Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

Solvability and topological index criteria for thermal management in liquid flow networks

A-K. Baum, M. Kolmbauer

RICAM-Report 2015-21

Solvability and topological index criteria for thermal management in liquid flow networks

Ann-Kristin Baum*† Michael Kolmbauer†‡

December 7, 2015

Abstract

This work is devoted to the analysis of a model for the thermal management in a liquid pipe network. In contrast to previous works, the underlying model equation for the liquid flow is not restricted to the equation of motion and the continuity equation, describing the mass transfer through the pipes, but also includes thermodynamic effects in order to cover cooling and heating prozesses. The resulting model gives rise to a differential-algebraic equation (DAE), for which a proof of unique solvability and an index analysis is presented. For the index analysis, the concepts of the *Strangeness Index* is pursued. Finally the index analysis is linked to topological conditions imposed on the underlying network and the practical realization of those conditions in simulation tools is discussed.

Keywords: Differential-algebraic equations, topological index criteria, hydraulic network.

AMS(MOS) subject classification: 65L80, 94C15

1 Introduction

The simulation and optimization of hydraulic networks has been studied in various works, including [15, 14, 30, 2, 7] and the references therein. The considered models are motivated by drinking water supply systems, where the main target is to circulate an amount of water at any time, assuring a certain pressure at extraction points. The aim of this work is to consider and analyze hydraulic networks used for thermal management systems. Examples in automotive applications are cooling systems for combustion engines or battery racks. In contrast to water transportation networks, the primary interest is not the pressure distribution across the whole system, but the temperature distribution. Consequently, the models have to be equipped with energy balance laws in order to model the thermodynamic effects. Basically, the intention of this work is to extend the results, which are already available for water transportation networks [15, 14] to cooling and heating systems used for thermal management.
The model under consideration is a quasi-stationary pipe network, cp. [15], equipped with energy balance laws. This model is suited to describe circuits, which are filled with incompressible fluids (e.g. water). Here incompressible means, that density changes with respect to temperature changes or pressure changes are neglected. Especially the extension with the energy balance laws is not straight forward, since the well-known concepts of bi-directional dependent flows of state-flow networks are not valid any more for energy flow networks.
While general networks consist of various types of elements (pipes, pumps, valves) [30], the model here is restricted to pipes only. Despite this simplification, the demanding issues are caused by the arbitrary network structure of the underlying model.

*Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenberger Str. 69, 4040 Linz, Austria
†MathConsult GmbH, Altenberger Str. 69, 4040 Linz, Austria
‡AVL List GmbH, Hans-List-Platz 1, 8020 Graz, Austria

State-of-the-art modeling and simulation packages such as Dymola[1], Matlab/Simulink[2], Flowmaster[3], Amesim[4], SimulationX[5] or Cruise M[6] offer many excellent concepts for the automatic generation of dynamic system models, including hydraulic networks. Modeling is done in a modularized way, based on a network of subsystems which again consists of simple standardized subcomponents. The network structure (topology) carries the core information of the network properties and therefore is predestinated to be exploited for the analysis and numerical simulation of those. In many applications the network describing equations are differential-algebraic equations (DAEs). Hence the analysis of existence and uniqueness of solutions, as well as rank considerations are a delicate issue.

Topology based index analysis for networks connects the research fields of *Analysis for DAEs* [29] and *Graph Theory* [8] in order to provide the appropriate base to analyze DAEs stemming from automatic generated system models. So far it has been established for various types of networks, including electric circuits [32], gas supply networks [12] and water supply networks [15, 14, 30]. Although all those networks share some similarities, an individual investigation is required due to their different physical nature. Recently, a unified modeling approach for different types of flow networks has been introduced in [16], aiming for a unified topology based index analysis for the different physical domains on an abstract level.

The structure of this work is the following. In Section 2, the main two concepts required for the basic ingredients of the analysis are introduced. A short introduction to graph theory and differential algebraic equations is given. The network model and arising DAE is formulated in Section 3 and Section 4. The analysis of the DAE is split into two parts: First a sub problem is considered, which neglects the thermodynamic effects. This sub model is formulated and analyzed in Section 3. Indeed, this intermediate step agrees with the result of [15], although the proof differs due to the usage of a different index concept. Furthermore, it provides the technical base for the investigation of the full model in the second part in Section 4. A summary of the results with comments on their practical relevance in commercial simulation software concludes the paper in Section 5.

