb{R}^3\backslash\left(\overset{M}{\underset{m=1}{\cup}}\bar{D}_m\right), \tag{4.4}$$

where V_n^t is the total field corresponding to the background modeled by the index of refraction n, see (1.15). We use the single layer potentials defined by the Greens' function G_κ instead of the fundamental function

Φ_κ. The main tools used in justifying Proposition 3.1 are the invertibility properties of the corresponding integral operators (i.e. the Fredholm property) and the jumps of the double layer potentials defined by Φ_κ. These two tools are satisfied also when we use G_κ instead of Φ_κ due to the error bounds in (4.2) and (4.3).

The a priori estimates on the densities σ_m's derived in section 3.2 are quantitative versions of the result in Proposition 3.1. Due to error bounds in (4.2) and (4.3), those estimates can then be translated to the densities used in the representation (4.4). Of course, in Lemma 3.3 one needs to replace the equalities by inequalities to estimate the properties of the operators defined by G_κ, on the scaled obstacles D_ϵ, in terms of the properties of the operators defined by Φ_κ on the original obstacles B.

Finally, to do the same analysis as in section 3.3, one needs to split G_κ as $G_\kappa = \Phi_\kappa + (G_\kappa - \Phi_\kappa)$ and use the results derived in section 3.3 and again error bounds in (4.2) and (4.3).

5 Justification of Remark 1.4

We show only the main changes needed in the proof of Theorem 1.2. This change occurs in the proof of Proposition 3.10 and precisely in evaluating the term C, i.e. (3.72). We rewrite C as

$$C = \int_{\partial D_m}\left(\int_{\partial D_m}\Phi_0(s_m,s)ds_m\right)\sigma_m(s)ds + \int_{\partial D_m}\left(\int_{\partial D_m}\Phi_\kappa(s_m,s)-\Phi_0(s_m,s)ds_m\right)\sigma_m(s)ds.$$

Since

$$\begin{aligned}\left|\int_{\partial D_m}\left(\int_{\partial D_m}[\Phi_\kappa(s_m,s)-\Phi_0(s_m,s)])ds_m\right)\sigma_m(s)ds\right| &< \frac{1}{2}\sum_{l=1}^{\infty}\frac{\kappa^l a^{l-1}}{l!}|\partial D_m|\int_{\partial D_m}|\sigma_m(s)|ds \\ &= O\left(\kappa a^{4-\beta}\right)\end{aligned}$$

then we can write C as follows;

$$\begin{aligned}C &= \int_{\partial D_m}\left(\int_{\partial D_m}\Phi_0(s_m,s)ds_m\right)\sigma_m(s)ds + O\left(\kappa a^{4-\beta}\right) \\ &= I_m Q_m + O\left(\kappa a^{4-\beta}\right),\end{aligned} \tag{5.1}$$

where $I_m = \int_{\partial D_m}\Phi_0(s_m,t)ds_m$, is a constant for each $t\in\partial D_m$ if D_m is a ball. Indeed, for the single layer potential $\left(\mathcal{S}^{i_\kappa}_{D_m}\psi\right)(s) = \int_{\partial D_\epsilon}\Phi_0(s,t)\psi(t)dt$, we have the jump condition as $\frac{\partial}{\partial\nu}\left(\mathcal{S}^{i_\kappa}_{D_m}\psi\right)(s)\big|_- = \left(\frac{1}{2}\mathbf{I}+\mathbf{K}^{i^*_\kappa}_{D_m}\right)\psi(s)$ recalling that $\mathbf{K}^{i^*_\kappa}_{D_m}$ is the adjoint of the double layer operator. Since $\mathbf{K}^{i^*_\kappa}_{D_m}[1] = \mathbf{K}^{i_\kappa}_{D_m}[1] = -1/2$, then we have $\frac{\partial}{\partial\nu^-}\mathcal{S}^{i_\kappa}_{D_m}[1] = 0$ on ∂D_m. Hence $\mathcal{S}^{i_\kappa}_{D_m}[1]$ is a constant, namely I_m, as it satisfies $\Delta\,\mathcal{S}^{i_\kappa}_{D_m}[1] = 0$ in D_m and has zero normal derivative on ∂D_m.

