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A. Mantile, A. Posilicano, M. Sini

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SELF-ADJOINT ELLIPTIC OPERATORS WITH BOUNDARY CONDITIONS ON NOT CLOSED HYPERSURFACES

ANDREA MANTILE, ANDREA POSILICANO, AND MOURAD SINI

Abstract. The theory of self-adjoint extensions of symmetric operators is used to construct self-adjoint realizations of a second-order elliptic differential operator on $\mathbb{R}^n$ with linear boundary conditions on (a relatively open part of) a compact hypersurface. Our approach allows to obtain Kreĭn-like resolvent formulae where the reference operator coincides with the "free" operator with domain $H^2(\mathbb{R}^n)$; this provides an useful tool for the scattering problem from a hypersurface. Concrete examples of this construction are developed in connection with the standard boundary conditions, Dirichlet, Neumann, Robin, δ and δ'-type, assigned either on a $(n-1)$ dimensional compact boundary $\Gamma = \partial\Omega$ or on a relatively open part $\Sigma \subset \Gamma$. Schatten-von Neumann estimates for the difference of the powers of resolvents of the free and the perturbed operators are also proven; these give existence and completeness of the wave operators of the associated scattering systems.

1. Introduction.

This work is concerned with the self-adjoint realization of symmetric, second-order elliptic operators

$$Au(x) = \sum_{1\le i,j\le n} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u(x)) - V(x)u(x)\,, \quad x \equiv (x_1,\dots,x_n) \in \mathbb{R}^n\,, \tag{1.1}$$

with boundary conditions on (relatively open parts of) hypersurfaces which can be realized as boundaries of bounded sets $\Omega \subset \mathbb{R}^n$ (see Section 4 for the precise regularity assumptions on the coefficients of A and on the set Ω). When defined on the domain $\mathrm{dom}(A) = H^2(\mathbb{R}^n)$, the operator A is self-adjoint and bounded from above. We then consider the same differential operator A but now acting on a domain characterized by linear boundary conditions on Γ or on a relatively open part $\Sigma \subset \Gamma$. Using the abstract theory of self-adjoint extensions of symmetric operators developed in [56]-[59], we construct these models as singular perturbations of the "free operator" with domain $H^2(\mathbb{R}^n)$. This allows us to describe all possible linear boundary conditions within an unified framework where the corresponding self-adjoint elliptic operators $A_{\Pi,\Theta}$ are parametrized through couples (Π,Θ), where Π is an orthogonal projector on the Hilbert trace space $H^{3/2}(\Gamma) \oplus H^{1/2}(\Gamma)$ and Θ is a self-adjoint operator in the Hilbert space given by the range of Π. Our approach naturally yields to Kreĭn-type formulae expressing the resolvent of the self-adjoint extension $A_{\Pi,\Theta}$ in terms of the unperturbed resolvent $(-A+z)^{-1}$, plus a non-perturbative term; under suitable regularity assumptions of the parameters (Π,Θ), the difference $(-A_{\Pi,\Theta}+z)^{-k} - (-A+z)^{-k}$ is of trace class (for sufficiently large k) and the Birman-Kato criterion allows to consider $\{A, A_{\Pi,\Theta}\}$ as a scattering system provided with the corresponding wave operators.

Singular perturbations supported on manifolds of lower dimension have been the object of a large number of investigations (see for instance [1]-[4], [8]-[12], [13]-[15], [19]-[31], [33], [34],

[39], [47]-[50], [55], [56] and references therein). These have mainly concerned the case of δ-perturbed Schrödinger operators and are generally motivated by the quantum dynamical modelling, as the case of leaky quantum graphs, or the quantum interaction with charged surfaces.

Covering a wider class of models, the analysis developed in our work have been inspired by the scattering problem from a compact hypersurface with abstract boundary conditions. When these conditions are encoded by the extension $A_{\Pi,\Theta}$, the scattered field u_{sc} corresponding to an incident wave u_{in} is expected to be related to a limit absorption principle for $(-A_{\Pi,\Theta}+z)^{-1}$. In particular, the result obtained in the simpler case of point scatterers (see [43]) suggests the relation

$$u_{\text{sc}} = \lim_{\mathbb{C}_+ \ni z \to \lambda \in \mathbb{R}} \left((-A_{\Pi,\Theta}+z)^{-1}(-A+z)u_{\text{in}} \right) - u_{\text{in}} \,,$$

where the limit is to be understood in an appropriate operator topology. This, using the Kreĭn resolvent identity for $(-A_{\Pi,\Theta}+z)^{-1} - (-A+z)^{-1}$, would lead to an explicit characterization of the scattered field in terms of a factorized formula depending on the incident wave. Different applications of this type of formulas can be foreseen. In the most standard cases (Dirichlet, Neumann and impedance boundary conditions on Γ or $\Sigma \subset \Gamma$), they have been exploited in the analysis of the corresponding inverse scattering problem for surfaces reconstruction (see [46] for an introduction to the factorization method). In this connection, our result could provide an unified method to derive factorized formulas for the scattered field for a large class of scattering problems with rather general linear boundary conditions.

The first part of this work is devoted to the construction of self-adjoint elliptic operators with abstract boundary conditions on Γ. In Section 2 we briefly recall the main results of the extension theory of symmetric operators according to [56]-[59], while, in Section 3 the mapping properties of the trace operators and of the single and double layer operators, related to the surface Γ and to the operator A, are reviewed. In the Section 4, we introduce our model through the symmetric operator S given by the restriction of $A : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, to the dense linear set $\{u \in H^2(\mathbb{R}^n) : u|\Gamma = \partial_{\underline{a}} u|\Gamma = 0\}$, being $\partial_{\underline{a}}$ the co-normal derivative on Γ. The construction of the self-adjoint extensions of S, parametrized through couples (Π, Θ) on the trace space $H^{3/2}(\Gamma) \oplus H^{1/2}(\Gamma)$, is then given in Theorem 4.3 and Corollary 4.5, where a Kreĭn-like resolvent formula is also provided. The Schatten-von Neumann type estimates for the difference of the powers of resolvents, together with the spectral properties of these extensions and the existence and completeness of the wave operators are then given in Theorem 4.6 and Corollary 4.7.

The second part of the work is devoted to applications. In Section 5 we construct the standard models, i.e. Dirichlet, Neumann, impedance (or Robin), δ and δ'-type boundary conditions on Γ, in terms of extensions of S. The main issue concerns the determination of the parameters (Π, Θ) corresponding to the required constraints. In the case of global conditions on Γ, this task is simplified by the nature of Π, which, in the above mentioned cases, identifies with the projection onto the $H^{3/2}(\Gamma)$ component for the Dirichlet case, with the projection onto the $H^{1/2}(\Gamma)$ component for the Neumann case and with $\Pi = 1$ in the Robin case; then, the determination of Θ for the corresponding boundary conditions easily follows (almost) from algebraic arguments. The case of Dirichlet, Neumann, impedance, δ and δ'-type conditions assigned only on a relatively open subset $\Sigma \subset \Gamma$ is more complex and requires further work: in particular the analysis of compressed operators on the subspaces $H^{3/2}_{\Sigma^c}(\Gamma)^\perp$ and $H^{1/2}_{\Sigma^c}(\Gamma)^\perp$ is needed. This point is developed in Section 6, see Theorems 6.1-6.9, and in the Appendix. At least to our knowledge, the Kreĭn formulae we provide in the case of boundary conditions on not closed hypersurfaces $\Sigma \subset \Gamma$ do not appear in the past literature.

2. Preliminaries: Self-adjoint extensions of symmetric operators.

Given the self-adjoint operator

$$A : \mathrm{dom}(A) \subseteq \mathscr{H} \to \mathscr{H}$$

in the Hilbert space $\mathscr{H}$ (equipped with the scalar product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$), let

$$\tau : \mathrm{dom}(A) \to \mathfrak{h}$$

be continuous (w.r.t. the graph norm in $\mathrm{dom}(A)$) and surjective onto the auxiliary Hilbert space $\mathfrak{h}$ (equipped with the scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$). We further assume that $\ker(\tau)$ is dense in $\mathscr{H}$ and introduce the densely defined, closed, symmetric operator

$$S := A|\ker(\tau)\,.$$

Our aim is to provide all self-adjoint extensions of S; here we use the approach developed in [56]-[59] to which we refer for proofs and for the connections with other well known approaches to this problem (von Neumann's theory and boundary triples theory) .

For notational convenience we do not identify $\mathfrak{h}$ with its dual $\mathfrak{h}'$ and we denote by $J : \mathfrak{h} \to \mathfrak{h}'$ the duality mapping (a bijective isometry) given by the canonical isomorphism from $\mathfrak{h}$ onto $\mathfrak{h}'$, i.e. $J(\varphi)$ is the differential of the function $\varphi \mapsto \frac{1}{2}\|\varphi\|^2_{\mathfrak{h}}$ (see e.g. [5, Section 3.1]); $\mathfrak{h}'$ inherits a Hilbert space structure by the scalar product $\langle\phi_1,\phi_2\rangle_{\mathfrak{h}'} := \langle J^{-1}\phi_1, J^{-1}\phi_2\rangle_{\mathfrak{h}}$, so that J becomes an unitary map, and we denote by $\langle\cdot,\cdot\rangle_{\mathfrak{h}'\mathfrak{h}}$ the $\mathfrak{h}'$-$\mathfrak{h}$ duality

$$\langle\phi,\varphi\rangle_{\mathfrak{h}'\mathfrak{h}} := \langle J^{-1}\phi,\varphi\rangle_{\mathfrak{h}}\,.$$

The inverse $J^{-1} : \mathfrak{h}' \to \mathfrak{h}$ gives the duality mapping from $\mathfrak{h}'$ to its dual $\mathfrak{h}'' \equiv \mathfrak{h}$; we denote by $\langle\cdot,\cdot\rangle_{\mathfrak{h}\mathfrak{h}'}$ the $\mathfrak{h}$-$\mathfrak{h}'$ duality defined by $\langle\varphi,\phi\rangle_{\mathfrak{h}\mathfrak{h}'} := \langle J\varphi,\phi\rangle_{\mathfrak{h}'} = \langle\varphi, J^{-1}\phi\rangle_{\mathfrak{h}}$.

Given a densely defined linear operator

$$\Xi : \mathrm{dom}(\Xi) \subseteq \mathfrak{h} \to \mathfrak{h}$$

we denote by Ξ' the dual operator

$$\Xi' : \mathrm{dom}(\Xi') \subseteq \mathfrak{h}' \to \mathfrak{h}'\,, \qquad \Xi\phi := \phi'\,,$$

$$\mathrm{dom}(\Xi') := \{\phi \in \mathfrak{h}' : \exists\phi' \in \mathfrak{h}' \text{ such that } \langle\phi',\varphi\rangle_{\mathfrak{h}'\mathfrak{h}} = \langle\phi,\Xi\varphi\rangle_{\mathfrak{h}'\mathfrak{h}} \text{ for all } \varphi \in \mathrm{dom}(\Xi)\}\,;$$

this is related to the (Hilbert) adjoint $\Xi^* : \mathrm{dom}(\Xi^*) \subseteq \mathfrak{h} \to \mathfrak{h}$ by

$$\Xi^* = J^{-1}\Xi' J\,, \qquad \mathrm{dom}(\Xi^*) = J^{-1}(\mathrm{dom}(\Xi'))\,.$$

In the case $\Xi : \mathrm{dom}(\Xi) \subseteq \mathfrak{h}' \to \mathfrak{h}$, the dual operator is defined in a similar way:

$$\Xi' : \mathrm{dom}(\Xi') \subseteq \mathfrak{h}' \to \mathfrak{h}\,, \qquad \Xi'\phi := \varphi\,,$$

$$\mathrm{dom}(\Xi') := \{\phi \in \mathfrak{h}' : \exists\varphi \in \mathfrak{h} \text{ such that } \langle\varphi,\psi\rangle_{\mathfrak{h}\mathfrak{h}'} = \langle\phi,\Xi\psi\rangle_{\mathfrak{h}'\mathfrak{h}} \text{ for all } \psi \in \mathrm{dom}(\Xi)\}\,,$$

or, equivalently,

$$\Xi' : \mathrm{dom}(\Xi') \subseteq \mathfrak{h}' \to \mathfrak{h}\,, \qquad \Xi'\phi := \varphi\,,$$

$$\mathrm{dom}(\Xi') := \{\phi \in \mathfrak{h}' : \exists \varphi \in \mathfrak{h} \text{ such that } \langle \varphi, J^{-1}\psi\rangle_{\mathfrak{h}} = \langle J^{-1}\phi, \Xi\psi\rangle_{\mathfrak{h}} \text{ for all } \psi \in \mathrm{dom}(\Xi)\}\,.$$

The latter definition shows that $\Xi = \Xi'$ if and only if $\tilde{\Xi} := \Xi J : J^{-1}(\mathrm{dom}(\Xi)) \subseteq \mathfrak{h} \to \mathfrak{h}$ is self-adjoint, i.e $\tilde{\Xi}^* = \tilde{\Xi}$; by a slight abuse of terminology we say that Ξ is self-adjoint (resp. symmetric) whenever $\Xi = \Xi'$ (resp. $\Xi \subset \Xi'$).

For any $z \in \rho(A)$ we define $R_z \in \mathsf{B}(\mathscr{H}, \mathrm{dom}(A))$ and $G_z \in \mathsf{B}(\mathfrak{h}', \mathscr{H})$ by

$$R_z := (-A + z)^{-1}\,, \qquad G_z : \mathfrak{h}' \to \mathscr{H}\,, \quad G_z := (\tau R_{\bar z})'\,,$$

i.e.

$$\forall \phi \in \mathfrak{h}'\,, \forall u \in \mathscr{H}\,, \quad \langle G_z\phi, u\rangle_{\mathscr{H}} = \langle \phi, \tau(-A + \bar z)^{-1}u\rangle_{\mathfrak{h}'\mathfrak{h}}\,. \tag{2.1}$$

By our hypotheses on the map τ one gets (see [56, Remark 2.9])

$$\mathrm{ran}(G_z) \cap \mathrm{dom}(A) = \{0\}\,, \tag{2.2}$$

and (see [56, Lemma 2.1])

$$(z - w)\, R_w G_z = G_w - G_z\,, \tag{2.3}$$

so that

$$\mathrm{ran}(G_w - G_z) \subseteq \mathrm{dom}(A) \tag{2.4}$$

and

$$A(G_z - G_w) = z\, G_z - w\, G_w\,. \tag{2.5}$$

Now, in order to simplify the exposition and since such an hypothesis holds true in the applications further considered, we suppose that A has a spectral gap, i.e.

$$\rho(A) \cap \mathbb{R} \neq \emptyset\,.$$

Then we pose

$$G := G_{\lambda_\circ}\,, \quad \lambda_\circ \in \rho(A) \cap \mathbb{R}$$

and define

$$M_z := \tau(G - G_z) \equiv (z - \lambda_\circ) G' G_z : \mathfrak{h}' \to \mathfrak{h}\,. \tag{2.6}$$

Given an orthogonal projection $\Pi : \mathfrak{h} \to \mathfrak{h}$, the dual map $\Pi' : \mathfrak{h}' \to \mathfrak{h}'$ is an orthogonal projection in $\mathfrak{h}'$ (since $\Pi' = J\Pi J^{-1}$ and J is unitary) and, by [5, Proposition 3.5.1], $\mathrm{ran}(\Pi)' = \mathfrak{h}'/\mathrm{ran}(\Pi')^\perp = \mathrm{ran}(\Pi')$ and $\mathrm{ran}(\Pi')' = \mathfrak{h}/\mathrm{ran}(\Pi)^\perp = \mathrm{ran}(\Pi)$. Thus, for a densely defined linear map $\Xi : \mathrm{dom}(\Xi) \subseteq \mathrm{ran}(\Pi') \to \mathrm{ran}(\Pi)$, one has $\Xi' : \mathrm{dom}(\Xi') \subseteq \mathrm{ran}(\Pi') \to \mathrm{ran}(\Pi)$. As in the case $\mathrm{ran}(\Pi) = \mathfrak{h}$ we say that Ξ is symmetric whenever $\Xi \subset \Xi'$ and that is self-adjoint whenever $\Xi = \Xi'$; one has that $\Xi = \Xi'$ (resp. $\Xi \subseteq \Xi'$) if and only if $\tilde{\Xi} = \tilde{\Xi}^*$ (resp. $\tilde{\Xi} \subseteq \tilde{\Xi}^*$), where $\tilde{\Xi} := \Xi J$, $\mathrm{dom}(\tilde{\Xi}) := J^{-1}(\mathrm{dom}(\Xi))$. Finally, given a self-adjoint $\Theta : \mathrm{dom}(\Theta) \subseteq \mathrm{ran}(\Pi') \to \mathrm{ran}(\Pi)$, we define

$$Z_{\Pi,\Theta} := \{z \in \rho(A) : \Theta + \Pi M_z \Pi' \text{ has a bounded inverse on } \mathrm{ran}(\Pi) \text{ to } \mathrm{ran}(\Pi')\}\,.$$

Theorem 2.1. *Let $\Pi:\mathfrak{h}\to\mathfrak{h}$ be a orthogonal projection and let $\Theta:\mathrm{dom}(\Theta)\subseteq\mathrm{ran}(\Pi')\to\mathrm{ran}(\Pi)$ be a self-adjoint operator. Any self-adjoint extension of $S=A|\ker(\tau)$ is of the kind $A_{\Pi,\Theta}$, where*

$$\mathrm{dom}(A_{\Pi,\Theta}):=\{u=u_\circ+G\phi\,,\ u_\circ\in\mathrm{dom}(A)\,,\ \phi\in\mathrm{dom}(\Theta)\,,\ \Pi\tau u_\circ=\Theta\phi\}\,,$$

$$A_{\Pi,\Theta}u:=Au_\circ+\lambda_\circ G\phi\,.$$

Moreover $Z_{\Pi,\Theta}$ is not void, $\mathbb{C}\backslash\mathbb{R}\subseteq Z_{\Pi,\Theta}\subseteq\rho(A_{\Pi,\Theta})$, and the resolvent of the self-adjoint extension $A_{\Pi,\Theta}$ is given by the Kreĭn's type formula

$$(-A_{\Pi,\Theta}+z)^{-1}=R_z+G_z\Pi'(\Theta+\Pi M_z\Pi')^{-1}\Pi\tau R_z\,,\quad z\in Z_{\Pi,\Theta}\,. \tag{2.7}$$

Proof. Let us pose $\tilde\Theta:=\Theta J:J^{-1}(\mathrm{dom}(\Theta))\subseteq\mathrm{ran}(\Pi)\to\mathrm{ran}(\Pi)$, $\tilde G_z:=G_zJ:\mathfrak{h}\to\mathscr{H}$ and $\tilde M_z:=M_zJ:\mathfrak{h}\to\mathfrak{h}$. Thus $\tilde\Theta$ is self-adjoint in $\mathrm{ran}(\Pi)$,

$$\begin{aligned}G_z\Pi'(\Theta+\Pi M_z\Pi')^{-1}\Pi\tau R_z=&G_zJ\Pi J^{-1}((\Theta J+\Pi\tilde M_z\Pi)J^{-1})^{-1}\Pi\tau R_z\\=&\tilde G_z\Pi(\tilde\Theta+\Pi\tilde M_z\Pi)^{-1}\Pi\tau G_z\end{aligned}$$

and $Z_{\Pi,\Theta}=\{z\in\rho(A):0\in\rho(\tilde\Theta+\Pi\tilde M_z\Pi)\}$. Therefore, by [59, Theorem 2.1] (see Remark 2.2 below), the linear operator

$$\hat A_{\Pi,\Theta}:\mathrm{dom}(\tilde A_{\Pi,\Theta})\subseteq\mathscr{H}\to\mathscr{H}\,,\quad(-\hat A_{\Pi,\Theta}+z):=(-A+z)u_z\,,$$

$$\mathrm{dom}(\hat A_{\Pi,\Theta})=\{u=u_z+G_z\Pi'(\Theta+\Pi M_z\Pi')^{-1}\Pi\tau u_z,\ u_z\in\mathrm{dom}(A)\,,\ z\in Z_{\Pi,\Theta}\}$$

is a z-independent self-adjoint extension of $A|\ker(\tau)$; moreover its resolvent is given by (2.7). Let us now show that $\hat A_{\Pi,\Theta}=A_{\Pi,\Theta}$. At first we pose $\phi_z:=(\Theta+\Pi M_z\Pi')^{-1}\Pi\tau u_z$, so that, since the definition of $\hat A_{\Pi,\Theta}$ is z-independent, $u\in\mathrm{dom}(\hat A_\Theta)$ if and only if for any $z\in Z_{\Pi,\Theta}$ there exists $u_z\in\mathrm{dom}(A_\circ)$, $\Pi\tau u_z=(\Theta+\Pi M_z\Pi)\phi_z$, such that $u=u_z+G_z\phi_z$. Therefore, by (2.3),

$$u_z-u_w=G_w\phi_w-G_z\phi_z=G_z(\phi_w-\phi_z)+(z-w)R_zG_w\phi_w\,.$$

By (2.2), one obtains $G_z(\phi_w-\phi_z)=0$. Since G_z is injective (it is the adjoint of a surjective map), this gives $\phi_z=\phi_w$, i.e. the definition of ϕ_z is z-independent. Thus, posing $u_\circ:=u_z+(G_z-G)\phi$, one has $u=u_\circ+G\phi$, with $u_\circ\in\mathrm{dom}(A)$ and

$$\Pi\tau u_\circ=\Pi\tau u_z+\Pi\tau(G_z-G)\phi=(\Theta+\Pi M_z\Pi)\phi-\Pi M_z\Pi\phi=\Theta\phi\,.$$

Then, by (2.5),

$$A_{\Pi,\Theta}u=Au_z+zG_z\phi=Au_\circ-A(G_z-G)\phi+zG_z\phi=Au_\circ+\lambda_\circ G\phi\,.$$

Finally, by [59, Corollary 3.2] (also see [57, Theorem 4.3]), any self-adjoint extension of $A|\ker(\tau)$ is of the kind $A_{\Pi,\Theta}$ for some couple (Π,Θ). □

Remark 2.2. Let us notice that the operators denoted by G_z and Γ_z in [58] and [59] here correspond to $\tilde G_z$ and $\tilde M_z$ respectively.

Let us remark that we have not used neither the adjoint S^* nor the defect space $\ker(S^*-z)$. However these can be readily obtained:

Lemma 2.3.

$$\mathrm{dom}(S^*)=\{u=u_\circ+G\phi\,,\ u_\circ\in\mathrm{dom}(A)\,,\ \phi\in\mathfrak{h}'\}\,,$$
$$S^*u=Au_\circ+\lambda_\circ G\phi\,,$$

and

$$\ker(S^*-z)=\{G_z\phi\,,\ \phi\in\mathfrak{h}'\}\,,\quad z\in\rho(A)\,.$$

Proof. By [58, Theorem 3.1] (see Remark 2.2) one has

$$\mathrm{dom}(S^*)=\{u=u_*+\frac{1}{2}\,(G_i+G_{-i})\phi\,,\ u_*\in\mathrm{dom}(A)\,,\ \phi\in\mathfrak{h}'\}$$
$$S^*u=Au_*+\frac{i}{2}\,(G_i-G_{-i})\phi\,.$$

Thus, posing $u_\circ:=u_*+\frac{1}{2}\,(G_i-G)\phi+\frac{1}{2}\,(G_{-i}-G)\phi$, one has

$$\mathrm{dom}(S^*)=\{u=u_\circ+G\phi\,,\ u_\circ\in\mathrm{dom}(A)\,,\ \phi\in\mathfrak{h}'\}$$

and

$$S^*u=Au_\circ-\frac{1}{2}\,A(G_i-G)\phi-\frac{1}{2}\,A(G_{-i}-G)\phi+\frac{i}{2}\,(G_i-G_{-i})\phi\,.$$

By (2.5) one then obtains $S^*u=Au_\circ+\lambda_\circ G\phi$.