2 Preliminaries

For the subsequent analysis, we need the ideas of graph theory, including the notion of a graphs incidence matrix, and basic concepts of DAE theory.

2.1 Graph theory

For a detailed introduction to graph theory, we refer the reader to [8].

A *graph* $\mathcal{G}$ is a pair $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ of subsets $\mathcal{V}, \mathcal{E} \subset \mathbb{N}$, such that $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, i.e., each element $e_j \in \mathcal{E}$ corresponds to a pair $(v_{i_1}, v_{i_2}) \in \mathcal{V} \times \mathcal{V}$, cp. [8, p. 2], [6, p. 1]. Graphically, this relation is realized by representing the *edges* $e_1, ..., e_{n_\mathcal{E}} \in \mathcal{E}$ as lines between the *vertices* $v_1, ..., v_{n_\mathcal{V}} \in \mathcal{V}$, cp. [8, p. 2], [6, p. 1].

We consider *simple* graphs, i.e., $\mathcal{G}$ contains no self-loops or parallel edges, i.e., if $e_j = (v_{i_1}, v_{i_2})$ then $v_{i_1} \neq v_{i_2}$ and if $e_j = (v_{i_1}, v_{i_2})$ and $e_\ell = (v_{i_1}, v_{i_2})$, then $e_j = e_k$, cp.[6, p. 2], [8, p. 25]. For a given numbering $\{v_1, ..., v_{n_\mathcal{V}}\}$, $\{e_1, ..., e_{n_\mathcal{E}}\}$ of $\mathcal{V}, \mathcal{E}$ with index sets $\mathcal{I}_\mathcal{V} = \{1, ..., n_\mathcal{V}\}, \mathcal{I}_\mathcal{E} = \{1, ..., n_\mathcal{E}\}$, we can define a function $b\colon I_\mathcal{V} \times I_\mathcal{V} \to I_\mathcal{E}$ relating two nodal indices $i_1, i_2 \in I_\mathcal{V}$ with the index j of the corresponding edge $e_j = (v_{i_1}, v_{i_2})$, i.e., $b(i_1, i_2) = j$. If $(v_{i_1}, v_{i_2}) \notin \mathcal{V}$, we set $b(i_1, i_2) = 0$.

Two edges $e_{j_1}, e_{j_2} \in \mathcal{E}$ are called *adjacent* if they are incident with a common vertex v_i, i.e., if $e_{j_1} = (v_{i_1}, v_i)$ and $e_{j_2} = (v_i, v_{i_2})$ [6, p. 7]. We summarize the indices of edges that are adjacent via the vertex v_i in the set $\hat{J}_{v_i} = \{\ell \in \mathcal{I}_\mathcal{E} \mid \exists v_k \colon e_\ell = (v_i, v_k)\}$.

[1]http://www.dynasim.com
[2]http://www.mathworks.com
[3]http://www.mentor.com
[4]http://www.plm.automation.siemens.com
[5]http://www.iti.de
[6]http://www.avl.com