With this correction at hand the estimate (3.73) becomes

$$\begin{aligned}(-1+\lambda_m I_m)Q_m &+ \sum_{j\neq m}^{M}\Phi_\kappa(z_m,z_j)\lambda_m|\partial D_m|Q_j \\ &= -\lambda_m|\partial D_m|e^{i\kappa\theta\cdot z_m} + O\left((|\lambda_m|+\kappa)\kappa a^3\right) + \lambda_m O\left(a^{4-\beta}\right), \\ &\quad +O\left(\kappa^2 a^{4-\beta}\right) + O\left(2\kappa^2\frac{a^{5-\beta}}{d^{2\alpha}}[\frac{6}{d^\alpha}+7]\right) + \lambda_m O\left(2\frac{a^{5-\beta}}{d^{2\alpha}}\left[7\kappa+\frac{6\kappa+13}{d^\alpha}\right]\right)\end{aligned} \tag{5.2}$$

and the system (3.74) as

$$\frac{-1+\lambda_m I_m}{\lambda_m|\partial D_m|}Q_m = -e^{i\kappa\theta\cdot z_m} - \sum_{j\neq m}^{M}\Phi_\kappa(z_m,z_j)Q_j + Err \tag{5.3}$$

with

$$
\begin{aligned}
Err &:= \frac{1}{\lambda_m |\partial D_m|}\Bigg[O\left((\lambda_+ + \kappa\epsilon^\beta)\kappa a^{3-\beta}\right) + O\left(\kappa^2 a^{4-\beta}\right) + O\left(2\kappa^2\frac{a^{5-\beta}}{d^{2\alpha}}[\frac{6}{d^\alpha}+7]\right) \\
&\quad + O\left(2\frac{a^{5-2\beta}}{d^{2\alpha}}\left[7\kappa + \frac{6\kappa+13}{d^\alpha}\right]\right) + O\left(a^{4-2\beta}\right)\Bigg] \\
&= \frac{1}{\lambda_m |\partial D_m|} O\left(a^{3-\beta} + \frac{a^{5-2\beta}}{d^{3\alpha}}\right) \\
&= O\left(a + \frac{a^{3-\beta}}{d^{3\alpha}}\right). \qquad (5.4)
\end{aligned}
$$

Finally, the estimate (3.86) becomes

$$
\begin{aligned}
U^\infty(\hat{x}) &= \sum_{m=1}^{M}[e^{-i\kappa\hat{x}\cdot z_m}Q_m + O(\kappa\, a^{3-\beta})] \\
&= \sum_{m=1}^{M}\left[e^{-i\kappa\hat{x}\cdot z_m}[\bar{Q}_m + (Q_m - \bar{Q}_m)] + O(\kappa\, a^{3-\beta})\right] \\
&= \sum_{m=1}^{M} e^{-i\kappa\hat{x}\cdot z_m}\bar{Q}_m + O(Ma^{2-\beta}[Err + (\kappa\, a)]) \\
&\underset{(5.4)}{=} \sum_{m=1}^{M} e^{-i\kappa\hat{x}\cdot z_m}\bar{Q}_m + O\left(Ma^{2-\beta}\left(a + \frac{a^{3-\beta}}{d^{3\alpha}}\right)\right) \\
&= \sum_{m=1}^{M} e^{-i\kappa\hat{x}\cdot z_m}\bar{Q}_m + O\left(a^{3-s-\beta}\right) \qquad (5.5)
\end{aligned}
$$

since, as $s = 3t\alpha \leq 2-\beta$, we have $\frac{a^{3-\beta}}{d^{3\alpha}} = O(a^{3-\beta-3t\alpha}) = O(a)$.

