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The equivalent mass density for the elastic scattering by many small rigid bodies and applications

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Abstract

We deal with the elastic scattering by a large number M of rigid bodies of arbitrary shapes with maximum radius a, $0 < a << 1$ with constant Lamé coefficients λ and μ. We show that, when these rigid bodies are distributed arbitrarily (not necessarily periodically) in a bounded region Ω of $\mathbb{R}^3$ where their number is $M := M(a) := O(a^{-1})$ and the minimum distance between them is $d := d(a) \approx a^t$ with t in some appropriate range, as $a \to 0$, the generated far-field patterns approximate the far-field patterns generated by an equivalent (possibly variable) mass density. This mass density is described by two coefficients: one modeling the local distribution of the small bodies and the other one by their geometries. In particular, if the distributed bodies have a uniform spherical shape then the equivalent mass density is isotropic while for general shapes it might be anisotropic. In addition, we can distribute the small bodies in such a way that the equivalent mass density is negative. Finally, if the background density ρ is variable in Ω and $\rho = 1$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, then if we remove from Ω appropriately distributed small bodies then the equivalent density will be equal to unity in $\mathbb{R}^3$, i.e. the obstacle Ω characterized by ρ is approximately cloaked.

Keywords: Elastic wave scattering, Small-scatterers, Effective mass density.

1 Introduction and statement of the results

1.1 The background

Let $B_1, B_2, \dots, B_M$ be M open, bounded and simply connected sets in $\mathbb{R}^3$ with Lipschitz boundaries, containing the origin. We assume that their sizes and Lipschitz constants are uniformly bounded. We set $D_m := \epsilon B_m + z_m$ to be the small bodies characterized by the parameter $\epsilon > 0$ and the locations $z_m \in \mathbb{R}^3$, $m = 1, \dots, M$.

Assume that the Lamé coefficients λ and μ are constants satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$ and the mass density ρ to be a constant that we normalize to a unity. Let U^i be a solution of the Navier equation $(\Delta^e + \omega^2)U^i = 0$ in $\mathbb{R}^3$, $\Delta^e := (\mu\Delta + (\lambda + \mu)\nabla \operatorname{div})$. We denote by U^s the elastic field scattered by the M small bodies $D_m \subset \mathbb{R}^3$ due to the incident field U^i. We restrict ourselves to the scattering by rigid bodies. Hence the total field $U^t := U^i + U^s$ satisfies the following exterior Dirichlet problem of the elastic waves

$$(\Delta^e + \omega^2)U^t = 0 \text{ in } \mathbb{R}^3 \setminus \left(\bigcup_{m=1}^{M} \bar{D}_m \right), \tag{1.1}$$

$$U^t|_{\partial D_m} = 0,\ 1 \leq m \leq M \tag{1.2}$$

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with the Kupradze radiation conditions (K.R.C)

$$\lim_{|x|\to\infty}|x|(\frac{\partial U_p}{\partial|x|}-i\kappa_{p^\omega}U_p)=0,\ \text{and}\ \lim_{|x|\to\infty}|x|(\frac{\partial U_s}{\partial|x|}-i\kappa_{s^\omega}U_s)=0, \tag{1.3}$$

where the two limits are uniform in all the directions $\hat{x}:=\frac{x}{|x|}\in\mathbb{S}^2$. Also, we denote $U_p:=-\kappa_{p^\omega}^{-2}\nabla(\nabla\cdot U^s)$ to be the longitudinal (or the pressure or P) part of the field U^s and $U_s:=\kappa_{s^\omega}^{-2}\nabla\times(\nabla\times U^s)$ to be the transversal (or the shear or S) part of the field U^s corresponding to the Helmholtz decomposition $U^s=U_p+U_s$. The constants $\kappa_{p^\omega}:=\frac{\omega}{c_p}$ and $\kappa_{s^\omega}:=\frac{\omega}{c_s}$ are known as the longitudinal and transversal wavenumbers, $c_p:=\sqrt{\lambda+2\mu}$ and $c_s:=\sqrt{\mu}$ are the corresponding phase velocities, respectively and ω is the frequency.

The scattering problem (1.1-1.3) is well posed in the Hölder or Sobolev spaces, see [8,9,12,13] for instance, and the scattered field U^s has the following asymptotic expansion:

$$U^s(x):=\frac{e^{i\kappa_{p^\omega}|x|}}{|x|}U_p^\infty(\hat{x})+\frac{e^{i\kappa_{s^\omega}|x|}}{|x|}U_s^\infty(\hat{x})+O(\frac{1}{|x|^2}),\ |x|\to\infty \tag{1.4}$$

uniformly in all directions $\hat{x}\in\mathbb{S}^2$. The longitudinal part of the far-field, i.e. $U_p^\infty(\hat{x})$ is normal to $\mathbb{S}^2$ while the transversal part $U_s^\infty(\hat{x})$ is tangential to $\mathbb{S}^2$. We set $U^\infty:=(U_p^\infty,U_s^\infty)$.

As usual, we use plane incident waves of the form $U^i(x,\theta):=\alpha\theta\, e^{i\kappa_{p^\omega}\theta\cdot x}+\beta\theta^\perp\, e^{i\kappa_{s^\omega}\theta\cdot x}$, where $\theta^\perp$ is any direction in $\mathbb{S}^2$ perpendicular to the incident direction $\theta\in\mathbb{S}^2$, α,β are arbitrary constants. The functions $U_p^\infty(\hat{x},\theta):=U_p^\infty(\hat{x})$ and $U_s^\infty(\hat{x},\theta):=U_s^\infty(\hat{x})$ for $(\hat{x},\theta)\in\mathbb{S}^2\times\mathbb{S}^2$ are called the P-part and the S-part of the far-field pattern respectively.

Definition 1.1. *We define*

1. $a:=\max\limits_{1\leq m\leq M} diam(D_m)\ \ \left[=\epsilon\max\limits_{1\leq m\leq M} diam(B_m)\right]$,
2. $d:=\min\limits_{\substack{m\neq j\\1\leq m,j\leq M}} d_{mj}$, *where* $d_{mj}:=dist(D_m,D_j)$. *We assume that* $0<d\leq d_{\max}$, *and* $d_{\max}$ *is given.*
3. $\omega_{\max}$ *as the upper bound of the used frequencies, i.e.* $\omega\in[0,\omega_{\max}]$.
4. Ω *to be a bounded domain in* $\mathbb{R}^3$ *containing the small bodies* D_m, $m=1,\dots,M$.

1.2 The results for a homogeneous elastic background

We assume that $D_m=\epsilon B_m+z_m, m=1,\dots,M$, with the same diameter a, are non-flat Lipschitz obstacles, i.e. D_m's are Lipschitz obstacles and there exist constants $t_m\in(0,1]$ such that $B^3_{t_m\frac{a}{2}}(z_m)\subset D_m\subset B^3_{\frac{a}{2}}(z_m)$, where t_m are assumed to be uniformly bounded from below by a positive constant. In [7], we have shown that there exist two positive constants a_0 and c_0 depending only on the size of Ω, the Lipschitz character of $B_m, m=1,\dots,M$, $d_{\max}$ and $\omega_{\max}$ such that if

$$a\leq a_0\ \ \text{and}\ \ \sqrt{M-1}\frac{a}{d}\leq c_0 \tag{1.5}$$

then we have the following asymptotic expansion for the P-part, $U_p^\infty(\hat{x},\theta)$, and the S-part, $U_s^\infty(\hat{x},\theta)$, of the far-field pattern:

$$U_p^\infty(\hat{x},\theta)=\ \ \frac{1}{4\pi\, c_p^2}(\hat{x}\otimes\hat{x})\left[\sum_{m=1}^M e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_m}Q_m\right.$$

$$+O\left(M\left[a^2+\frac{a^3}{d^{5-3\alpha}}+\frac{a^4}{d^{9-6\alpha}}\right]+M(M-1)\left[\frac{a^3}{d^{2\alpha}}+\frac{a^4}{d^{4-\alpha}}+\frac{a^4}{d^{5-2\alpha}}\right]+M(M-1)^2\frac{a^4}{d^{3\alpha}}\right)\Bigg], \tag{1.6}$$

$$U_s^\infty(\hat{x},\theta) = \frac{1}{4\pi\, c_s^2}(\mathbf{I}-\hat{\mathrm{x}}\otimes\hat{\mathrm{x}})\left[\sum_{\mathrm{m}=1}^{\mathrm{M}}\mathrm{e}^{-\mathrm{i}\frac{\omega}{\mathrm{c_s}}\hat{\mathrm{x}}\cdot\mathrm{z_m}}\mathrm{Q_m}\right.$$

$$+O\left(M\left[a^2+\frac{a^3}{d^{5-3\alpha}}+\frac{a^4}{d^{9-6\alpha}}\right]+M(M-1)\left[\frac{a^3}{d^{2\alpha}}+\frac{a^4}{d^{4-\alpha}}+\frac{a^4}{d^{5-2\alpha}}\right]+M(M-1)^2\frac{a^4}{d^{3\alpha}}\right)\Bigg], \tag{1.7}$$

where α, $0<\alpha\leq 1$, is a parameter describing the relative distribution of the small bodies.

The vector coefficients Q_m, $m=1,...,M$, are the solutions of the following linear algebraic system

$$C_m^{-1}Q_m = -U^i(z_m,\theta)-\sum_{\substack{j=1\\ j\neq m}}^{M}\Gamma^\omega(z_m,z_j)Q_j, \tag{1.8}$$

for $m=1,...,M$, with Γ^ω denoting the Kupradze matrix of the fundamental solution to the Navier equation with frequency ω, $C_m:=\int_{\partial D_m}\sigma_m(s)ds$ and σ_m is the solution matrix of the integral equation of the first kind

$$\int_{\partial D_m}\Gamma^0(s_m,s)\sigma_m(s)ds = \mathbf{I},\ \mathrm{s_m}\in\partial\mathrm{D_m}, \tag{1.9}$$

with $\mathbf{I}$ the identity matrix of order 3.

Consider now the special case $d_{min}a^t\leq d\leq d_{max}a^t$ and $M\leq M_{max}a^{-s}$ with $t,s>0$, d_{min}, d_{max} and M_{max} are positive. Then the asymptotic expansions (1.6-1.7) can be rewritten as

$$U_p^\infty(\hat{x},\theta) = \frac{1}{4\pi\, c_p^2}(\hat{x}\otimes\hat{x})\Big[\sum_{m=1}^{M}e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_m}Q_m$$

$$+O\left(a^{2-s}+a^{3-s-5t+3t\alpha}+a^{4-s-9t+6t\alpha}+a^{3-2s-2t\alpha}+a^{4-3s-3t\alpha}+a^{4-2s-5t+2t\alpha}\right)\Big], \tag{1.10}$$

$$U_s^\infty(\hat{x},\theta) = \frac{1}{4\pi\, c_s^2}(\mathbf{I}-\hat{\mathrm{x}}\otimes\hat{\mathrm{x}})\Big[\sum_{\mathrm{m}=1}^{\mathrm{M}}\mathrm{e}^{-\mathrm{i}\frac{\omega}{\mathrm{c_s}}\hat{\mathrm{x}}\cdot\mathrm{z_m}}\mathrm{Q_m}$$

$$+O\left(a^{2-s}+a^{3-s-5t+3t\alpha}+a^{4-s-9t+6t\alpha}+a^{3-2s-2t\alpha}+a^{4-3s-3t\alpha}+a^{4-2s-5t+2t\alpha}\right)\Big]. \tag{1.11}$$

As the diameter a tends to zero the error term tends to zero for t and s such that

$$0<t<1 \text{ and } 0<s<\min\{2(1-t),\frac{7-5t}{4},\frac{12-9t}{7},\frac{20-15t}{12},\frac{4}{3}-t\alpha\}. \tag{1.12}$$

In [7], we have shown that $Q_m\approx a$, then we have the upper bound

$$|\sum_{m=1}^{M}e^{-i\kappa\hat{x}\cdot z_m}Q_m|\leq M\sup_{m=1,...,M}|Q_m|=O(a^{1-s}). \tag{1.13}$$

Hence if the number of obstacles is $M:=M(a):=O(a^{-s}),\ s<1$ and t satisfies (1.12), $a\to 0$, then from (1.10, 1.11), we deduce that

$$U^\infty(\hat{x},\theta)\to 0,\ \text{as } a\to 0,\ \text{uniformly in terms of } \theta \text{ and } \hat{x} \text{ in } \mathbb{S}^2. \tag{1.14}$$

This means that this collection of obstacles has no effect on the homogeneous medium as $a\to 0$.