Two vertices $v_{i_1}, v_{i_2} \in \mathcal{V}$ are called *adjacent* if there exists an edge $e_j \in \mathcal{E}$ such that $e_j = (v_{i_1}, v_{i_2})$ [6, p. 7]. We summarize the indices of vertices that are adjacent to v_i in the set $\hat{I}_{v_i} = \{k \in \mathcal{I}_\mathcal{V} \,|\, \exists e_j \colon e_j = (v_i, v_k)\}$. The *degree* $d(v_i)$ of a vertex v_i is the number of edges e_{i_j} that are incident with v_i, [6, p. 7]. If $d(v_i) = 0$, then v_i is called *isolated* and if $d(v_i) = 1$, then v_i is called an *end vertex* [6, p. 8].
A graph $\mathcal{G}$ is called *oriented* if every edge e_j is associated with an *ordered* pair (v_{i_1}, v_{i_2}), i.e., $(v_{i_1}, v_{i_2}) \neq (v_{i_2}, v_{i_1})$. If $\mathcal{G}$ is oriented and $e_j = (v_{i_1}, v_{i_2})$, we call v_{i_1} the originating and v_{i_2} the terminating vertex of e_j, [8, p. 25].
A pair $\mathcal{G}_1 = \{\mathcal{V}_1, \mathcal{E}_1\}$ of subsets $\mathcal{V}_1 \subset \mathcal{V}, \mathcal{E}_1 \subset \mathcal{E}$ is called a subgraph of $\mathcal{G}$ [6, p. 16]. For example, a single vertex or a single edge with its adjacent vertices are subgraphs [6, p. 17]. A subgraph $\mathcal{G}_1$ *spanns* $\mathcal{G}$ if the nodal sets agree, i.e., if $\mathcal{V}_1 = \mathcal{V}$. For a vertex $v_i \in \mathcal{V}_1$, we call $d_{\mathcal{V}_1}(v_i)$ the degree of v_i as an element of the subgraph $\mathcal{G}_1$.
A subgraph $\mathcal{P} = \{\mathcal{V}_\mathcal{P}, \mathcal{E}_\mathcal{P}\}$ is called a *path* if there exists a numbering $\{v_{i_1}, ..., v_{i_{p_v}}\}$, $\{e_{j_1}, ..., e_{j_{p_e}}\}$ of $\mathcal{V}_\mathcal{P}, \mathcal{E}_\mathcal{P}$, such that, for $k = 1, ..., p_e - 1$, the edge e_{j_k} is adjacent to $e_{j_{k+1}}$, and $d_\mathcal{P}(v_i) = 2$ for $i = 2, ..., i_{p_N - 1}$ and $d_\mathcal{P}(v_1) = d_\mathcal{P}(v_{p_N}) = 1$. A graph $\mathcal{G}$ is called *connected* if any pair of vertices v_1, v_2 can be connected by a path.
A subgraph $\mathcal{C} = \{\mathcal{V}_\mathcal{C}, \mathcal{E}_\mathcal{C}\}$ is called a *cycle*, if there exists a numbering $\{v_{i_1}, ..., v_{i_{c_v}}\}$, $\{e_{j_1}, ..., e_{j_{c_e}}\}$ of $\mathcal{V}_\mathcal{C}, \mathcal{E}_\mathcal{C}$, such that $(\{v_{i_1}, ..., v_{i_{c_e - 1}}\}, \{e_{j_1}, ..., e_{j_{c_e - 1}}\})$ is a path and $e_{j_{c_e}} = (v_{i_{c_v}}, v_{i_1})$.
A connected graph $\mathcal{G}$ that contains no cycles is called a *tree* [6, p. 39]. Equivalently, a connected graph $\mathcal{G}$ is a tree if any two vertices can be connected by a unique path, or if $\mathcal{G}$ becomes disconnected by removing any edge in $\mathcal{G}$ [8, p. 12].
A subgraph $\mathcal{T} \subset \mathcal{G}$ that is a tree and spans $\mathcal{G}$, i.e., $\mathcal{V}_1 = \mathcal{V}$, is called a *spanning tree* [6, p. 55]. Every connected graph has at least one spanning tree [6, p. 56]. The complement $\mathcal{T}^c := \mathcal{G} \setminus \mathcal{T}$ of a spanning tree $\mathcal{T}$ is called the *cotree* or *chord set* [6, p. 56]. The edges of a spanning tree $\mathcal{T}$ are called *branches*, while the edges in the cotree $\mathcal{T}^c$ are called *chords* [6, p. 56].
For an oriented graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ consisting of $n_\mathcal{V}$ vertices $\mathcal{V} = \{v_1, ..., v_{n_\mathcal{V}}\}$ and $n_\mathcal{E}$ edges $\mathcal{E} = \{e_1, ..., e_{n_\mathcal{E}}\}$ edges, the *incidence matrix* A is defined as

$$A_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is the left vertex of } e_j, \\ -1, & \text{if } v_i \text{ is the right vertex of } e_j, \\ 0, & \text{else.} \end{cases}$$