The vector $u=u_\circ+G\phi\in\mathrm{dom}(S^*)$ belong to $\ker(S^*-z)$ if and only if $(\lambda_\circ-z)G\phi=(-A+z)u_\circ$. This gives $u=(\lambda_\circ-z)R_zG\phi+G\phi$; by (2.3) one gets $u=G_z\phi$. □

Remark 2.4. By Lemma 2.3 and Theorem 2.1, any self-adjoint extension of S is of the kind

$$A_{\Pi,\Theta}:=S^*|\mathrm{dom}(A_{\Pi,\Theta})\,,$$
$$\mathrm{dom}(A_{\Pi,\Theta})=\{u\in\mathrm{dom}(S^*):\beta_0u\in\mathrm{dom}(\Theta)\,,\ \Pi\beta_1u=\Theta\beta_0u\}\,,$$

where

$$\beta_0:\mathrm{dom}(S^*)\to\mathfrak{h}'\,,\quad \beta_0u:=\phi\,.$$
$$\beta_1:\mathrm{dom}(S^*)\to\mathfrak{h}\,,\quad \beta_1u:=\tau u_\circ\,,$$

Moreover (using [58, Theorem 3.1]) one has the abstract Green's identity

$$\langle S^*u,v\rangle_{\mathscr{H}}-\langle u,S^*v\rangle_{\mathscr{H}}=\langle\beta_1u,\beta_0v\rangle_{\mathfrak{h}\mathfrak{h}'}-\langle\beta_0u,\beta_1v\rangle_{\mathfrak{h}'\mathfrak{h}}$$

Let us also notice that $G_z\phi$ solves the adjoint (abstract) boundary value problem

$$\begin{cases}S^*u=z\,u\\ \beta_0u=\phi\,.\end{cases}$$

Remark 2.5. By [58, Theorem 3.1], the triple $(\mathfrak{h},\Gamma_1,\Gamma_2)$, where $\Gamma_1:=-J^{-1}\beta_0$ and $\Gamma_2:=\beta_1$ is a boundary triplet for S^*, i.e. Γ_1 and Γ_2 are surjective and

$$\langle S^*u,v\rangle_{\mathscr{H}}-\langle u,S^*v\rangle_{\mathscr{H}}=\langle\Gamma_1u,\Gamma_2v\rangle_{\mathfrak{h}}-\langle\Gamma_2u,\Gamma_1v\rangle_{\mathfrak{h}}$$

holds true. The Weyl function of the boundary triple $(\mathfrak{h},\Gamma_1,\Gamma_1)$ is the bounded linear operator $M_zJ:\mathfrak{h}\to\mathfrak{h}$, where M_z is defined in (2.6) (see [58, Theorem 3.1]). For Boundary Triple Theory we refer to [18], [61] and references therein.

We conclude the section with the following result:

Lemma 2.6. *Given the linear operator* $\Xi : \mathrm{dom}(\Xi) \subseteq \mathrm{ran}(\Pi') \to \mathrm{ran}(\Pi)$*, let us define the linear operator* $A_{\Pi,\Xi}$ *by*

$$A_{\Pi,\Xi} := S^*|\mathrm{dom}(A_{\Pi,\Xi}) , \quad \mathrm{dom}(A_{\Pi,\Xi}) := \{u \in \mathrm{dom}(S^*) : \beta_0 u \in \mathrm{dom}(\Xi) , \ \Pi\beta_1 u = \Xi\beta_0 u\} .$$

Then $A_{\Pi,\Xi}$ *is self-adjoint if and only if* Ξ *is self-adjoint. In the particular case* $\Pi = 1$ *and* Ξ *self-adjoint,* Ξ *is semibounded whenever* $A_{1,\Xi}$ *is semibounded.*

Proof. Since we can re-write $\mathrm{dom}(A_{\Pi,\Xi})$ as

$$\mathrm{dom}(A_{\Pi,\Xi}) = \{u \in \mathrm{dom}(S^*) : \Gamma_1 u \in J^{-1}(\mathrm{dom}(\Xi)) , \ \Pi\Gamma_2 u = -\Xi J\Gamma_1 u\} ,$$

and since $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ is a boundary triple, by e.g. [61, Lemma 14.6 and Theorem 14.7] one obtains that $A_{\Pi,\Xi}$ is self-adjoint if and only if $\Xi J : J^{-1}(\mathrm{dom}(\Xi)) \subseteq \mathrm{ran}(\Pi)) \to \mathrm{ran}(\Pi))$ is self-adjoint. The latter is equivalent to Ξ self-adjoint. Semiboundedness of ΞJ (and hence of Ξ) is consequence of Theorem 3 in [18]. □

3. Preliminaries: Sobolev spaces and boundary-layer operators

3.1. **Sobolev spaces and trace maps.** Let Ω be a non-empty open subset of $\mathbb{R}^n$; $H^k(\Omega)$, $k \in \mathbb{N}$, denotes the usual Sobolev-Hilbert spaces $H^k(\Omega) := \{u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega) , \ |\alpha| \leq k\}$, where $\partial^\alpha : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ denotes the distributional partial derivatives of order $|\alpha|$. In the case $\Omega = \mathbb{R}^n$, the scale of Sobolev-Hilbert spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is defined by $H^s(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\hat{u}(\kappa)|^2(|\kappa|^2 + 1)^s d\kappa < +\infty\}$, where $\mathscr{S}'(\mathbb{R}^n)$ is the space of tempered distributions and $\hat{u}$ denotes Fourier's transform. $H^{-s}(\mathbb{R}^n)$ identifies with the dual space of $H^s(\mathbb{R}^n)$; we denote by $\langle\cdot,\cdot\rangle_{-s,s}$ the H^{-s}-H^s duality pairing.

Let us now suppose that Ω is bounded and let Γ denote its boundary: $\Gamma = \partial\Omega$. We further suppose that $\Omega \subset \mathbb{R}^n$ is of class $\mathcal{C}^{k,1}$, $k \geq 0$, i.e we suppose that its boundary Γ is a manifold of dimension $n-1$ whose local maps are Lipschitz-continuous, together with their inverses, up to the order k. In the particular case $\mathcal{C}^{0,1}$, Ω is referred to as a Lipschitz domain. The scale of Sobolev-Hilbert space $H^s(\Omega)$, $s \in \mathbb{R}$, is then defined by $H^s(\Omega) := \{u|\Omega : u \in H^s(\mathbb{R}^n)\}$, $u|\Omega$ denoting the restriction of u to Ω. In the case $s \in \mathbb{N}$, this definition reproduces the previous one (see e.g. [36, 1.4.3.1]).

The Sobolev spaces of L^2-functions on Γ, next denoted with $H^s(\Gamma)$, are defined by using an atlas of Γ and the Sobolev space on flat, open, bounded, $(n-1)$-dimensional domains (see e.g. [36, Section 1.3.3], [53, Chapter 3], [63, Section 3.1]); they are well defined up to the order $|s| = k+1$, and $H^{-s}(\Gamma)$ identifies with the dual space of $H^s(\Gamma)$. We denote by $\langle\cdot,\cdot\rangle_{-s,s}$ the H^{-s}-H^s duality pairing. If Γ is a $\mathcal{C}^2$ manifold, considering the Riemannian structure inherited from $\mathbb{R}^n$, we have that $H^s(\Gamma)$ identifies with $\mathrm{dom}((-\Delta_\Gamma)^{s/2})$ with respect of the scalar product

$$\langle\phi, \varphi\rangle_{H^s(\Gamma)} := \langle\Lambda^s\phi, \Lambda^s\varphi\rangle_{L^2(\Gamma)} , \quad \Lambda := (-\Delta_\Gamma + 1)^{\frac{1}{2}} , \tag{3.1}$$

being Δ_Γ the self-adjoint operator in $L^2(\Gamma)$ corresponding to the Laplace-Beltrami operator on the complete Riemannian manifold Γ (see e.g. [54, Remark 7.6, Chapter 1]). According to this definition, Λ^r is self-adjoint in $H^s(\Gamma)$ with domain $H^{s+r}(\Gamma)$ and acts as a unitary map $\Lambda^r : H^s(\Gamma) \to H^{s-r}(\Gamma)$. In particular $\Lambda^{2s} : H^s(\Gamma) \to H^{-s}(\Gamma)$ plays the role of the duality mapping J introduced in Section 2, and one has $\langle\phi, \psi\rangle_{-s,s} = \langle\Lambda^{-s}\phi, \Lambda^s\psi\rangle_{L^2(\Gamma)}$. In the case Γ is not $\mathcal{C}^2$ one can use the definition (which works in the case Ω is of class $\mathcal{C}^{0,1}$)

$\Delta_\Gamma := \mathrm{Div}\circ\nabla_{\tan}$ provided in [35, Theorem 1.2]. The two definitions coincide in the case Γ is $\mathcal{C}^2$ (see [35, Section 7]).

Denoting by $0=\lambda_0<\lambda_1\le\cdots\le\lambda_k\le\lambda_{k+1}\le\ldots$, the increasing sequence of the eigenvalues of the self-adjoint operator $-\Delta_\Gamma : \mathrm{dom}(-\Delta_\Gamma)\subset L^2(\Gamma)\to L^2(\Gamma)$ and by $\{\varphi_k\}_{k=1}^\infty$ the corresponding normalized eigenfunctions, one has

$$\|\phi\|^2_{H^s(\Gamma)}=\sum_{k=0}^{\infty}(\lambda_k+1)^s|\langle\phi,\varphi_k\rangle_{L^2(\Gamma)}|^2\,.$$

Thus, given $r<s<t$, for any $\epsilon>0$ there exists $c_\epsilon>0$, $c_\epsilon\uparrow\infty$ as $\epsilon\downarrow 0$, such that

$$\|\phi\|_{H^s(\Gamma)}\le\epsilon\,\|\phi\|_{H^t(\Gamma)}+c_\epsilon\|\phi\|_{H^r(\Gamma)}\,. \tag{3.2}$$

In the sequel, we shall also use some closed subspaces of $H^s(\Gamma)$: given $\Sigma\subset\Gamma$ and $s>0$, we define

$$H^s_\Sigma(\Gamma):=\{\phi\in H^s(\Gamma):\mathrm{supp}(\phi)\subseteq\Sigma\}\equiv H^s(\Gamma)\cap L^2_\Sigma(\Gamma)\,,$$

where $L^2_\Sigma(\Gamma)$ denotes the space of square-integrable functions with essential support contained in Σ, and

$$H^{-s}_\Sigma(\Gamma):=\{\phi\in H^{-s}(\Gamma):\langle\phi,\psi\rangle_{-s,s}=0,\ \text{for any }\psi\in H^s_{\Sigma^c}(\Gamma)\}\,.$$

Therefore

$$\forall\phi\in H^{-s}_\Sigma(\Gamma),\,\forall\varphi\in H^s_{\Sigma^c}(\Gamma)\,,\quad\langle\phi,\varphi\rangle_{-s,s}=0\,. \tag{3.3}$$

If $\Sigma\subset\Gamma$ is relatively open, then $H^s(\Sigma)$, $|s|\le k+1$, is constructed using an atlas of Σ and the Sobolev spaces on flat $(n-1)$-dimensional domains. If moreover Σ is of class $\mathcal{C}^{0,1}$, i.e. if its boundary is a Lipschitz manifold, then a continuation map allows the identification $H^s(\Sigma)=\{\phi|\Sigma:\phi\in H^s(\Gamma)\}$ (see e.g. [36, Theorem 1.4.3.1], [16, Definition 3.6]); moreover one has the identifications $H^s(\Sigma)'=H^{-s}_\Sigma(\Gamma)$ and $H^s_\Sigma(\Gamma)'=H^{-s}(\Sigma)$ (see [16, Proposition 3.5 and remarks at page 111]). By the continuos (with dense range) embeddings $H^s(\Gamma)\hookrightarrow H^r(\Gamma)$, $r<s$, one gets

$$H^r_\Sigma(\Gamma)\cap H^s(\Gamma)=H^s_\Sigma(\Gamma)\,,\quad H^s_\Sigma(\Gamma)\subset H^r_\Sigma(\Gamma)\ \text{(dense inclusion)}\,.$$

We shall also need the Hilbert orthogonal

$$H^s_\Sigma(\Gamma)^\perp=\{\phi\in H^s(\Gamma):\langle\Lambda^{2s}\phi,\psi\rangle_{-s,s}=0,\ \text{for any }\psi\in H^s_\Sigma(\Gamma)\}=\Lambda^{-2s}H^{-s}_{\Sigma^c}(\Gamma)\,. \tag{3.4}$$

For a bounded open domain Ω of class $\mathcal{C}^{k,1}$, we pose

$$\Omega_+:=\mathbb{R}^n\backslash\overline{\Omega}\,,\qquad\Omega_-:=\Omega\,,$$

while ν denotes the outward normal vector on Γ. The one-sided, zero-order, trace operators $\gamma_0^\pm$ act on a smooth function $u\in\mathcal{C}^\infty\left(\overline{\Omega}_\pm\right)$ as $\gamma_0^\pm u=u|\Gamma$, where $\varphi|\Gamma$ is the restriction to Γ. These maps uniquely extend to bounded linear operators (see e.g. [53, Theorem 3.37])

$$\gamma_0^\pm\in\mathsf{B}(H^s(\Omega_\pm),H^{s-\frac12}(\Gamma))\,,\qquad\frac12<s\le k+1\,. \tag{3.5}$$

Then, given $a_{ij}\in\mathcal{C}^\infty(\mathbb{R}^n)$, $a_{ij}(x)=a_{ji}(x)$ such that

$$\forall x,\xi\in\mathbb{R}^n\,,\quad\sum_{1\le i,j\le n}a_{ij}(x)\xi_i\xi_j\ge c_\circ|\xi|^2\,,\quad c_\circ>0\,, \tag{3.6}$$

we define one-sided, first-order, trace operators

$$\gamma_1^\pm \in \mathsf{B}(H^s(\Omega_\pm), H^{s-\frac{3}{2}}(\Gamma)), \qquad \frac{3}{2} < s \le k+1, \tag{3.7}$$

by the zero-order trace of the co-normal derivative:

$$\gamma_1^\pm u := \sum_{1\le i,j\le n} \nu_i \gamma_0^\pm (a_{ij}\partial_{x_j} u) \tag{3.8}$$

Using the maps $\gamma_0^\pm$ and $\gamma_1^\pm$ we define the two-sided, bounded, trace operators

$$\gamma_0 : H^s(\Omega_-) \oplus H^s(\Omega_+) \to H^{s-1/2}(\Gamma), \quad \gamma_0(u_- \oplus u_+) := \frac{1}{2}(\gamma_0^+ u_+ + \gamma_0^- u_-),$$

$$\gamma_1 : H^s(\Omega_-) \oplus H^s(\Omega_+) \to H^{s-3/2}(\Gamma), \quad \gamma_1(u_- \oplus u_+) := \frac{1}{2}(\gamma_1^+ u_+ + \gamma_1^- u_-)$$

and

$$[\gamma_0] : H^s(\Omega_-) \oplus H^s(\Omega_+) \to H^{s-1/2}(\Gamma), \quad [\gamma_0](u_- \oplus u_+) := \gamma_0^+ u_+ - \gamma_0^- u_-,$$

$$[\gamma_1] : H^s(\Omega_-) \oplus H^s(\Omega_+) \to H^{s-3/2}(\Gamma), \quad [\gamma_1](u_- \oplus u_+) := \gamma_1^+ u_+ - \gamma_1^- u_-.$$

Notice that $u_- \oplus u_+ \in H^1(\Omega_-) \oplus H^1(\Omega_+)$ belongs to $H^1(\mathbb{R}^n)$ if and only if

$$[\gamma_0](u_- \oplus u_+) = 0$$

and $u_- \oplus u_+ \in H^2(\Omega_-) \oplus H^2(\Omega_+)$ belongs to $H^2(\mathbb{R}^n)$ if and only if

$$[\gamma_0](u_- \oplus u_+) = [\gamma_1](u_- \oplus u_+) = 0.$$

Hence

$$H^1(\mathbb{R}^n) = (H^1(\Omega_-) \oplus H^1(\Omega_+)) \cap \ker([\gamma_0]), \tag{3.9}$$

$$H^2(\mathbb{R}^n) = (H^2(\Omega_-) \oplus H^2(\Omega_+)) \cap \ker([\gamma_0]) \cap \ker([\gamma_1]). \tag{3.10}$$

3.2. **Boundary-layer operators.** In what follows A denotes the 2nd order, symmetric, elliptic partial differential operator

$$A : \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n), \quad Au := \sum_{1\le i,j\le n} \partial_{x_i}(a_{ij}\partial_{x_j} u) - Vu,$$

where we suppose a_{ij}, $\partial_{x_i} a_{ij}$, $V \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, $a_{ij}(x) = a_{ji}(x)$ and that (3.6) holds true. When restricted to $H^1(\mathbb{R}^n)$, A provides a bounded operator in $\mathsf{B}(H^1(\mathbb{R}^n), H^{-1}(\mathbb{R}^n))$ by the identity

$$\langle Au, v\rangle_{-1,1} = -\sum_{1\le i,j\le n} \langle a_{ij}\partial_{x_i} u, \partial_{x_j} v\rangle_{L^2(\mathbb{R}^n)} + \langle Vu, v\rangle_{L^2(\mathbb{R}^n)}.$$

Moreover the sesquilinear form

$$F : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to \mathbb{R}, \quad F(u,v) := -\langle Au, v\rangle_{-1,1}$$

satisfies

$$\forall u \in H^1(\mathbb{R}^n), \quad F(u,u) \ge c_\circ \|\nabla u\|^2_{H^1(\mathbb{R}^n)} - \|V_{\text{neg}}\|_\infty \|u\|^2_{L^2(\mathbb{R}^n)}, \tag{3.11}$$

where $V_{\rm neg}$ denotes the negative part of V; therefore F is closed and semi-bounded. By (7.1), the corresponding self-adjoint operator is then given by the restriction of A to the domain D_A,

$$D_A := \{u \in H^1(\mathbb{R}^n) : Au \in L^2(\mathbb{R}^n)\} \equiv H^2(\mathbb{R}^n)\,.$$

Thus $A : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is self-adjoint, $(-A+z)^{-1} \in \mathsf{B}(L^2(\mathbb{R}^n), H^2(\mathbb{R}^n))$ for any $z \in \rho(A)$ and $(\|V_{\rm neg}\|_\infty, +\infty) \subset \rho(A)$. By (3.11), for any $\lambda > \|V_{\rm neg}\|_\infty$ one obtains

$$\forall u \in H^1(\mathbb{R}^n)\,, \quad \|(-A+\lambda)u\|_{H^{-1}(\mathbb{R}^n)} \ge c\,\|u\|_{H^1(\mathbb{R}^n)}\,, \tag{3.12}$$

and so

$$(-A+\lambda)^{-1} \in \mathsf{B}(H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n))\,. \tag{3.13}$$

By (3.13) and by elliptic regularity, see e.g. [37, 6.22], $(-A+\lambda) \in \mathsf{B}(H^{m+2}(\mathbb{R}^n), H^m(\mathbb{R}^n))$ is a bijection; thus, by the inverse mapping theorem, one has

$$(-A+\lambda)^{-1} \in \mathsf{B}(H^m(\mathbb{R}^n), H^{m+2}(\mathbb{R}^n)\,, \quad m \ge -1\,. \tag{3.14}$$

Given the bounded open set $\Omega \subset \mathbb{R}^n$ of class $\mathcal{C}^{k,1}$, $k \ge 0$, the single and double-layer operators

$$SL_z : H^{-3/2}(\Gamma) \to L^2(\mathbb{R}^n)\,, \qquad DL_z : H^{-1/2}(\Gamma) \to L^2(\mathbb{R}^n)\,,$$

related to A and Γ are defined by

$$\langle SL_z\phi, u\rangle_{L^2(\mathbb{R}^n)} := \langle \phi, \gamma_0(-A+\bar z)^{-1}u\rangle_{-\frac{3}{2},\frac{3}{2}}\,, \quad u \in L^2(\mathbb{R}^n)\,, \tag{3.15}$$

$$\langle DL_z\varphi, u\rangle_{L^2(\mathbb{R}^n)} := \langle \varphi, \gamma_1(-A+\bar z)^{-1}u\rangle_{-\frac{1}{2},\frac{1}{2}}\,, \quad u \in L^2(\mathbb{R}^n)\,. \tag{3.16}$$

Let $g_z(x,y)$ be the integral kernel of the resolvent $(-A+z)^{-1}$; it is a smooth function for $x \ne y$ (see e.g. [53, Lemma 6.3]). Therefore (3.15) and (3.16) give, if $x \notin \Gamma$ and $\phi \in L^2(\Gamma)$,

$$SL_z\phi(x) = \int_\Gamma g_z(x,y)\,\phi(y)\,d\sigma_\Gamma(y)\,, \tag{3.17}$$

and

$$DL_z\phi = \sum_{1\le i,j\le n} \int_\Gamma \nu_i(y)a_{ij}(y)\partial_{x_j}g_z(x,y)\,\varphi(y)\,d\sigma_\Gamma(y)\,, \tag{3.18}$$

where σ_Γ denotes the surface measure. We need the following mapping properties:

Lemma 3.1. *For any $\lambda > \|V_{neg}\|_\infty$ one has*

$$SL_\lambda \in \mathsf{B}(H^{-1/2}(\Gamma), H^1(\mathbb{R}^n))\,, \qquad DL_\lambda \in \mathsf{B}(H^{1/2}(\Gamma), H^1(\Omega_\pm))$$

Proof. By (3.13), (3.15) and by $\gamma_0 \in \mathsf{B}(H^1(\mathbb{R}^n), H^{1/2}(\Gamma))$, if $\phi \in H^{-1/2}(\Gamma)$ then

$$|\langle SL_\lambda\phi, u\rangle_{L^2(\mathbb{R}^n)}| \le \|\phi\|_{H^{-1/2}(\Gamma)}\|\gamma_0(-A+\lambda)^{-1}u\|_{H^{1/2}(\Gamma)} \le c\,\|\phi\|_{H^{-1/2}(\Gamma)}\|u\|_{H^{-1}(\mathbb{R}^n)}\,.$$

By [53, Lemma 4.3] and [53, Theorem 3.30] one has

$$\|\gamma_1(-A+\lambda)^{-1}u\|_{H^{-1/2}(\Gamma)} \le c\left(\|(-A+\lambda)^{-1}u\|_{H^1(\Omega_\pm)} + \|u\|_{H^1(\Omega_\pm)'}\right)\,.$$

Thus, by (3.13) and (3.16), if $\varphi \in H^{1/2}(\Gamma)$ there follows

$$|\langle DL_\lambda\varphi, u\rangle_{L^2(\Omega_\pm)}| \le \|\varphi\|_{H^{1/2}(\Gamma)}\|\gamma_1(-A+\lambda)^{-1}1_{\Omega_\pm}u\|_{H^{-1/2}(\Gamma)} \le c\,\|\phi\|_{H^{1/2}(\Gamma)}\|u\|_{H^1(\Omega_\pm)'}\,.$$