Let us consider the case when $s=1$. We set Ω to be a bounded domain, say of unit volume, containing the obstacles $D_m, m=1,...,M$. Given a positive and continuous function $K:\mathbb{R}^3\to\mathbb{R}$, we divide Ω into

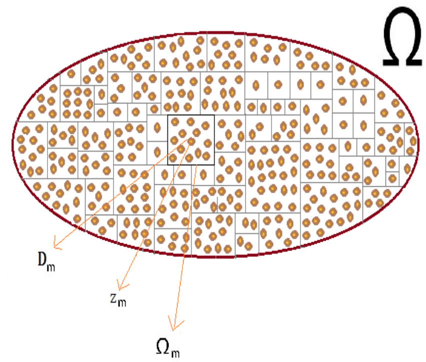

Figure 1: An example on how the obstacles are distributed in Ω.

$[a^{-1}]$ subdomains $\Omega_m,\ m = 1, ..., [a^{-1}]$, each of volume $a\frac{[K(z_m)+1]}{K(z_m)+1}$, with $z_m \in \Omega_m$ as its center and contains $[K(z_m) + 1]$ obstacles, see Fig 1.2. We set $K_{max} := \sup_{z_m}(K(z_m) + 1)$, hence $M = \sum_{j=1}^{[a^{-1}]}[K(z_m) + 1] \leq K_{max}[a^{-1}] = O(a^{-1})$.

Theorem 1.2. *Let the small obstacles be distributed in a bounded domain Ω, say of unit volume, with their number $M := M(a) := O(a^{-1})$ and their minimum distance $d := d(a) := O(a^t)$, $\frac{1}{3} \leq t < \frac{7}{12}$, as $a \to 0$, as described above.*

1. *If the obstacles are distributed arbitrarily in Ω, i.e. with different capacitances, then there exists a potential $\mathbf{C}_0 \in \cap_{p\geq 1} L^p(\mathbb{R}^3)$ with support in Ω such that*

$$\lim_{a\to 0} U^\infty(\hat{x}, \theta) = U_0^\infty(\hat{x}, \theta) \text{ uniformly in terms of } \theta \text{ and } \hat{x} \text{ in } \mathbb{S}^2 \tag{1.15}$$

where $U_0^\infty(\hat{x}, \theta)$ is the farfield corresponding to the scattering problem

$$(\Delta^e + \omega^2 - (K + 1)\mathbf{C}_0)U_0^t = 0 \text{ in } \mathbb{R}^3, \tag{1.16}$$

$$U_0^t|_{\partial D_m} = 0,\ 1 \leq m \leq M \tag{1.17}$$

with the radiation conditions

$$\lim_{|x|\to\infty} |x|(\frac{\partial U_{0,p}}{\partial |x|} - i\kappa_{p^\omega} U_{0,p}) = 0, \text{ and } \lim_{|x|\to\infty} |x|(\frac{\partial U_{0,s}}{\partial |x|} - i\kappa_{s^\omega} U_{0,s}) = 0 \tag{1.18}$$

2. *If in addition $K\mid_\Omega$ is in $C^{0,\gamma}(\Omega)$, $\gamma \in (0, 1]$ and the obstacles have the same capacitances C, then*

$$U^\infty(\hat{x}, \theta) = U_0^\infty(\hat{x}, \theta) + O(a^{\min\{\gamma, \frac{1}{3}, \frac{3}{2}-3t\}}) \text{ uniformly in terms of } \theta \text{ and } \hat{x} \text{ in } \mathbb{S}^2 \tag{1.19}$$

where $\mathbf{C}_0 = C$ in Ω and $\mathbf{C}_0 = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$.

1.3 The results for variable background elastic mass density

Assume that the Lamé coefficients λ and μ are constants satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$ and the mass density ρ to be a measurable and bounded function which is equal to a constant that we normalize to a unity outside of a bounded domain Ω. We set ρ_{max} to be the upper bound of ρ.

In this case, the total field $U^t_\rho := U^i + U^s_\rho$ satisfies the following exterior Dirichlet problem of the elastic waves

$$(\Delta^e + \omega^2\rho)U^t_\rho = 0 \text{ in } \mathbb{R}^3 \backslash \left(\overset{M}{\underset{m=1}{\cup}} \bar{D}_m\right), \tag{1.20}$$

$$U^t_\rho|_{\partial D_m} = 0,\ 1 \leq m \leq M \tag{1.21}$$

with the Kupradze radiation conditions (K.R.C)

$$\lim_{|x|\to\infty} |x|(\frac{\partial U_{\rho,p}}{\partial |x|} - i\kappa_{p^\omega} U_{\rho,p}) = 0, \text{ and } \lim_{|x|\to\infty} |x|(\frac{\partial U_{\rho,s}}{\partial |x|} - i\kappa_{s^\omega} U_{\rho,s}) = 0, \tag{1.22}$$

where the two limits are uniform in all the directions $\hat{x} := \frac{x}{|x|} \in \mathbb{S}^2$ and $U_{\rho,p}$ and $U_{\rho,s}$ are respectively the P-part and S-part of the scattered field $U^s_{\rho,p}$

The scattering problem (1.20-1.22) is well posed in the Hölder or Sobolev spaces, see [8, 9, 12, 13] for instance, and the scattered field U^s has the following asymptotic expansion:

$$U^s_\rho(x) := \frac{e^{i\kappa_{p^\omega}|x|}}{|x|} U^\infty_{\rho,p}(\hat{x}) + \frac{e^{i\kappa_{s^\omega}|x|}}{|x|} U^\infty_{\rho,s}(\hat{x}) + O(\frac{1}{|x|^2}),\ |x| \to \infty \tag{1.23}$$

uniformly in all directions $\hat{x} \in \mathbb{S}^2$. The longitudinal part of the far-field, i.e. $U^\infty_{\rho,p}(\hat{x})$ is normal to $\mathbb{S}^2$ while the transversal part $U^\infty_s(\hat{x})$ is tangential to $\mathbb{S}^2$. We set $U^\infty_\rho := (U^\infty_{\rho,p}, U^\infty_{\rho,s})$.

As in the case of constant background mass density, there exist two positive constants a_0 and c_0 depending only on the size of Ω, the Lipschitz character of $B_m, m = 1, \dots, M$, $d_{\max}$, $\omega_{\max}$, ρ_{max} and ρ_{max} such that if

$$a \leq a_0 \quad \text{and} \quad \sqrt{M-1}\frac{a}{d} \leq c_0 \tag{1.24}$$

then we have the following asymptotic expansion for the P-part, $U^\infty_{\rho,p}(\hat{x}, \theta)$, and the S-part, $U^\infty_{\rho,s}(\hat{x}, \theta)$, of the far-field pattern:

$$\begin{aligned} U^\infty_{\rho,p}(\hat{x}, \theta) = \quad & V^\infty_{\rho,p}(\hat{x}, \theta) \ + \ \left[\sum_{m=1}^{M} G^\infty_{\rho,p}(\hat{x}, z_m) Q_{\rho,m} \right. \\ & \left. + O\left(M\left[a^2 + \frac{a^3}{d^{5-3\alpha}} + \frac{a^4}{d^{9-6\alpha}}\right] + M(M-1)\left[\frac{a^3}{d^{2\alpha}} + \frac{a^4}{d^{4-\alpha}} + \frac{a^4}{d^{5-2\alpha}}\right] + M(M-1)^2\frac{a^4}{d^{3\alpha}}\right)\right], \end{aligned} \tag{1.25}$$

$$\begin{aligned} U^\infty_{\rho,s}(\hat{x}, \theta) = \quad & V^\infty_{\rho,s}(\hat{x}, \theta) \ + \ \left[\sum_{m=1}^{M} G^\infty_{\rho,s}(\hat{x}, z_m) Q_{\rho,m} \right. \\ & \left. + O\left(M\left[a^2 + \frac{a^3}{d^{5-3\alpha}} + \frac{a^4}{d^{9-6\alpha}}\right] + M(M-1)\left[\frac{a^3}{d^{2\alpha}} + \frac{a^4}{d^{4-\alpha}} + \frac{a^4}{d^{5-2\alpha}}\right] + M(M-1)^2\frac{a^4}{d^{3\alpha}}\right)\right], \end{aligned} \tag{1.26}$$

where $G^\infty_{\rho,p}(\hat{x}, z_m)$ and $G^\infty_{\rho,s}(\hat{x}, z_m)$ are the P-part and S-part of the farfields of the Green's function $G_\rho(x, z)$, of the operator $\Delta^e + \omega^2\rho$ in the whole space $\mathbb{R}^3$, evaluated in the direction $\hat{x}$ and the source point z_m.

The vector coefficients $Q_{\rho,m}$, $m = 1, ..., M$, are the solutions of the following linear algebraic system

$$C_m^{-1} Q_{\rho,m} \quad = \quad -V_\rho(z_m, \theta) - \sum_{\substack{j=1 \\ j\neq m}}^{M} G_\rho(z_m, z_j) Q_{\rho,j}, \tag{1.27}$$

for $m = 1, ..., M$, with $V_\rho(\cdot, \theta) := V^s_\rho(\cdot, \theta) + U^i(\cdot, \theta)$ is the total field satisfying

$$(\Delta^e + \omega^2\rho)V^t_\rho = 0 \text{ in } \mathbb{R}^3, \tag{1.28}$$

and the scattered field $V^s_\rho(\cdot, \theta)$ the Kupradze radiation conditions (K.R.C).

Corollary 1.3. *Let the small obstacles be distributed in a bounded domain Ω, say of unit volume, with their number $M := M(a) := O(a^{-1})$ and their minimum distance $d := d(a) := O(a^t)$, $\frac{1}{3} \leq t < \frac{7}{12}$, as $a \to 0$, as described above.*

1. *If the obstacles are distributed arbitrarily in Ω, i.e. with different capacitances, then there exists a potential $\mathbf{C}_0 \in \cap_{p\geq 1} L^p(\mathbb{R}^3)$ with support in Ω such that*

$$\lim_{a\to 0} U^\infty_\rho(\hat{x}, \theta) = U^\infty_{\rho,0}(\hat{x}, \theta) \text{ uniformly in terms of } \theta \text{ and } \hat{x} \text{ in } \mathbb{S}^2 \tag{1.29}$$

where $U^\infty_{\rho,0}(\hat{x}, \theta)$ is the farfield corresponding to the scattering problem

$$(\Delta^e + \omega^2 \rho - (K+1)\mathbf{C}_0) U^t_{\rho,0} = 0 \text{ in } \mathbb{R}^3, \tag{1.30}$$

$$U^t_{\rho,0}|_{\partial D_m} = 0,\ 1 \leq m \leq M \tag{1.31}$$

with the radiation conditions.

2. *If in addition $K\,|_\Omega$ is in $C^{0,\gamma}(\Omega)$, $\gamma \in (0, 1]$ and the obstacles have the same capacitances, then*

$$U^\infty_\rho(\hat{x}, \theta) = U^\infty_{\rho,0}(\hat{x}, \theta) + O(a^{\min\{\gamma, \frac{1}{3}, \frac{3}{2}-3t\}}) \text{ uniformly in terms of } \theta \text{ and } \hat{x} \text{ in } \mathbb{S}^2 \tag{1.32}$$

where $\mathbf{C}_0 = C$ in Ω and $\mathbf{C}_0 = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$.

1.4 Applications of the results and a comparison to the literature

The main contribution of this work is to have shown that by removing from a bounded region of an elastic background, modeled by constant Lamé coefficients λ and μ and a possibly variable density ρ, a number $M := M(a) \sim a^{-1}$ small and rigid bodies of radius a distant from each other of at least $d := d(a) \sim a^t$, $\frac{1}{3} \leq t \leq \frac{7}{12}$, then the 'perforated' medium behaves, as $a \to 0$, as a new elastic medium modeled by the same Lamé coefficients λ and μ but with a mass density-like $\rho - (K+1)\mathbf{C}_0\omega^{-2}$. The coefficient K models the local distribution (or the local number) of the bodies while the coefficient $\mathbf{C}_0$, coming from the capacitance of the bodies, describes the geometry of the small bodies as well as their elastic directional diffusion properties (i.e. the anisotropy character). In addition, we provide explicit error estimates between the far-fields corresponding to the perforated medium and the equivalent one. From this result we can make the following conclusions:

1. Assume that the removed bodies have spherical shapes. For these shapes the corresponding elastic capacitance C is of the form cI_3 (i.e. a scalar multiplied by the identity matrix). In section 4, we describe a more general set of shapes satisfying this property. Hence the equivalent mass density $\rho - (K+1)c\omega^{-2}$ is isotropic while for general shapes it might be anisotropic. To achieve anisotropic densities, a possible choice of the shapes might be an ellipse.

2. If we choose the local number of bodies K large enough or the shapes of the reference bodies, B_m, $m = 1, ..., M$, having a large capacitance (i.e. a relative large radius) so that $\rho - (K+1)c\omega^{-2} < 0$, then we design elastic materials having negative mass densities.