Let A be the incidence matrix of G. If $\mathcal{G}$ is connected, then $\operatorname{rank}(A) = n_\mathcal{V} - 1$.
If $\mathcal{G}_1 \subset \mathcal{G}$ is a subgraph, then the incidence matrix $A_{\mathcal{G}_1}$ associated with $\mathcal{G}_1$ is a (possibly permuted) sub matrix of $A_\mathcal{G}$ [6, p. 141].
If $\mathcal{G}_1 \subset \mathcal{G}$ is a subgraph, where $\mathcal{G}$ is connected and $n_{\mathcal{V}_1} < n_\mathcal{V}$, then the incidence matrix $A_{\mathcal{G}_1}$ has full row rank, i.e., $\operatorname{rank}(A_{\mathcal{G}_1}) = n_{\mathcal{V}_1}$. The incidence matrix $A_{\mathcal{G}_1}$ is nonsingular if and only if $\mathcal{G}_1$ is a spanning tree [6, p. 141, Thm 7-3].
Removing any row $E_i^T A_\mathcal{G}$ associated with a vertex v_i from the incidence matrix $A_\mathcal{G}$, leads to a *reduced incidence matrix.* The associated vertex is called the reference or ground vertex [6, p. 141].
A directed graph $\mathcal{G}$ is strongly connected if every pair of vertices v_i, v_k is connected by a directed path from v_i to v_k and a directed path from v_k to v_i, cp. [21, p. 528]. Every vertex is strongly connected to itself [31, p. 100]. Strong connectivity defines an equivalence relation on the set of vertices [28, p. 100] and if $\mathcal{G}$ is not strongly connected, then it is composed of strongly connected components, which are strongly connected subgraphs of maximal size [28, p. 100].
Contracting the elements of the strongly connected components into single vertices, respectively, results in the condensed graph [31, p. 101]. The condensed graph is acyclic, i.e., contains no circles. There may be several edges connecting two strongly connected components $\mathcal{C}_1, \mathcal{C}_2$. However, these edges need to be directed in the same direction, e.g. from $\mathcal{C}_1$ to $\mathcal{C}_2$. Else, they would form a loop between $\mathcal{C}_1, \mathcal{C}_2$ implying that $\mathcal{C}_1 \cup \mathcal{C}_2$ is a larger strongly connected component. Contracting these edges to a single edge between $\mathcal{C}_1, \mathcal{C}_2$, the condensed graph again is simple.
For a matrix $A \in \mathbb{R}^{n \times n}$, the *graph* $\mathcal{G}(A)$ *of* A is defined as $\mathcal{G}(A) = \big\{\{v_1, ..., v_n\}, \{(v_i, v_j) | a_{ij} \neq 0\}\big\}$, i.e., whenever the ij-th entry is nonzero, there is an edge from vertex v_i to v_j. Note that $\mathcal{G}(A)$ contains self-loops associated with $a_{ii} \neq 0$ and multiple edges if $|a_{ij}|, |a_{ji}| > 1$, cp. [21, p. 528].

2.2 Differential-algebraic equations

There are several approaches studying the solvability of DAEs, e.g., derivative arrays [1, 4, 3], projector chains [11, 22, 23, 27] or structural analysis [25, 26], that differ in the regularity assumptions and the way the differential and algebraic components are separated. Related with these approaches are different index concepts, e.g., the differentiation index, the strangeness index, the tractability index or the structural index, which, roughly spoken, measures the complexity of solving a given DAE. A comparison of the different index concepts is given in [5, 24]. We follow the concept of derivative arrays and the strangeness index as developed in [17, 18, 19, 20], as it is applicable to a large class of DAEs and allows for efficient numerical integration of the considered systems.

We consider the DAE

$$F(t, x, \dot{x}) = 0 \tag{1}$$

with sufficiently smooth system function $F : \mathbb{D} \to \mathbb{R}^n$ defined on an open set $\mathbb{D} = \mathcal{I} \times \Omega_x \times \Omega_{\dot{x}} \subset \mathbb{R}_0^+ \times \mathbb{R}^n \times \mathbb{R}^n$. For $\ell \in \mathbb{N}$, the *derivative array* of size ℓ is the inflated DAE

$$\mathcal{F}_{F,\ell}(t, x, \dot{x}, ..., x^{(\ell+1)}) := \begin{bmatrix} F(t,x,\dot{x}) \\ \frac{d}{dt}F(t,x,\dot{x}) \\ \vdots \\ \frac{d^\ell}{dt^\ell}F(t,x,\dot{x}) \end{bmatrix} = 0, \tag{2}$$