□

Since $g_z(x,y)$ is a smooth function for $x \neq y$, one has $SL_z\phi$, $DL_z\phi \in \mathcal{C}^\infty(\mathbb{R}^n\backslash\Gamma)$ and (see [53, eqs. (6.18) and (6.19)])

$$\forall x \notin \Gamma\,, \qquad A\,SL_z\phi(x) = z\,SL_z\phi(x)\,, \quad A\,DL_z\phi(x) = z\,DL_z\varphi(x)\,. \tag{3.19}$$

Therefore, setting

$$SL_z^\pm\phi := SL_z\phi|\Omega_\pm\,, \quad DL_z^\pm\varphi := DL_z\varphi|\Omega_\pm\,,$$

one has

$$SL_z^\pm\phi \in \ker(A_\pm^{\max} - z)\,, \qquad DL_z^\pm\varphi \in \ker(A_\pm^{\max} - z)\,, \tag{3.20}$$

where

$$A_\pm^{\max} = A|\mathrm{dom}(A_\pm^{\max})\,, \quad \mathrm{dom}(A_\pm^{\max}) := \{u_\pm \in L^2(\Omega_\pm) : Au_\pm \in L^2(\Omega_\pm)\}\,. \tag{3.21}$$

In the case Ω is of class $\mathcal{C}^{1,1}$, by proceeding as in the proof of theorem 6.5 in [54, Chapter 6] (see the comment in [36] before Theorem 1.5.3.4), the maps $\gamma_0^\pm$ and $\gamma_1^\pm$ can be extended to

$$\hat\gamma_0^\pm \in \mathsf{B}(\mathrm{dom}(A_\pm^{\max}), H^{-1/2}(\Gamma))\,, \quad \hat\gamma_1^\pm \in \mathsf{B}(\mathrm{dom}(A_\pm^{\max}), H^{-3/2}(\Gamma))$$

(here $\mathrm{dom}(A^\pm_{\max})$ has the graph norm), which in turn provide us with the bounded maps

$$\hat\gamma_0 : \mathrm{dom}(A_-^{\max}) \oplus \mathrm{dom}(A_+^{\max}) \to H^{-1/2}(\Gamma)\,, \quad \hat\gamma_0(u_- \oplus u_+) := \frac{1}{2}(\hat\gamma_0^+ u_+ + \hat\gamma_0^- u_-)\,,$$

$$\hat\gamma_1 : \mathrm{dom}(A_-^{\max}) \oplus \mathrm{dom}(A_+^{\max}) \to H^{-3/2}(\Gamma)\,, \quad \hat\gamma_1(u_- \oplus u_+) := \frac{1}{2}(\hat\gamma_1^+ u_+ + \hat\gamma_1^- u_-)$$

and

$$[\hat\gamma_0] : \mathrm{dom}(A_-^{\max}) \oplus \mathrm{dom}(A_+^{\max}) \to H^{-1/2}(\Gamma)\,, \quad [\hat\gamma_0](u_- \oplus u_+) := \hat\gamma_0^+ u_+ - \hat\gamma_0^- u_-\,,$$

$$[\hat\gamma_1] : \mathrm{dom}(A_-^{\max}) \oplus \mathrm{dom}(A_+^{\max}) \to H^{-3/2}(\Gamma)\,, \quad [\hat\gamma_1](u_- \oplus u_+) := \hat\gamma_1^+ u_+ - \hat\gamma_1^- u_-\,.$$

These maps, together with [53, Theorem 7.2], give, whenever Ω is of class $\mathcal{C}^{k,1}$, $k \geq 1$, and for any $|s| \leq k$, the bounded operators

$$\hat\gamma_0^\pm SL_z \in \mathsf{B}(H^{s-1/2}(\Gamma), H^{s+1/2}(\Gamma))\,, \qquad \hat\gamma_1^\pm SL_z \in \mathsf{B}(H^{s-1/2}(\Gamma), H^{s-1/2}(\Gamma))\,, \tag{3.22}$$

$$\hat\gamma_0^\pm DL_z \in \mathsf{B}(H^{s+1/2}(\Gamma), H^{s+1/2}(\Gamma))\,, \qquad \hat\gamma_1^\pm DL_z \in \mathsf{B}(H^{s+1/2}(\Gamma), H^{s-1/2}(\Gamma))\,. \tag{3.23}$$

Moreover the single and double layer operators satisfy the jump relations

$$[\hat\gamma_0]SL_z\phi = [\hat\gamma_1]DL_z\phi = 0\,, \quad [\hat\gamma_1]SL_z\phi = -[\hat\gamma_0]DL_z\phi = -\phi\,. \tag{3.24}$$

By the first relation in (3.24) one gets

$$\hat\gamma_0^\pm SL_z = \hat\gamma_0 SL_z\,, \qquad \hat\gamma_1^\pm DL_z = \hat\gamma_1 DL_z\,.$$

Notice that in (3.22), (3.23) and (3.24) the extended trace operators coincide with the usual ones whenever the range spaces are Sobolev space on Γ of strictly positive index.

For any $\lambda \in \rho(A) \cap \mathbb{R}$ both the bounded operators $\gamma_0 SL_\lambda$ and $\hat\gamma_1 DL_\lambda$ are symmetric w.r.t. the $H^{-1/2}(\Gamma)$-$H^{1/2}(\Gamma)$ pairing (see [53, Theorems 6.15 and 6.17]:

$$\forall \phi, \varphi \in H^{-1/2}(\Gamma)\,, \quad \langle \phi, \gamma_0 SL_\lambda\varphi\rangle_{-\frac{1}{2},\frac{1}{2}} = \langle \gamma_0 SL_\lambda\phi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}}\,.$$

$$\forall \phi, \varphi \in H^{1/2}(\Gamma)\,, \quad \langle \phi, \hat\gamma_1 DL_\lambda\varphi\rangle_{\frac{1}{2},-\frac{1}{2}} = \langle \hat\gamma_1 DL_\lambda\phi, \varphi\rangle_{-\frac{1}{2},\frac{1}{2}}\,.$$

Moreover these operators are coercive:

Lemma 3.2. *Let $\lambda > \max(V_{neg})$. Then there exist $c_0 > 0$ and $c_1 > 0$ such that*

$$\forall\phi \in H^{-1/2}(\Gamma)\,, \quad \langle \phi, \gamma_0 SL_\lambda \phi\rangle_{-\frac12,\frac12} \ge c_0\, \|\phi\|^2_{H^{-1/2}(\Gamma)} \tag{3.25}$$

and

$$\forall\varphi \in H^{1/2}(\Gamma)\,, \quad -\langle \hat\gamma_1 DL_\lambda \varphi, \varphi\rangle_{-\frac12,\frac12} \ge c_1\, \|\varphi\|^2_{H^{1/2}(\Gamma)}\,. \tag{3.26}$$

Proof. In the case $A = \Delta$ and $\lambda = 1$, the proof is given in [46, Lemma 1.14 (c)] as regards $\hat\gamma_0 SL_\lambda$ and in [46, Theorem 1.26 (e)] as regards $\hat\gamma_1 DL_\lambda$. Here we provide an alternative proof which adapts to our hypotheses.

By [53, Lemma 4.3] for any $u_\pm \in \mathrm{dom}(A^{\max}_\pm) \cap H^1(\Omega_\pm)$ one has $\hat\gamma_1^\pm u_\pm \in H^{-\frac12}(\Gamma)$ and

$$\|\hat\gamma_1^\pm u_\pm\|^2_{H^{-1/2}(\Gamma)} \le c\left(\|u_\pm\|^2_{H^1(\Omega_\pm)} + \|A^{\max}_\pm u_\pm\|^2_{H^{-1}(\Omega_\pm)}\right)\,; \tag{3.27}$$

moreover for such a $u_\pm$ and for any $v_\pm \in H^1(\Omega_\pm)$ the "half" Green's formula holds (see [53, Theorem 4.4]):

$$\begin{aligned}&\langle(-A^{\max}_\pm + \lambda)u_\pm, v_\pm\rangle_{L^2(\Omega_\pm)}\\ =&\sum_{1\le i,j\le n}\langle a_{ij}\partial_i u_\pm, \partial_j v_\pm\rangle_{L^2(\Omega_\pm)} + \langle (V+\lambda)u_\pm, v_\pm\rangle_{L^2(\Omega_\pm)} \pm \langle \hat\gamma_1^\pm u_\pm, \gamma_0^\pm v_\pm\rangle_{-\frac12,\frac12}\,.\end{aligned}$$

Thus, posing $u_\pm = v_\pm = SL^\pm_\lambda\phi$, by $(-A^\pm_{\max} + \lambda)u_\pm = 0$ and by (3.24), one gets

$$\begin{aligned}0=&\sum_{1\le i,j\le n}\langle a_{ij}\partial_i SL_\lambda\phi, \partial_j SL_\lambda\phi\rangle_{L^2(\Omega)} + \langle (V+\lambda)SL_\lambda\phi, SL_\lambda\phi\rangle_{L^2(\Omega)} + \langle [\hat\gamma_1]SL_\lambda\phi, \gamma_0 SL_\lambda\phi\rangle_{-\frac12,\frac12}\\ \ge& c_\circ\|\nabla SL_\lambda\phi\|^2_{L^2(\Omega)} + (\lambda - \max(V_{\rm neg}))\,\|SL_\lambda\phi\|^2_{L^2(\Omega)} - \langle\phi, \gamma_0 SL_z\phi\rangle_{-\frac12,\frac12}\,.\end{aligned}$$

Therefore, setting $\kappa_\circ := \min\{c_\circ, \lambda - \max(V_{\rm neg})\}$, one obtains

$$\langle\phi, \gamma_0 SL_\lambda\phi\rangle_{-\frac12,\frac12} \ge \kappa_\circ\,\|SL_\lambda\phi\|^2_{H^1(\Omega)}\,.$$

By $[\hat\gamma_1]SL_\lambda\phi = -\phi$, by (3.27) and by the continuous embedding $H^1(\Omega_\pm) \hookrightarrow H^{-1}(\Omega_\pm)$ one gets

$$\begin{aligned}&\|\phi\|^2_{H^{-1/2}(\Gamma)}\\ \le& c\left(\|SL^-_\lambda\phi\|^2_{H^1(\Omega_-)} + \lambda^2\|SL^-_\lambda\phi\|^2_{H^{-1}(\Omega_-)} + \|SL^+_\lambda\phi\|^2_{H^1(\Omega_+)} + \lambda^2\|SL^+_\lambda\phi\|^2_{H^{-1}(\Omega)}\right)\\ \le& c\,(1+\lambda^2)\|SL_\lambda\phi\|^2_{H^1(\Omega)}\end{aligned}$$

and so (3.25) follows by posing $c_0 = \kappa_\circ(c\,(1+\lambda^2))^{-1}$.

The proof of (3.26) proceeds along the same lines by inserting $u_\pm = v_\pm = DL^\pm_\lambda\varphi$ in the Green's formula above: in this case one obtains

$$-\langle\hat\gamma_1 DL_\lambda\varphi, \varphi\rangle_{-\frac12,\frac12} \ge \kappa_0\,\|DL_\lambda\varphi\|^2_{H^1(\Omega)}\,.$$

By $[\gamma_0]DL_\lambda\varphi = \varphi$, denoting by $N_\pm$ the norm of $\gamma_0^\pm \in \mathsf{B}(H^1(\Omega_\pm), H^{1/2}(\Gamma))$, one obtains

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} \le N_-^2\|DL^-_\lambda\varphi\|^2_{H^1(\Omega_-)} + N_+^2\|DL^+_\lambda\varphi\|^2_{H^1(\Omega_+)} \le \max\{N_-^2, N_+^2\}\|DL_\lambda\varphi\|^2_{H^1(\Omega)}\,,$$

and so (3.25) follows by posing $c_1 = \kappa_\circ(\max\{N_-^2, N_+^2\})^{-1}$. □

Lemma 3.3. *Let $\lambda > \max(V_{neg})$ and Ω be of class $\mathcal{C}^{k,1}$, $k \ge 1$. Then*

$$\hat\gamma_0 SL_\lambda \phi \in H^{s+1/2}(\Gamma) \iff \phi \in H^{s-1/2}(\Gamma)\,, \quad |s| \le k-1\,, \tag{3.28}$$

$$\hat\gamma_1 DL_\lambda \phi \in H^{s-1/2}(\Gamma) \iff \phi \in H^{s+1/2}(\Gamma)\,, \qquad |s| \le k\,. \tag{3.29}$$

Proof. By (3.22) and (3.23) we only need to prove the $\Rightarrow$ implications. By [53, Theorem 7.17], both the maps $\hat\gamma_0 SL_\lambda$ and $\hat\gamma_1 DL_\lambda$ are Fredholm with index zero and s-independent kernel. By (3.25) and (3.26) such maps are injective and therefore bijective. Thus, by (3.22), (3.23) and by the inverse mapping theorem, $(\hat\gamma_0 SL_\lambda)^{-1} \in \mathsf{B}(H^{s+1/2}(\Gamma), H^{s-1/2}(\Gamma))$ and $(\hat\gamma_1 DL_\lambda)^{-1} \in \mathsf{B}(H^{s-1/2}(\Gamma), H^{s+1/2}(\Gamma))$. □

We conclude this section with a result which shows that the mapping properties of $\hat\gamma_0 SL_\lambda$ and $\hat\gamma_1 DL_\lambda$ hold locally:

Lemma 3.4. *Let $\lambda > \max(V_{neg})$, let Ω be of class $\mathcal{C}^{k,1}$, $k \ge 1$, and let $\Sigma \subset \Gamma$ be relatively open. If $\phi \in H^{-1/2}_\Sigma(\Gamma)$, then*

$$(\hat\gamma_0 SL_\lambda \phi)|\Sigma \in H^{s+1/2}(\Sigma) \iff \phi \in H^{s-1/2}(\Gamma)\,, \quad 0 \le s \le k-1\,. \tag{3.30}$$

If $\phi \in H^{1/2}_\Sigma(\Gamma)$, then

$$(\hat\gamma_1 DL_\lambda \phi)|\Sigma \in H^{s-1/2}(\Sigma) \iff \phi \in H^{s+1/2}(\Gamma)\,, \quad 0 \le s \le k\,. \tag{3.31}$$

Proof. By (3.22) and (3.23) we only need to prove that $\phi \notin H^{s-1/2}(\Gamma)$ implies $(\hat\gamma_0 SL_\lambda \phi)|\Sigma \notin H^{s+1/2}(\Sigma)$ and that $\phi \notin H^{s+1/2}(\Gamma)$ implies $(\hat\gamma_1 DL_\lambda \phi)|\Sigma \notin H^{s-1/2}(\Sigma)$.

We begin proving (3.30). Let $\phi \notin H^{s-1/2}(\Gamma)$; since $\phi \in H^{-1/2}_\Sigma(\Gamma)$, one has $(\hat\gamma_0 SL_\lambda \phi)|\Sigma \ne 0$ by (3.25). By (3.28), one has $\hat\gamma_0 SL_\lambda \phi \notin H^{s+1/2}(\Gamma)$; thus $SL_\lambda \phi|\Omega \notin H^{s+1}(\Omega)$. Since $\operatorname{supp}(\phi) \subset \Sigma$, for any couple Ω_1, $\Omega_2 \subset \Omega$ of open bounded domains of class $\mathcal{C}^{k,1}$ such that

$$\operatorname{supp}(\phi) \subseteq \Gamma_1 \cap \Gamma \subset \Sigma\,, \quad \Sigma^c \subset \Gamma_2 \cap \Gamma \subset (\operatorname{supp}(\phi))^c\,, \quad \Gamma_1 = \partial\Omega_1\,,\ \Gamma_2 = \partial\Omega_2\,,$$

by [53, Theorem 6.13] one has $SL_\lambda \phi|\Omega_1 \notin H^{s+1}(\Omega_1)$ and $SL_\lambda \phi|\Omega_2 \in H^{s+1}(\Omega_2)$. Since $A\, SL_\lambda \phi|\Omega_i = \lambda SL_\lambda \phi|\Omega_i$, $i = 1, 2$, by elliptic regularity the zero-order trace of $SL_\lambda \phi|\Omega_1$ along Γ_1 does not belong to $H^{s+1/2}(\Gamma_1)$ and conversely the trace of $SL_\lambda \phi|\Omega_2$ along Γ_2 belongs to $H^{s+1/2}(\Gamma_2)$. In conclusion, choosing $\Omega_2 \subset \Omega_1$ in such a way that $\Gamma_1 = (\Gamma_1 \cap \Sigma) \cup (\Gamma_1 \cap \Gamma_2)$, one obtains $(\hat\gamma_0 SL_\lambda \phi)|\Sigma \notin H^{s+1/2}(\Sigma)$.

The proof of (3.31) is similar to the one for (3.30). Let $\phi \notin H^{s+1/2}(\Gamma)$; since $\phi \in H^{1/2}_\Sigma(\Gamma)$, one has $\hat\gamma_1 DL_\lambda \phi|\Sigma \ne 0$ by (3.26). By (3.29), one has $\hat\gamma_1 DL_\lambda \phi \notin H^{s-1/2}(\Gamma)$; thus $DL_\lambda \phi|\Omega \notin H^{s+1}(\Omega)$. Since $\operatorname{supp}(\Lambda\psi) \subset \Sigma$, for any couple Ω_1, $\Omega_2 \subset \Omega_-$ of open bounded domains as above, by [53, Theorem 6.13], one has $DL_\lambda \phi|\Omega_1 \notin H^{s+1}(\Omega_1)$ and $DL_\lambda \Lambda\psi|\Omega_2 \in H^{s+1}(\Omega_2)$. Since $A\, DL_\lambda \phi|\Omega_i = \lambda DL_\lambda \phi|\Omega_i$, $i = 1, 2$, by elliptic regularity the first-order trace of $DL_\lambda \phi|\Omega_1$ along Γ_1 does not belong to $H^{s-1/2}(\Gamma_1)$ and conversely the trace of $DL_\lambda \phi|\Omega_2$ along Γ_2 belongs to $H^{s-1/2}(\Gamma_2)$. In conclusion, choosing $\Omega_2 \subset \Omega_1$ in such a way that $\Gamma_1 = (\Gamma_1 \cap \Sigma) \cup (\Gamma_1 \cap \Gamma_2)$, one obtains $\hat\gamma_1 DL_\lambda \phi|\Sigma \notin H^{s-1/2}(\Sigma)$. □

Let us now introduce the following notation: we write $\psi \in M^s(\Gamma)$, $s \ge 0$, whenever ψ is a multiplier in $H^s(\Gamma)$, i.e.

$$\begin{aligned} M^s(\Gamma) :=& \{\psi \in H^s(\Gamma) : \psi\phi \in H^s(\Gamma) \text{ for any } \phi \in H^s(\Gamma)\} \\ =& \{\psi \in M^s(\Gamma) : \exists\, m_\psi \ge 0 \text{ s.t. } \|\psi\phi\|_{H^s(\Gamma)} \le m_\psi \|\phi\|_{H^s(\Gamma)}\}\,. \end{aligned} \tag{3.32}$$

The equality holds, by the closed graph theorem, since the map $\phi \mapsto \psi\phi$ is closed and everywhere defined. Notice that $M^s(\Gamma) \subseteq L^\infty(\Gamma)$:

$$\|\psi\|_{L^\infty(\Gamma)} = \lim_{k\to+\infty} \|\psi\|_{L^{2k}(\Gamma)} = \lim_{k\to+\infty} \|\psi^k\|^{1/k}_{L^2(\Gamma)} \le \lim_{k\to+\infty} \|\psi^k\|^{1/k}_{H^s(\Gamma)} \le m_\psi \lim_{k\to+\infty} \|1\|^{1/k}_{H^s(\Gamma)} = m_\psi\,.$$

By the same kind of proofs which hold in the flat case (see [52, Proposition 3.5.1, Corollary 3.5.7]) one has

$$M^s(\Gamma) \subseteq M^r(\Gamma)\,, \quad r \le s\,,$$

and

$$\psi \in M^s(\Gamma) \text{ and } 1/\psi \in L^\infty(\Gamma) \quad \Rightarrow \quad 1/\psi \in M^s(\Gamma)\,.$$

Corollary 3.5. *Let λ, Ω, Σ be as in Lemma 3.4. If $\phi \in H^{-1/2}_\Sigma(\Gamma)$, then*

$$(\psi\phi + \hat\gamma_0 SL_\lambda\phi)|\Sigma \in H^{s-1/2}(\Sigma) \quad \Longleftrightarrow \quad \phi \in H^{s-1/2}(\Gamma)\,, \quad \frac{1}{2} \le s \le k-1\,, \tag{3.33}$$

whenever $\psi \in M^{s-1/2}(\Gamma)$ and $1/\psi \in L^\infty(\Gamma)$. If $\phi \in H^{1/2}_\Sigma(\Gamma)$, then

$$(\psi\phi + \hat\gamma_1 DL_\lambda\phi)|\Sigma \in H^{s-1/2}(\Sigma) \quad \Longleftrightarrow \quad \phi \in H^{s+1/2}(\Gamma)\,, \quad \frac{1}{2} \le s \le k\,, \tag{3.34}$$

whenever $\psi \in M^{s-1/2}(\Gamma)$.

Proof. By (3.32), (3.22) and (3.23), we only need to prove the $\Rightarrow$ implications.

At first we prove (3.33). Let $\phi \in H^{-1/2}(\Gamma)\backslash H^{s-1/2}(\Gamma)$. If $(\psi\phi + \hat\gamma_0 SL_\lambda\phi)|\Sigma \in H^{s-1/2}(\Sigma)$, then, since $\psi\phi|\Sigma \notin H^{s-1/2}(\Sigma)$, we have $(\hat\gamma_0 SL_\lambda)\phi)|\Sigma \notin H^{s-1/2}(\Sigma)$. However, by (3.28), $(\hat\gamma_0 SL_\lambda)\phi)|\Sigma \in H^1(\Sigma)$, so that $s > 1$, otherwise the proof is concluded. Then, by (3.30), $\phi \notin H^{s-3/2}(\Gamma)$; repeating the same arguments we get $\phi \notin H^{s-5/2}(\Gamma)$. If $s \le 2$ we get a contradiction and so $(\psi\phi + \hat\gamma_1 SL_\lambda)\phi)|\Sigma \notin H^{s-1/2}(\Sigma)$. Otherwise we can iterate n times such kind of reasonings till we get $s - n \le 0$.