3. Assume that the background medium is modeled by variable mass density $\rho > 1$ in Ω. If we remove small bodies from Ω with appropriate K and/ or capacitance $\mathbf{C}_0$ so that $\rho - (K+1)c\omega^{-2} = 1$, then the new elastic material will behave every where in $\mathbb{R}^3$ as the background medium. Hence the new material will not scatter the sent incident waves, i.e. the region Ω modeled by ρ will be cloaked.

The 'equivalent' behavior between a collection of, appropriately dense, small holes and an extended penetrable obstacle modeled by an additive potential was already observed by Cioranescu and Murat [10,11] and also the references therein, where the coefficient K is reduced to zero since locally they have only one hole. Their analysis is based on the homogenization theory for which they assume that the obstacles are distributed periodically, see also [5] and [16].

In the results presented here, we do not need such periodicity and no homogenization is used. Instead, the analysis is based on the invertibility properties of the algebraic system (1.8) and the precise treatment of the summation in the dominant terms of (1.6)-(1.7). This analysis was already tested for the acoustic model in [1]. Compared to [1], here, in addition to the difficulties coming from the vector character of the Lamé system, we improved the order of the error estimate, i.e. $O(a^{\min\{\gamma, \frac{1}{3}, \frac{3}{2}-3t\}})$ instead of $O(a^{\min\{\gamma, \frac{1}{3}-\frac{4}{5}t\}})$ which , for $t := \frac{1}{3}$ for instance, reduce to $O(a^{\min\{\gamma, \frac{1}{3}\}})$ and $O(a^{\min\{\gamma, \frac{1}{15}\}})$ respectively.

Let us finally mention that a result similar to (1.29), for the acoustic model, is also derived by Ramm in several of his papers, see for instance [17], but without error estimates. Compared to his results, and as we said earlier in addition to the vector character of Lame model, we provide the approximation by improved explicit error estimates without any other assumptions while, as shown in [17, 18] for instance, in addition to some formal arguments, he needs extra assumptions on the distribution of the obstacles.

The rest of the paper is organized as follows. In section 2, we give the detailed proof of Theorem 1.2. In section 3, we describe the one of Corollary 1.3 by discussing the main changes one needs to make in the proof of Theorem 1.2. Finally, in section 4, we discuss some invariant properties of the elastic capacitance to characterize the shapes that have a 'scalar' capacitance.

2 Proof of Theorem 1.2

2.1 The fundamental solution

The Kupradze matrix $\Gamma^\omega = (\Gamma^\omega_{ij})^3_{i,j=1}$ of the fundamental solution to the Navier equation is given by

$$\Gamma^\omega(x,y) = \frac{1}{\mu}\Phi_{\kappa_{s^\omega}}(x,y)\mathbf{I} + \frac{1}{\omega^2}\nabla_{\rm x}\nabla_{\rm x}^\top[\Phi_{\kappa_{s^\omega}}({\rm x,y}) - \Phi_{\kappa_{p^\omega}}({\rm x,y})], \tag{2.1}$$

where $\Phi_\kappa(x,y) = \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$ denotes the free space fundamental solution of the Helmholtz equation $(\Delta+\kappa^2)\,u = 0$ in $\mathbb{R}^3$. The asymptotic behavior of Kupradze tensor at infinity is given as follows

$$\Gamma^\omega(x,y) = \frac{1}{4\pi\, c_p^2}\hat{x}\otimes\hat{x}\frac{e^{i\kappa_{p^\omega}|x|}}{|x|}e^{-i\kappa_{p^\omega}\hat{x}\cdot y} + \frac{1}{4\pi\, c_s^2}(\mathbf{I}-\hat{\rm x}\otimes\hat{\rm x})\frac{{\rm e}^{{\rm i}\kappa_{s^\omega}|{\rm x}|}}{|{\rm x}|}{\rm e}^{-{\rm i}\kappa_{s^\omega}\hat{\rm x}\cdot{\rm y}} + {\rm O}(|{\rm x}|^{-2}) \tag{2.2}$$

with $\hat{x} = \frac{x}{|x|} \in \mathbb{S}^2$ and $\mathbf{I}$ being the identity matrix in $\mathbb{R}^3$, see [2] for instance. As mentioned in [4], (2.1) can also be represented as

$$\begin{aligned}\Gamma^\omega(x,y) \quad &= \quad \frac{1}{4\pi}\sum_{l=0}^\infty \frac{i^l}{l!(l+2)}\frac{1}{\omega^2}\left((l+1)\kappa_{s^\omega}^{l+2} + \kappa_{p^\omega}^{l+2}\right)|x-y|^{l-1}\mathbf{I} \\ &\quad -\frac{1}{4\pi}\sum_{l=0}^\infty \frac{i^l}{l!(l+2)}\frac{(l-1)}{\omega^2}\left(\kappa_{s^\omega}^{l+2} - \kappa_{p^\omega}^{l+2}\right)|x-y|^{l-3}(x-y)\otimes(x-y),\end{aligned} \tag{2.3}$$

from which we can get the gradient

$$\begin{aligned}\nabla_y\Gamma^\omega(x,y) \quad &= \quad -\frac{1}{4\pi}\sum_{l=0}^\infty \frac{i^l}{l!(l+2)}\frac{(l-1)}{\omega^2}\Big[\left((l+1)\kappa_{s^\omega}^{l+2} + \kappa_{p^\omega}^{l+2}\right)|x-y|^{l-3}(x-y)\otimes\mathbf{I} \\ &\quad - \left(\kappa_{s^\omega}^{l+2} - \kappa_{p^\omega}^{l+2}\right)|x-y|^{l-3}\left((l-3)|x-y|^{-2}\otimes^3(x-y) + \mathbf{I}\otimes({\rm x-y}) + ({\rm x-y})\otimes\mathbf{I}\right)\Big].\end{aligned} \tag{2.4}$$

Using the formulas (2.3) and (2.4) we can have the following estimates, for $x, y \in \Omega, x \neq y$, see [7];

$$|\Gamma^{\omega}(x,y)| \leq \frac{1}{4\pi}\left[\frac{C_7}{|\,x-y|}+C_8\right], \qquad |\nabla_y \Gamma^{\omega}(x,y)| \leq \frac{1}{4\pi}\left[\frac{C_9}{|\,x-y|^2}+C_{10}\right], \tag{2.5}$$

with

$$C_7 := \left[\frac{1}{c_s^2}+\frac{2}{c_p^2}\right], \qquad C_9 := 3\left(\frac{1}{c_s^2}+\frac{1}{c_p^2}\right),$$

$$C_8 := 2\frac{\kappa_{s^\omega}}{c_s^2}\left(\frac{1-\left(\frac{1}{2}\kappa_{s^\omega} diam(\Omega)\right)^{N_\Omega}}{1-\frac{1}{2}\kappa_{s^\omega} diam(\Omega)}+\frac{1}{2^{N_\Omega-1}}\right)+\frac{\kappa_{p^\omega}}{c_p^2}\left(\frac{1-\left(\frac{1}{2}\kappa_{p^\omega} diam(\Omega)\right)^{N_\Omega}}{1-\frac{1}{2}\kappa_{p^\omega} diam(\Omega)}+\frac{1}{2^{N_\Omega-1}}\right),$$

$$C_{10} := 2\frac{\omega^2}{c_s^4}\left(\frac{1}{8}+\frac{1-\left(\frac{1}{2}\kappa_{s^\omega} diam(\Omega)\right)^{N_\Omega}}{1-\frac{1}{2}\kappa_{s^\omega} diam(\Omega)}+\frac{1}{2^{N_\Omega-1}}\right)+\frac{\omega^2}{c_p^4}\left(\frac{1}{4}+\frac{1-\left(\frac{1}{2}\kappa_{p^\omega} diam(\Omega)\right)^{N_\Omega}}{1-\frac{1}{2}\kappa_{p^\omega} diam(\Omega)}+\frac{1}{2^{N_\Omega-1}}\right),$$

and $N_\Omega = [2diam(\Omega)\max\{\kappa_{s^\omega}, \kappa_{p^\omega}\}e^2]$ where we assume the wave number ω and the Lamé parameters λ and μ to satisfy the condition $\max\{\kappa_{s^\omega},\ \kappa_{p^\omega}\} < \frac{2}{diam(\Omega)}$.

The estimates (2.5) can be written as

$$|\Gamma^{\omega}(x,y)| \leq \frac{\mathring{c}}{4\pi|\,x-y|}, \qquad |\nabla_y \Gamma^{\omega}(x,y)| \leq \frac{\mathring{c}}{4\pi|\,x-y|^2}, \tag{2.6}$$

for different points $x, y \in \Omega$, where

$$\mathring{c} := \max\{C_7, C_8\ diam(\Omega), C_8, C_{10}\ diam(\Omega)\}. \tag{2.7}$$

2.2 The relative distribution of the small bodies

The following observation will be useful for the proof of Theorem 1.2. For $m = 1, \dots, M$ fixed, we distinguish between the obstacles D_j, $j \neq m$ by keeping them into different layers based on their distance from D_m. Let us first assume that $K(z_m) = 0$ for every z_m. Hence each Ω_m has the (same) volume a and contains only one obstacle D_m. Without loss of generality, we can take the Ω_m's as cubes. Hence we can suppose that these cubes are arranged in cuboids, for example unit rubics cube, in different layers such that the total cubes upto the n^{th} layer consists $(2n+1)^3$ cubes for $n = 0, \dots, [\left(a^{\frac{1}{3}} - \frac{a}{2}\right)^{-1}]$, and Ω_m is located on the center, see Fig 1.2. Hence the number of obstacles located in the n^{th}, $n \neq 0$ layer will be $[(2n+1)^3 - (2n-1)^3] = 24n^2 + 2$ and their distance from D_m is more than $n\left(a^{\frac{1}{3}} - \frac{a}{2}\right)$. Observe that, $\frac{1}{2}a^{\frac{1}{3}} \leq \left(a^{\frac{1}{3}} - \frac{a}{2}\right) \leq\ a^{\frac{1}{3}}$.

Now, we come back to the case where $K(z_m) \neq 0$. First observe that $\frac{1}{2} \leq \frac{[K(z_m)+1]}{K(z_m)+1} \leq 1$. Hence with such Ω_m's, the total cubes located in the n'th layer n consists of at most the double of $[(2n+1)^3 - (2n-1)^3]$, i.e. $48n^2 + 4$.

2.3 Solvability of the linear-algebraic system (1.8)

We start with the following lemma on the uniform bounds of the elastic capacitances, see [7, Lemma 3.1] and the references there in.

Lemma 2.1. *Let $\lambda_{eig_m}^{min}$ and $\lambda_{eig_m}^{max}$ be the minimal and maximal eigenvalues of the elastic capacitance matrices $\bar{C}_m$, for $m = 1, 2, \dots, M$. Denote by C_m^a the capacitance of each scatterer in the acoustic case,*[1] *then we have the following estimate;*

$$\frac{\mu C(B_m)}{\max\limits_{1\leq m\leq M} diam(B_m)}\,a = \mu\, C_m^a\ \leq\ \lambda_{eig_m}^{min}\ \leq\ \lambda_{eig_m}^{max}\ \leq\ (\lambda+2\mu)\, C_m^a = \frac{(\lambda+2\mu)C(B_m)}{\max\limits_{1\leq m\leq M} diam(B_m)}\,a, \tag{2.8}$$

[1] Recall that, for $m = 1, \dots, M$, $C_m^a := \int_{\partial D_m}\sigma_m(s)ds$ and σ_m is the solution of the integral equation of the first kind $\int_{\partial D_m}\frac{\sigma_m(s)}{4\pi|t-s|}ds = 1,\ t \in \partial D_m$, see [6].

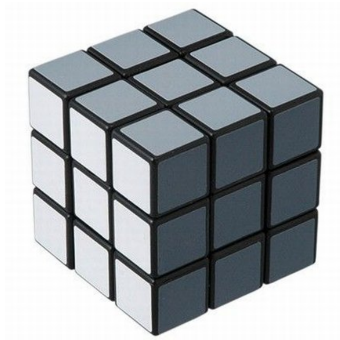

Figure 2: Rubics cube consisting of two layers

for $m = 1, 2, \ldots, M$.