6 Appendix: *Proof of Lemma 3.11*

We start by factorizing $\mathbf{B}$ as $\mathbf{B} = -(\mathbf{C}^{-1} + \mathbf{B}_n)$ where $\mathbf{C} := Diag(\bar{C}_1, \bar{C}_2, \dots, \bar{C}_M) \in \mathbb{R}^{M\times M}$, I is the identity matrix and $\mathbf{B}_n := -\mathbf{C}^{-1} - \mathbf{B}$. We have $\mathbf{B} : \mathbb{C}^M \to \mathbb{C}^M$, so it is enough to prove the injectivity in order to prove its invetibility. For this purpose, let X, Y are vectors in $\mathbb{C}^M$ and consider the system

$$(\mathbf{C}^{-1} + \mathbf{B}_n)X = Y. \qquad (6.1)$$

Let $(\cdot)^{real}$ and $(\cdot)^{img}$ denotes the real and the imaginary parts of the corresponding complex number/vector/matrix. For convenience, let us denote $\mathbf{C}^{-1}$ by $\mathbf{C}_I$. Now, the following can be written from (6.1);

$$
\begin{aligned}
(\mathbf{C}_I^{real} + \mathbf{B}_n^{real})X^{real} - (\mathbf{C}_I^{img} + \mathbf{B}_n^{img})X^{img} &= Y^{real}, \qquad (6.2) \\
(\mathbf{C}_I^{real} + \mathbf{B}_n^{real})X^{img} + (\mathbf{C}_I^{img} + \mathbf{B}_n^{img})X^{real} &= Y^{img}, \qquad (6.3)
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\langle (\mathbf{C}_I^{real} + \mathbf{B}_n^{real})X^{real}, X^{real}\rangle - \langle (\mathbf{C}_I^{img} + \mathbf{B}_n^{img})X^{img}, X^{real}\rangle &= \langle Y^{real}, X^{real}\rangle, \qquad (6.4) \\
\langle (\mathbf{C}_I^{real} + \mathbf{B}_n^{real})X^{img}, X^{img}\rangle + \langle (\mathbf{C}_I^{img} + \mathbf{B}_n^{img})X^{real}, X^{img}\rangle &= \langle Y^{img}, X^{img}\rangle. \qquad (6.5)
\end{aligned}
$$

By summing up (6.4) and (6.5) will give

$$\begin{aligned} \langle\, \mathbf{C}_I^{real} X^{real}, X^{real}\rangle \,+ \langle\, \mathbf{B}_n^{real} X^{real}, X^{real}\rangle \,+ \langle\, \mathbf{C}_I^{real} X^{img}, X^{img}\rangle \,+ \langle\, \mathbf{B}_n^{real} X^{img}, X^{img}\rangle \\ = \langle\, Y^{real}, X^{real}\rangle + \langle\, Y^{img}, X^{img}\rangle. \end{aligned} \tag{6.6}$$

We can observe that, the right-hand side in (6.6) does not exceed

$$\begin{aligned} \langle\, X^{real}, X^{real}\rangle^{1/2} \langle\, Y^{real}, Y^{real}\rangle^{1/2} + \langle\, X^{img}, X^{img}\rangle^{1/2} \langle\, Y^{img}, Y^{img}\rangle^{1/2} \\ \leq 2 \langle\, X^{|\cdot|}, X^{|\cdot|}\rangle^{1/2} \langle\, Y^{|\cdot|}, Y^{|\cdot|}\rangle^{1/2}. \end{aligned} \tag{6.7}$$

At this stage, we divide the proof into two cases.