obtained by successive differentiation. Every sufficiently smooth solution of (1) solves the inflated system (2). Vice versa, if $(t, x, \dot{x}, ..., x^{(\ell+1)})$ solves the derivative array (2), then $(t, x, \dot{x})$ also solves (1). For a suitable size ℓ, the idea of the strangeness index is to filter out a set of differential and algebraic equations that uniquely determines the x-part of this solution $(t, x, \dot{x}, ..., x^{(\ell+1)})$. Since this may include algebraic equations for derivatives of x, we consider (2) formally as an algebraic equation for the variable $z_\ell := (t, x, v_1, ..., v_{\ell+1})$, $v_k = x^{(k)}(t)$, $k = 1, ..., \ell+1$. The algebraic solution set is denoted by

$$\mathcal{F}_{F,\ell}^{-1}(0) = \{z_\ell \in \mathcal{I} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \mid \mathcal{F}_{F,\ell}(z_\ell) = 0\}. \tag{3}$$

To solve the derivative array (2) locally for $(t, x, \dot{x}, x^{(2)}, ..., x^{(\ell+1)})$, we make the following assertions on the Jacobians

$$M_\ell(z_\ell) := \partial_{v_1,...,v_{\ell+1}} \mathcal{F}_{F,\ell}(z_\ell), \quad N_\ell(z_\ell) := \partial_x \mathcal{F}_{F,\ell}(z_\ell),$$

containing the partial derivatives of $\mathcal{F}_{F,\ell}(z_\ell)$ with respect to the variables $v_1, ..., v_{\ell+1}$ and x, respectively, cp. [20, p. 155].

Hypothesis 2.1 ([20]). *Consider $F : \mathbb{D} \to \mathbb{R}^n$. Let there exist $\mu, d, a \in \mathbb{N}_0$, $n = d + a$, such that $F \in C^{\mu+1}(\mathbb{D}, \mathbb{R}^n)$, $\mathcal{F}_{F,\mu}^{-1}(0) \neq \emptyset$ and for every $z_{\mu,0} \in \mathcal{F}_{F,\mu}^{-1}(0)$, there exists a sufficiently small neighborhood $\mathcal{U}(z_{\mu,0})$, such that the following properties hold:*

(i) On $\mathcal{U}(z_{\mu,0}) \cap \mathcal{F}_{F,\mu}^{-1}(0)$, $\operatorname{rank}(M_\mu(z_\mu)) = (\mu+1)n - a$ and there exists a pointwise orthogonal matrix function $Z_2 \in C^\mu(\mathcal{U}(z_{\mu,0}), \mathbb{R}^{(\mu+1)n \times a})$ with $\operatorname{rank}(Z_2(z_\mu)) = a$ and $(Z_2^T M_\mu)(z_\mu) = 0$.

(ii) On $\mathcal{U}(z_{\mu,0}) \cap \mathcal{F}_{F,\mu}^{-1}(0)$, $\operatorname{rank}(Z_2^T \bar{N}_\mu(z_\mu)) = a$, where $\bar{N}_\mu = N_\mu[I_n\, 0]$, and there exists a pointwise orthogonal matrix function $T_1 \in C^\mu(\mathcal{U}(z_{\ell,0}), \mathbb{R}^{n \times d})$ with $\operatorname{rank}(T_1(z_\mu)) = d$ and $(Z_2^T \bar{N}_\mu T_1)(z_\mu) = 0$.

(iii) On $\mathcal{U}(z_{\mu,0}) \cap \mathcal{F}_{F,\ell}^{-1}(0)$, $\operatorname{rank}(F_{\dot{x}}(t, x, \dot{x}) T_1(z_\mu)) = d$ and there exists an orthogonal matrix $Z_1 \in \mathbb{R}^{n \times d}$ with $\operatorname{rank}(Z_1) = d$ and $\operatorname{rank}(Z_1^T F_{\dot{x}} T_1(z_\mu)) = d$.

The minimal μ for which F satisfies Hypothesis 2.1 on $\mathbb{D}$, is called the *strangeness index (s-index)* of (1). If F has s-index μ and satisfies Hypothesis 2.1 with $\mu+1, d, a$, we say that (1) has *regular s-index* μ. If F has (regular) s-index $\mu = 0$, then F is called (regular and) *s-free*. To match the smoothness assumptions of Hypothesis 2.1, we can reduce the domain of definition $\mathbb{D}$.