The constant $C(B_m)$ is the acoustic capacitance of the reference body B_m which can be estimated above and below by the Lipschitz character of B_m, [6]. The following lemma provides us with the needed estimate on the invertibility of the algebraic system (1.8) whose coefficient matrix '**B**' is given by;

$$\mathbf{B} := \begin{pmatrix} -\bar{C}_1^{-1} & -\Gamma^\omega(z_1, z_2) & -\Gamma^\omega(z_1, z_3) & \cdots & -\Gamma^\omega(z_1, z_M) \\ -\Gamma^\omega(z_2, z_1) & -\bar{C}_2^{-1} & -\Gamma^\omega(z_2, z_3) & \cdots & -\Gamma^\omega(z_2, z_M) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\Gamma^\omega(z_M, z_1) & -\Gamma^\omega(z_M, z_2) & \cdots & -\Gamma^\omega(z_M, z_{M-1}) & -\bar{C}_M^{-1} \end{pmatrix} \tag{2.9}$$

Lemma 2.2. *The matrix* $\mathbf{B}$ *is invertible and the solution vector* Q_m $m = 1, ..., M$, *of* (1.8) *satisfies the estimate:*

$$\sum_{m=1}^{M} \|Q_m\|_2^2 \leq \left((\lambda + 2\mu)^{-1} - \frac{\sqrt{26 M_{max}}\, \mathring{c}\, (\max_m C_m^a)\ a^{-1}}{\pi} \right)^{-2} (C_m^a)^2 \sum_{m=1}^{M} \|U^i(z_m)\|_2^2, \tag{2.10}$$

if $\max C_m^a\ a^{-1} < \frac{\pi}{\sqrt{26 M_{max}}\, \mathring{c}(\lambda + 2\mu)}$. *In addition,*

$$\sum_{m=1}^{M} \|Q_m\|_1 \leq \left((\lambda + 2\mu)^{-1} - \frac{\sqrt{26 M_{max}}\, \mathring{c}\, (\max_m C_m^a)\ a^{-1}}{\pi} \right)^{-1} \max C_m^a\ M \max_{m=1}^{M} \|U^i(z_m)\|_2. \tag{2.11}$$

Since $C_m^a = C^a(B_m)\epsilon = \frac{C^a(B_m)}{\max_m diam(B_m)} a$, then

$$\max_m C_m^a\ a^{-1} < \frac{\pi}{\sqrt{26 M_{max}}\, \mathring{c}(\lambda + 2\mu)} \tag{2.12}$$

makes sense if $\frac{\max_m C^a(B_m)}{\max_m diam(B_m)} < \frac{\pi}{\sqrt{26 M_{max}}\, \mathring{c}(\lambda + 2\mu)}$. As $C_m(B_m)$ is proportional to the radius of B_m, then (2.12) will be satisfied if ω and the Lamé parameters λ and μ satisfy the condition

$$\mathring{c}(\lambda + 2\mu) < \frac{\pi}{\sqrt{26 M_{max}}} \frac{\max_m diam(B_m)}{\max_m C^a(B_m)} \tag{2.13}$$

recalling that $\mathring{c}$ is defined in (2.7). Finally, let us observe that the right hand side of (2.13) depends only $M_{max}$and the Lipschitz character of the reference obstacles B_m's.

Proof of Lemma 2.2. We start by factorizing $\mathbf{B}$ as $\mathbf{B} = -(\mathbf{C}^{-1}+\mathbf{B}_n)$ where $\mathbf{C} := Diag(\bar{C}_1, \bar{C}_2, \dots, \bar{C}_M) \in \mathbb{R}^{M\times M}$, I is the identity matrix and $\mathbf{B}_n := -\mathbf{C}^{-1} - \mathbf{B}$. We have $\mathbf{B} : \mathbb{C}^{3M} \to \mathbb{C}^{3M}$, so it is enough to prove the injectivity in order to prove its invetibility. For this purpose, let X, Y are vectors in $\mathbb{C}^{3M}$ and consider the system

$$(\mathbf{C}^{-1} + \mathbf{B}_n)X \quad = \quad Y. \tag{2.14}$$

We denote by $(\cdot)^{real}$ and $(\cdot)^{img}$ the real and the imaginary parts of the corresponding complex vector/matrix. we also set $\mathbf{C}^{-1}$ by $\mathbf{C}_I$. From (2.14) we derive the following two identities:

$$(\mathbf{C}_I + \mathbf{B}_n^{real})X^{real} - (\mathbf{C}_I + \mathbf{B}_n^{img})X^{img} \quad = \quad Y^{real}, \tag{2.15}$$

$$(\mathbf{C}_I + \mathbf{B}_n^{real})X^{img} + (\mathbf{C}_I + \mathbf{B}_n^{img})X^{real} \quad = \quad Y^{img}, \tag{2.16}$$

and then

$$\langle (\mathbf{C}_I + \mathbf{B}_n^{real})X^{real}, X^{real}\rangle \; - \langle (\mathbf{C}_I + \mathbf{B}_n^{img})X^{img}, X^{real}\rangle \quad = \quad \langle Y^{real}, X^{real}\rangle, \tag{2.17}$$

$$\langle (\mathbf{C}_I + \mathbf{B}_n^{real})X^{img}, X^{img}\rangle \; + \langle (\mathbf{C}_I + \mathbf{B}_n^{img})X^{real}, X^{img}\rangle \quad = \quad \langle Y^{img}, X^{img}\rangle. \tag{2.18}$$

Summing up (2.17) and (2.18) we obtain

$$\begin{aligned}\langle \mathbf{C}_I^{real}X^{real}, X^{real}\rangle + \langle \mathbf{B}_n^{real}X^{real}, X^{real}\rangle + \langle \mathbf{C}_I^{real}X^{img}, X^{img}\rangle + \langle \mathbf{B}_n^{real}X^{img}, X^{img}\rangle \\ = \langle Y^{real}, X^{real}\rangle + \langle Y^{img}, X^{img}\rangle.\end{aligned} \tag{2.19}$$

The right-hand side in (2.19) can be estimated as

$$\begin{aligned}\langle X^{real}, X^{real}\rangle^{1/2}\langle Y^{real}, Y^{real}\rangle^{1/2} + \langle X^{img}, X^{img}\rangle^{1/2}\langle Y^{img}, Y^{img}\rangle^{1/2} \\ \leq 2\langle X^{|\cdot|}, X^{|\cdot|}\rangle^{1/2}\langle Y^{|\cdot|}, Y^{|\cdot|}\rangle^{1/2}.\end{aligned} \tag{2.20}$$

Let us now consider the right hand side in (2.19). First we have

$$|\langle B_n^{real}X^{real},\; X^{real}\rangle| \leq \|B_n^{real}\|_2 |X^{real}|_2^2 \tag{2.21}$$

where $\|B_n^{real}\|_2^2 := \sum_{i,\, j=1}^{M} |(B_n^{real})_{i,j}|_2^2$ and $(B_n^{real})_{i,j} := \Re\; \Gamma^{\omega}(z_i, z_j)$ if $i \neq j$ and $(B_n^{real})_{i,i} := 0$ for $i, j = 1, ..., M$. Hence from (2.3) $|(B_n^{real})_{i,j}|_2 \leq \frac{\mathring{c}}{4\pi|z_i - z_j|}$, $i \neq j$. From the observation before Lemma 2.1, we deduce that

$$\sum_{i,j=1}^{M} (B_n^{real})_{i,j}^2 \leq M \sum_{n=1}^{[2a^{-\frac{1}{3}}]} 2[(2n+1)^3 - (2n-1)^3] \frac{\mathring{c}^2}{(4\pi)^2 n^2 \left(\frac{a^{\frac{1}{3}}}{2}\right)^2} \leq 13\mathring{c}^2 M \frac{a^{-\frac{2}{3}}}{\pi^2} \sum_{n=1}^{[2a^{-\frac{1}{3}}]} 1 = \frac{26 M_{max}\mathring{c}^2 a^{-2}}{\pi^2} \tag{2.22}$$

or

$$\|B_n^{real}\|_2 \leq \frac{\sqrt{26 M_{max}}\,\mathring{c}}{\pi} a^{-1} \tag{2.23}$$

and then

$$|\langle B_n^{real}X^{real},\; X^{real}\rangle| + |\langle B_n^{real}X^{img},\; X^{img}\rangle| \leq \left(\frac{\sqrt{26 M_{max}}\,\mathring{c}}{\pi} a^{-1}\right)^2 |X|_2^2 \tag{2.24}$$

Using Lemma 2.1, we deduce that

$$\langle \mathbf{C}_I X^{real}, X^{real}\rangle \; + \langle \mathbf{C}_I X^{img}, X^{img}\rangle \; \geq (\lambda + 2\mu)^{-1} (\max C_m^a)^{-1} |X|_2^2 \tag{2.25}$$

From (2.19), (2.20), (2.24) and (2.25), we deduce that

$$\sum_{m=1}^{M}|X_m|^2 \leq \left((\lambda+2\mu)^{-1}(\max C_m^a)^{-1} - \frac{\sqrt{26M_{max}}\,\mathring{c}\,a^{-1}}{\pi}\right)^{-2}\sum_{m=1}^{M}|Y_m|^2 \tag{2.26}$$

if $(\lambda+2\mu)^{-1}(\max C_m^a)^{-1} > \frac{\sqrt{26M_{max}}\,\mathring{c}\,a^{-1}}{\pi}$ and then the matrix $\mathbf{B}$ in algebraic system (1.8) is invertible.

□

2.4 The limiting model

From the function K, we define a bounded function $K_a : \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$K_a(x) := K_a(z_m) := \begin{cases} K(z_m)+1 & \text{if} \quad x \in \Omega_m \\ 0 & \text{if} \quad x \notin \Omega_m \text{ for any } m = 1,\dots,[a^{-1}]. \end{cases} \tag{2.27}$$

Hence each Ω_m contains $[K_a(z_m)]$ obstacles and $K_{max} := \sup_{z_m} K_a(z_m)$.

Let $\mathbf{C}_a$ be the 3×3 matrix having entries as piecewise constant functions such that $\mathbf{C}_a|_{\Omega_m} = \bar{C}_m$ for all $m = 1,\dots,M$ and vanishes outside Ω. Here, $\bar{C}_m$ are the capacitances of B_m's. From [7], we can observe that $\bar{C}_m$ are defined through defined through $C_m := \bar{C}_m a$, and are independent of a.

We set

$$\mathcal{C} := \max_{1\leq m\leq M} \|\bar{C}_m\|_\infty. \tag{2.28}$$

Consider the Lippmann-Schwinger equation

$$Y_a(z) + \int_\Omega \Gamma^\omega(z,y)K_a(y)\mathbf{C}_a(y)Y_a(y)dy \quad = \quad -U^i(z,\theta), z\in\Omega \tag{2.29}$$

and set the Lamé potential

$$V(Y)(x) := \int_\Omega \Gamma^\omega(x,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy, \qquad x\in\mathbb{R}^3. \tag{2.30}$$

The coefficients K_a and $\mathbf{C}_a$ are uniformly bounded. The next lemma concerns the mapping properties of the Lamé potential. These properties are proved for the scalar Poisson potential in [8], for instance. Similar arguments are applicable for the Lamé potential as well, so we omit to give the details.

Lemma 2.3. *The operator $V : L^2(\Omega) \to H^2(\Omega)$ is well defined and it is a bounded operator for any bounded domain Ω in $\mathbb{R}^3$, i.e. there exists a positive constant c_0 such that*

$$\|V(Y)\|_{H^2(\Omega)} \leq c_0\|Y\|_{L^2(\Omega)}. \tag{2.31}$$

We have also the following lemma.

Lemma 2.4. *There exists one and only one solution Y of the Lippmann-Schwinger equation (2.29) and it satisfies the estimate*

$$\|Y\|_{L^\infty(\Omega)} \leq C\|U^i\|_{H^2(\Omega)} \quad \text{and} \quad \|\nabla Y\|_{L^\infty(\Omega)} \leq C'\|U^i\|_{H^2(\tilde{\Omega})}, \tag{2.32}$$

where $\tilde{\Omega}$ being a large bounded domain which contains $\bar{\Omega}$.

Proof of Lemma 2.4

Using the Lemma 2.3, we see that $I+V : L^2(\Omega) \longmapsto L^2(\Omega)$ is Fredholm with index zero and then we can apply the Fredholm alternative to $I+V : L^2(\Omega) \longmapsto L^2(\Omega)$. The uniqueness is a consequence of the uniqueness of the scattering problem corresponding to the model

$$(\Delta^e + \omega^2 I - K_a \mathbf{C}_a)Y = 0, \quad \text{in } \mathbb{R}^3 \tag{2.33}$$

where $Y := Y^i + Y^s$ and Y^s satisfies the Kupradze radiation conditions and Y^i is an incident field.

The estimate (2.32) can be derived, as it is done in [1] for the acoustic case, by coupling the invertibility of $I+V : L^2(\Omega) \longmapsto L^2(\Omega)$ and the $W^{2,p}-$ interior estimates of the solutions of the system $(\Delta^e + \omega^2 I - K_a \mathbf{C}_a)Y = 0$.