1. If $\Re(\lambda_{m,0}) < 0$ for each m. In this case $\Re \bar{C}_m > 0$. We know that:

$$|\langle\, B_n^{real} X^{real},\ X^{real}\rangle| \leq \|B_n^{real}\|_2 |X^{real}|_2^2 \tag{6.8}$$

   where $\|B_n^{real}\|_2^2 := \sum_{i,\, j=1}^{M} (B_n^{real})_{i,j}^2$ and $(B_n^{real})_{i,j} := \Re\ \Phi(z_i, z_j)$ if $i \neq j$ and $(B_n^{real})_{i,i} := 0$ for $i, j = 1, ..., M$. Hence $|(B_n^{real})_{i,j}| \leq \frac{1}{4\pi |z_i - z_j|}$, $i \neq j$. Arguing as in (3.46), we see that

$$\sum_{i,j=1}^{M} (B_n^{real})_{i,j}^2 \leq M \sum_{n=1}^{[d^{-\alpha}]} [(2n+1)^3 - (2n-1)^3] \frac{1}{(4\pi)^2 n^2 d^{2\alpha}} \leq M \frac{2d^{-2\alpha}}{\pi^2} \sum_{n=1}^{[d^{-\alpha}]} 1 = \frac{2Md^{-3\alpha}}{\pi^2}. \tag{6.9}$$

   Observing that $M = O(a^{-s}) \leq M_{max} a^{-s}$ and that $s = 3t\alpha$, we obtain:

$$\|B_n^{real}\|_2 \leq \frac{\sqrt{2M_{max}}}{\pi} a^{-\frac{s}{t}}. \tag{6.10}$$

   From (6.6) and (6.7), we deduce that

$$\left(\frac{\min\limits_{1 \leq m \leq M} \Re \bar{C}_m}{(\max\limits_{1 \leq m \leq M} |\bar{C}_m|)^2} - \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}} \right) \sum_{m=1}^{M} |X_m|^2 \leq 2 \left(\sum_{m=1}^{M} |X_m|^2 \right)^{1/2} \left(\sum_{m=1}^{M} |Y_m|^2 \right)^{1/2}, \tag{6.11}$$

   which yields

$$\sum_{m=1}^{M} |X_m|^2 \leq 4 \left(\frac{\min\limits_{1 \leq m \leq M} \Re(\bar{C}_m)}{(\max\limits_{1 \leq m \leq M} |\bar{C}_m|)^2} - \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}} \right)^{-2} \sum_{m=1}^{M} |Y_m|^2 . \tag{6.12}$$

   Thus, if $\frac{\min\limits_{1 \leq m \leq M} \Re(\bar{C}_m)}{(\max\limits_{1 \leq m \leq M} |\bar{C}_m|)^2} > \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}}$, then the matrix $\mathbf{B}$ in algebraic system (3.77) is invertible.

2. If $\Re(\lambda_{m,0}) > 0$ for each m. In this case $\Re \bar{C}_m < 0$ and so to prove the invertibility of $\mathbf{B}$ in algebraic system (3.77), consider $(-\mathbf{C}^{-1} - \mathbf{B}_n)X = Y$ in place of (6.1) and proceed in the way as it done for the case $\Re \lambda_m \geq 0$ for each m. Then we can get the following estimate

$$\sum_{m=1}^{M} |X_m|^2 \leq 4 \left(\frac{\min\limits_{1 \leq m \leq M} \Re(-\bar{C}_m)}{(\max\limits_{1 \leq m \leq M} |\bar{C}_m|)^2} - \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}} \right)^{-2} \sum_{m=1}^{M} |Y_m|^2 . \tag{6.13}$$

   and the invertibility of the matrix $\mathbf{B}$, under the assumption that $\frac{\min\limits_{1 \leq m \leq M} \Re(-\bar{C}_m)}{(\max\limits_{1 \leq m \leq M} |\bar{C}_m|)^2} > \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}}$.

3. We are left with the case where $\Re(\lambda_{m,0}) > 0$ for few m's. In this case we multiply every line of the system (6.1) corresponding to $\Re(\lambda_{m,0}) > 0$ by -1. Hence, the system (6.1) becomes

$$(\tilde{\mathbf{C}}^{-1} + \tilde{\mathbf{B}}_n)X = \tilde{Y}. \tag{6.14}$$

where now every component of the diagonal matrix $\tilde{\mathbf{C}}^{-1}$ is positive and $\|\tilde{\mathbf{B}}_n\|_2 = \|\mathbf{B}_n\|_2$. Then we are in the case (1).