The proof of (3.34) is similar to the one for (3.33). Let $\phi \in H^{1/2}(\Gamma)\backslash H^{s+1/2}(\Gamma)$. If $(\psi\phi + \hat\gamma_1 DL_\lambda\phi)|\Sigma \in H^{s-1/2}(\Sigma)$, then, since, by (3.31), $(\hat\gamma_1 DL_\lambda\phi)|\Sigma \notin H^{s-1/2}(\Sigma)$, we have $\psi\phi|\Sigma \notin H^{s-1/2}(\Sigma)$ and so $\phi \notin H^{s-1/2}(\Gamma)$; if $s \le 1$ we get a contradiction and therefore $(\psi\phi + \hat\gamma_1 DL)\phi)|\Sigma \notin H^{s-1/2}(\Sigma)$. Otherwise we can iterate n times such kind of reasonings till we get $s - n \le 0$. □

Remark 3.6. Here we recall some relatively simple sufficient conditions in order that a given function belongs to $M^s(\Gamma)$. By [53, Theorem 3.20],

$$W^{k,\infty}(\Gamma) \subseteq M^s(\Gamma)\,, \qquad k \ge \max\{1, s\}\,,$$

where $W^{k,\infty}(\Gamma)$ denotes the set of functions in $\mathcal{C}^{k-1}(\Gamma)$ with k-order distributional derivatives in $L^\infty(\Gamma)$. By [63, Proposition 4.5] this result can be improved to

$$A^s(\Gamma) \subseteq M^s(\Gamma)\,,$$

where (see [63, Section 4.1]) $A^k(\Gamma) = W^{k,\infty}(\Gamma)$ whenever $s = k$ and, in case $s > 0$ is not an integer,

$$A^s(\Gamma) := \left\{\phi \in W^{[s],\infty}(\Gamma) : q_{i,\lambda,\phi} \in L^\infty(f_i(U_i))\,,\ i \in I\,,\ \lambda = s - [s]\right\}\,,$$

$$q_{\phi,\lambda,i}(x) := \sum_{|\alpha|\le[s]} \int_{f_i(U_i)} \frac{|\partial^\alpha\phi_i(x) - \partial^\alpha\phi_i(y)|^2}{\|x-y\|^{n-1+2\lambda}}\,dy\,, \quad \phi_i := (\varphi_i\phi)\circ f_i^{-1}\,.$$

Here $\{(U_i, f_i)\}_{i\in I}$ is an admissible atlas of Γ and $\{\varphi_i\}_{i\in I}$ is a subordinate partition of unity.

Notice that, for any $s-[s]<\kappa<1$, one has $\tilde{\mathcal{C}}^{[s],\kappa}(\Gamma)\subseteq A^s(\Gamma)$, where $\tilde{\mathcal{C}}^{k,\kappa}(\Gamma)$ denotes the set of functions in $W^{k,\infty}(\Gamma)$ having Hölder continuos (of exponent κ) k-order derivatives.

In the case Γ is smooth manifold, by[17, Theorem 24] one gets

$$H^s(\Gamma)=M^s(\Gamma)\,,\qquad s>\frac{1}{2}\,(n-1)\,.$$

By the embedding $H^s(\Gamma)\hookrightarrow L^q(\Gamma)$, $\frac{1}{q}=\frac{1}{2}-\frac{s}{n-1}$, which holds whenever $s<\frac{1}{2}\,(n-1)$, and by [17, Theorem 27], one gets

$$L^{(n-1)/s}_s(\Gamma)\cap L^\infty(\Gamma)\subseteq M^s(\Gamma)\,,\qquad s<\frac{1}{2}\,(n-1)\,,$$

where

$$L^q_s(\Gamma):=\{\phi\in L^q(\Gamma):(-\Delta_\Gamma)^{s/2}\phi\in L^q(\Gamma)\}\,.$$

4. Self-adjoint realizations of singular perturbations supported on hypersurfaces.

Let $A:H^2(\mathbb{R}^n)\subset L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ be the elliptic, self-adjoint operator defined in Subsection 3.2, i.e.

$$Au:=\sum_{1\le i,j\le n}\partial_{x_i}(a_{ij}\partial_{x_j}u)-Vu\,,$$

with $a_{ij},\partial_{x_i}a_{ij},V\in\mathcal{C}^\infty_b(\mathbb{R}^n)$, the symmetric matrix $\underline{a}\equiv[a_{ij}(x)]$ satisfying (3.6).

Given Ω open, bounded and of class $\mathcal{C}^{1,1}$, posing

$$\tau:H^2(\mathbb{R}^n)\to H^{\frac{3}{2}}(\Gamma)\oplus H^{\frac{1}{2}}(\Gamma)\,,\quad \tau u:=\gamma_0u\oplus\gamma_1u\,,$$

one has

Lemma 4.1. *The map τ is bounded, surjective and* $\ker(\tau)$ *is dense in* $L^2(\mathbb{R}^n)$.

Proof. Let $u\in H^2(\mathbb{R}^n)$. Then, by $\|u\|^2_{H^2(\mathbb{R}^n)}=\|u|\Omega_-\|^2_{H^2(\Omega_-)}+\|u|\Omega_+\|^2_{H^2(\Omega_+)}$, τ is bounded since both $\gamma_0^\pm$ and $\gamma_1^\pm$ are bounded; $\ker(\tau)$ is dense since it contains the dense set $\mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^n\backslash\Gamma)$. By [36, Remark 2.5.1.2], both

$$\tau^\pm:H^2(\Omega_\pm)\to H^{\frac{3}{2}}(\Gamma)\oplus H^{\frac{1}{2}}(\Gamma)\,,\quad \tau^\pm u_\pm:=\gamma_0^\pm u_\pm\oplus\gamma_1^\pm u_\pm$$

are surjective. Given $\phi\oplus\varphi\in H^{\frac{3}{2}}(\Gamma)\oplus H^{\frac{1}{2}}(\Gamma)$, let $u_\pm\in H^2(\Omega_\pm)$ such that $\tau^\pm u_\pm=\phi\oplus\varphi$. Let $\tilde u_\pm\in H^2(\mathbb{R}^n)$ be extensions of $u_\pm$. Then $\tau\left(\frac{1}{2}\,(\tilde u_-+\tilde u_+)\right)=\phi\oplus\varphi$ and so τ is surjective. □

By the definition of the map τ we have that

$$S:=A|\ker(\tau)=A^{\min}_-\oplus A^{\min}_+\,,$$

where

$$A^{\min}_\pm:=A|H^2_0(\Omega_\pm)\,,\quad H^2_0(\Omega_\pm):=\{u_\pm\in H^2(\Omega_\pm):\gamma_0^\pm u_\pm=\gamma_1^\pm u_\pm=0\}\,.$$

Since it is known that $(A^{\min}_\pm)^*=A^{\max}_\pm$, one obtains

$$\text{dom}(S^*)=\text{dom}(A^{\max}_-)\oplus\text{dom}(A^{\max}_+)\,,\qquad S^*=A^{\max}_-\oplus A^{\max}_+\,.\tag{4.1}$$

Thus we have the well-defined bounded (w.r.t. the graph norm in $\text{dom}(S^*)$) maps

$$\hat\gamma_0,\,[\hat\gamma_0]\in\mathsf{B}(\text{dom}(S^*),H^{-1/2}(\Gamma))\,,\quad \hat\gamma_1,\,[\hat\gamma_1]\in\mathsf{B}(\text{dom}(S^*),H^{-3/2}(\Gamma))\,.$$

By Lemma 4.1 we can apply the results of Section 2 to A and so find all self-adjoint extensions of the closed symmetric operator $A|\ker(\tau) = A_-^{\min}\oplus A_+^{\min}$. To this end we need to determine the operator G_z and M_z (see definitions (2.1) and (2.6)). By (3.15) and (3.16) this is immediate:

$$G_z : H^{-3/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\to L^2(\mathbb{R}^n)\,,\quad G_z(\phi\oplus\varphi):=SL_z\phi+DL_z\varphi \tag{4.2}$$

and

$$M_z : H^{-3/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\to H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)\,,$$

$$M_z := \begin{bmatrix}\gamma_0(SL-SL_z) & \gamma_0(DL-DL_z)\\ \gamma_1(SL-SL_z) & \gamma_1(DL-DL_z)\end{bmatrix}, \tag{4.3}$$

where

$$SL:=SL_{\lambda_\circ}\,,\qquad DL:=DL_{\lambda_\circ}\,,\qquad \lambda_\circ>\max(V_{\text{neg}})\,.$$

Next lemma provides a representation of $A_-^{\max}\oplus A_+^{\max}$ and of its domain. Before giving the precise statement we need some definitions. The distribution $\delta_\Gamma\in\mathscr{D}'(\mathbb{R}^n)$ is defined as usual by

$$\forall u\in\mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^n)\,,\quad (\delta_\Gamma,u)=\int_\Gamma u(x)\,d\sigma_\Gamma(x)\,.$$

Given $f\in H^{-s}(\Gamma)$, we then define $f\delta_\Gamma\in\mathscr{D}'(\mathbb{R}^n)$ and $f\partial_{\underline{a}}\delta_\Gamma\in\mathscr{D}'(\mathbb{R}^n)$ by

$$\forall u\in\mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^n)\,,\quad (f\delta_\Gamma,u)=\langle\bar f,u|\Gamma\rangle_{-s,s}$$

and

$$\forall u\in\mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^n)\,,\quad (f\partial_{\underline{a}}\delta_\Gamma,u):=-\sum_{1\le i,j\le n}(f\nu_i\delta_\Gamma,a_{ij}\partial_{x_j}u)\,.$$

Notice that if Ω is of class $\mathcal{C}^{1,1}$ then ν is Lipschitz continuous and so the product $f\nu$ is a well-defined vector in $H^{-r}(\Gamma)$, $r=\min\{1,s\}$.

Lemma 4.2.

$$\begin{aligned}&\text{dom}(A_-^{\max})\oplus\text{dom}(A_+^{\max})\\ =&\{u=u_\circ+SL\phi+DL\varphi\,,\ u_\circ\in H^2(\mathbb{R}^n)\,,\ \phi\oplus\varphi\in H^{-3/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\}\\ \equiv&\{u=u_\circ-SL[\hat\gamma_1]u+DL[\hat\gamma_0]u\,,\ u_\circ\in H^2(\mathbb{R}^n)\}\,,\end{aligned}$$

and

$$(A_-^{\max}\oplus A_+^{\max})u=Au-[\hat\gamma_1]u\,\delta_\Gamma-[\hat\gamma_0]u\,\partial_{\underline{a}}\delta_\Gamma\,.$$

Proof. By (4.1), Lemma 2.3 and (4.2) one has

$$\begin{aligned}&\text{dom}(A_-^{\max})\oplus\text{dom}(A_+^{\max})\\ =&\{u=u_\circ+G(\phi\oplus\varphi)\,,\ u_\circ\in H^2(\mathbb{R}^n)\,,\ \phi\oplus\varphi\in H^{-3/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\}\,.\end{aligned}$$

Since $[\gamma_0]u_\circ=[\gamma_1]u_\circ=0$, the proof of the first statement follows by using the jump relations (3.24). As regards the second statement, in the case $A=\Delta$ the proof has been given in [19, Theorem 3.1] (be aware that there the jumps of the trace maps have been defined with opposite signs). The proof in the more general case discussed here proceeds along the same lines and is left to the reader. □

From now on

$$\Pi : H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)\to H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)\,,$$

denotes an orthogonal projector,

$$\Pi' : H^{-3/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\to H^{-3/2}(\Gamma)\oplus H^{-1/2}(\Gamma)\,,$$

denotes the orthogonal projector defined as the dual of Π, so that $\Pi'=(\Lambda^3\oplus\Lambda)\Pi(\Lambda^{-3}\oplus\Lambda^{-1})$, and

$$\Theta : \mathrm{dom}(\Theta)\subseteq \mathrm{ran}(\Pi')\to \mathrm{ran}(\Pi)$$

denotes a self-adjoint operator. By Theorem 2.1 and Lemma 4.2, one readily obtains all self-adjoint extension of S:

Theorem 4.3. *Any self-adjoint extension of $A^{\min}_-\oplus A^{\min}_+$ is of the kind $A_{\Pi,\Theta}$, where*

$$A_{\Pi,\Theta} : \mathrm{dom}(A_{\Pi,\Theta})\subseteq L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)\,,\qquad A_{\Pi,\Theta}:=(A^{\max}_-\oplus A^{\max}_+)|\mathrm{dom}(A_{\Pi,\Theta})\,,$$

$$\begin{aligned}&\mathrm{dom}(A_{\Pi,\Theta}):=\{u\in \mathrm{dom}(A^{\max}_-)\oplus\mathrm{dom}(A^{\max}_+) : (-[\hat\gamma_1]u)\oplus[\hat\gamma_0]u\in\mathrm{dom}(\Theta)\,,\\ &\Pi(\gamma_0(u+SL[\hat\gamma_1]u-DL[\hat\gamma_0]u)\oplus\gamma_1(u+SL[\hat\gamma_1]u-DL[\hat\gamma_0]u))=\Theta((-[\hat\gamma_1]u)\oplus[\hat\gamma_0]u)\}\,.\end{aligned}$$

The set

$$Z_{\Pi,\Theta}:=\{z\in\rho(A) : \Theta+\Pi M_z\Pi' \text{ has a bounded inverse}\}$$

is not void; in particular $\mathbb{C}\backslash\mathbb{R}\subseteq Z_{\Pi,\Theta}\subseteq\rho(A_{\Pi,\Theta})$ and for any $z\in Z_{\Pi,\Theta}$ the resolvent of $A_{\Pi,\Theta}$ is given by

$$\begin{aligned}&(-A_{\Pi,\Theta}+z)^{-1}u\\ =&(-A+z)^{-1}u+G_z\Pi'(\Theta+\Pi M_z\Pi')^{-1}\Pi(\gamma_0((-A+z)^{-1}u)\oplus\gamma_1((-A+z)^{-1}u))\,,\end{aligned}\tag{4.4}$$

where G_z and M_z are defined in (4.2) *and* (4.3) *respectively.*

Remark 4.4. Let us notice that the self-adjoint extension A corresponds to the choice $\Pi=0$. By Lemma 4.2, the choice $\Pi=\Pi_1\oplus 0$ gives $[\hat\gamma_0]u=0$ and so produces self-adjoint extension ("δ-type" interactions) of the kind $Au-[\hat\gamma_1]u\,\delta_\Gamma$ while the choice $\Pi=0\oplus\Pi_2$ gives $[\hat\gamma_1]u=0$ and so produces self-adjoint extension ("δ'-type" interactions) of the kind $Au-[\hat\gamma_0]u\,\partial_{\underline{a}}\delta_\Gamma$; different Π's give combinations of δ and δ' interactions.

Corollary 4.5. *Suppose that* $\mathrm{dom}(\Theta)\subseteq\mathrm{ran}(\Pi')\cap(H^{1/2}(\Gamma)\oplus H^{3/2}(\Gamma))$ *and define*

$$B_\Theta : \mathrm{dom}(\Theta)\subseteq\mathrm{ran}(\Pi')\to\mathrm{ran}(\Pi)\,,\quad B_\Theta:=\Theta+\Pi B\Pi'\,,$$

$$B : H^{1/2}(\Gamma)\oplus H^{3/2}(\Gamma)\to H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)\,,\quad B:=\begin{bmatrix}\gamma_0 SL & \gamma_0 DL\\ \gamma_1 SL & \gamma_1 DL\end{bmatrix}.$$

Then

$$\begin{aligned}&\mathrm{dom}(A_{\Pi,\Theta})\\ =&\{u\in H^2(\mathbb{R}^n\backslash\Gamma) : (-[\gamma_1]u)\oplus[\gamma_0]u\in\mathrm{dom}(\Theta)\,,\ \Pi(\gamma_0 u\oplus\gamma_1 u)=B_\Theta((-[\gamma_1]u)\oplus[\gamma_0]u)\}\end{aligned}$$

and

$$\begin{aligned}&(-A_{\Pi,\Theta}+z)^{-1}u\\ =&(-A+z)^{-1}u+G_z\Pi'(B_\Theta-\Pi M^\circ_z\Pi')^{-1}\Pi(\gamma_0(-A+z)^{-1}u\oplus\gamma_1(-A+z)^{-1}u)\,,\end{aligned}\tag{4.5}$$

where

$$M_z^\circ := \begin{bmatrix} \gamma_0 SL_z & \gamma_0 DL_z \\ \gamma_1 SL_z & \gamma_1 DL_z \end{bmatrix}.$$

Proof. Let $u = u_\circ + SL\phi + DL\varphi = u_\circ - SL[\hat\gamma_1]u + DL[\hat\gamma_0]u$ be in $\mathrm{dom}(A_{\Pi,\Theta})$. Thus $\phi\oplus\varphi \in \mathrm{dom}(\Theta) \subseteq H^{1/2}(\Gamma)\oplus H^{3/2}(\Gamma)$. Therefore, by the mapping properties of single and double layer operators (see (3.22) and (3.23)),

$$\hat\gamma_0^\pm(SL\phi + DL\varphi)\oplus\hat\gamma_1^\pm(SL\phi+DL\varphi) \in H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma).$$

Therefore $\hat\gamma_0 u\oplus\hat\gamma_1 u \in H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)$ and so, by elliptic regularity, $u\in H^2(\Omega_-)\oplus H^2(\Omega_+)$. Thus $\Pi(\gamma_0 u\oplus\gamma_1 u)$ is well defined and, by the definition of $\mathrm{dom}(A_{\Pi,\Theta})$ given in Theorem 4.3,

$$\begin{aligned}&\Theta(\phi\oplus\varphi) = \Pi(\gamma_0(u - SL\phi - DL\varphi)\oplus\gamma_1(u - SL\phi - DL\varphi))\\ =&\Pi(\gamma_0 u\oplus\gamma_1 u) - \Pi(\gamma_0(SL\phi+DL\varphi)\oplus\gamma_1(SL\phi+DL\varphi)).\end{aligned}$$

The proof is then concluded by the identity $\Theta + \Pi M_z\Pi' = B_\Theta - \Pi M_z^\circ\Pi'$. □

Let us recall some definitions: $\mathfrak{S}_{p,\infty}(H_1,H_2)$, $p>0$ ($\mathfrak{S}_{p,\infty}(H) := \mathfrak{S}_{p,\infty}(H,H)$), denote the operator ideals of compact operators T on the Hilbert space H_1 to the Hilbert space H_2 such that $s_k(T) = O(k^{-1/p})$, where the the singular values $s_k(T)$ are defined as the eigenvalues of the non-negative compact operator $(T^*T)^{1/2}$. One has $T_2T_1\in\mathfrak{S}_{p,\infty}(H_1,H_2)$ whenever $T_1\in\mathfrak{S}_{p_1,\infty}(H_1,H_0)$, $T_2\in\mathfrak{S}_{p_2,\infty}(H_0,H_2)$ and $\frac1p = \frac1{p_1}+\frac1{p_2}$. Notice that if $p<q$ then $\mathfrak{S}_{p,\infty}(H_1,H_2)\subset\mathfrak{S}_q(H_1,H_2)$, where $\mathfrak{S}_q(H_1,H_2)$ denotes the Schatten-von Neumann ideal of compact operator with q-summable singular values; in particular $T\in\mathfrak{S}_{p,\infty}(H_1,H_2)$ is trace class whenever $p<1$.

The next results apply to the self-adjoint extensions given in Corollary 4.5 (and so to all the operators appearing in the next two sections). In the proofs we follow the same arguments as in [51] and [12].

Theorem 4.6. *Suppose Γ is smooth. If $\mathrm{dom}(\Theta)\subseteq\mathrm{ran}(\Pi')\cap(H^{s_1}(\Gamma)\oplus H^{s_2}(\Gamma))$ with $s = \min\left\{s_1+\frac32, s_2+\frac12\right\}\ge 2$, then for any integer $k\ge1$ and for any $z\in\rho(A)\cap\rho(A_{\Pi,\Theta})$ one has*

$$(-A_{\Pi,\Theta}+z)^{-k} - (-A+z)^{-k} \in \mathfrak{S}_{\frac{n-1}{2(k-1)+s},\infty}(L^2(\mathbb{R}^n)). \tag{4.6}$$

Proof. By (4.4) and by the definition of G_z one has

$$(-A_{\Pi,\Theta}+z)^{-1} - (-A+z)^{-1} = (\tau(-A+\bar z)^{-1})^*\Pi(\tilde\Theta + \Pi\tilde M_z\Pi)^{-1}\Pi\tau(-A+z)^{-1}, \tag{4.7}$$

where

$$\tilde\Theta := \Theta(\Lambda^3\oplus\Lambda) : (\Lambda^3\oplus\Lambda)^{-1}(\mathrm{dom}(\Theta))\subseteq\mathrm{ran}(\Pi)\to\mathrm{ran}(\Pi),$$

and

$$\tilde M_z := M_z(\Lambda^3\oplus\Lambda) : H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)\to H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma).$$

By (3.14), one has $(-A+z_\circ)^{-(m+1)}\in\mathsf{B}(L^2(\mathbb{R}^n), H^{2m+2}(\mathbb{R}^n))$, $z_\circ\in\mathbb{C}\backslash\mathbb{R}$, $\mathrm{Re}(z_\circ)\ge\lambda_\circ$; therefore $\tau(-A+z_\circ)^{-(m+1)}$ has range in $H^{2m+3/2}(\Gamma)\oplus H^{2m+1/2}(\Gamma)$ and so

$$\tau(-A+z_\circ)^{-(m+1)} = (\Lambda^{-2m}\oplus\Lambda^{-2m})\,T,$$

where

$$T := (\Lambda^{2m}\oplus\Lambda^{2m})\tau(-A+z_\circ)^{-(m+1)}\in\mathsf{B}(L^2(\mathbb{R}^n), H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)).$$

By the asymptotic $\lambda_j \sim c\, j^{2/(n-1)}$ which holds for the eigenvalues λ_j of the Laplace-Beltrami operator $-\Delta_\Gamma$ on the compact smooth Riemannian manifold Γ (see e.g. [62, Theorem 3.1, Chapter 8]), one gets

$$\Lambda^{-r} \in \mathfrak{S}_{\frac{n-1}{r},\infty}(H^s(\Gamma)) \tag{4.8}$$

and so

$$\Pi\tau(-A+z_\circ)^{-1}(-A+z_\circ)^{-m} \in \mathfrak{S}_{\frac{n-1}{2m},\infty}(L^2(\mathbb{R}^n),\operatorname{ran}(\Pi))\,.$$

That also gives

$$(-A+z_\circ)^{-m}(\tau(-A+\bar z_\circ)^{-1})^*\Pi = (\Pi\tau(-A+\bar z_\circ)^{-(m+1)})^* \in \mathfrak{S}_{\frac{n-1}{2m},\infty}(\operatorname{ran}(\Pi),L^2(\mathbb{R}^n))\,.$$

Since $(\tilde\Theta+\Pi\tilde M_{z_\circ}\Pi)^{-1} \in \mathsf{B}(\operatorname{ran}(\Pi))$ and its range is contained in $H^{s_1+3}(\Gamma)\oplus H^{s_2+1}(\Gamma)$, one gets

$$(\tilde\Theta+\Pi\tilde M_{z_\circ}\Pi)^{-1} = \left(\Lambda^{-(s_1+\frac32)}\oplus\Lambda^{-(s_2+\frac12)}\right)\Xi\,,$$

where

$$\Xi := (\Lambda^{s_1+\frac32}\oplus\Lambda^{s_2+\frac12})(\tilde\Theta+\Pi\tilde M_{z_\circ}\Pi)^{-1} \in \mathsf{B}(\operatorname{ran}(\Pi),H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma))\,.$$

Thus, by (4.8),

$$(\tilde\Theta+\Pi\tilde M_{z_\circ}\Pi)^{-1} \in \mathfrak{S}_{\frac{n-1}{s},\infty}(\operatorname{ran}(\Pi))\,,$$

and so

$$(-A+z_\circ)^{-m}(\tau(-A+\bar z_\circ)^{-1})^*\Pi(\tilde\Theta+\Pi\tilde M_{z_\circ}\Pi)^{-1} \in \mathfrak{S}_{\frac{n-1}{2m+s},\infty}(\operatorname{ran}(\Pi),L^2(\mathbb{R}^n))\,.$$

The proof is then concluded by [12, Lemma 2.3]. □

Corollary 4.7. *Let Γ and Θ satisfy the same hypotheses as in Theorem 4.6. Then*

$$\sigma_{ess}(A_{\Pi,\Theta}) = \sigma_{ess}(A)\,,\qquad \sigma_{ac}(A_{\Pi,\Theta}) = \sigma_{ac}(A)\,.$$

Moreover the wave operators

$$W_\pm(-A_{\Pi,\Theta},-A) = \text{s-}\lim_{t\to\pm\infty} e^{-itA_{\Pi,\Theta}}e^{itA}P_{ac}(-A)\,,$$

$$W_\pm(-A,-A_{\Pi,\Theta}) = \text{s-}\lim_{t\to\pm\infty} e^{-itA}e^{itA_{\Pi,\Theta}}P_{ac}(-A_{\Pi,\Theta})$$

exist and are complete, i.e. the limits exist everywhere and the ranges coincide with with the absolutely continuous subspaces.