□

2.4.1 Case when the obstacles are arbitrarily distributed

The capacitances of the obstacles B_j, i.e. C_j are bounded by their Lipschitz constants, see [6], and we assumed that these Lipschitz constants are uniformly bounded. Hence $\mathbf{C}_a$ is bounded in $L^2(\Omega)$ and then there exists a function $\mathbf{C}_0$ in $L^2(\Omega)$ (actually in every $L^p(\Omega)$) such that $\mathbf{C}_a$ converges weakly to $\mathbf{C}_0$ in $L^2(\Omega)$. Now, since K is continuous hence K_a converges to $(K+1)$ in $L^\infty(\Omega)$ and hence in $L^2(\Omega)$. Then we can show that $K_a\mathbf{C}_a$ converges to $(K+1)\mathbf{C}_0$ in $L^2(\Omega)$.

Since $K\mathbf{C}_a$ is bounded in $L^\infty(\Omega)$, then from the invertibility of the Lippmann-Schwinger equation and the mapping properties of the Lamé potential, see Lemma 2.4, we deduce that $\|U^t_a\|_{H^2(\Omega)}$ is bounded and in particular, up to a sub-sequence, U^t_a tends to U^t_0 in $L^2(\Omega)$. From the convergence of $K_a\mathbf{C}_a$ to $(K+1)\mathbf{C}_0$ and the one of U^t_a to U^t_0 and (2.29), we derive the following equation satisfied by $U^t_0(x)$

$$U^t_0(x) + \int_\Omega (K_a)(y)\Gamma^\omega(x,y)\mathbf{C}_0(y)U^t_0(y)dy = -U^i(x,\theta) \text{ in } \Omega.$$

This is the Lippmann-Schwinger equation corresponding to the scattering problem $(\Delta^e + \omega^2 - (K+1)\mathbf{C}_0)U^t_0 = 0$ in $\mathbb{R}^3$, $U^t_0 = U^s_0 + U^i$, and U^s satisfies the Kupradze radiation conditions. As the corresponding farfields are of the form

$$U^\infty_{0,p}(\hat{x},\theta) = \int_\Omega \frac{1}{4\pi\, c_p^2}(\hat{x}\otimes\hat{x})e^{-i\frac{\omega}{c_p}\hat{x}\cdot y}(K+1)\mathbf{C}_0(y)U^t_0(y)dy$$

and the ones of U^t_a are of the form

$$U^\infty_{a,p}(\hat{x},\theta) = \int_\Omega \frac{1}{4\pi\, c_p^2}(\hat{x}\otimes\hat{x})e^{-i\frac{\omega}{c_p}\hat{x}\cdot y}K_a\mathbf{C}_a(y)U^t_a(y)dy$$

we deduce that

$$U^\infty_{a,p}(\hat{x},\theta) - U^\infty_{0,p}(\hat{x},\theta) = o(1), \ a \to 0, \text{ uniformly in terms of } \hat{x},\theta \ \in \mathbb{S}^2.$$

2.4.2 Case when K is Hölder continuous

If we assume that $K \in C^{0,\gamma}(\Omega)$, $\gamma \in (0,1]$, then we have the estimate $\|(K+1) - K_a\|_{L^\infty(\Omega)} \leq Ca^\gamma$, $a << 1$. Since the capacitances of the obstacles are assumed to be equal, we set $\mathbf{C}_0$ to be a constant in Ω and $\mathbf{C}_0 = 0$ in $\mathbb{R}^3 \setminus \Omega$. Recall that U_0 and U_a are solutions of the Lippmann-Schwinger equations

$$U_0 + \int_\Omega (K+1)\Gamma^\omega(x,y)\mathbf{C}_0(y)U^t_0(y)dy = U^i$$

and

$$U_a + \int_\Omega K_a\Gamma^\omega(x,y)\mathbf{C}_0(y)U^t_a(y)dy = U^i.$$

From the estimate $\|(K+1)-K_a\|_{L^\infty(\Omega)} \leq Ca^\gamma,\ a << 1$, we derive the estimate

$$U_0^\infty(\hat{x},\theta) - U_a^\infty(\hat{x},\theta) = O(a^\gamma),\ a << 1,\ \text{uniformly in terms of } \hat{x},\theta\ \in \mathbb{S}^2. \tag{2.34}$$

2.5 The approximation by the algebraic system

For each $m = 1, \dots, M$, we rewrite the equation (2.29) as follows

$$\begin{aligned} U_a(z_m) &+ \sum_{\substack{j=1\\ j\neq m}}^{M} \Gamma^\omega(z_m,z_j)\bar{C}_j U_a(z_j) a \\ &= U^i(z_m,\theta) + \sum_{\substack{j=1\\ j\neq m}}^{M} \Gamma^\omega(z_m,z_j)\bar{C}_j U_a(z_j) a - \sum_{\substack{j=1\\ j\neq m}}^{[a^{-1}]} \Gamma^\omega(z_m,z_j)K_a(z_j)\bar{C}_j U_a(z_j) Vol(\Omega_j) \\ &\quad + \sum_{\substack{j=1\\ j\neq m}}^{[a^{-1}]} \Gamma^\omega(z_m,z_j)K_a(z_j)\bar{C}_j U_a(z_j) Vol(\Omega_j) - \int_\Omega \Gamma^\omega(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy. \end{aligned} \tag{2.35}$$

Let us estimate the following quantities:

$$A := \sum_{\substack{j=1\\ j\neq m}}^{[a^{-1}]} \Gamma^\omega(z_m,z_j)K_a(z_j)\bar{C}_j U_a(z_j) Vol(\Omega_j) - \int_\Omega \Gamma^\omega(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy$$

and

$$B := \sum_{\substack{j=1\\ j\neq m}}^{M} \Gamma^\omega(z_m,z_j)\bar{C}_j U_a(z_j) a - \sum_{\substack{j=1\\ j\neq m}}^{[a^{-1}]} \Gamma^\omega(z_m,z_j)K_a(z_j)\bar{C}_j U_a(z_j) Vol(\Omega_j).$$

2.5.1 Estimate of A

By the decomposition of Ω, $\Omega := \cup_{l=1}^{[a^{-1}]}$, we have

$$\int_\Omega \Gamma^\omega(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy = \sum_{l=1}^{[a^{-1}]} \int_{\Omega_l} \Gamma^\omega(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy. \tag{2.36}$$

$$\begin{aligned} \text{Hence,}\quad A := &\int_{\Omega_m} \Gamma^\omega(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy \\ &+ \sum_{\substack{j=1\\ j\neq m}}^{[a^{-1}]} \left[\Gamma^\omega(z_m,z_j)K_a(z_j)\bar{C}_j U_a(z_j) Vol(\Omega_j) - \int_{\Omega_j} \Gamma^\omega(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy \right]. \end{aligned} \tag{2.37}$$

For $l \neq m$, we have

$$\begin{aligned} &\int_{\Omega_l} \Gamma^\omega(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy - \Gamma^\omega(z_m,z_l)K_a(z_l)\bar{C}_l U_a(z_l) Vol(\Omega_l) \\ &\qquad = K_a(z_l)\bar{C}_l \int_{\Omega_l} \left[\Gamma^\omega(z_m,y)U_a(y) - \Gamma^\omega(z_m,z_l)U_a(z_l)\right] dy. \end{aligned} \tag{2.38}$$

We set $f(z_m,y)=\Gamma^\omega(z_m,y)U_a(y)$ then every component $f_i(z_m,y)$ of $f(z_m,y)$ satisfies

$$f_i(z_m,y)-f_i(z_m,z_l)=(y-z_l)R_l^i(z_m,y)$$

where

$$\begin{aligned}
R_l^i(z_m,y) &= \int_0^1 \nabla_y f_i(z_m,y-\beta(y-z_l))\,d\beta \\
&= \int_0^1 \sum_{j=1}^3 \nabla_y\left[\Gamma^\omega_{i,j}(z_m,y-\beta(y-z_l))U_{a,j}(y-\beta(y-z_l))\right]\,d\beta \\
&= \int_0^1 \sum_{j=1}^3 \left[\nabla_y\Gamma^\omega_{i,j}(z_m,y-\beta(y-z_l))\right]U_{a,j}(y-\beta(y-z_l))\,d\beta \\
&\quad + \int_0^1 \sum_{j=1}^3 \Gamma^\omega_{i,j}(z_m,y-\beta(y-z_l))\left[\nabla_y U_{a,j}(y-\beta(y-z_l))\right]\,d\beta.
\end{aligned} \tag{2.39}$$

From (2.6) and from Section 2.2, we derive for $l\neq m$

$$|\Gamma^\omega_{i,j}(z_m,y-\beta(y-z_l))|\leq \frac{\mathring{c}}{4\pi\; n\frac{a^{\frac{1}{3}}}{2}}, \quad \text{and} \quad |\nabla_y\Gamma^\omega_{i,j}(z_m,y-\beta(y-z_l))|\leq \frac{\mathring{c}}{4\pi\; n^2\left(\frac{a^{\frac{1}{3}}}{2}\right)^2}$$

where $\mathring{c}$ depends only on ω and some universal constants. Then

$$|R_l(z_m,y)| \leq \frac{\mathring{c}}{2\pi\; na^{\frac{1}{3}}}\left(\frac{1}{na^{\frac{1}{3}}}\int_0^1 |U_a(y-\beta(y-z_l))|d\beta+\int_0^1 |\nabla_y U_a(y-\beta(y-z_l))|d\beta\right). \tag{2.40}$$

Then, for $l\neq m$, (2.38) and (2.40) and observing that $\bar{C}_l$ is a constant matrix in Ω_l, imply the estimate

$$\begin{aligned}
&\Big|\int_{\Omega_l} \Gamma^\omega(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy-\Gamma^\omega(z_m,z_l) \quad K_a(z_l)\bar{C}_lU_a(z_l)Vol(\Omega_l)\Big| \\
&\leq \tfrac{\mathring{c}\bar{C}_lK_a(z_l)}{\pi\; n^2a^{\frac{2}{3}}}\int_{\Omega_l}\left[\int_0^1 |U_a(y-\beta(y-z_l))|\,d\beta\right] \quad |y-z_l|dy \\
&\qquad +\frac{\mathring{c}\bar{C}_lK_a(z_l)}{2\pi\; na^{\frac{1}{3}}}\int_{\Omega_l}\left[\int_0^1 |\nabla_y U_a(y-\beta(y-z_l))|d\beta\right]|y-z_l|dy \\
&\underset{(2.32)}{\leq} c_1\tfrac{[K_a(z_l)]\bar{C}_l}{n^2a^{\frac{2}{3}}}\,a^{\frac{4}{3}} \underset{(2.28)}{\leq} c_1\tfrac{K_{\max}\mathcal{C}}{n^2}\,a^{\frac{2}{3}},
\end{aligned} \tag{2.41}$$

for a suitable constant c_1.

Regarding the integral $\int_{\Omega_m}\Gamma^\omega(z_m,y)\mathbf{C}_a(y)U_a(y)dy$ we do the following estimates:

$$\begin{aligned}
&\Big|\int_{\Omega_m} \Gamma^\omega(z_m,y)K_a(y)\mathbf{C}_a(y)U_a(y)dy\Big|_\infty \\
&\underset{(2.32)}{\leq} c_2K_a(z_m)\bar{C}_m\int_{\Omega_m}|\Gamma^\omega(z_m,y)|_\infty dy \\
&\underset{(2.6)}{\leq} \tfrac{\mathring{c}}{4\pi}c_2K_a(z_m)\bar{C}_m\Big(\int_{B(z_m,r)}\tfrac{1}{|z_m-y|} \quad dy+\int_{\Omega_m\setminus B(z_m,r)}\frac{1}{|z_m-y|}dy\Big) \\
&\qquad \left(\text{here, } \frac{1}{|z_m-y|}\in L^1(B(z_m,r)), r<\frac{a^{\frac{1}{3}}}{2}\right) \\
&\leq \tfrac{\mathring{c}c_2}{4\pi}K_a(z_m)\bar{C}_m\Big(\sigma(\mathbb{S}^{3-1})\int_0^r \tfrac{1}{s}s^{3-1} \quad ds+\frac{1}{r}Vol(\Omega_m\setminus B(z_m,r))\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\left(2\pi r^2 + \frac{1}{r}\left[a - \frac{4}{3}\pi r^3\right]\right)}_{=:lm(r,a)} \tfrac{\mathring{c}}{4\pi} c_2 \bar{C}_m K_a \quad (z_m) \\
&\leq \tfrac{\mathring{c}}{4\pi} c_2 K_a(z_m) \bar{C}_m \; lm(r^c, a),
\end{aligned}
$$

here r^c is the value of r where $lm(r,a)$ attains maximum.