□

Remark 6.1. *Assume that $\Re(\lambda_{m,0}) < 0$. Following the computations in [15, Lemma 2.22] and assuming that $\frac{5\pi}{3}\frac{\min\limits_{1\leq m\leq M}\Re\bar{C}_m}{(\max\limits_{1\leq m\leq M}|\bar{C}_m|)^2} > \frac{\gamma}{d}$ and $\gamma := \min\limits_{j\neq m, 1\leq j,m\leq M}\cos(\kappa|z_m - z_j|) \geq 0$, we can prove that*

$$\left(\frac{\min\limits_{1\leq m\leq M}\Re\bar{C}_m}{(\max\limits_{1\leq m\leq M}|\bar{C}_m|)^2} - \frac{3\gamma}{5\pi\, d}\right)\sum_{m=1}^{M}|X_m|^2 \leq 2\left(\sum_{m=1}^{M}|X_m|^2\right)^{1/2}\left(\sum_{m=1}^{M}|Y_m|^2\right)^{1/2},$$

which yields

$$\sum_{m=1}^{M}|X_m|^2 \leq 4\left(\frac{\min\limits_{1\leq m\leq M}\Re\bar{C}_m}{(\max\limits_{1\leq m\leq M}|\bar{C}_m|)^2} - \frac{3\gamma}{5\pi\, d}\right)^{-2}\sum_{m=1}^{M}|Y_m|^2.$$

and thus the invertibility of the matrix $\mathbf{B}$ in the algebraic system (3.77). Observe that the condition $\frac{\min\limits_{1\leq m\leq M}\Re(\bar{C}_m)}{(\max\limits_{1\leq m\leq M}|\bar{C}_m|)^2} > \frac{\sqrt{2M_{max}}}{\pi\, d^{\frac{s}{t}}}$ is satisfied if $t < 2-\beta$.

We can see that if the number of the scatterers is $O(a^{-s})$ such that $s \leq 2-\beta$, then we do not need upper bound on t, which allow as to have very close scatterers. However, if we want to have larger number of scatterers, i.e. s limited only by the bound $s \leq 3$, then we need a condition on t, i.e. $t \leq 2-\beta$, which makes more restrictions on the minimum distance between the scatterers.

References

[1] B. Ahmad, D. P. Challa, M. Kirane, and M. Sini. The equivalent refraction index for the acoustic scattering by many small obstacles: with error estimates. *J. Math. Anal. Appl.*, 424(1):563–583, 2015.

[2] A. Alsaedi, B. Ahmed, D. P. Challa, M. Kirane, M. Sini, *A cluster of many small holes with negative imaginary surface impedance's may generate a negative refraction index. Preprint, arXiv:1504.06947*

[3] A. Alsaedi, F. Alzahrani, D. P. Challa, M. Kirane, M. Sini, *Extraction of the index of refraction by embedding multiple and close small inclusions Preprint arXiv:1505.07236*

[4] H. Ammari; Y. Capdeboscq; F. de Gournay; A. Rozanova-Pierrat; F. Triki, Microwave imaging by elastic deformation. *SIAM J. Appl. Math.* 71 (2011), no. 6, 21122130.

[5] H. Ammari, E. Bretin, J. Garnier, H. Kang, H. Lee, and A. Wahab. *Mathematical Methods in Elasticity Imaging.* Princeton Series in Applied Mathematics. Princeton University Press, Princeton, Newjersy, 2015.

[6] H. Ammari and H. Kang. *Polarization and moment tensors*, volume 162 of *Applied Mathematical Sciences*. Springer, New York, 2007. With applications to inverse problems and effective medium theory.

[7] H. Ammari. *An Introduction to Mathematics of Emerging Biomedical Imaging.* Springer-Verlag, Berlin, 2008.