Proof. By Theorem 4.6 the resolvent difference (4.7) (with $k=1$) is a compact operator and so $\sigma_{ess}(A_{\Pi,\Theta}) = \sigma_{ess}(A)$ by Weyl's theorem on the preservation of the essential spectrum under compact perturbations. Again by Theorem 4.6, the resolvent difference (4.7) (with $k > \frac12(n+1-s)$) is trace class; so, by the Birman-Kato criterion, one obtains the existence and completeness of the wave operators; thus $\sigma_{ac}(A_{\Pi,\Theta}) = \sigma_{ac}(A)$. □

Remark 4.8. In the case $A \le 0$ and $A_{\Pi,\Theta} \le 0$, by Corollary 4.7 and [44, Sections 8 and 9], one also gets the existence and completeness of wave operators for the pairs of wave equations $\partial^2_{tt}u = A_{\Pi,\Theta}u$ and $\partial^2_{tt}u = Au$.

Remark 4.9. Under additional hypotheses on the behavior at infinity of the coefficients of A, the spectral results in Corollary 4.7 can be specified. Let us suppose that

$$a_{ij}(x) = a^\circ_{ij} + b_{ij}(x)\,, \quad b_{ij}(x) = O(1/\|x\|^\delta)\,, \tag{4.9}$$

$$\partial_{x_i} b_{ij}(x) = O(1/\|x\|^\delta)\,, \quad V(x) = O(1/\|x\|^\delta)\,, \tag{4.10}$$

for some $\delta > 1$, as $\|x\| \to +\infty$. Let $A_\circ$ be the differential operator with constant coefficients

$$A_\circ : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,, \quad A_\circ u = \sum_{1\le i,j\le n} a^\circ_{ij}\partial^2_{x_i x_j} u\,.$$

One has $\sigma(A_\circ) = \sigma_{ac}(A_\circ) = \sigma_{ess}(A_\circ) = (-\infty, 0]$. Since, by [7, Theorem 5.3], $\sigma_{ess}(A) = \sigma_{ess}(A_\circ)$ and, by [41, Theorem 2.1], [42, Chapter XIV], $\sigma_{ac}(A) = \sigma_{ac}(A_\circ)$, Corollary 4.7 gives

$$\sigma_{ess}(A_{\Pi,\Theta}) = \sigma_{ac}(A_{\Pi,\Theta}) = (-\infty, 0]\,.$$

5. Applications: boundary conditions on Γ.

Using the scheme provided by Theorem 4.3 and Corollary 4.5, we next give the construction of some standard models of elliptic operators with boundary conditions on Γ, the boundary of a bounded domain Ω of class $\mathcal{C}^{1,1}$.

5.1. **Dirichlet boundary conditions on Γ.** Let us consider the self-adjoint extension A^D corresponding to Dirichlet boundary conditions on the whole Γ; it is given by the direct sum $A^D = A^D_- \oplus A^D_+$, where the self-adjoint operators $A^D_\pm$ are defined by $A^D_\pm := A|\mathrm{dom}(A^D_\pm)$, $\mathrm{dom}(A^D_\pm) = \{u_\pm \in H^2(\Omega_\pm) : \gamma^\pm_0 u_\pm = 0\}$. Since

$$\begin{aligned}\mathrm{dom}(A^D_-) \oplus \mathrm{dom}(A^D_+) =& \{u = u_- \oplus u_+ \in H^2(\Omega_-) \oplus H^2(\Omega_+) : [\gamma_0]u = 0\,, \gamma_0 u = 0\}\\ =& \{u \in H^1(\mathbb{R}^n) \cap H^2(\mathbb{R}^n\backslash\Gamma) : \gamma_0 u = 0\}\,,\end{aligned}$$

that corresponds, in Corollary 4.5, to the choice $\Pi = \Pi_1$, where $\Pi_1(\phi \oplus \varphi) := \phi \oplus 0$, and $B_\Theta = 0$, i.e. $\Theta(\phi \oplus \varphi) := (-\Theta_D\phi) \oplus 0$, where Θ_D is the (necessarily self-adjoint, by Lemma 2.6) operator

$$\Theta_D = \gamma_0 SL : H^{1/2}(\Gamma) \subseteq H^{-3/2}(\Gamma) \to H^{3/2}(\Gamma)\,. \tag{5.1}$$

Thus

$$(A^D_- \oplus A^D_+)u = Au - [\gamma_1]u\,\delta_\Gamma$$

and, for any $z \in \rho(A) \cap \rho(A^D_-) \cap \rho(A^D_+)$,

$$(-(A^D_- \oplus A^D_+) + z)^{-1} = (-A + z)^{-1} - SL_z(\gamma_0 SL_z)^{-1}\gamma_0(-A + z)^{-1}\,. \tag{5.2}$$

Let $z \in \rho(A) \cap \rho(A^D_-) \cap \rho(A^D_+)$, so that, by (5.2), $(\gamma_0 SL_z)^{-1} \in \mathsf{B}(H^{3/2}(\Gamma), H^{1/2}(\Gamma))$. Given $\varphi \in H^{3/2}(\Gamma)$, let us define $\phi := (\gamma_0 SL_z)^{-1}\varphi$ and $u_\pm := SL^\pm_z\phi$. Then $A^{\max}_\pm u_\pm = z\,u_\pm$ and $\gamma^\pm_0 u_\pm = \varphi$ and so one gets $u_\pm = K^\pm_z\varphi$, where $K^\pm_z \in \mathsf{B}(H^s(\Gamma), \mathrm{dom}(A^{\max}_\pm))$, $s \ge -\frac{1}{2}$, is the Poisson operator which solves the Dirichlet boundary value problem

$$\begin{cases}(A^{\max}_\pm - z)K^\pm_z\psi = 0\\ \hat\gamma^\pm_0 K^\pm_z\psi = \psi\,.\end{cases} \tag{5.3}$$

To $K_z^\pm$ one associates the Dirichlet-to-Neumann operator $P_z^\pm \in \mathsf{B}(H^s(\Gamma), H^{s-1}(\Gamma))$, $s \ge -\frac{1}{2}$, defined by $P_z^\pm := \hat\gamma_1^\pm K_z^\pm$. Thus, since $[\gamma_1]SL_z\phi = -\phi$, one has

$$\forall z \in \rho(A) \cap \rho(A_-^D) \cap \rho(A_+^D)\,, \qquad (\gamma_0 SL_z)^{-1} = P_z^- - P_z^+\,. \tag{5.4}$$

Therefore, by (5.2),

$$(-(A_-^D \oplus A_+^D) + z)^{-1} = (-A+z)^{-1} - SL_z(P_z^- - P_z^+)\gamma_0(-A+z)^{-1}\,.$$

5.2. **Neumann boundary conditions on Γ.** Let us consider the self-adjoint extension A^N corresponding to Neumann boundary conditions on the whole Γ; it is given by the direct sum $A^N = A_-^N \oplus A_+^N$, where the self-adjoint operators $A_\pm^N$ are defined by $A_\pm^N := A|\text{dom}(A_\pm^N)$, $\text{dom}(A_\pm^N) = \{u_\pm \in H^2(\Omega_\pm) : \gamma_1^\pm u_\pm = 0\}$. Since

$$\text{dom}(A_-^N) \oplus \text{dom}(A_+^N) = \{u \in H^2(\mathbb{R}^n\backslash\Gamma) : [\gamma_1]u = 0\,, \gamma_1 u = 0\}\,,$$

that corresponds, in Corollary 4.5, to the choice $\Pi = \Pi_2$, where $\Pi_2(\phi \oplus \varphi) := 0 \oplus \varphi$, and $B_\Theta = 0$, i.e. $\Theta(\phi \oplus \varphi) := 0 \oplus (-\Theta_N\phi)$, where Θ_N is the (necessarily self-adjoint, by Lemma 2.6) operator

$$\Theta_N = \gamma_1 DL : H^{3/2}(\Gamma) \subseteq H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)\,. \tag{5.5}$$

Thus

$$(A_-^N \oplus A_+^N)u = Au - [\gamma_0]u\,\partial_{\underline{a}}\delta_\Gamma\,,$$

and, for any $z \in \rho(A) \cap \rho(A_-^N) \cap \rho(A_+^N)$

$$(-(A_-^N \oplus A_+^N) + z)^{-1} = (-A+z)^{-1} - DL_z(\gamma_1 DL_z)^{-1}\gamma_1(-A+z)^{-1}\,. \tag{5.6}$$

Let $z \in \rho(A) \cap \rho(A_-^N) \cap \rho(A_+^N)$, so that, by (5.6), $(\gamma_1 DL_z)^{-1} \in \mathsf{B}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$. Given $\phi \in H^{1/2}(\Gamma)$, let us define $\varphi := (\gamma_1 DL_z)^{-1}\phi$ and $u_\pm := SL_z^\pm\varphi$. Then $A_\pm^{\max}u_\pm = z\,u_\pm$ and $\gamma_1^\pm u_\pm = \phi$ and so one gets $u_\pm = \tilde K_z^\pm\phi$, where $\tilde K_z^\pm \in \mathsf{B}(H^s(\Gamma), \text{dom}(A_\pm^{\max}))$, $s \ge -\frac{3}{2}$, solves the boundary value problem

$$\begin{cases}(A_\pm^{\max} - z)\tilde K_z^\pm\psi = 0\\ \hat\gamma_1^\pm \tilde K_z^\pm\psi = \psi\,.\end{cases}$$

To $\tilde K_z^\pm$ one associates the Neumann-to-Dirichlet operator $Q_z^\pm \in \mathsf{B}(H^s(\Gamma), H^{s+1}(\Gamma))$, $s \ge -\frac{3}{2}$, defined by $Q_z^\pm := \hat\gamma_0^\pm \tilde K_z^\pm$. Thus, since $[\gamma_0]DL_z\varphi = \varphi$, one has

$$\forall z \in \rho(A) \cap \rho(A_-^N) \cap \rho(A_+^N)\,, \qquad (\gamma_1 DL_z)^{-1} = Q_z^+ - Q_z^-\,. \tag{5.7}$$

Therefore, by (5.6),

$$(-(A_-^N \oplus A_+^N) + z)^{-1} = (-A+z)^{-1} + DL_z(Q_z^- - Q_z^+)\gamma_1(-A+z)^{-1}\,.$$

5.3. **Robin boundary conditions on Γ.** Let us consider the linear operator A^R corresponding to Robin boundary conditions on the whole Γ; it is given by the direct sum $A^R = A^R_- \oplus A^R_+$, where

$$A^R_\pm := A|\text{dom}(A^R_\pm)\,, \qquad \text{dom}(A^R_\pm) = \{u_\pm \in \text{dom}(A^{\max}_\pm) : \gamma_1^\pm u_\pm = b_\pm\, \gamma_0^\pm u_\pm\}\,.$$

We suppose that $b_\pm \in M^{1/2}(\Gamma)$ and that the functions $b_\pm$ are real-valued. Hence the operators $A^R_\pm$ are self-adjoint and $\text{dom}(A^\pm_R) \subseteq H^2(\Omega_\pm)$ (use e.g [38, Theorem 11]). In case $b_+(x) \neq b_-(x)$ for a.e. $x \in \Gamma$, the domain of $A^R_- \oplus A^R_+$ represents as

$$\begin{aligned}\text{dom}(A^R_- \oplus A^R_+) &= \{u \in H^2(\mathbb{R}^n\backslash\Gamma) : \gamma_1^\pm u_\pm = b_\pm\, \gamma_0^\pm u_\pm\} \\ &= \Big\{u \in H^2(\mathbb{R}^n\backslash\Gamma) : (b_+ - b_-)\gamma_0 u = [\gamma_1]u - \frac{1}{2}(b_+ + b_-)[\gamma_0]u\,, \\ &\qquad (b_+ - b_-)\gamma_1 u = \frac{1}{2}(b_+ + b_-)[\gamma_1]u - b_+ b_-[\gamma_0]u\Big\}\,. \end{aligned} \tag{5.8}$$

Then, according to Corollary 4.5, the self-adjoint operator $A^R_- \oplus A^R_+$ corresponds to the choice $\Pi = 1$ and $B_\Theta = B_R$, where

$$B_R = -\frac{1}{[b]}\begin{bmatrix} 1 & \langle b\rangle \\ \langle b\rangle & b_+ b_- \end{bmatrix}\,, \quad \langle b\rangle := \frac{1}{2}(b_+ + b_-)\,, \qquad [b] := b_+ - b_-\,, \tag{5.9}$$

provided that the operator

$$\Theta = -\Theta_R\,, \quad \Theta_R := \begin{bmatrix} 1/[b] + \gamma_0 SL & \langle b\rangle/[b] + \gamma_0 DL \\ \langle b\rangle/[b] + \gamma_1 SL & b_+ b_-/[b] + \gamma_1 DL \end{bmatrix} \tag{5.10}$$

is self-adjoint. This follows by the next lemma:

Lemma 5.1. *If $b_\pm \in M^{3/2}(\Gamma)$ and $1/[b] \in L^\infty(\Gamma)$, then*

$$\Theta_R : H^{3/2}(\Gamma) \times H^{3/2}(\Gamma) \subset H^{-3/2}(\Gamma) \oplus H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \oplus H^{1/2}(\Gamma)$$

is a semibounded self-adjoint operator.

Proof. Θ_R is a well-defined linear operator by (3.22) and (3.23). By (5.10) and (5.8), setting $u_\circ := u + SL[\gamma_1]u - DL[\gamma_0]u$, one has

$$\begin{aligned}&\{u \in H^2(\mathbb{R}^n\backslash\Gamma) : (-[\gamma_1]u) \oplus [\gamma_0]u \in \text{dom}(\Theta_R)\,,\ \gamma_0 u_\circ \oplus \gamma_1 u_\circ = -\Theta_R((-[\gamma_1]u) \oplus [\gamma_0]u)\} \\ =&\big\{u \in H^2(\mathbb{R}^n\backslash\Gamma) : \gamma_0 u = ([\gamma_1]u - \langle b\rangle[\gamma_0]u)/[b]\,,\ \gamma_1 u = (\langle b\rangle[\gamma_1]u - b_+ b_-[\gamma_0]u)/[b]\big\} \\ =&\text{dom}(A^R_- \oplus A^R_+)\,.\end{aligned}$$

Since $A^R_- \oplus A^R_+$ is self-adjoint, Θ_R is self-adjoint by Lemma 2.6. By Green's formula and Ehrling's lemma (here we procede as in the proof of [12, Proposition 3.15]) one has, for $\epsilon > 0$

sufficiently small and $\frac{1}{2} < s < 1$,

$$\begin{aligned}&\langle(-A^R_\pm+\lambda_\circ)u_\pm,u_\pm\rangle_{L^2(\Omega_\pm)}\\ \geq& c_\circ\|\nabla u_\pm\|^2_{L^2(\Omega_\pm)}-\|b_\pm\|_{L^\infty(\Gamma)}\|\gamma_0^\pm u_\pm\|^2_{L^2(\Gamma)}\\ \geq& c_\circ\|\nabla u_\pm\|^2_{L^2(\Omega_\pm)}-\|b_\pm\|_{L^\infty(\Gamma)}\|u_\pm\|^2_{H^s(\Omega_\pm)}\\ \geq& c_\circ\|\nabla u_\pm\|^2_{L^2(\Omega_\pm)}-\epsilon\,\|b_\pm\|_{L^\infty(\Gamma)}\|u_\pm\|^2_{H^1(\Omega_\pm)}-c^\pm_\epsilon\|b_\pm\|_{L^\infty(\Gamma)}\|u_\pm\|^2_{L^2(\Omega_\pm)}\\ \geq& -\kappa^\pm_\epsilon\|u_\pm\|^2_{L^2(\Omega_\pm)}\,.\end{aligned}$$

Hence Θ_R is semibounded by Lemma 2.6. □

According to this result, the Corollary 4.5 applies and we get

$$(A^R_-\oplus A^R_+)u=Au-\frac{4}{[b]}\left(\left(\langle b\rangle\,\gamma_1u-b_+b_-\gamma_0u\right)\delta_\Gamma+\left(\gamma_1u-\langle b\rangle\,\gamma_0u\right)\partial_{\underline{a}}\delta_\Gamma\right)\,.$$

Moroever, for any $z\in\rho(A)\cap\rho(A^R_-)\cap\rho(A^R_+)$,

$$\begin{aligned}&(-(A^R_-\oplus A^R_+)+z)^{-1}u\\ =&(-A+z)^{-1}u-G_z\begin{bmatrix}1/[b]+\gamma_0SL_z & \langle b\rangle/[b]+\gamma_0DL_z\\ \langle b\rangle/[b]+\gamma_1SL_z & b_+b_-/[b]+\gamma_1DL_z\end{bmatrix}^{-1}\begin{bmatrix}\gamma_0(-A+z)^{-1}u\\ \gamma_1(-A+z)^{-1}u\end{bmatrix},\end{aligned}$$

where G_z is defined in (4.2). Let us notice that the case in which one has the same Robin boundary conditions on both sides of Γ corresponds to the choice $b_+=b=-b_-$. Thus in this case one has

$$(A^R_-\oplus A^R_+)u=Au-2b\,\gamma_0u\,\delta_\Gamma-(2/b)\,\gamma_1u\,\partial_{\underline{a}}\delta_\Gamma$$

and

$$\begin{aligned}&(-(A^R_-\oplus A^R_+)+z)^{-1}u\\ =&(-A+z)^{-1}u-G_z\begin{bmatrix}1/(2b)+\hat\gamma_0SL_z & \hat\gamma_0DL_z\\ \hat\gamma_1SL_z & -b/2+\hat\gamma_1DL_z\end{bmatrix}^{-1}\begin{bmatrix}\gamma_0(-A+z)^{-1}u\\ \gamma_1(-A+z)^{-1}u\end{bmatrix}.\end{aligned}$$

5.4. **δ-interactions on Γ.** Let $\Pi(\phi\oplus\varphi)=\Pi_1(\phi\oplus\varphi):=\phi\oplus0$ and $\Theta(\phi\oplus\varphi)=(-\Theta_{\alpha,D})\phi\oplus0$, where $\Theta_{\alpha,D}:=1/\alpha+\Theta_D=1/\alpha+\gamma_0SL$ is the compression to $\mathrm{ran}(\Pi_1)$ of Θ_R (here we consider the case $b_+=-b_-=\alpha/2$). This gives the boundary condition $\alpha\gamma_0u=[\gamma_1]u$ and so one obtains the self-adjoint extensions usually called "δ-interactions on Γ" (see [12], [14] and references therein). In order to apply Corollary 4.5 we need the following

Lemma 5.2. *If $\alpha\in M^{3/2}(\Gamma)$ and $1/\alpha\in L^\infty(\Gamma)$, then the compression $\Theta_{\alpha,D}: H^{3/2}(\Gamma)\subseteq H^{-3/2}(\Gamma)\to H^{3/2}(\Gamma)$ is self-adjoint.*

Proof. By $1/\alpha\in M^{3/2}(\Gamma)$, the linear operator $(1/\alpha): H^{3/2}(\Gamma)\subset H^{-3/2}(\Gamma)\to H^{3/2}(\Gamma)$ is well-defined; by $\alpha\in M^{3/2}(\Gamma)$ it is not difficult to check that it is self-adjoint. By (3.22), $\gamma_0SL\alpha\in\mathsf{B}(H^{1/2}(\Gamma),H^{3/2}(\Gamma))$. Thus, by (3.2), for any $\epsilon>0$,

$$\begin{aligned}\|\gamma_0SL\phi\|_{H^{3/2}(\Gamma)}=&\|\gamma_0SL\alpha(1/\alpha)\phi\|_{H^{3/2}(\Gamma)}\leq c\,\|(1/\alpha)\phi\|_{H^{1/2}(\Gamma)}\\ \leq& c\,\epsilon\,\|(1/\alpha)\phi\|_{H^{3/2}(\Gamma)}+c_\epsilon\,\|(1/\alpha)\phi\|_{H^{-3/2}(\Gamma)}\\ \leq& c\,\left(\epsilon\,\|(1/\alpha)\phi\|_{H^{3/2}(\Gamma)}+c_\epsilon\,\|\phi\|_{H^{-3/2}(\Gamma)}\right)\end{aligned}$$

and so the self-adjoint operator $\gamma_0 SL : H^{1/2}(\Gamma) \subset H^{-3/2}(\Gamma) \to H^{3/2}(\Gamma)$ is infinitesimally $(1/\alpha)$-bounded. The proof is then concluded by [40, Corollary 1]. $\square$

Therefore, by Corollary 4.5 one gets the self-adjoint extension

$$A_{\alpha,\delta}\, u = Au - \alpha\gamma_0 u\, \delta_\Gamma\,,$$

$$\mathrm{dom}(A_{\alpha,\delta}) := \{u \in H^1(\mathbb{R}^n) \cap H^2(\mathbb{R}^n\backslash\Gamma) : \alpha\gamma_0 u = [\gamma_1]u\}\,.$$

By (4.5) and (5.4), its resolvent is given by

$$\begin{aligned}&(-A_{\alpha,\delta} + z)^{-1} = (-A + z)^{-1} - SL_z((1/\alpha) + \gamma_0 SL_z)^{-1}\gamma_0(-A + z)^{-1}\\ =&(-A + z)^{-1} - SL_z(P_z^- - P_z^+)(\alpha + P_z^- - P_z^+)^{-1}\alpha\gamma_0(-A + z)^{-1}\,.\end{aligned}$$

5.5. **δ'-interactions on Γ.** Let $\Pi(\phi\oplus\varphi) = \Pi_2(\phi\oplus\varphi) := 0\oplus\varphi$ and $\Theta(\phi\oplus\varphi) = 0\oplus(-\Theta_{\beta,N}\varphi)$, where $\Theta_{\beta,N} := -1/\beta + \Theta_N = -1/\beta + \gamma_1 DL$ is the compression to $\mathrm{ran}(\Pi_2)$ of Θ_R (here we consider the case $b_+ = -b_- = 2/\beta$). This gives the boundary condition $\beta\gamma_1 u = [\gamma_0]u$ and so one obtains the self-adjoint extensions usually called "δ'-interactions on Γ" (see [12]and references therein). In order to apply Corollary 4.5 we need the following

Lemma 5.3. *If $\beta \in M^{1/2}(\Gamma)$ and $1/\beta \in L^\infty(\Gamma)$, then the compression $\Theta_{\beta,N} : H^{3/2}(\Gamma) \subseteq H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is self-adjoint.*

Proof. By (3.26), one has $(\hat\gamma_1 DL)^{-1} \in \mathsf{B}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))$. Thus, by (3.2) and (3.23), one gets, for any $\epsilon > 0$,

$$\begin{aligned}\|(1/\beta)\varphi\|_{H^{1/2}(\Gamma)} =&\|(1/\beta)(\hat\gamma_1 DL)^{-1}\hat\gamma_1 DL\varphi\|_{H^{1/2}(\Gamma)} \le c\,\|\hat\gamma_1 DL\varphi\|_{H^{-1/2}(\Gamma)}\\ \le& c\,\epsilon\,\|\hat\gamma_1 DL\varphi\|_{H^{1/2}(\Gamma)} + c_\epsilon\,\|\hat\gamma_1 DL\varphi\|_{H^{-3/2}(\Gamma)}\\ \le& c\left(\epsilon\,\|\hat\gamma_1 DL\varphi\|_{H^{1/2}(\Gamma)} + c_\epsilon\,\|\varphi\|_{H^{-1/2}(\Gamma)}\right)\end{aligned}$$

and so the operator $(1/\beta) : H^{3/2}(\Gamma) \subseteq H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is infinitesimally $\hat\gamma_1 DL$-bounded. Since $\hat\gamma_1 DL : H^{3/2}(\Gamma) \subseteq H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is self-adjoint, the proof is then concluded by [40, Corollary 1]. $\square$

Therefore, by Corollary 4.5 one gets the self-adjoint extension

$$A_{\alpha,\delta'} u = Au - \beta\gamma_1 u\, \partial_{\underline{a}}\delta_\Gamma\,,$$

$$\mathrm{dom}(A_{\alpha,\delta'}) := \{u \in H^2(\mathbb{R}^n\backslash\Gamma) : [\gamma_1]u = 0\,,\ \beta\gamma_1 u = [\gamma_0]u\}\,.$$

By (4.5) and (5.7), its resolvent is given by

$$\begin{aligned}&(-A_{\beta,\delta'} + z)^{-1} = (-A + z)^{-1} + DL_z((1/\beta) - \gamma_1 DL_z)^{-1}\gamma_1(-A + z)^{-1}\\ =&(-A + z)^{-1} + DL_z(Q_z^+ - Q_z^-)(-\beta + Q_z^+ - Q_z^-)^{-1}\beta\gamma_1(-A + z)^{-1}\,.\end{aligned}$$

6. Applications: boundary conditions on $\Sigma \subset \Gamma$.

Using the scheme provided by Theorem 4.3 and Corollary 4.5, we next give the construction of some models of elliptic operators with boundary conditions on a relatively open part $\Sigma \subset \Gamma$ of class $\mathcal{C}^{0,1}$. In such cases we need more regularity hypotheses on Ω (with respect to the ones used in the previous section); these are needed in the proofs, in order that Sobolev spaces of appropriate order can be properly defined.