$$\partial_r lm(r,a) = 0 \Rightarrow 4\pi r - \frac{a}{r^2} - \frac{8}{3}\pi r = 0 \Rightarrow r_c = \left(\frac{3}{4}\pi a\right)^{\frac{1}{3}}$$

$$
\begin{aligned}
lm(r_c, a) &= 2\pi \left(\tfrac{3}{4}\pi\right)^{\frac{2}{3}} a^{\frac{2}{3}} + \left(\tfrac{4}{3\pi}\right)^{\frac{1}{3}} a^{\frac{2}{3}} - \tfrac{4}{3}\pi \left(\tfrac{3}{4}\pi\right)^{\frac{2}{3}} a^{\frac{2}{3}} \\
&= \left[\tfrac{2}{3\pi}\left(\tfrac{3}{4}\pi\right)^{\frac{2}{3}} + \left(\tfrac{4}{3\pi}\right)^{\frac{1}{3}}\right] a^{\frac{2}{3}} = \tfrac{3}{2}\left(\tfrac{4}{3\pi}\right)^{\frac{1}{3}} a^{\frac{2}{3}}
\end{aligned}
$$

$$\leq \tfrac{3}{8\pi} \mathring{c} c_2 K_{max} \mathcal{C} \left(\tfrac{4}{3\pi}\right)^{\frac{1}{3}} a^{\frac{2}{3}}. \tag{2.42}$$

From (2.37), we can have

$$
\begin{aligned}
|A|_\infty \leq &\left| \int_{\Omega_m} \Gamma^\omega(z_m, y) K_a(y) \mathbf{C}_a(y) U_a(y) dy \right|_\infty \\
&+ \sum_{\substack{j=1 \\ j\neq m}}^{[a^{-1}]} \left[\left|\Gamma^\omega(z_m, z_j) K_a(z_j) \bar{C}_j U_a(z_j) Vol(\Omega_j) - \int_{\Omega_j} \Gamma^\omega(z_m, y) K_a(y) \mathbf{C}_a(y) U_a(y) dy\right|_\infty \right].
\end{aligned}
$$

which we can estimate by

$$
\begin{aligned}
|A|_\infty \leq \sum_{n=1}^{[2a^{-\frac{1}{3}}]} 2[(2n+1)^3 - (2n-1)^3] \Big[&|\Gamma^\omega(z_m, z_j) K_a(z_j) \bar{C}_j U_a(z_j) Vol(\Omega_j) \\
&- \int_{\Omega_j} \Gamma^\omega(z_m, y) K_a(y) \mathbf{C}_a(y) U_a(y) dy|_\infty \Big] + \left| \int_{\Omega_m} \Gamma^\omega(z_m, y) K_a(y) \mathbf{C}_a(y) U_a(y) dy \right|_\infty.
\end{aligned}
$$

and then

$$|A|_\infty \leq c_3 \mathcal{C} K_{max} [a^{\frac{2}{3}} + a^{\frac{1}{3}}].$$

Finally

$$|A| \leq c_4 \mathcal{C} K_{max} a^{\frac{1}{3}}.$$

2.5.2 Estimate of B

$$
\begin{aligned}
&\sum_{\substack{j=1 \\ j\neq m}}^{M} \Gamma^\omega(z_m, z_j) \bar{C}_j U_a(z_j) a - \sum_{\substack{j=1 \\ j\neq m}}^{[a^{-1}]} \Gamma^\omega(z_m, z_j) K_a(z_j) \bar{C}_j U_a(z_j) Vol(\Omega_j) \\
&= \sum_{\substack{l=1 \\ l\neq m \\ z_l \in \Omega_m}}^{[K_a(z_m)]} \Gamma^\omega(z_m, z_l) \bar{C}_l U_a(z_l) a + \sum_{\substack{j=1 \\ j\neq m}}^{[a^{-1}]} \sum_{\substack{l=1 \\ z_l \in \Omega_j}}^{[K_a(z_j)]} \Gamma^\omega(z_m, z_l) \bar{C}_l U_a(z_l) a - \sum_{\substack{j=1 \\ j\neq m}}^{[a^{-1}]} \Gamma^\omega(z_m, z_j) K_a(z_j) \bar{C}_j U_a(z_j) Vol(\Omega_j) \\
&= \bar{C}_m a \sum_{\substack{l=1 \\ l\neq m \\ z_l \in \Omega_m}}^{[K_a(z_m)]} \Gamma^\omega(z_m, z_l) U_a(z_l) + \sum_{\substack{j=1 \\ j\neq m}}^{[a^{-1}]} \bar{C}_j a \Big[\Big(\sum_{\substack{l=1 \\ z_l \in \Omega_j}}^{[K_a(z_j)]} \Gamma^\omega(z_m, z_l) U_a(z_l) \Big) - \Gamma^\omega(z_m, z_j)[K_a(z_j)] U_a(z_j) \Big],
\end{aligned}
$$

since $Vol(\Omega_j) = a\frac{[K_a(z_j)]}{K_a(z_j)}$ and $\bar{C}_l = \bar{C}_j,$ for $l = 1, ..., K_a(z_j)$. We write,

$$E_1^j := \sum_{\substack{l=1\\ l\neq m\\ z_l\in\Omega_m}}^{[K_a(z_m)]} \Gamma^\omega(z_m, z_l) U_a(z_l) \tag{2.43}$$

and

$$\begin{aligned} E_2^j &:= \Big[\Big(\sum_{\substack{l=1\\ z_l\in\Omega_j}}^{[K_a(z_j)]} \Gamma^\omega(z_m, z_l) U_a(z_l)\Big) - \Gamma^\omega(z_m, z_j)[K_a(z_j)] U_a(z_j)\Big] \\ &= \sum_{\substack{l=1\\ z_l\in\Omega_j}}^{[K_a(z_j)]} \Big(\Gamma^\omega(z_m, z_l) U_a(z_l) - \Gamma^\omega(z_m, z_j) U_a(z_j)\Big). \end{aligned} \tag{2.44}$$

We need to estimate $\bar{C}_m a E_1^j$ and $\sum_{\substack{j=1\\ j\neq m}}^{[a^{-1}]} \bar{C}_j a E_2^j$.

Now by writing $f'(z_m, y) := \Gamma^\omega(z_m, y) U_a(y)$. For $z_l \in \Omega_j$, $j \neq m$, using Taylor series, we can write

$$f'(z_m, z_j) - f'(z_m, z_l) = (z_j - z_l) R'(z_m; z_j, z_l),$$

with

$$R'(z_m; z_j, z_l) = \int_0^1 \nabla_y f'(z_m, z_j - \beta(z_j - z_l))\, d\beta. \tag{2.45}$$

By doing the computations similar to the ones we have performed in (2.39-2.40) and by using Lemma 2.4, we obtain

$$|\sum_{\substack{j=1\\ j\neq m}}^{[a^{-1}]} \bar{C}_j a E_2^j| \leq c_4 \mathcal{C} K_{max} a^{\frac{1}{3}} \tag{2.46}$$

One can easily see that,

$$|\bar{C}_m a E_1^j| \leq \frac{\mathring{c} c_2 (K_{max} - 1)\mathcal{C}}{4\pi} \frac{a}{d} = \frac{\mathring{c} c_2 (K_{max} - 1)\mathcal{C}}{4\pi} a^{1-t}. \tag{2.47}$$

Substitution of (2.36) in (2.35) and using the estimates (2.41) and (2.42) associated to A and the estimates (2.46) and (2.47) associated to B gives us

$$U_a(z_m) + \sum_{\substack{j=1\\ j\neq m}}^{M} \Gamma^\omega(z_m, z_j) \bar{C}_j U_a(z_j) a = U^i(z_m, \theta) + O\left(c_4 K_{max} a^{\frac{1}{3}}\right) + O\left(\frac{\mathring{c} c_2 (K_{max} - 1)\mathcal{C}}{4\pi} a^{1-t}\right). \tag{2.48}$$

We rewrite the algebraic system (1.8) as

$$U_{a,m} + \sum_{\substack{j=1\\ j\neq m}}^{M} \Gamma^\omega(z_m, z_j) \bar{C}_j U_{a,j} a = U^i(z_m) \tag{2.49}$$

where we set $U_{a,m} := -C_m^{-1}Q_m$, recalling that $C_m = \bar{C}_m\, a$.

Taking the difference between (2.48) and (2.49) produces the algebraic system

$$(U_{a,m} - U_a(z_m)) + \sum_{\substack{j=1 \\ j\neq m}}^{M} \Gamma^\omega(z_m, z_j)\bar{C}_j(U_{a,j} - U_a(z_j))a \quad = \quad O\left(\mathcal{C}K_{max}(a^{\frac{1}{3}} + a^{1-t})\right).$$

Comparing this system with (1.8) and by using Lemma 2.2, we obtain the estimate

$$\sum_{m=1}^{M}(U_{a,m} - U_a(z_m)) \quad = \quad O\left(\mathcal{C}K_{max}M(a^{\frac{1}{3}} + a^{1-t})\right). \tag{2.50}$$

For the special case $d = a^t$, $M = O(a^{-1})$ with $t > 0$, we have the following approximation of the far-field from the Foldy-Lax asymptotic expansion (1.6) and from the definitions $U_{a,m} := C_m^{-1}Q_m$ and $C_m := \bar{C}_m a$, for $m = 1, \dots, M$:

$$\begin{aligned} 4\pi c_p^2 U_p^\infty(\hat{x},\theta)\cdot\hat{x} \quad &= \quad \sum_{j=1}^{M} e^{-i\kappa\hat{x}\cdot z_j}\bar{C}_j U_{a,j}\cdot\hat{x}\, a \\ &\quad + O\left(a + a^{2-5t+3t\alpha} + a^{3-9t+6t\alpha} + a^{1-2t\alpha} + a^{1-3t\alpha} + a^{2-5t+2t\alpha}\right). \end{aligned} \tag{2.51}$$

Consider the far-field of type:

$$U^\infty_{\mathbf{C}_a}(\hat{x},\theta) \quad = \quad \frac{1}{4\pi c_p^2}(\hat{x}\otimes\hat{x})\int_\Omega e^{-i\frac{\omega}{c_p}\hat{x}\cdot y}K_a(y)\mathbf{C}_a(y)U_a(y)dy + \frac{1}{4\pi c_s^2}(I - \hat{x}\otimes\hat{x})\int_\Omega e^{-i\frac{\omega}{c_s}\hat{x}\cdot y}K_a(y)\mathbf{C}_a(y)U_a(y)dy.$$

corresponding to the scattering problem (2.33) and set

$$U^\infty_{\mathbf{C}_a,p}(\hat{x},\theta) := \frac{1}{4\pi c_p^2}(\hat{x}\otimes\hat{x})\int_\Omega e^{-i\frac{\omega}{c_p}\hat{x}\cdot y}K_a(y)\mathbf{C}_a(y)U_a(y)dy \tag{2.52}$$

and

$$U^\infty_{\mathbf{C}_a,s}(\hat{x},\theta) := \frac{1}{4\pi c_s^2}(I - \hat{x}\otimes\hat{x})\int_\Omega e^{-i\frac{\omega}{c_s}\hat{x}\cdot y}K_a(y)\mathbf{C}_a(y)U_a(y)dy \tag{2.53}$$

Taking the difference between (2.52) and (2.51) we have:

$$\begin{aligned} & 4\pi c_p^2(U^\infty_{\mathbf{C}_a,p}(\hat{x},\theta) - U_p^\infty(\hat{x},\theta))\cdot\hat{x} \\ = \quad & \int_\Omega e^{-i\kappa\hat{x}\cdot y}K_a(y)\mathbf{C}_a(y)U_a(y)\cdot\hat{x}dy - \sum_{j=1}^{M} e^{-i\kappa\hat{x}\cdot z_j}\bar{C}_j U_{a,j}\cdot\hat{x}a \\ & + O\left(a + a^{2-5t+3t\alpha} + a^{3-9t+6t\alpha} + a^{1-2t\alpha} + a^{1-3t\alpha} + a^{2-5t+2t\alpha}\right) \\ = \quad & \sum_{j=1}^{[a^{-1}]}\int_{\Omega_j} e^{-i\frac{\omega}{c_p}\hat{x}\cdot y}K_a(y)\mathbf{C}_a(y)U_a(y)\cdot\hat{x}dy - \sum_{j=1}^{[a^{-1}]}\sum_{\substack{l=1 \\ z_l\in\Omega_j}}^{[K_a(z_j)]} e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_l}\bar{C}_l U_{a,l}\cdot\hat{x}a \\ & + O\left(a + a^{2-5t+3t\alpha} + a^{3-9t+6t\alpha} + a^{1-2t\alpha} + a^{1-3t\alpha} + a^{2-5t+2t\alpha}\right) \\ = \quad & \sum_{j=1}^{[a^{-1}]} K_a(z_j)\bar{C}_j\int_{\Omega_j}\left[e^{-i\frac{\omega}{c_p}\hat{x}\cdot y}U_a(y)\cdot\hat{x} - e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_j}U_a(z_j)\cdot\hat{x}\right]dy \end{aligned}$$