The set of functions satisfying Hypothesis 2.1 with integers μ, d, a and $\mu+1, d, a$ is denoted by

$$C^{\ell}_{\mu,d,a,\mathrm{reg}}(\mathbb{D}, \mathbb{R}^n) := \big\{F \in C^{\ell}(\mathbb{D}, \mathbb{R}^n) \mid F \text{ satisfies Hypothesis 2.1} \\ \text{with } \mu, d, a \text{ and } \mu+1, d, a \ \big\}.$$

Initial values that are part of a vector in the algebraic solution set are summarized in the *set of consistent initial values*

$$\mathcal{C}_{\mu}(F) := \big\{(t_0, x_0) \in \mathcal{I} \times \Omega \mid \exists\ (v_1, ..., v_{\mu+1}) \in \Omega_{\dot{x}} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \\ : (t_0, x_0, v_1, ..., v_{\mu+1}) \in \mathcal{F}^{-1}_{F,\mu}(0)\big\}. \tag{4}$$

Similarly, tuples $(t_0, x_0, \dot{x}_0)$ part of a vector in $F^{-1}_{\mu}(0)$ are summarized in the *set of consistent initializations*

$$\mathcal{L}_{\mu}(F) := \big\{(t_0, x_0, v_1) \in F^{-1}(0) \mid \exists\ (v_2, ..., v_{\mu+1}) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n \\ : (t_0, x_0, v_1, v_2 ..., v_{\mu+1}) \in \mathcal{F}^{-1}_{F,\mu}(0)\big\}. \tag{5}$$

For functions $F \in C^{\mu+1}_{\mu,d,a,\mathrm{reg}}(\mathbb{D}, \mathbb{R}^n)$, the DAE (1) is uniquely solvable for every initial value $(t_0, x_0) \in \mathcal{C}_{\mu+1}(F)$ and the solution is maximally extendable on $\mathcal{C}_{\mu+1}$.

Theorem 2.1. *[19, 20] If $F \in C^{\mu+1}_{\mu,d,a,reg}(\mathbb{D}, \mathbb{R}^n)$, then the DAE* (1) *is uniquely solvable for every $(t_0, x_0) \in \mathcal{C}_{\mu+1}$. The solution is $x \in C^1([t_0, \hat{t}^+_0), \mathbb{R}^n)$, where $\hat{t}^+_0 = \sup\{t \geq t_0 \mid (t, x(t)) \in \mathcal{C}_{\mu+1}\}$.*

In the following sections, Theorem 2.1 in combination with Hypothesis 2.1 is used to provide existence and uniqueness results, as well as to derive the index analysis for the resulting DAEs. This is done by checking the three properties *(i)*, *(ii)* and *(iii)* consecutively.
To check the rank assertions of Hypothesis 2.1, we make frequently use of the following result on the rank of a block matrix, cp. [13, p. 25].

Lemma 2.1. *Consider*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are matrices of suitable size. If A is nonsingular, then

$$\operatorname{rank}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \operatorname{rank}(A) + \operatorname{rank}(D - BA^{-1}C).$$

If S, T are nonsinuglar matrices, then $\operatorname{rank}(SMT) = \operatorname{rank}(M)$.

3 A quasi-stationary network model for incompressible flow

This section is devoted to the analysis of the standard incompressible flow model in order to provide the framework for the analysis of the full model in Section 4. We follow the model approach used in [15] and use the classification of flow network elements, as introduced in [16].
In contrast to the work in [15], our analysis is based on the *strangeness index* as it allows to treat DAEs in the most general form (1) and thus gives more flexibility in the application.