6.1. **Dirichlet boundary conditions on** $\Sigma \subset \Gamma$**.** Given $\Sigma \subset \Gamma$ relatively open of class $\mathcal{C}^{0,1}$, we denote by Π_Σ the orthogonal projector in the Hilbert space $H^{3/2}(\Gamma)$ such that $\text{ran}(\Pi_\Sigma) = H^{3/2}_{\Sigma^c}(\Gamma)^\perp$, i.e. (see (3.4)) $\text{ran}(\Pi_\Sigma) = \Lambda^{-3} H^{-3/2}_{\Sigma}(\Gamma)$.

Let $\Theta_D = \gamma_0 SL : H^{1/2}(\Gamma) \subseteq H^{-3/2}(\Gamma) \to H^{3/2}(\Gamma)$ be the self-adjoint operator we introduced in Example 5.1. We define then its compression to $\text{ran}(\Pi_\Sigma)$ by

$$\Theta_{D,\Sigma} := \Pi_\Sigma \Theta_D \Pi'_\Sigma : \text{ran}(\Pi'_\Sigma) \cap H^{1/2}(\Gamma) \subseteq \text{ran}(\Pi'_\Sigma) \to \text{ran}(\Pi_\Sigma) .$$

Since the dual projection Π'_Σ is given by $\Pi'_\Sigma = \Lambda^3 \Pi_\Sigma \Lambda^{-3}$, one has $\text{ran}(\Pi'_\Sigma) = H^{-3/2}_\Sigma(\Gamma)$ and $\text{dom}(\Theta_{\Sigma,D}) = H^{1/2}_\Sigma(\Gamma)$. If such a compression was self-adjoint then it will then provide a self-adjoint extension corresponding to Dirichlet boundary conditions supported on Σ. The known results about self-adjointness of compressions give a positive answer only under the hypothesis of finite codimensionality of the range of the projection (see e.g. [6] and references therein), an hypothesis that does not hold true in our case. Thus we need to provide a direct proof:

Theorem 6.1. *Let Ω be of class $\mathcal{C}^{3,1}$, then the compression*

$$\Theta_{D,\Sigma} := \Pi_\Sigma \hat\gamma_0 SL \Pi'_\Sigma : H^{1/2}_\Sigma(\Gamma) \subseteq H^{-3/2}_\Sigma(\Gamma) \to H^{3/2}_{\Sigma^c}(\Gamma)^\perp ,$$

is self-adjoint.

Proof. Equivalently we prove the self-adjointness of the compression to $\text{ran}(\Pi_\Sigma)$

$$\tilde\Theta_{D,\Sigma} := \Pi_\Sigma \tilde\Theta_D \Pi_\Sigma : \text{ran}(\Pi_\Sigma) \cap H^{7/2}(\Gamma) \subseteq \text{ran}(\Pi_\Sigma) \to \text{ran}(\Pi_\Sigma)$$

of the self-adjoint operator in $H^{3/2}(\Gamma)$ defined by

$$\tilde\Theta_D := \hat\gamma_0 SL \Lambda^3 : H^{7/2}(\Gamma) \subseteq H^{3/2}(\Gamma) \to H^{3/2}(\Gamma) .$$

Notice that

$$\tilde\Theta_{D,\Sigma} = \Theta_{D,\Sigma}\Lambda^3 , \quad \text{dom}(\tilde\Theta_{D,\Sigma}) = \Lambda^{-3}\text{dom}(\Theta_{D,\Sigma}) ,$$

and

$$\tilde\Theta_{D,\Sigma} : \Lambda^{-3} H^{1/2}_\Sigma(\Gamma) \subseteq H^{3/2}_{\Sigma^c}(\Gamma)^\perp \to H^{3/2}_{\Sigma^c}(\Gamma)^\perp .$$

Let f_D be the densely defined sesquilinear form in the Hilbert space $H^{3/2}(\Gamma)$

$$f_D : H^{5/2}(\Gamma) \times H^{5/2}(\Gamma) \subseteq H^{3/2}(\Gamma) \times H^{3/2}(\Gamma) \to \mathbb{R}$$

$$f_D(\phi_1, \phi_2) := \langle \Lambda^3 \phi_1, \hat\gamma_0 SL \Lambda^3 \phi_2 \rangle_{-\frac12,\frac12} = \langle \hat\gamma_0 SL \Lambda^3 \phi_1, \Lambda^3 \phi_2 \rangle_{\frac12,-\frac12} .$$

By (3.25),

$$f_D(\phi,\phi) \ge c_0 \, \|\Lambda^3 \phi\|^2_{-1/2} = c_0 \, \|\phi\|^2_{H^{5/2}(\Gamma)}$$

and so f_D is strictly positive and closed. Since, for any $\phi_1 \in H^{7/2}(\Gamma)$ and for any $\phi_2 \in H^{5/2}(\Gamma)$,

$$f_D(\phi_1,\phi_2)=\langle \Lambda^{3/2}\hat\gamma_0 SL\Lambda^3\phi_1,\Lambda^{3/2}\phi_2,\rangle_{L^2(\Gamma)}=\langle\tilde\Theta_D\phi_1,\phi_2\rangle_{H^{3/2}(\Gamma)}\,,$$

f_D is the sesquilinear form associated with the self-adjoint operator $\tilde\Theta_D$. Since

$$\mathrm{dom}(f_D)\cap\mathrm{ran}(\Pi_\Sigma)=H^{5/2}(\Gamma)\cap H^{3/2}_{\Sigma^c}(\Gamma)^\perp=H^{5/2}(\Gamma)\cap\Lambda^{-3}H^{-3/2}_{\Sigma}(\Gamma)=\Lambda^{-3}H^{-1/2}_{\Sigma}(\Gamma)$$

is dense in $\Lambda^{-3}H^{-3/2}_{\Sigma}(\Gamma)$, we can use Lemma 7.1 and so we need to determine the operator $\breve\Theta_D:=(\tilde\Theta_D)^{\breve{}}$ and the subspace $K_\Sigma:=\mathfrak{k}_{\Pi_\Sigma}$ (we refer to the Appendix for the notations). Let H_D be the Hilbert space given by $\mathrm{dom}(f_D)=H^{5/2}(\Gamma)$ endowed with the scalar product $\langle\phi_1,\phi_2\rangle_D:=f_D(\phi_1,\phi_2)$; let H_D' denote its dual space. Since

$$f_D(\phi_1,\phi_2)=\langle\hat\gamma_0 SL\Lambda^3\phi_1,\Lambda^3\phi_2\rangle_{\frac12,-\frac12}=\langle\Lambda^{1/2}\hat\gamma_0 SL\Lambda^3\phi_1,\Lambda^{5/2}\phi_2\rangle_{L^2(\Gamma)}\,,$$

one has

$$H_D'=H^{1/2}(\Gamma)\,,\quad\langle\varphi,\phi\rangle_{H_D',H_D}=\langle\Lambda^{1/2}\varphi,\Lambda^{5/2}\phi\rangle_{L^2(\Gamma)}$$

and

$$\breve\Theta_D:H^{5/2}(\Gamma)\to H^{1/2}(\Gamma)\,,\quad\breve\Theta_D=\hat\gamma_0 SL\Lambda^3\,.$$

Moreover

$$\begin{aligned}K_\Sigma=&\{\varphi\in H^{1/2}(\Gamma):\forall\phi\in\Lambda^{-3}H^{-1/2}_\Sigma(\Gamma)\,,\quad\langle\Lambda^{1/2}\varphi,\Lambda^{5/2}\phi\rangle_{L^2(\Gamma)}=0\}\\=&\{\varphi\in H^{1/2}(\Gamma):\forall\phi\in H^{-1/2}_\Sigma(\Gamma)\,,\ \langle\Lambda^{-1/2}\phi,\Lambda^{1/2}\varphi\rangle_{L^2(\Gamma)}=0\}\\=&\{\varphi\in H^{1/2}(\Gamma):\forall\phi\in H^{-1/2}_\Sigma(\Gamma)\,,\ \langle\phi,\varphi\rangle_{-\frac12,\frac12}=0\}\\=&H^{1/2}_{\Sigma^c}(\Gamma)\,.\end{aligned}$$

Therefore, by Lemma 7.1, $\tilde\Theta_{D,\Sigma}$ is self-adjoint if and only if

$$D_\Sigma:=\{\phi\in\Lambda^{-3}H^{-1/2}_\Sigma(\Gamma):\exists\tilde\phi\in\mathrm{ran}(\Pi_\Sigma)\text{ s.t. }\hat\gamma_0 SL\Lambda^3\phi-\tilde\phi\in H^{1/2}_{\Sigma^c}(\Gamma)\}\subseteq H^{7/2}(\Gamma)\,.$$

Suppose that such an inclusion does not hold, so that there exists $\phi\in D_\Sigma$ such that $\Lambda^3\phi\notin H^{1/2}(\Gamma)$. Then, by (3.30), one gets $\hat\gamma_0 SL\Lambda^3\phi|\Sigma\notin H^{3/2}(\Sigma)$. Therefore, by $\tilde\phi\in\mathrm{ran}(\Pi_\Sigma)\subset H^{3/2}(\Gamma)$ and $\hat\gamma_0 SL\Lambda^3\phi|\Sigma=\tilde\phi|\Sigma$, one gets a contradiction. In conclusion $\tilde\Theta_{D,\Sigma}$ (and hence $\Theta_{D,\Sigma}$) is self-adjoint. □

By Theorem 6.1 and Corollary 4.5, taking $\Pi(\phi\oplus\varphi)=\Pi_\Sigma\phi\oplus0$ and $\Theta(\phi\oplus\varphi)=(-\Theta_{D,\Sigma}\phi)\oplus0$ one gets the self-adjoint extension $A_{D,\Sigma}:=(A_-^{\max}\oplus A_+^{\max})|\mathrm{dom}(A_{D,\Sigma})$ with domain

$$\begin{aligned}\mathrm{dom}(A_{D,\Sigma})=&\{u\in H^2(\mathbb{R}^n\backslash\Gamma):[\gamma_0]u=0\,,\ [\gamma_1]u\in H^{1/2}_\Sigma(\Gamma)\,,\ \Pi_\Sigma\gamma_0u=0\}\\=&\{u\in H^1(\mathbb{R}^n)\cap H^2(\mathbb{R}^n\backslash\Gamma):\gamma_0u\in H^{3/2}_{\Sigma^c}(\Gamma)\,,\ [\gamma_1]u\in H^{1/2}_\Sigma(\Gamma)\}\\=&\{u\in H^1(\mathbb{R}^n)\cap H^2(\mathbb{R}^n\backslash\Sigma):\gamma_0u|\Sigma=0\}\,.\end{aligned}$$

The resolvent of $A_{D,\Sigma}$ is given by

$$(-A_{D,\Sigma}+z)^{-1}=(-A+z)^{-1}-SL_z\Pi_\Sigma'(\Pi_\Sigma\gamma_0SL_z\Pi_\Sigma')^{-1}\Pi_\Sigma\gamma_0(-A+z)^{-1}\,,$$

where Π_Σ is the orthogonal projection onto $H^{3/2}_{\Sigma^c}(\Gamma)^\perp$ and Π_Σ' is the orthogonal projection onto $H^{-3/2}_\Sigma(\Gamma)$.

Remark 6.2. Since $\mathrm{supp}([\gamma_1]u)\subset\Sigma$ for any $u\in\mathrm{dom}(A_{D,\Sigma})$, one has $[\gamma_1]u\,\delta_\Gamma=[\gamma_1]u\,\delta_\Sigma$; thus

$$A_{D,\Sigma}u=Au-[\gamma_1]u\,\delta_\Sigma$$

and so $A_{D,\Sigma}u(x)=Au(x)$ for a.e. x in $\mathbb{R}^n\backslash\Sigma$. This also shows that $A_{D,\Sigma}$ depends only on Σ and not on the whole Γ: one would obtain the same operator by considering any other bounded domain $\Omega_\circ$ with boundary $\Gamma_\circ$ such that $\Sigma\subset\Gamma_\circ$.

6.2. **Neumann boundary conditions on $\Sigma\subset\Gamma$.** Given $\Sigma\subset\Gamma$ relatively open of class $\mathcal{C}^{0,1}$, here we denote by Π_Σ the orthogonal projector in the Hilbert space $H^{1/2}(\Gamma)$ such that $\mathrm{ran}(\Pi_\Sigma)=H^{1/2}_{\Sigma^c}(\Gamma)^\perp$, i.e. (see (3.4)) $\mathrm{ran}(\Pi_\Sigma)=\Lambda^{-1}H^{-1/2}_{\Sigma}(\Gamma)$; one has $\mathrm{ran}(\Pi'_\Sigma)=H^{-1/2}_\Sigma(\Gamma)$. Similarly to Example 6.1 one has the following

Theorem 6.3. *Let Ω be of class $\mathcal{C}^{2,1}$, then the compression*

$$\Theta_{\Sigma,N}:=\Pi_\Sigma\hat\gamma_1 DL\Pi'_\Sigma:H^{3/2}_\Sigma(\Gamma)\subseteq H^{-1/2}_\Sigma(\Gamma)\to H^{1/2}_{\Sigma^c}(\Gamma)^\perp\,,$$

is self-adjoint.

Proof. The proof is almost the same as the one given in Example 6.1. Equivalently we prove the self-adjointness of the compression

$$\tilde\Theta_{\Sigma,N}:=\Pi_\Sigma\tilde\Theta_N\Pi_\Sigma:\Lambda^{-1}H^{3/2}_\Sigma(\Gamma)\subseteq H^{1/2}_{\Sigma^c}(\Gamma)^\perp\to H^{1/2}_{\Sigma^c}(\Gamma)^\perp\,,$$

where

$$\tilde\Theta_N:=\hat\gamma_1 DL\Lambda:H^{5/2}(\Gamma)\subset H^{1/2}(\Gamma)H^{1/2}(\Gamma)\,.$$

Let f_N be the densely defined sesquilinear form in the Hilbert space $H^{1/2}(\Gamma)$

$$f_N:H^{3/2}(\Gamma)\times H^{3/2}(\Gamma)\subseteq H^{1/2}(\Gamma)\times H^{1/2}(\Gamma)\to\mathbb{R}$$

$$f_N(\varphi_1,\varphi_2):=\langle\hat\gamma_1 DL\Lambda\varphi_1,\Lambda\varphi_2\rangle_{-\frac12,\frac12}\,.$$

By (3.26),

$$-f_N(\varphi,\varphi)\ge c_1\,\|\Lambda\varphi\|^2_{1/2}=c_1\,\|\varphi\|^2_{H^{3/2}(\Gamma)}$$

and so f_N is strictly negative and closed. Since, for any $\varphi_1\in H^{5/2}(\Gamma)$ and for any $\phi_2\in H^{3/2}(\Gamma)$,

$$f_N(\varphi_1,\varphi_2)=\langle\Lambda^{1/2}\hat\gamma_1 DL\Lambda\varphi_1,\Lambda^{1/2}\varphi_2,\rangle_{L^2(\Gamma)}=\langle\tilde\Theta_N\varphi_1,\varphi_2\rangle_{H^{1/2}(\Gamma)}\,,$$

f_N is the sesquilinear form associated with the self-adjoint operator $\tilde\Theta_N$. Since

$$\mathrm{dom}(f_N)\cap\mathrm{ran}(\Pi_\Sigma)=H^{3/2}(\Gamma)\cap H^{1/2}_{\Sigma^c}(\Gamma)^\perp=H^{3/2}(\Gamma)\cap\Lambda^{-1}H^{-1/2}_\Sigma(\Gamma)=\Lambda^{-1}H^{1/2}_\Sigma(\Gamma)$$

is dense in $\Lambda_{-1}H^{-1/2}_\Sigma(\Gamma)$, we can use Lemma 7.1 and so we need to determine the operator $\breve\Theta_N:=(\tilde\Theta_N)^\smile$ and the subspace $K_\Sigma:=\mathfrak{k}_{\Pi_\Sigma}$ (we refer to the Appendix for the notations). Let H_N be the Hilbert space given by $\mathrm{dom}(f_N)=H^{3/2}(\Gamma)$ endowed with the scalar product $\langle\varphi_1,\varphi_2\rangle_N:=-f_N(\varphi_1,\varphi_2)$; let H'_N denote its dual space. Since

$$f_N(\varphi_1,\varphi_2)=\langle\hat\gamma_1 DL\Lambda\varphi_1,\Lambda\varphi_2\rangle_{-\frac12,\frac12}=\langle\Lambda^{-1/2}\hat\gamma_1 DL\Lambda\varphi_1,\Lambda^{3/2}\varphi_2\rangle_{L^2(\Gamma)}\,,$$

one has

$$H'_N=H^{-1/2}(\Gamma)\,,\quad\langle\phi,\varphi\rangle_{H'_N,H_N}=\langle\Lambda^{-1/2}\phi,\Lambda^{3/2}\varphi\rangle_{L^2(\Gamma)}$$

and

$$\breve\Theta_N:H^{-1/2}(\Gamma)\to H^{3/2}(\Gamma)\,,\quad\breve\Theta_N=\hat\gamma_1 DL\Lambda\,.$$

Moreover

$$\begin{aligned}K_\Sigma =&\{\phi\in H^{-1/2}(\Gamma):\forall\varphi\in\Lambda^{-1}H^{1/2}_\Sigma(\Gamma)\,,\quad\langle\Lambda^{-1/2}\phi,\Lambda^{3/2}\varphi\rangle_{L^2(\Gamma)}=0\}\\ =&\{\phi\in H^{-1/2}(\Gamma):\forall\varphi\in H^{1/2}_\Sigma(\Gamma)\,,\ \langle\Lambda^{-1/2}\phi,\Lambda^{1/2}\varphi\rangle_{L^2(\Gamma)}=0\}\\ =&\{\phi\in H^{-1/2}(\Gamma):\forall\varphi\in H^{1/2}_\Sigma(\Gamma)\,,\ \langle\phi,\varphi\rangle_{-\frac12,\frac12}=0\}\\ =&H^{-1/2}_{\Sigma^c}(\Gamma)\,.\end{aligned}$$

Therefore, by Lemma 5.1, $\tilde\Theta_{N,\Sigma}$ is self-adjoint if and only if

$$N_\Sigma:=\{\varphi\in\Lambda^{-1}H^{1/2}_\Sigma(\Gamma):\exists\tilde\varphi\in\text{ran}(\Pi_\Sigma)\text{ s.t. }\hat\gamma_1DL\Lambda\varphi-\tilde\varphi\in H^{-1/2}_{\Sigma^c}(\Gamma)\}\subseteq H^{5/2}(\Gamma)\,.$$

Suppose that such an inclusion does not hold, so that there exists $\varphi\in N_\Sigma$ such that $\Lambda\varphi\notin H^{3/2}(\Gamma)$. Then, by (3.31), one gets $\hat\gamma_1DL\Lambda\varphi|\Sigma\notin H^{1/2}(\Sigma)$. Therefore, by $\tilde\varphi\in\text{ran}(\Pi_\Sigma)\subset H^{1/2}(\Gamma)$ and $\hat\gamma_1DL\Lambda\varphi|\Sigma=\tilde\varphi|\Sigma$, one gets a contradiction. In conclusion $\tilde\Theta_{N,\Sigma}$ (and hence $\Theta_{N,\Sigma}$) is self-adjoint. □

By Theorem 6.1 and Corollary 4.5, taking $\Pi(\phi\oplus\varphi)=0\oplus\Pi_\Sigma\varphi$ and $\Theta(\phi\oplus\varphi)=0\oplus(-\Theta_{N,\Sigma}\varphi)$, one gets the self-adjoint extension $A_{N,\Sigma}:=(A^{\max}_-\oplus A^{\max}_+)|\text{dom}(A_{N,\Sigma})$ with domain

$$\begin{aligned}\text{dom}(A_{N,\Sigma})=&\{u\in H^2(\mathbb{R}^n\backslash\Gamma):[\gamma_0]u\in H^{3/2}_\Sigma(\Gamma)\,,\ [\gamma_1]u=0\,,\ \Pi_\Sigma\gamma_1u=0\}\\ =&\{u\in H^2(\mathbb{R}^n\backslash\Gamma):[\gamma_0]u\in H^{3/2}_\Sigma(\Gamma)\,,\ [\gamma_1]u=0\,,\ \gamma_1u\in H^{1/2}_{\Sigma^c}(\Gamma)\}\\ =&\{u\in H^2(\mathbb{R}^n\backslash\Sigma):\gamma_1u|\Sigma=0\}\,.\end{aligned}$$

The resolvent of $A_{N,\Sigma}$ is given by

$$(-A_{N,\Sigma}+z)^{-1}=(-A+z)^{-1}-DL_z\Pi'_\Sigma(\Pi_\Sigma\gamma_1DL_z\Pi'_\Sigma)^{-1}\Pi_\Sigma\gamma_1(-A+z)^{-1}\,,$$

where Π_Σ is the orthogonal projection onto $H^{1/2}_{\Sigma^c}(\Gamma)^\perp$ and Π'_Σ is the orthogonal projection onto $H^{-1/2}_\Sigma(\Gamma)$.