$$
\begin{aligned}
& +\sum_{j=1}^{[a^{-1}]} \bar{C}_j a \left[\sum_{\substack{l=1 \\ z_l \in \Omega_j}}^{[K_a(z_j)]} \left(e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_j} U_a(z_j)\cdot\hat{x} - e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_l} U_a(z_l) \right) + \sum_{\substack{l=1 \\ z_l \in \Omega_j}}^{[K_a(z_j)]} e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_l} \left(U_a(z_l) - U_{a,l}\right)\cdot\hat{x} \right] \\
& +O\left(a + a^{2-5t+3t\alpha} + a^{3-9t+6t\alpha} + a^{1-2t\alpha} + a^{1-3t\alpha} + a^{2-5t+2t\alpha}\right) \\
= \quad & \sum_{j=1}^{[a^{-1}]} \int_{\Omega_j} K_a(z_j)\bar{C}_j \left[e^{-i\frac{\omega}{c_p}\hat{x}\cdot y} U_a(y)\cdot\hat{x} - e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_j} U_a(z_j)\cdot\hat{x} \right] dy \\
& +\sum_{j=1}^{[a^{-1}]} \bar{C}_j a \sum_{\substack{l=1 \\ z_l \in \Omega_j}}^{[K_a(z_j)]} \left(e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_j} U_a(z_j)\cdot\hat{x} - e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_l} U_a(z_l)\cdot\hat{x} \right) + \sum_{j=1}^{M} e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_j} \bar{C}_j a \left[U_a(z_j) - U_{a,j}\right]\cdot\hat{x} \\
& +O\left(a + a^{2-5t+3t\alpha} + a^{3-9t+6t\alpha} + a^{1-2t\alpha} + a^{1-3t\alpha} + a^{2-5t+2t\alpha}\right) \\
\overset{(2.50)}{=} \quad & \sum_{j=1}^{[a^{-1}]} K_a(z_j)\bar{C}_j \int_{\Omega_j} \left[e^{-i\frac{\omega}{c_p}\hat{x}\cdot y} U_a(y)\cdot\hat{x} - e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_j} U_a(z_j)\cdot\hat{x} \right] dy \\
& +\sum_{j=1}^{[a^{-1}]} \bar{C}_j a \sum_{\substack{l=1 \\ z_l \in \Omega_j}}^{[K_a(z_j)]} \left(e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_j} U_a(z_j)\cdot\hat{x} - e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_l} U_a(z_l)\cdot\hat{x} \right) + O\left(\mathcal{C}^2 K_{max}(a^{\frac{1}{3}} + a^{1-t})\right) \\
& +O\left(a + a^{2-5t+3t\alpha} + a^{3-9t+6t\alpha} + a^{1-2t\alpha} + a^{1-3t\alpha} + a^{2-5t+2t\alpha}\right). \qquad (2.54)
\end{aligned}
$$

Now, let us estimate the difference $\sum_{j=1}^{[a^{-1}]} K_a(z_j)\bar{C}_j \int_{\Omega_j} \left[e^{-i\frac{\omega}{c_p}\hat{x}\cdot y} U_a(y) - e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_j} U_a(z_j) \right] dy$. Write, $f_1(y) = e^{-i\frac{\omega}{c_p}\hat{x}\cdot y} U_a(y)$. Using Taylor series, we can write

$$f_1(y) - f_1(z_j) = (y - z_j) R_j(y),$$

with

$$
\begin{aligned}
(R_j(y))_{k,l} &= \int_0^1 \nabla_y (f_1)_{k,l}(y - \beta(y - z_j))\, d\beta \\
&= \int_0^1 \left[\nabla_y \left[e^{-i\frac{\omega}{c_p}\hat{x}\cdot(y-\beta(y-z_j))} U_a(y - \beta(y - z_j)) \right] \right]_k d\beta \\
&= \int_0^1 \left[\nabla_y e^{-i\frac{\omega}{c_p}\hat{x}\cdot(y-\beta(y-z_j))} \right]_k U_a(y - \beta(y - z_j))\, d\beta \\
&\quad + \int_0^1 e^{-i\frac{\omega}{c_p}\hat{x}\cdot(y-\beta(y-z_j))} \left[\nabla_y U_a(y - \beta(y - z_j)) \right]_k d\beta. \qquad (2.55)
\end{aligned}
$$

We have $\nabla_y e^{-i\frac{\omega}{c_p}\hat{x}\cdot y} = -i\frac{\omega}{c_p}\hat{x} e^{-i\frac{\omega}{c_p}\hat{x}\cdot y}$ then

$$|R_j(y)|_\infty \leq \left(\frac{\omega}{c_p} \int_0^1 |U_a(y - \beta(y - z_j))|_\infty\, d\beta + \int_0^1 |\nabla_y U_a(y - \beta(y - z_j))|_\infty\, d\beta \right). \qquad (2.56)$$

Using (2.56) we get the estimate

$$\left| \sum_{j=1}^{[a^{-1}]} K_a(z_j)\bar{C}_j \int_{\Omega_j} \left[e^{-i\frac{\omega}{c_p}\hat{x}\cdot y}(y) U_a(y) - e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_j} U_a(z_j) \right] dy \right|_\infty \leq \qquad (2.57)$$

$$\sum_{j=1}^{[a^{-1}]} K_a(z_j)\bar{C}_j \left(\frac{\omega}{c_p} \int_{\Omega_j} |y - z_j|_\infty \int_0^1 |U_a(y - \beta(y - z_j))|\, d\beta\, dy \right)$$

$$+\sum_{j=1}^{[a^{-1}]} K_a(z_j)\bar{C}_j \left(\int_{\Omega_j} |y-z_j| \int_0^1 |\nabla_y U_a(y-\beta(y-z_j))|_\infty \, d\beta \, dy \right)$$

$$\leq \sum_{j=1}^{[a^{-1}]} K_a(z_j)\bar{C}_j c_1 \, a \, a^{\frac{1}{3}} \left(\frac{\omega}{c_p} + c_5 \right) \leq K_{max}\mathcal{C}c_1 \left(\frac{\omega}{c_p} + c_5 \right) a^{\frac{1}{3}}.$$

In the similar way, using (2.50), we have,

$$\left| \sum_{j=1}^{[a^{-1}]} \bar{C}_j a \sum_{\substack{l=1 \\ z_l \in \Omega_j}}^{[K_a(z_j)]} \left(e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_j} U_a(z_j) - e^{-i\frac{\omega}{c_p}\hat{x}\cdot z_l} U_a(z_l) \right) \right|_\infty \quad \leq \quad O\left(K_{max}\mathcal{C}(a^{\frac{1}{3}} + a^{1-t}) \right). \tag{2.58}$$

Using the estimates (2.57) and (2.58) in (2.54), we obtain

$$\begin{aligned} \frac{1}{4\pi c_p^2} U^\infty_{\mathbf{C}_a,p}(\hat{x},\theta) - U^\infty_p(\hat{x},\theta)\cdot\hat{x} \quad &= \quad O\left(K_{max} a^{\frac{1}{3}} \mathcal{C} c_1 \left(\frac{\omega}{c_p} + c_5 \right) \right) + O(\mathcal{C}(\mathcal{C}+1)\mathcal{C}K_{max}(a^{\frac{1}{3}} + a^{1-t})) \\ &\quad + O\left(a + a^{2-5t+3t\alpha} + a^{3-9t+6t\alpha} + a^{1-2t\alpha} + a^{1-3t\alpha} + a^{2-5t+2t\alpha} \right) \\ &= \quad O\left(a^{\frac{1}{3}} + a^{1-t} + a^{3-9t+6\alpha t} + a^{1-3\alpha t} + a^{2-5t+2t\alpha} \right). \end{aligned} \tag{2.59}$$

Since $Vol(\Omega)$ is of order $a^{-1}(\frac{a}{2}+\frac{d}{2})^3$, and d is of the order a^t, we should have $t \geq \frac{1}{3}$. Hence

$$t \geq \frac{1}{3}; \qquad 1-t>0; 3-9t+6t\alpha>0; \qquad 1-3t\alpha>0; \qquad 2-5t+2t\alpha>0;$$

Hence for $\frac{1}{3} \leq t < 1$, we have

$$t\alpha < \frac{1}{3}; \qquad 2-5t+2t\alpha>0; \qquad 3-9t+6t\alpha>0.$$

Equating $2-5t+2t\alpha = 3-9t+6t\alpha$, we find that $\alpha t = t - \frac{1}{4}$ and then $2-5t+2t\alpha = 3-9t+6t\alpha = \frac{3}{2} - 3t$ and $1-3\alpha t = \frac{1}{4} - 3t$. In addition, since $\alpha t \leq \frac{1}{3}$, then $t \leq \frac{7}{12}$. Hence, the error is

$$\frac{1}{4\pi c_p^2}\left[U^\infty_{\mathbf{C}_a,p}(\hat{x},\theta) - U^\infty_p(\hat{x},\theta) \right]\cdot\hat{x} \quad = \quad O\left(a^{\frac{1}{3}} + a^{\frac{3}{2}-3t} \right), \quad \frac{1}{3} \leq t \leq \frac{7}{12}. \tag{2.60}$$

2.6 End of the proof of Theorem 1.2

Combining the estimates (2.60) and (2.34), we deduce that

$$\frac{1}{4\pi c_p^2}\left[U^\infty_p(\hat{x},\theta) - U^\infty_{0,p}(\hat{x},\theta) \right]\cdot\hat{x} = O(a^{\min\{\gamma, \frac{1}{3}, \frac{3}{2}-3t\}}), \ a << 1, \quad \frac{1}{3} \leq t \leq \frac{7}{12} \tag{2.61}$$

uniformly in terms of $\hat{x}, \theta \in \mathbb{S}^2$.

3 Justification of Corollary 1.3

For an obstacle D_ϵ of radius ϵ, $\mathcal{S}(\phi)(s) := \int_{\partial D_\epsilon} \Gamma^\omega(s,t)\phi(t)dt$ and $\mathcal{D}(\phi)(s) := \int_{\partial D_\epsilon} \frac{\partial \Gamma^\omega(s,t)}{\partial \nu(t)}\phi(t)dt$. Similarly, we set $\mathcal{S}_G(\phi)(s) := \int_{\partial D_\epsilon} G_\rho(s,t)\phi(t)dt$ and $\mathcal{D}_G(\phi)(s) := \int_{\partial D_\epsilon} \frac{\partial G_\rho(s,t)}{\partial \nu(t)}\phi(t)dt$. We see that $W_\kappa(x,z) := G_\rho(x,z) - \Gamma^\omega(x,z)$ satisfies

$$(\Delta + \omega^2)W_\kappa = \omega^2(1-\rho)\Gamma^\omega, \quad \text{in } \mathbb{R}^3 \tag{3.1}$$

with the Kupradze radiation conditions. Since $\Gamma^\omega(\cdot, z)$, $z \in \mathbb{R}^3$ is bounded in $L^p(\Omega)$, for $p < 3$, by interior estimates, we deduce that $W(\cdot, z)$, $z \in \mathbb{R}^3$ is bounded in $W^{2,p}(\Omega)$, for $p < 3$, and hence, in particular, the normal traces are bounded in $L^2(\partial D_\epsilon)$. Then we can show that the norms of the operators

$$\mathcal{S}_G - \mathcal{S} : (L^2(\partial D_\epsilon))^3 \to (H^1(\partial D_\epsilon))^3 \tag{3.2}$$

and

$$\mathcal{D}_G - \mathcal{D} : (L^2(\partial D_\epsilon))^3 \to (L^2(\partial D_\epsilon))^3 \tag{3.3}$$

are of the order $O(\epsilon)$ at least.

1. Using these properties and arguing as in [7], we derive the asymptotic expansions (1.25)-(1.26). Indeed, apart from the computations done in [7], the main arguments needed to extend those results to the case of variable density is the Fredholm alternative for the corresponding integral operators and the application of the Neumann series expansions. After splitting G_ρ as $G_\rho = \Gamma^\omega + (G_\rho - \Gamma^\omega)$, these two arguments are applicable as soon as we have (3.2)-(3.3).

2. The justification of the invertibility of the algebraic system (1.8) depends only on (1) the distribution of small bodies and (2) the background medium through the singularities of the fundamental solution (of the form $|\Gamma^\omega(s,t)| \leq c|s-t|^{-1}$). However, this type of singularity is true for general background elastic media [2]. Then the same arguments can be used to justify the invertibiliy of the algebraic system (1.27). Using the above mentioned decomposition of the Green's function G_ρ, the properties of the Lippmann Schwinger integral equation are also valid replacing Γ^ω by G_ρ, and hence the results in section 2.4 are valid. Finally, and again using the decomposition of G_ρ, the computations in section 2.5 can be carried out using G_ρ.

4 The elastic capacitance

We start with the following lemma on the symmetry structure of the elastic capacitance.