[8] H. Ammari, E. Bossy, J. Garnier, and L. Seppecher, *Acousto-electromagnetic tomography.* SIAM J. Appl. Math. 72 (2012), no. 5, 1592-1617.

[9] G. Bal; E. Bonnetier; F. Monard; F. Triki, Inverse diffusion from knowledge of power densities. *Inverse Probl. Imaging* 7 (2013), no. 2, 353375.

[10] G. Bal, Hybrid inverse problems and redundant systems of partial differential equations. *Inverse problems and applications,* 1547, Contemp. Math., 615, Amer. Math. Soc., Providence, RI, 2014.

[11] A. Bendali, P.-H. Cocquet, and S. Tordeux. Scattering of a scalar time-harmonic wave by n small spheres by the method of matched asymptotic expansions. *Numerical Analysis and Applications*, 5(2):116–123, 2012.

[12] A Bendali, PH Cocquet, S Tordeux, Approximation by Multipoles of the Multiple Acoustic Scattering by Small Obstacles in Three Dimensions and Application to the Foldy Theory of Isotropic Scattering. Arch. Rational Mech. Anal, (DOI) 10.1007/s00205-015-0915-5

[13] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam, 1978.

[14] M. Cassier and C. Hazard. Multiple scattering of acoustic waves by small sound-soft obstacles in two dimensions: mathematical justification of the Foldy-Lax model. *Wave Motion*, 50(1):18–28, 2013.

[15] D. P. Challa and M. Sini. On the justification of the Foldy-Lax approximation for the acoustic scattering by small rigid bodies of arbitrary shapes. *Multiscale Model. Simul.*, 12(1):55–108, 2014.

[16] D. P. Challa and M. Sini. The foldy-lax approximation of the scattered waves by many small bodies for the lamé system. *Math. Nachr.*, 2015. To appear,`arXiv:1308.3072`.

[17] D. Cioranescu and F. Murat. A strange term coming from nowhere [MR0652509 (84e:35039a); MR0670272 (84e:35039b)]. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 45–93. Birkhäuser Boston, Boston, MA, 1997.

[18] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences.* Springer-Verlag, Berlin, second edition, 1998.

[19] D. Colton and R. Kress. *Integral equation methods in scattering theory.* Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1983. A Wiley-Interscience Publication.

[20] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals.* Springer-Verlag, Berlin, 1994.

[21] V. A. Marchenko and E. Y. Khruslov. *Homogenization of partial differential equations*, volume 46 of *Progress in Mathematical Physics.* Birkhäuser Boston Inc., Boston, MA, 2006.

[22] P. A. Martin. *Multiple scattering*, volume 107 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2006. Interaction of time-harmonic waves with N obstacles.

[23] O. Mendez; W. Reichel, Electrostatic characterization of spheres. *Forum Math.* 12 (2000), no. 2, 223245.

[24] V. Mazýa and A. Movchan. Asymptotic treatment of perforated domains without homogenization. *Math. Nachr.*, 283(1):104–125, 2010.

[25] V. Mazýa, A. Movchan, and M. Nieves. Mesoscale asymptotic approximations to solutions of mixed boundary value problems in perforated domains. *Multiscale Model. Simul.*, 9(1):424–448, 2011.

[26] V. Maz'ya, A. Movchan, and M. Nieves. *Green's kernels and meso-scale approximations in perforated domains*, volume 2077 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2013.

[27] W. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[28] D. Mitrea. The method of layer potentials for non-smooth domains with arbitrary topology. *Integral Equations Operator Theory*, 29(3):320–338, 1997.

[29] J.-C. Nédélec. *Acoustic and electromagnetic equations*, volume 144 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001. Integral representations for harmonic problems.

[30] A. G. Ramm. *Inverse problems*. Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005. Mathematical and analytical techniques with applications to engineering, With a foreword by Alan Jeffrey.

[31] A. G. Ramm. Many-body wave scattering by small bodies and applications. *J. Math. Phys.*, 48(10):103511, 29, 2007.