Remark 6.4. Since $\text{supp}([\gamma_0]u)\subset\Sigma$ for any $u\in\text{dom}(A_{N,\Sigma})$, one has $[\gamma_0]u\,\partial_{\underline{a}}\delta_\Gamma=[\gamma_0]u\,\partial_{\underline{a}}\delta_\Sigma$; thus

$$A_{N,\Sigma}u=Au-[\gamma_0]u\,\partial_{\underline{a}}\delta_\Sigma$$

and so $A_{N,\Sigma}u(x)=Au(x)$ for a.e. x in $\mathbb{R}^n\backslash\Sigma$. This also shows that $A_{N,\Sigma}$ depends only on Σ and not on the whole Γ: one would obtain the same operator by considering any other bounded domain $\Omega_\circ$ with boundary $\Gamma_\circ$ such that $\Sigma\subset\Gamma_\circ$.

6.3. **Robin boundary conditions on $\Sigma\subset\Gamma$.** Given $\Sigma\subset\Gamma$ relatively open of class $\mathcal{C}^{0,1}$, we denote by Π_Σ the orthogonal projector in the Hilbert space $H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)$ such that $\text{ran}(\Pi_\Sigma)=H^{3/2}_{\Sigma^c}(\Gamma)^\perp\oplus H^{1/2}_{\Sigma^c}(\Gamma)^\perp$, i.e. (see (3.4)) $\text{ran}(\Pi_\Sigma)=\Lambda^{-3}H^{-3/2}_\Sigma(\Gamma)\oplus\Lambda^{-1}H^{-1/2}_\Sigma(\Gamma)$.

Theorem 6.5. *Let Ω be of class $\mathcal{C}^{4,1}$ and let Θ_R be defined according to* (5.10)*. Under the assumptions of Lemma 5.1, the compression*

$$\Theta_{R,\Sigma}:=\Pi_\Sigma\Theta_R\Pi'_\Sigma:H^{3/2}_\Sigma(\Gamma)\times H^{3/2}_\Sigma(\Gamma)\subseteq H^{-3/2}_\Sigma(\Gamma)\oplus H^{-1/2}_\Sigma(\Gamma)\to H^{3/2}_{\Sigma^c}(\Gamma)^\perp\oplus H^{1/2}_{\Sigma^c}(\Gamma)^\perp\,,$$

is self-adjoint.

Proof. Equivalently we prove the self-adjointness of the compression to $\text{ran}(\Pi_\Sigma)$ of the self-adjoint operator in $H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)$ given by

$$\tilde\Theta_R:=\Theta_R(\Lambda^3\oplus\Lambda):H^{9/2}(\Gamma)\times H^{5/2}(\Gamma)\subseteq H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)\to H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma)\,,$$

its compression being defined by

$$\tilde\Theta_{R,\Sigma}:=\Pi_\Sigma\tilde\Theta_R\Pi_\Sigma:\Lambda^{-3}H^{3/2}_{\Sigma}(\Gamma)\times\Lambda^{-1}H^{3/2}_{\Sigma}(\Gamma)\subseteq H^{3/2}_{\Sigma^c}(\Gamma)^\perp\oplus H^{1/2}_{\Sigma^c}(\Gamma)^\perp\to H^{3/2}_{\Sigma^c}(\Gamma)^\perp\oplus H^{1/2}_{\Sigma^c}(\Gamma)^\perp\,.$$

Let f_R be the densely defined, semibounded, closed sesquilinear form associated with $\tilde\Theta_R$, i.e.

$$f_R:(H^3(\Gamma)\times H^{3/2}(\Gamma))\times(H^3(\Gamma)\times H^{3/2}(\Gamma))\subset(H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma))\times(H^{3/2}(\Gamma)\oplus H^{1/2}(\Gamma))\to\mathbb{R}\,,$$

$$\begin{aligned}f_R((\phi_1,\varphi_1),(\phi_2,\varphi_2)):=&\langle(1/[b]+\hat\gamma_0SL)\Lambda^3\phi_1+(\langle b\rangle/[b]+\hat\gamma_0DL)\Lambda\varphi_1,\Lambda^3\phi_2\rangle_{L^2(\Gamma)}\\&+\langle(\langle b\rangle/[b]+\hat\gamma_1SL)\Lambda^3\phi_1+(b_+b_-/[b]+\hat\gamma_1DL)\Lambda\varphi_1,\Lambda\varphi_2\rangle_{-\frac12,\frac12}.\end{aligned}$$

Since $H^3(\Gamma)\cap H^{3/2}_{\Sigma^c}(\Gamma)^\perp=H^3(\Gamma)\cap\Lambda^{-3}H^{-3/2}_{\Sigma}(\Gamma)=\Lambda^{-3}L^2_\Sigma(\Gamma)$ is dense in $\Lambda^{-3}H^{-3/2}_\Sigma(\Gamma)$ and $H^{3/2}(\Gamma)\cap H^{1/2}_{\Sigma^c}(\Gamma)^\perp=H^{3/2}(\Gamma)\cap\Lambda^{-1}H^{-1/2}_\Sigma(\Gamma)=\Lambda^{-1}H^{1/2}_\Sigma(\Gamma)$ is dense in $\Lambda^{-1}H^{-1/2}_\Sigma(\Gamma)$, we can use Lemma 7.1 and so we need to determine the operator $\breve\Theta_R:=(\tilde\Theta_R)^\smile$ and the subspace $K_\Sigma:=\mathfrak{k}_{\Pi_\Sigma}$ (we refer to the Appendix for the notations). Let H_R be the Hilbert space given by $\text{dom}(f_R)=H^3(\Gamma)\times H^{3/2}(\Gamma)$ endowed with the scalar product

$$\langle(\phi_1,\varphi_1),(\phi_2,\varphi_2)\rangle_R:=f_R((\phi_1,\varphi_1),(\phi_2,\varphi_2))+\lambda_R\left(\langle\phi_1,\phi_2\rangle_{H^{3/2}(\Gamma)}+\langle\varphi_1,\varphi_2\rangle_{H^{1/2}(\Gamma)}\right)\,,$$

where λ_R is chosen in such a way to have $f_R+\lambda_R>0$. Let H'_R denote its dual space. Since

$$\begin{aligned}&f_R((\phi_1,\varphi_1),(\phi_2,\varphi_2))=\langle(1/[b]+\hat\gamma_0SL)\Lambda^3\phi_1+(\langle b\rangle/[b]+\hat\gamma_0DL)\Lambda\varphi_1,\Lambda^3\phi_2\rangle_{L^2(\Gamma)}\\&+\langle\Lambda^{-1/2}((\langle b\rangle/[b]+\hat\gamma_1SL)\Lambda^3\phi_1+(b_+b_-/[b]+\hat\gamma_1DL))\Lambda\varphi_1,\Lambda^{3/2}\varphi_2\rangle_{L^2(\Gamma)}\,,\end{aligned}$$

one has

$$H'_R=L^2(\Gamma)\times H^{-1/2}(\Gamma)\,,\quad\langle(\phi',\varphi'),(\phi,\varphi)\rangle_{H'_R,H_R}=\langle\phi',\Lambda^3\phi\rangle_{L^2(\Gamma)}+\langle\Lambda^{-1/2}\varphi',\Lambda^{3/2}\varphi\rangle_{L^2(\Gamma)}\,,$$

and

$$\breve\Theta_R:H^3(\Gamma)\times H^{3/2}(\Gamma)\to L^2(\Gamma)\times H^{-1/2}(\Gamma)\,,$$

$$\breve\Theta_R(\phi,\varphi)=\big((1/[b]+\hat\gamma_0SL)\Lambda^3\phi+(\langle b\rangle/[b]+\hat\gamma_0DL)\Lambda\varphi,(\langle b\rangle/[b]+\hat\gamma_1SL)\Lambda^3\phi+(b_+b_-/[b]+\hat\gamma_1DL)\Lambda\varphi\big).$$

Moreover

$$\begin{aligned}&K_\Sigma\\=&\{(\phi',\varphi')\in L^2(\Gamma)\times H^{-1/2}(\Gamma):\forall(\phi,\varphi)\in\Lambda^{-3}L^2_\Sigma(\Gamma)\times\Lambda^{-1}H^{1/2}_\Sigma(\Gamma)\,,\ \langle(\varphi',\phi'),(\phi,\varphi)\rangle_{H'_R,H_R}=0\}\\=&\{(\phi',\varphi')\in L^2(\Gamma)\times H^{-1/2}(\Gamma):\forall\,(\phi,\varphi)\in L^2_\Sigma(\Gamma)\times H^{1/2}_\Sigma(\Gamma)\,,\ \langle\phi',\phi\rangle_{L^2(\Gamma)}+\langle\varphi',\varphi\rangle_{-\frac12,\frac12}=0\}\\=&L^2_{\Sigma^c}(\Gamma)\times H^{-1/2}_{\Sigma^c}(\Gamma)\,.\end{aligned}$$

Therefore, by Lemma 5.1, $\tilde\Theta_{R,\Sigma}$ is self-adjoint if and only if

$$\begin{aligned}R_\Sigma := \big\{&(\phi,\varphi)\in\Lambda^{-3}L^2_\Sigma(\Gamma)\times\Lambda^{-1}H^{1/2}_\Sigma(\Gamma) : \exists\,\tilde\phi\oplus\tilde\varphi\in\operatorname{ran}(\Pi_\Sigma)\text{ s.t.}\\ &(1/[b]+\hat\gamma_0 SL)\Lambda^3\phi+(\langle b\rangle/[b]+\hat\gamma_0 DL)\Lambda\varphi-\tilde\phi\in L^2_{\Sigma^c}(\Gamma)\text{ and}\\ &(\langle b\rangle/[b]+\hat\gamma_1 SL)\Lambda^3\phi+(b_+b_-/[b]+\hat\gamma_1 DL)\Lambda\varphi-\tilde\varphi\in H^{-1/2}_{\Sigma^c}(\Gamma)\big\}\\ \subseteq\, &H^{9/2}(\Gamma)\times H^{5/2}(\Gamma)\,.\end{aligned}$$

Suppose that such an inclusion does not hold, so that there exists $(\phi,\varphi)\in R_\Sigma$ such that $(\Lambda^3\phi,\Lambda\varphi)\notin H^{3/2}(\Gamma)\times H^{3/2}(\Gamma)$. By the definition of R_Σ one has

$$\big((1/[b]+\hat\gamma_0 SL)\Lambda^3\phi+(\langle b\rangle/[b]+\hat\gamma_0 DL)\Lambda\varphi\big)|\Sigma\in H^{3/2}(\Sigma) \tag{6.1}$$

$$\big((\langle b\rangle/[b]+\hat\gamma_1 SL)\Lambda^3\phi+(b_+b_-/[b]+\hat\gamma_1 DL)\Lambda\varphi\big)|\Sigma\in H^{1/2}(\Sigma)\,. \tag{6.2}$$

Suppose that $\Lambda\varphi\notin H^{3/2}(\Gamma)$; by (6.2) and by (3.34) one gets $(\langle b\rangle/[b]+\hat\gamma_1 SL)\Lambda^3\phi\notin H^{1/2}(\Gamma)$; thus, by the mapping properties of $\hat\gamma_1 SL$, there follows $\Lambda^3\phi\notin H^{1/2}(\Gamma)$. Therefore, by (3.33), one gets $((1/[b]+\hat\gamma_0 SL)\Lambda^3\phi)|\Sigma\notin H^{1/2}(\Sigma)$ and so, by (6.1), $((\langle b\rangle/[b]+\hat\gamma_0 DL)\Lambda\varphi)|\Sigma\notin H^{1/2}(\Sigma)$; thus, by the mapping properties of $\hat\gamma_0 DL$, there follows $\Lambda\varphi\notin H^{1/2}(\Gamma)$. That contradicts $(\phi,\varphi)\in R_\Sigma$; therefore $\Lambda\varphi\in H^{3/2}(\Gamma)$. Suppose now that $\Lambda^3\phi\notin H^{3/2}(\Gamma)$ and $\Lambda\varphi\in H^{3/2}(\Gamma)$. By (3.33), one gets $((1/[b]+\hat\gamma_0 SL)\Lambda^3\phi)|\Sigma\notin H^{3/2}(\Sigma)$; since $(\langle b\rangle/[b]+\hat\gamma_0 DL)\Lambda\varphi\in H^{3/2}(\Gamma)$, this contradicts (6.1). In conclusion $\tilde\Theta_{R,\Sigma}$ (and hence $\Theta_{R,\Sigma}$) is self-adjoint. □

By Theorem 6.5 and Corollary 4.5, taking $\Pi=\Pi_\Sigma$ and $\Theta=-\Theta_{R,\Sigma}$ one gets the self-adjoint extension $A_{R,\Sigma}:=(A_-^{\max}\oplus A_+^{\max})|\operatorname{dom}(A_{R,\Sigma})$ with domain

$$\begin{aligned}&\operatorname{dom}(A_{R,\Sigma})\\ =&\big\{u\in H^2(\mathbb{R}^n\backslash\Gamma):[\gamma_0]u\oplus[\gamma_1]u\in H^{3/2}_\Sigma(\Gamma)\oplus H^{3/2}_\Sigma(\Gamma),\ \Pi_\Sigma\gamma_0 u\oplus\gamma_1 u=B_{R,\Sigma}(-[\gamma_1]u)\oplus[\gamma_0]u\big\},\end{aligned}$$

where

$$B_{R,\Sigma}:=\Pi_\Sigma B_R\Pi'_\Sigma : H^{3/2}_\Sigma(\Gamma)\times H^{3/2}_\Sigma(\Gamma)\subseteq H^{-3/2}_\Sigma(\Gamma)\oplus H^{-1/2}_\Sigma(\Gamma)\to H^{3/2}_{\Sigma^c}(\Gamma)^\perp\oplus H^{1/2}_{\Sigma^c}(\Gamma)^\perp$$

is the compression to $\operatorname{ran}(\Pi_\Sigma)$ of the linear operator B_R defined in (5.9). Thus

$$\begin{aligned}\operatorname{dom}(A_{R,\Sigma})=&\{u\in H^2(\mathbb{R}^n\backslash\Gamma):[\gamma_0]u\in H^{3/2}_\Sigma(\Gamma)\,,\ [\gamma_1]u\in H^{3/2}_\Sigma(\Gamma)\,,\\ &[b]\gamma_0 u-[\gamma_1]u+\langle b\rangle[\gamma_0]u\in H^{3/2}_{\Sigma^c}(\Gamma)\,,\ [b]\gamma_1 u-\langle b\rangle[\gamma_1]u+b_+b_-[\gamma_0]u\in H^{1/2}_{\Sigma^c}(\Gamma)\}\\ =&\{u\in H^2(\mathbb{R}^n\backslash\Sigma):(\gamma_1^\pm u-b_\pm\gamma_0^\pm u)|\Sigma=0\}\,.\end{aligned}$$

The resolvent of $A_{R,\Sigma}$ is given by

$$\begin{aligned}&(-A_{R,\Sigma}+z)^{-1}u\\ =&(-A+z)^{-1}u-G_z\Pi'_\Sigma\left(\Pi_\Sigma\begin{bmatrix}1/[b]+\gamma_0 SL_z & \langle b\rangle/[b]+\gamma_0 DL_z\\ \langle b\rangle/[b]+\gamma_1 SL_z & b_+b_-/[b]+\gamma_1 DL_z\end{bmatrix}\Pi'_\Sigma\right)^{-1}\Pi_\Sigma\begin{bmatrix}\gamma_0(-A+z)^{-1}u\\ \gamma_1(-A+z)^{-1}u\end{bmatrix},\end{aligned}$$

where Π_Σ is the orthogonal projection onto $H^{3/2}_{\Sigma^c}(\Gamma)^\perp\oplus H^{1/2}_{\Sigma^c}(\Gamma)^\perp$, Π'_Σ is the orthogonal projection onto $H^{-3/2}_\Sigma(\Gamma)\oplus H^{-1/2}_\Sigma(\Gamma)$ and G_z is defined in (4.2).

Remark 6.6. Since $\text{supp}([\gamma_0]u)\subset\Sigma$ and $\text{supp}([\gamma_1]u)\subset\Sigma$, for any $u\in\text{dom}(A_{R,\Sigma})$, one has

$$\begin{aligned}A_{R,\Sigma}u&=Au-[\gamma_1]u\,\delta_\Sigma-[\gamma_0]u\,\partial_{\underline{a}}\delta_\Sigma\\ &=Au-\frac{4}{[b]}\left(\left(\langle b\rangle\,\gamma_1u-b_+b_-\gamma_0u\right)\delta_\Sigma+\left(\gamma_1u-\langle b\rangle\,\gamma_0u\right)\partial_{\underline{a}}\delta_\Sigma\right)\end{aligned}$$

and so $A_{R,\Sigma}u(x)=Au(x)$ for a.e. x in $\mathbb{R}^n\backslash\Sigma$. This also shows that $A_{R,\Sigma}$ depends only on Σ and $b_\pm|\Sigma$ and not on the whole Γ: one would obtain the same operator by considering any other bounded domain $\Omega_\circ$ with boundary $\Gamma_\circ$ such that $\Sigma\subset\Gamma_\circ$.

6.4. **δ-interactions on $\Sigma\subset\Gamma$.** Given $\Sigma\subset\Gamma$ relatively open of class $\mathcal{C}^{0,1}$, we denote by Π_Σ the orthogonal projector in the Hilbert space $H^{3/2}(\Gamma)$ such that $\text{ran}(\Pi_\Sigma)=H^{3/2}_{\Sigma^c}(\Gamma)^\perp$, i.e. (see (3.4)) $\text{ran}(\Pi_\Sigma)=\Lambda^{-3}H^{-3/2}_\Sigma(\Gamma)$.

Theorem 6.7. *Let Ω be of class $\mathcal{C}^{4,1}$ and let $\Theta_{\alpha,D}$ be defined according to Lemma 5.2. Then the compression*

$$\Theta_{\alpha,D,\Sigma}:=\Pi_\Sigma\Theta_{\alpha,D}\Pi'_\Sigma:H^{3/2}_\Sigma(\Gamma)\subseteq H^{-3/2}_\Sigma(\Gamma)\to H^{3/2}_{\Sigma^c}(\Gamma)^\perp\,,$$

is self-adjoint.

Proof. Equivalently we prove the self-adjointness of the compression to $\text{ran}(\Pi_\Sigma)$ of the self-adjoint operator in $H^{3/2}(\Gamma)$ given by

$$\tilde\Theta_{\alpha,D}:=\Theta_{\alpha,D}\Lambda^3:H^{9/2}(\Gamma)\subseteq H^{3/2}(\Gamma)\to H^{3/2}(\Gamma)\,,$$

its compression being defined by

$$\tilde\Theta_{\alpha,D,\Sigma}:=\Pi_\Sigma\tilde\Theta_{\alpha,D}\Pi_\Sigma:\Lambda^{-3}H^{3/2}_\Sigma(\Gamma)\subseteq H^{3/2}_{\Sigma^c}(\Gamma)^\perp\to H^{3/2}_{\Sigma^c}(\Gamma)^\perp\,.$$

Let $f_{\alpha,D}$ be the densely defined, semibounded, closed sesquilinear form associated with $\tilde\Theta_{\alpha,D}$, i.e.

$$f_{\alpha,D}:H^3(\Gamma)\times H^3(\Gamma)\subset H^{3/2}(\Gamma)\times H^{3/2}(\Gamma)\to\mathbb{R}\,,$$

$$f_{\alpha,D}(\phi_1\phi_2):=\langle(1/\alpha+\hat\gamma_0SL)\Lambda^3\phi_1,\Lambda^3\phi_2\rangle_{L^2(\Gamma)}\,.$$

Since $H^3(\Gamma)\cap H^{3/2}_{\Sigma^c}(\Gamma)^\perp=H^3(\Gamma)\cap\Lambda^{-3}H^{-3/2}_\Sigma(\Gamma)=\Lambda^{-3}L^2_\Sigma(\Gamma)$ is dense in $\Lambda^{-3}H^{-3/2}_\Sigma(\Gamma)$, we can use Lemma 7.1 and so we need to determine the operator $\breve\Theta_{\alpha,D}:=(\tilde\Theta_{\alpha,D})^\smile$ and the subspace $K_\Sigma:=\mathfrak{k}_{\Pi_\Sigma}$ (we refer to the Appendix for the notations). Let $H_{\alpha,D}$ be the Hilbert space given by $\text{dom}(f_{\alpha,D})=H^3(\Gamma)$ endowed with the scalar product

$$\langle\phi_1,\phi_2\rangle_{\alpha,D}:=f_{\alpha,D}(\phi_1,\phi_2)+\lambda_{\alpha,D}\langle\phi_1,\phi_2\rangle_{H^{3/2}(\Gamma)}\,,$$

where $\lambda_{\alpha,D}$ is chosen in such a way to have $f_{\alpha,D}+\lambda_{\alpha,D}>0$. Let $H'_{\alpha,D}$ denote its dual space. Since

$$f_{\alpha,D}(\phi_1,\phi_2)=\langle(1/\alpha+\hat\gamma_0SL)\Lambda^3\phi_1,\Lambda^3\phi_2\rangle_{L^2(\Gamma)}\,,$$

one has

$$H'_{\alpha,D}=L^2(\Gamma)\,,\quad\langle\varphi,\phi\rangle_{H'_{\alpha,D},H_{\alpha,D}}=\langle\varphi,\Lambda^3\phi\rangle_{L^2(\Gamma)}\,,$$

and

$$\breve\Theta_{\alpha,D}:H^3(\Gamma)\to L^2(\Gamma)\,,\quad\breve\Theta_{\alpha,D}=(1/\alpha+\hat\gamma_0SL)\Lambda^3\,.$$

Moreover

$$\begin{aligned} K_\Sigma =&\{\varphi\in L^2(\Gamma) : \forall\phi\in\Lambda^{-3}L^2_\Sigma(\Gamma)\,,\ \langle\varphi,\Lambda^3\phi\rangle_{L^2(\Gamma)}=0\}\\ =&\{\varphi\in L^2(\Gamma) : \forall\phi\in L^2_\Sigma(\Gamma)\,,\ \langle\varphi,\phi\rangle_{L^2(\Gamma)}=0\}\\ =&L^2_{\Sigma^c}(\Gamma)\,. \end{aligned}$$

Therefore, by Lemma 5.1, $\tilde\Theta_{\alpha,D,\Sigma}$ is self-adjoint if and only if

$$D_{\alpha,\Sigma}:=\{\phi\in\Lambda^{-3}L^2_\Sigma(\Gamma) : \exists\,\tilde\phi\in\operatorname{ran}(\Pi_\Sigma)\text{ s.t. }(1/\alpha+\hat\gamma_0 SL)\Lambda^3\phi-\tilde\phi\in L^2_{\Sigma^c}(\Gamma)\}\subseteq H^{9/2}(\Gamma)\,.$$

Suppose that such an inclusion does not hold, so that there exists $\phi\in D_{\alpha,\Sigma}$ such that $\Lambda^3\phi\notin L^2(\Gamma)$. By (3.33) one gets $((1/\alpha+\hat\gamma_0 SL)\Lambda^3\phi)|\Sigma\notin L^2(\Sigma)$. Therefore, by $\tilde\phi\in\operatorname{ran}(\Pi_\Sigma)\subset H^{3/2}(\Gamma)$ and $((1/\alpha+\hat\gamma_0 SL)\Lambda^3\phi)|\Sigma=\tilde\phi|\Sigma$, one gets a contradiction. In conclusion $\tilde\Theta_{\alpha,D,\Sigma}$ (and hence $\Theta_{\alpha,D,\Sigma}$) is self-adjoint. □

By Theorem 6.7 and Corollary 4.5, taking $\Pi=\Pi_\Sigma$ and $\Theta=-\Theta_{\alpha,D,\Sigma}$ one gets the self-adjoint extension $A_{\alpha,\delta,\Sigma}:=(A^{\max}_-\oplus A^{\max}_+)|\operatorname{dom}(A_{\alpha,\delta,\Sigma})$ with domain

$$\begin{aligned} \operatorname{dom}(A_{\alpha,\delta,\Sigma}) =&\{u\in H^1(\mathbb{R}^n)\cap H^2(\mathbb{R}^n\backslash\Gamma) : [\gamma_1]u\in H^{3/2}_\Sigma(\Gamma)\,,\ \Pi_\Sigma\,(\gamma_0 u-(1/\alpha)[\gamma_1]u)=0\}\\ =&\{u\in H^1(\mathbb{R}^n)\cap H^2(\mathbb{R}^n\backslash\Sigma) : (\alpha\gamma_0 u-[\gamma_1]u)|\Sigma=0\}\,. \end{aligned}$$

The resolvent of $A_{\alpha,\delta,\Sigma}$ is given by

$$(-A_{\alpha,\delta,\Sigma}+z)^{-1}u=(-A+z)^{-1}-SL_z\Pi'_\Sigma(\Pi_\Sigma(1/\alpha+\gamma_0 SL_z)\Pi'_\Sigma)^{-1}\Pi_\Sigma\gamma_0(-A+z)^{-1}\,,$$

where Π_Σ is the orthogonal projection onto $H^{3/2}_{\Sigma^c}(\Gamma)^\perp$ and Π'_Σ is the orthogonal projection onto $H^{-3/2}_\Sigma(\Gamma)$.