Lemma 4.1. *Let $C := (\int_{\partial D} \sigma_{i,j}(t)dt)_{i,j=1}^3$ be the elastic capacitance of a bounded and Lischitz regular set D and C^* be its adjoint. Then*

$$C = C^*. \tag{4.1}$$

Proof of Lemma 4.1. We know that the matrix $\sigma := (\sigma_{i,j})_{i,j=1}^3$ solves the invertible integral equation $\int_{\partial D} \Gamma_0(s,t)\sigma(t)dt = I_3$, or precisely $\int_{\partial D} \Gamma_0(s,t)\sigma_i(t)dt = e_i$, where $\sigma_i := (\sigma_{i,j})_{j=1}^3$ and e_i is the i^{th} column of I_3. Let a be any constant vector in $\mathbb{R}^3$, then the vector $\sigma\, a$ satisfies

$$\int_{\partial D} \Gamma_0(s,t)\,(\sigma(t)\, a)\; dt = a.$$

We set $\varphi^a := \int_{\partial D} \Gamma_0(s,t)\,(\sigma(t)\, a)\, dt$. Then φ^a satisfies the problem $\Delta^e \varphi^a = 0$, in D and $\varphi^a = a$ on ∂D. In addition, we have the jump relation $\partial_{\nu^+}\varphi^a - \partial_{\nu^-}\varphi^a = \sigma(t)\, a$ on ∂D where $\partial_\nu u := \lambda(\nabla \cdot u)\nu + \mu(\nabla u + \nabla u^\top)\nu$ is the elastic conormal derivative. Hence

$$\int_{\partial D} \partial_{\nu^+}\varphi^a - \partial_{\nu^-}\varphi^a\; dt = \int_{\partial D} \sigma(t)\; a dt = C\; a.$$

[2]Of course, it can be justify using the decomposition $G_\rho = \Gamma^\omega + (G_\rho - \Gamma^\omega)$ with the singularity of Γ^ω and the smoothness of $G_\rho - \Gamma^\omega$

Now, let a and b be arbitrary constant vectors in $\mathbb{R}^3$. To both a and b, we correspond φ^a and φ^b as above. Using the Green formulas inside and outside of D, we deduce that

$$(C\,a,b) = \int_{\partial D} (\partial_{\nu+}\varphi^a - \partial_{\nu-}\varphi^a(t)) \ \cdot \varphi^b(t)\,dt = \int_{\partial D} \left(\partial_{\nu+}\varphi^b - \partial_{\nu-}\varphi^b\right)\cdot \varphi^a dt = (C\,b,a) = (a, C\,b)$$

recalling that every quantity here is real valued.

□

The next lemma describes the elastic capacitance of a given bounded and Lipschitz regular domain with the one of its image by a unitary transform.

Lemma 4.2. *Let $\mathcal{R} = (r_{lm})$ be a unitary transform in $\mathbb{R}^d$, D be bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$ and $\tilde{D} = \mathcal{R}(D)$. Let C and $\tilde{C}$ be the corresponding elastic capacitance matrices due to the denisty matrices σ and $\tilde{\sigma}$, as defined in* (1.9)*, respectively. Then we have $\tilde{C} = \mathcal{R}\,C\,\mathcal{R}^{-1}$.*

Proof of Lemma 4.2. First recall the relation $\Gamma^0 \circ \mathcal{R}(\xi,\eta) = \mathcal{R}\Gamma^0(\xi,\eta)\mathcal{R}^{-1}$, see [3, Lemma 6.11]. From (1.9), we have that

$$\begin{array}{rll} & \int_{\partial\tilde{D}} \Gamma^0(\tilde{\xi},\tilde{\eta})\tilde{\sigma}(\tilde{\eta})d\tilde{\eta} & = \mathbf{I},\ \tilde{\eta}\in\partial\tilde{\mathrm{D}} \\ \Rightarrow & \int_{\partial D} \left(\Gamma^0\circ\mathcal{R}\right)(\xi,\eta)\left(\tilde{\sigma}\circ\mathcal{R}\right)(\eta)d\eta & = \mathbf{I},\ \eta\in\partial\mathrm{D} \\ \Rightarrow & \int_{\partial D} \mathcal{R}\Gamma^0(\xi,\eta)\mathcal{R}^{-1}\tilde{\sigma}\circ\mathcal{R}(\eta)d\eta & = \mathbf{I},\ \eta\in\partial\mathrm{D} \\ \Rightarrow & \int_{\partial D} \Gamma^0(\xi,\eta)\mathcal{R}^{-1}\left(\tilde{\sigma}\circ\mathcal{R}\right)(\eta)d\eta & = \mathcal{R}^{-1},\ \eta\in\partial D \\ \Rightarrow & \int_{\partial D} \Gamma^0(\xi,\eta)\mathcal{R}^{-1}\left(\tilde{\sigma}\circ\mathcal{R}\right)(\eta)\mathcal{R}d\eta & = \mathbf{I},\ \eta\in\partial\mathrm{D}. \end{array} \tag{4.2}$$

Now from the uniqueness of solutions of (1.9), we deduce that $\mathcal{R}^{-1}\left(\tilde{\sigma}\circ\mathcal{R}\right)(\cdot)\mathcal{R} = \sigma(\cdot)$ and then

$$\left(\tilde{\sigma}\circ\mathcal{R}\right)(\cdot) \ = \ \mathcal{R}\sigma(\cdot)\mathcal{R}^{-1}. \tag{4.3}$$

From the definition of the capacitance, see (1.9), and (4.3), we have

$$\tilde{C} = \int_{\partial\tilde{D}} \tilde{\sigma}(\tilde{\eta})d\tilde{\eta} = \int_{\partial D} \left(\tilde{\sigma}\circ\mathcal{R}\right)(\eta)d\eta = \int_{\partial D} \mathcal{R}\sigma(\eta)\mathcal{R}^{-1}d\eta = \mathcal{R}C\mathcal{R}^{-1}. \tag{4.4}$$

□

Proposition 4.3. *Let $\mathcal{R} = (r_{lm})$ be a unitary transform in $\mathbb{R}^d$, $d = 2, 3$, and let D be a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$, and $\tilde{D} = \mathcal{R}(D)$. Let C and $\tilde{C}$ be the corresponding elastic capacitance matrices due to the denisty matrices σ and $\tilde{\sigma}$, as defined in* (1.9)*, respectively.*

1. *2D-case. If the shape of D is rotationally invariant for any rotation by one angle $\theta \neq 0, \pi$, then C is a scalar multiplied by the identity matrix.*
2. *3D-case. If the shape of D is rotationally invariant for any two of the rotations around the x, y or z axis by one angle $\theta \neq 0, \pi$ and $\alpha \neq 0, \pi$ respectively, then C is a scalar multiplied by the identity matrix.*

Proof of Proposition 4.3. In 2**D** case, the rotation matrix by an angle θ is given by

$$\mathcal{R} \;=\; \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4.5}$$

As the shape is invariant by this rotation then $\tilde{C} = C$. Since $\mathcal{R}$ is unitary then $\mathcal{R}^{-1} = \mathcal{R}^\top$ and then (4.4) implies

$$C \quad = \tilde{C} = \mathcal{R}C\mathcal{R}^\top$$

$$
\begin{aligned}
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} C_{11}\cos\theta - C_{12}\sin\theta & C_{11}\sin\theta + C_{12}\cos\theta \\ C_{21}\cos\theta - C_{22}\sin\theta & C_{21}\sin\theta + C_{22}\cos\theta \end{pmatrix} \qquad (4.6) \\
&= \begin{pmatrix} C_{11}\cos^2\theta + C_{22}\sin^2\theta - (C_{12}+C_{21})\sin\theta\cos\theta & C_{12}\cos^2\theta - C_{21}\sin^2\theta + (C_{11}-C_{22})\sin\theta\cos\theta \\ C_{21}\cos^2\theta - C_{12}\sin^2\theta + (C_{11}-C_{22})\sin\theta\cos\theta & C_{11}\sin^2\theta + C_{22}\cos^2\theta + (C_{12}+C_{21})\sin\theta\cos\theta \end{pmatrix}.
\end{aligned}
$$

We deduce from (4.6) and the symmetry of matrix C the following relations:

$$C_{11} = C_{11}\cos^2\theta + C_{22}\sin^2\theta - 2C_{12}\sin\theta\cos\theta \qquad (4.7)$$

$$C_{22} = C_{11}\sin^2\theta + C_{22}\cos^2\theta + 2C_{12}\sin\theta\cos\theta \qquad (4.8)$$

$$C_{12} = C_{12}\cos^2\theta - C_{12}\sin^2\theta + (C_{11} - C_{22})\sin\theta\cos\theta. \qquad (4.9)$$

We rewrite (4.7) and (4.9) respectively as

$$(C_{11} - C_{22})\sin^2\theta + 2C_{12}\sin\theta\cos\theta = 0 \qquad (4.10)$$

$$(C_{11} - C_{22})\cos\theta\sin\theta - 2C_{12}\sin^2\theta = 0 \qquad (4.11)$$

Taking $\sin\theta$ as a multiplicative factor in (4.10) and (4.11) we see that if $\theta \neq 0, \pi$, i.e. $\sin\theta \neq 0$, then we have

$$C_{11} = C_{22} \text{ and } C_{12} = C_{21} = 0. \qquad (4.12)$$

Let us now consider the 3D case. First, let us assume that the shape is invariant under the rotation about the $x - axis$ and an angle $\theta \neq 0, \pi$. This rotation matrix is given by

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad (4.13)$$

Since $\mathcal{R}$ is unitary, $\mathcal{R}^{-1} = \mathcal{R}^\top$, then (4.4) gives us;

$$
\begin{aligned}
C = \tilde{C} &= \mathcal{R} C \mathcal{R}^\top \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} C_{11} & C_{12}\cos\theta - C_{13}\sin\theta & C_{12}\sin\theta + C_{13}\cos\theta \\ C_{21} & C_{22}\cos\theta - C_{23}\sin\theta & C_{22}\sin\theta + C_{23}\cos\theta \\ C_{31} & C_{32}\cos\theta - C_{33}\sin\theta & C_{32}\sin\theta + C_{33}\cos\theta \end{pmatrix} \qquad (4.14) \\
&= \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{cases}
T_{11} := C_{11}; \\
T_{12} := C_{12}\cos\theta - C_{13}\sin\theta; \\
T_{13} := C_{12}\sin\theta + C_{13}\cos\theta; \\
T_{21} := C_{21}\cos\theta - C_{31}\sin\theta; \\
T_{22} := C_{22}\cos^2\theta + C_{33}\sin^2\theta - (C_{22} + C_{33})\sin\theta\cos\theta; \\
T_{23} := C_{23}\cos^2\theta - C_{32}\sin^2\theta + (C_{22} - C_{33})\sin\theta\cos\theta; \\
T_{31} := C_{21}\sin\theta + C_{31}\cos\theta; \\
T_{32} := C_{32}\cos^2\theta - C_{23}\sin^2\theta + (C_{22} - C_{33})\sin\theta\cos\theta; \\
T_{33} := C_{22}\sin^2\theta + C_{33}\cos^2\theta + (C_{23} + C_{32})\sin\theta\cos\theta;
\end{cases} \qquad (4.15)
$$

We observe the equality of the 2×2 matrices:

$$\begin{pmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{pmatrix} \tag{4.16}$$
$$= \begin{pmatrix} C_{22}\cos^2\theta + C_{33}\sin^2\theta - (C_{22}+C_{33})\sin\theta\cos\theta & C_{23}\cos^2\theta - C_{32}\sin^2\theta + (C_{22}-C_{33})\sin\theta\cos\theta \\ C_{32}\cos^2\theta - C_{23}\sin^2\theta + (C_{22}-C_{33})\sin\theta\cos\theta & C_{22}\sin^2\theta + C_{33}\cos^2\theta + (C_{23}+C_{32})\sin\theta\cos\theta \end{pmatrix}.$$

These are similar to the matrices we obtained in the $2D$ case. Hence we deduce, as in the $2D$ case, that

$$C_{22} = C_{33} \text{ and } C_{23} = C_{32} = 0. \tag{4.17}$$

To show that C is scalar multiplied by the identity matrix we need to prove that $C_{11} = C_{22}$, for instance, and $C_{13} = C_{31} = 0$. For this purpose, we use another rotation. Taking the rotation around the z-axis [3] by one angle $\alpha \neq 0, \pi$ and proceeding as we did for the rotation about the x-axis, we show that

$$C_{11} = C_{33} \text{ and } C_{13} = C_{31} = 0. \tag{4.18}$$

□

From the above analysis, we have the following remark:

Remark 4.4. *1. For the spherical shapes, in particular, the capacitance is a scalar multiplied by the identity matrix.*

2. Ellipsoidal shapes are invariant only under rotations with angle π (or trivially 0). For these shapes, the capacitance might not be a scalar multiplied by the identity matrix but a diagonal matrix instead. To justify this property, the arguments in [3] can be useful.

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