Remark 6.8. Since $\operatorname{supp}([\gamma_0]u)\subset\Sigma$, for any $u\in\operatorname{dom}(A_{\alpha,\delta,\Sigma})$, one has

$$A_{\alpha,\delta,\Sigma}\,u=Au-\alpha\gamma_0 u\,\delta_\Sigma$$

and so $A_{\alpha,\delta,\Sigma}\,u(x)=Au(x)$ for a.e. x in $\mathbb{R}^n\backslash\Sigma$. This also shows that $A_{\alpha,\delta,\Sigma}$ depends only on Σ and $\alpha|\Sigma$ and not on the whole Γ: one would obtain the same operator by considering any other bounded domain $\Omega_\circ$ with boundary $\Gamma_\circ$ such that $\Sigma\subset\Gamma_\circ$.

6.5. **δ'-interaction on $\Sigma\subset\Gamma$.** Given $\Sigma\subset\Gamma$ relatively open of class $\mathcal{C}^{0,1}$, we denote by Π_Σ the orthogonal projector in the Hilbert space $H^{1/2}(\Gamma)$ such that $\operatorname{ran}(\Pi_\Sigma)=H^{1/2}_{\Sigma^c}(\Gamma)^\perp$, i.e. (see (3.4)) $\operatorname{ran}(\Pi_\Sigma)=\Lambda^{-1}H^{-1/2}_\Sigma(\Gamma)$.

Theorem 6.9. *Let Ω be of class $\mathcal{C}^{2,1}$ and let $\Theta_{\beta,N}$ be defined according to Lemma 5.3. Then the compression*

$$\Theta_{\beta,N,\Sigma}:=\Pi_\Sigma\Theta_{\beta,N}\Pi'_\Sigma : H^{3/2}_\Sigma(\Gamma)\subseteq H^{-1/2}_\Sigma(\Gamma)\to H^{1/2}_{\Sigma^c}(\Gamma)^\perp\,,$$

is self-adjoint.

Proof. Equivalently we prove the self-adjointness of the compression to $\operatorname{ran}(\Pi_\Sigma)$ of the self-adjoint operator in $H^{1/2}(\Gamma)$ given by

$$\tilde\Theta_{\beta,N}:=\Theta_{\beta,N}\Lambda : H^{5/2}(\Gamma)\subseteq H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)\,,$$

its compression being defined by

$$\tilde\Theta_{\beta,N,\Sigma}:=\Pi_\Sigma\tilde\Theta_{\beta,N}\Pi_\Sigma:\Lambda^{-1}H^{3/2}_\Sigma(\Gamma)\subseteq H^{1/2}_{\Sigma^c}(\Gamma)^\perp\to H^{1/2}_{\Sigma^c}(\Gamma)^\perp\,.$$

Let $f_{\beta,N}$ be the densely defined, semibounded, closed sesquilinear form associated with $\tilde\Theta_{\beta,N}$, i.e.

$$f_{\beta,N}:H^{3/2}(\Gamma)\times H^{3/2}(\Gamma)\subset H^{1/2}(\Gamma)\times H^{1/2}(\Gamma)\to\mathbb{R}\,,$$
$$f_{\beta,N}(\varphi_1,\varphi_2):=\langle(1/\beta+\hat\gamma_1 DL)\Lambda\varphi_1,\Lambda\varphi_2\rangle_{-\frac12,\frac12}.$$

Since $H^{3/2}(\Gamma)\cap H^{1/2}_{\Sigma^c}(\Gamma)^\perp=H^{3/2}(\Gamma)\cap\Lambda^{-1}H^{-1/2}_\Sigma(\Gamma)=\Lambda^{-1}H^{1/2}_\Sigma(\Gamma)$ is dense in $\Lambda^{-1}H^{-1/2}_\Sigma(\Gamma)$, we can use Lemma 7.1 and so we need to determine the operator $\breve\Theta_{\beta,N}:=(\tilde\Theta_{\beta,N})^{\breve{}}$ and the subspace $K_\Sigma:=\mathfrak{k}_{\Pi_\Sigma}$ (we refer to the Appendix for the notations). Let $H_{\beta,N}$ be the Hilbert space given by $\mathrm{dom}(f_{\beta,N})=H^{3/2}(\Gamma)$ endowed with the scalar product

$$\langle(\varphi_1,\varphi_2\rangle_{\beta,N}:=f_{\beta,N}(\varphi_1\varphi_2)+\lambda_{\beta,N}\langle\varphi_1,\varphi_2\rangle_{H^{1/2}(\Gamma)}\,,$$

where $\lambda_{\beta,N}$ is chosen in such a way to have $f_{\beta,N}+\lambda_{\beta,N}>0$. Let $H'_{\beta,N}$ denote its dual space. Since

$$f_{\beta,N}(\varphi_1,\varphi_2)=\langle\Lambda^{-1/2}(1/\beta-\hat\gamma_1 DL))\Lambda\varphi_1,\Lambda^{3/2}\varphi_2\rangle_{L^2(\Gamma)}\,,$$

one has

$$H'_{\beta,N}=H^{-1/2}(\Gamma)\,,\quad\langle(\phi,\varphi)\rangle_{H'_{\beta,N},H_{\beta,N}}=\langle\Lambda^{-1/2}\phi,\Lambda^{3/2}\varphi\rangle_{L^2(\Gamma)}\,,$$

and

$$\breve\Theta_{\beta,N}:H^{3/2}(\Gamma)\to H^{-1/2}(\Gamma)\,,\quad\breve\Theta_{\beta,N}=(1/\beta+\hat\gamma_1 DL)\Lambda\,.$$

Moreover

$$\begin{aligned}K_\Sigma=&\{\phi\in H^{-1/2}(\Gamma):\forall\varphi\in\Lambda^{-1}H^{1/2}_\Sigma(\Gamma)\,,\ \langle\Lambda^{-1/2}\phi,\Lambda^{3/2}\varphi\rangle_{L^2(\Gamma)}=0\}\\=&\{\phi\in H^{-1/2}(\Gamma):\forall\varphi\in H^{1/2}_\Sigma(\Gamma)\,,\ \langle\phi,\varphi\rangle_{-\frac12,\frac12}=0\}\\=&H^{-1/2}_{\Sigma^c}(\Gamma)\,.\end{aligned}$$

Therefore, by Lemma 5.1, $\tilde\Theta_{\beta,N,\Sigma}$ is self-adjoint if and only if

$$D'_{\beta,\Sigma}:=\{\varphi\in\Lambda^{-1}H^{1/2}_\Sigma(\Gamma):\exists\,\tilde\varphi\in\mathrm{ran}(\Pi_\Sigma)\text{ s.t. }(1/\beta+\hat\gamma_1 DL)\Lambda\varphi-\tilde\varphi\in H^{-1/2}_{\Sigma^c}(\Gamma)\}\subseteq H^{5/2}(\Gamma)\,.$$

Suppose that such an inclusion does not hold, so that there exists $\varphi\in D'_{\beta,\Sigma}$ such that $\Lambda\varphi\notin H^{3/2}(\Gamma)$. By (3.34), one gets $((1/\beta+\hat\gamma_1 DL)\Lambda\varphi)|\Sigma\notin H^{1/2}(\Sigma)$. Therefore, by $\tilde\varphi\in\mathrm{ran}(\Pi_\Sigma)\subset H^{1/2}(\Gamma)$ and $((1/\beta+\hat\gamma_1 DL)\Lambda\varphi)|\Sigma=\tilde\varphi|\Sigma$, one gets a contradiction. In conclusion $\tilde\Theta_{\beta,N,\Sigma}$ (and hence $\Theta_{\beta,N,\Sigma}$) is self-adjoint. □

By Theorem 6.9 and Corollary 4.5, taking $\Pi(\phi\oplus\varphi)=0\oplus\Pi_\Sigma\varphi$ and $\Theta(\phi\oplus\varphi)=0\oplus(-\Theta_{\beta,N,\Sigma}\varphi)$ one gets the self-adjoint extension $A_{\beta,\delta',\Sigma}:=(A^{\max}_-\oplus A^{\max}_+)|\mathrm{dom}(A_{\beta,\delta',\Sigma})$ with domain

$$\begin{aligned}\mathrm{dom}(A_{\beta,\delta',\Sigma})=&\{u\in H^2(\mathbb{R}^n\backslash\Gamma):[\gamma_0]u\in H^{3/2}_\Sigma(\Gamma),\ [\gamma_1]u=0,\ \Pi_\Sigma(\gamma_1u-(1/\beta)[\gamma_0]u)=0\}\\=&\{u\in H^2(\mathbb{R}^n\backslash\Sigma):[\gamma_1]u|\Sigma=0,\ (\beta\gamma_1u-[\gamma_0]u)|\Sigma=0\}\,.\end{aligned}$$

The resolvent of $A_{\beta,\delta',\Sigma}$ is given by

$$(-A_{\beta,\delta',\Sigma}+z)^{-1}=(-A+z)^{-1}+DL_z\Pi'_\Sigma(\Pi_\Sigma(1/\beta-\gamma_1DL_z)\Pi'_\Sigma)^{-1}\Pi_\Sigma\gamma_1(-A+z)^{-1}\,,$$

where Π_Σ is the orthogonal projection onto $H^{1/2}_{\Sigma^c}(\Gamma)^\perp$ and Π'_Σ is the orthogonal projection onto $H^{-1/2}_{\Sigma}(\Gamma)$.

Remark 6.10. Since $\text{supp}([\gamma_1]u) \subset \Sigma$, for any $u \in \text{dom}(A_{\beta,\delta',\Sigma})$, one has

$$A_{\beta,\delta',\Sigma}u = Au - \beta\gamma_1 u\,\partial_{\underline{a}}\delta_\Sigma$$

and so $A_{\beta,\delta',\Sigma}u(x) = Au(x)$ for a.e. x in $\mathbb{R}^n\backslash\Sigma$. This also shows that $A_{\beta,\delta',\Sigma}$ depends only on Σ and $\beta|\Sigma$ and not on the whole Γ: one would obtain the same operator by considering any other bounded domain $\Omega_\circ$ with boundary $\Gamma_\circ$ such that $\Sigma \subset \Gamma_\circ$.

7. Appendix. Some remarks on compressions of self-adjoint operators.

Let $\Theta : \text{dom}(\Theta) \subseteq \mathfrak{h} \to \mathfrak{h}$ be a semibounded self-adjoint operator on the Hilbert space $\mathfrak{h}$ and let $f : \text{dom}(f) \times \text{dom}(f) \subseteq \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ be the corresponding semibounded sesquilinear form. Without loss of generality, eventually by considering $(-\Theta)$ and/or adding a constant, we can suppose that Θ (and hence f) is strictly positive. Let $\mathfrak{h}_\Theta$ be the Hilbert space given by $\text{dom}(f)$ endowed with the scalar product $\langle\varphi_1, \varphi_2\rangle_{\mathfrak{h}_\Theta} := f(\varphi_1, \varphi_2)$. Let $\mathfrak{h}'_\Theta$ be its dual space and let $\iota : \mathfrak{h}_\Theta \to \mathfrak{h}'_\Theta$ be the injection defined by $\langle\iota\phi, \varphi\rangle_{\mathfrak{h}'_\Theta\mathfrak{h}_\Theta} = \langle\phi, \varphi\rangle_{\mathfrak{h}}$, where $\langle\cdot,\cdot\rangle_{\mathfrak{h}'_\Theta\mathfrak{h}_\Theta}$ denotes the $\mathfrak{h}'_\Theta$-$\mathfrak{h}_\Theta$ duality and $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ denotes the scalar product in $\mathfrak{h}$; by the identification $\iota\phi \equiv \phi$, we may regard $\mathfrak{h}_\Theta \subseteq \mathfrak{h} \subseteq \mathfrak{h}'_\Theta$. Let $\breve{\Theta} : \mathfrak{h}_\Theta \to \mathfrak{h}'_\Theta$ be the bounded operator defined by

$$\langle\breve{\Theta}\varphi_1, \varphi_2\rangle_{\mathfrak{h}'_\Theta\mathfrak{h}_\Theta} = \langle\varphi_1, \varphi_2\rangle_{\mathfrak{h}_\Theta}\,, \qquad \varphi_1, \varphi_2 \in \mathfrak{h}_\Theta\,.$$

Obviously $\breve{\Theta}|\text{dom}(\Theta) = \Theta$; moreover, by [45, Theorem 2.1],

$$\text{dom}(\Theta) = \mathfrak{d}_\Theta := \{\varphi \in \mathfrak{h}_\Theta : \breve{\Theta}\varphi \in \mathfrak{h}\}\,.$$

In conclusion one gets a well know characterization of Θ (see [32, Remark at page 13], [60, Proof of Theorem VIII.15]):

$$\Theta = \breve{\Theta}|\mathfrak{d}_\Theta\,. \tag{7.1}$$

Let now $\Pi : \mathfrak{h} \to \mathfrak{h}$ be an orthogonal projector such that $\text{dom}(f)\cap\text{ran}(\Pi)$ is dense in $\text{ran}(\Pi)$. Then the sesquilinear form

$$f_\Pi : \text{dom}(f_\Pi) \times \text{dom}(f_\Pi) \subseteq \text{ran}(\Pi) \times \text{ran}(\Pi) \to \mathbb{R}\,,$$

$$f_\Pi(\varphi_1, \varphi_2) := f(\Pi\varphi_1, \Pi\varphi_2) = f(\varphi_1, \varphi_2)\,, \quad \text{dom}(f_\Pi) := \text{dom}(f) \cap \text{ran}(\Pi)\,,$$

is densely defined, closed and strictly positive. Hence there exists a unique strictly positive self-adjoint operator Θ_Π in $\text{ran}(\Pi)$ corresponding to f_Π. By the above reasonings applied to Θ_Π, we know that $\Theta_\Pi = \breve{\Theta}_\Pi|\mathfrak{d}_{\Theta_\Pi}$. For any $\varphi_1 \in \text{dom}(\Theta) \cap \text{ran}(\Pi)$ and for any $\varphi_2 \in \text{dom}(f) \cap \text{ran}(\Pi)$ one has

$$\langle\breve{\Theta}_\Pi\varphi_1, \varphi_2\rangle_{\mathfrak{h}'_{\Theta_\Pi}\mathfrak{h}_{\Theta_\Pi}} = \langle\varphi_1, \varphi_2\rangle_{\mathfrak{h}_{\Theta_\Pi}} = f(\Pi\varphi_1, \Pi\varphi_1) = \langle\Theta\Pi\varphi_1, \Pi\varphi_2\rangle_{\mathfrak{h}} = \langle\Pi\Theta\Pi\varphi_1, \varphi_2\rangle_{\mathfrak{h}}\,.$$

Thus $\text{dom}(\Theta) \cap \text{ran}(\Pi) \subseteq \mathfrak{d}_{\Theta_\Pi}$ and $\text{C}_\Pi(\Theta) \subseteq \Theta_\Pi$, where $\text{C}_\Pi(\Theta)$ is the compression of Θ to $\text{ran}(\Pi)$ defined by

$$\text{C}_\Pi(\Theta) : \text{dom}(\text{C}_\Pi(\Theta)) \subseteq \text{ran}(\Pi) \to \text{ran}(\Pi)\,,$$

$$\text{dom}(\text{C}_\Pi(\Theta)) := \text{dom}(\Theta) \cap \text{ran}(\Pi)\,, \quad \text{C}_\Pi(\Theta)\phi := \Pi\Theta\Pi\phi = \Pi\Theta\phi\,.$$

Notice that $\text{C}_\Pi(\Theta)$ can be not self-adjoint; it is self-adjoint if and only if $\mathfrak{d}_{\Theta_\Pi} \subseteq \text{dom}(\Theta)$.

We now give a more explicit definition of $\breve{\Theta}_\Pi$. Let $\mathfrak{h}_{\Theta_\Pi}$ be the Hilbert space $\mathrm{dom}(f_\Pi)$ endowed with the scalar product

$$\langle\varphi_1,\varphi_2\rangle_{\mathfrak{h}_{\Theta_\Pi}} := f_\Pi(\varphi_1,\varphi_2) = f(\varphi_1,\varphi_2) = \langle\varphi_1,\varphi_2\rangle_{\mathfrak{h}_\Theta}$$

and let $\mathfrak{h}'_{\Theta_\Pi}$ denote its dual. Since $\mathfrak{h}_{\Theta_\Pi} = \mathfrak{h}_\Theta \cap \mathrm{ran}(\Pi)$, by [5, Proposition 3.5.1] one has

$$\mathfrak{h}'_{\Theta_\Pi} = \mathfrak{h}'_\Theta/\mathfrak{k}_\Pi\,,$$

where

$$\mathfrak{k}_\Pi := \{\phi \in \mathfrak{h}'_\Theta : \forall\varphi \in \mathfrak{h}_{\Theta_\Pi}\,,\ \langle\phi,\varphi\rangle_{\mathfrak{h}'_\Theta\mathfrak{h}_\Theta} = 0\ \}\,,$$

i.e.

$$\mathfrak{h}'_{\Theta_\Pi} = \{\,[\phi]\,,\ \phi\in\mathfrak{h}'_\Theta\}\,,\quad [\phi] := \{\psi\in\mathfrak{h}'_\Theta : \psi-\phi\in\mathfrak{k}_\Pi\}\,.$$

The $\mathfrak{h}'_{\Theta_\Pi}$-$\mathfrak{h}_{\Theta_\Pi}$ duality is then defined by $\langle[\phi],\varphi\rangle_{\mathfrak{h}'_{\Theta_\Pi}\mathfrak{h}_{\Theta_\Pi}} := \langle\phi,\varphi\rangle_{\mathfrak{h}'_\Theta\mathfrak{h}_\Theta}$. Let $\iota_\Pi : \mathfrak{h}_{\Theta_\Pi} \to \mathfrak{h}'_{\Theta_\Pi}$ be the injection defined by $\langle\iota_\Pi\phi,\varphi\rangle_{\mathfrak{h}'_{\Theta_\Pi}\mathfrak{h}_{\Theta_\Pi}} := \langle\phi,\varphi\rangle_{\mathfrak{h}_{\Theta_\Pi}}$. Since

$$\langle\phi,\varphi\rangle_{\mathfrak{h}_{\Theta_\Pi}} = \langle\phi,\varphi\rangle_{\mathfrak{h}_\Theta} = \langle\iota\phi,\varphi\rangle_{\mathfrak{h}'_\Theta\mathfrak{h}_\Theta} = \langle[\iota\phi],\varphi\rangle_{\mathfrak{h}'_{\Theta_\Pi}\mathfrak{h}_{\Theta_\Pi}}\,,$$

one gets $\iota_\Pi\varphi = [\iota\varphi]$. By the identification $\iota_\Pi\varphi \equiv \varphi$, we may regard $\mathfrak{h}_{\Theta_\Pi} \subseteq \mathrm{ran}(\Pi) \subseteq \mathfrak{h}'_{\Theta_\Pi}$. For any $\varphi_1,\varphi_2 \in \mathfrak{h}_{\Theta_\Pi}$, the bounded operator $\breve{\Theta}_\Pi : \mathfrak{h}_{\Theta_\Pi} \to \mathfrak{h}'_{\Theta_\Pi}$ satisfies the relations

$$\langle\breve{\Theta}_\Pi\varphi_1,\varphi_2\rangle_{\mathfrak{h}'_{\Theta_\Pi},\mathfrak{h}_{\Theta_\Pi}} = f_\Pi(\varphi_1,\varphi_2) = f(\varphi_1,\varphi_2) = \langle\breve{\Theta}\varphi_1,\varphi_2\rangle_{\mathfrak{h}'_\Theta\mathfrak{h}_\Theta} = \langle[\breve{\Theta}\varphi_1],\varphi_2\rangle_{\mathfrak{h}'_{\Theta_\Pi}\mathfrak{h}_{\Theta_\Pi}}\,.$$

Thus $\breve{\Theta}_\Pi\varphi = [\breve{\Theta}\varphi]$ and

$$\mathfrak{d}_{\Theta_\Pi} = \{\varphi\in\mathfrak{h}_{\Theta_\Pi} : [\breve{\Theta}\varphi]\in\mathrm{ran}(\Pi)\} = \{\varphi\in\mathfrak{h}_{\Theta_\Pi} : \exists\tilde{\varphi}\in\mathrm{ran}(\Pi)\ \text{s.t.}\,\breve{\Theta}\varphi-\tilde{\varphi}\in\mathfrak{k}_\Pi\}\,.$$

In conclusion we have the following

Lemma 7.1. *Let* $f : \mathrm{dom}(f)\times\mathrm{dom}(f) \subseteq \mathfrak{h}\times\mathfrak{h} \to \mathbb{R}$ *be the closed sesquilinear form corresponding to the semibounded self-adjoint operator* Θ*; let* $\Pi : \mathfrak{h}\to\mathfrak{h}$ *be an orthogonal projector such that* $\mathrm{dom}(f)\cap\mathrm{ran}(\Pi)$ *is dense in* $\mathrm{ran}(\Pi)$*. Then the compression* $\mathrm{C}_\Pi(\Theta)$ *is self-adjoint if and only if*

$$\{\varphi\in\mathrm{dom}(f)\cap\mathrm{ran}(\Pi) : \exists\tilde{\varphi}\in\mathrm{ran}(\Pi)\ \text{s.t.}\,\breve{\Theta}\varphi-\tilde{\varphi}\in\mathfrak{k}_\Pi\} \subseteq \mathrm{dom}(\Theta)\,.$$

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Laboratoire de Mathématiques, Université de Reims - FR3399 CNRS, Moulin de la Housse BP 1039, 51687 Reims, France

DiSAT, Sezione di Matematica, Università dell'Insubria, via Valleggio 11, I-22100 Como, Italy

RICAM, Austrian Academy of Sciences, Altenbergerstr. 69, A-4040 Linz, Austria
E-mail address: andrea.mantile@univ-reims.fr
E-mail address: posilicano@uninsubria.it
E-mail address: mourad.sini@oeaw.ac.at