3.1 Model

We consider a network $\mathcal{N}$ of pipes $\mathrm{Pi}_1, ..., \mathrm{Pi}_{n_{\mathrm{Pi}}}$ filled by an incompressible fluid, e.g. water. The pipes are connected by junctions $\mathrm{Jc}_1, ..., \mathrm{Jc}_{n_{\mathrm{Jc}}}$, in which the mass flow of the fluid is split or merged. The connection to the environment is modeled by reservoirs $\mathrm{Re}_1, ..., \mathrm{Re}_{n_{\mathrm{Re}}}$ and demand branches $\mathrm{De}_1, ..., \mathrm{De}_{n_{\mathrm{De}}}$ imposing predefined pressures $\bar{p}_{\mathrm{Re},1}, ..., \bar{p}_{\mathrm{Re},n_{\mathrm{Re}}}$ or mass flows $\bar{q}_{\mathrm{De},1}, ..., \bar{q}_{\mathrm{De},n_{\mathrm{De}}}$ into the network. The number of each element in $\mathcal{N}$ is denoted by $n_{\mathrm{Pi}}, n_{\mathrm{Jc}}, n_{\mathrm{De}}$ and n_{Re}, respectively, and we set $n := n_{\mathrm{Jc}} + n_{\mathrm{Re}} + n_{\mathrm{Pi}} + n_{\mathrm{De}}$.
Given boundary conditions $\bar{p}_{\mathrm{Re}} = [\bar{p}_{\mathrm{Re}_i}]_{i=1,...,n_{\mathrm{Re}}}$, $\bar{q}_{\mathrm{De}} = [\bar{q}_{\mathrm{De}_i}]_{i=1,...,n_{\mathrm{De}}}$, the task is to compute the mass flows $q_{\mathrm{Pi}} = [q_{\mathrm{Pi}_i}]_{i=1,...,n_{\mathrm{Pi}}}$, $q_{\mathrm{De}} = [q_{\mathrm{De}_i}]_{i=1,...,n_{\mathrm{De}}}$ in the pipes and the demand branches as well as the pressures $p_{\mathrm{Jc}} = [p_{\mathrm{Jc}_i}]_{i=1,...,n_{\mathrm{Jc}}}$ in the junctions.
To set up a mathematical model for the network $\mathcal{N}$, we first note that each network element $\mathrm{Pi}, \mathrm{De}, \mathrm{Jc}$ and Re is equipped with a characteristic relation for the associated flow and pressure or pressure difference, cp. e.g. [16]. In a pipe Pi_j, the mass flow $q_{\mathrm{Pi},j}$ is specified by the transient momentum equation

$$\dot{q}_{\mathrm{Pi},j} = c_{1,j}\Delta p_j + c_{2,j}\,\mathrm{sgn}(q_{\mathrm{Pi},j})q_{\mathrm{Pi},j}^2 + c_{3,j} \tag{6}$$

depending on the pressure difference $\Delta p_j = p_{j_1} - p_{j_2}$ between the adjacent nodes v_{j_1}, v_{j_2} and on constants $c_{1,j}, c_{2,j} > 0$ and $c_{3,j}$ depending on the pipe diameter, length, inclination angle and physical properties like the density.
In a junction Jc_i, the amount of mass entering and leaving Jc_i is equal due to mass conservation. Summarizing the indices of pipes and demand branches that are incident to Jc_i in the set $\hat{J}_i := \{j \mid e_j \text{ is incident to } \mathrm{Jc}_i\}$, we thus get that

$$\sum_{j\in\hat{J}_i} q_j = 0. \tag{7}$$

Note that equation (7) is a generalization of Kirchhoff's current law for fluids.
In a demand branch De_j, the mass flow q_j is set to the predefined value $\bar{q}_{\mathrm{De},j}$, i.e., $q_j = \bar{q}_{\mathrm{De},j}$. Similarly, in a reservoir Re_i, the pressure p_i is kept at a prescribed value $\bar{p}_{\mathrm{Re},i}$, i.e., $p_i = \bar{p}_{\mathrm{Re},i}$. We assume that each reservoir is connected to exactly one pipe.
Besides the individual element equations, the dynamics of the network depend on the connection of the elements. To include this structure, we represent $\mathcal{N}$ as a graph $\mathcal{G}_{\mathcal{N}}$. The pipes and the demand branches correspond to the edges of $\mathcal{G}_{\mathcal{N}}$ while the junctions and reservoirs serve as vertices, i.e., $\mathcal{E}_{\mathcal{N}} = \{\mathrm{Pi}_1, ..., \mathrm{Pi}_{n_{\mathrm{Pi}}}, \mathrm{De}_1, ..., \mathrm{De}_{n_{\mathrm{De}}}\}$ and $\mathcal{V}_{\mathcal{N}} = \{\mathrm{Jc}_1, ..., \mathrm{Jc}_{n_{\mathrm{Jc}}}, \mathrm{Re}_1, ..., \mathrm{Re}_{n_{\mathrm{Re}}}\}$. We set $n_{\mathcal{E}} = n_{\mathrm{Pi}} + n_{\mathrm{De}}$ and $n_{\mathcal{V}} = n_{\mathrm{Jc}} + n_{\mathrm{Re}}$. Assigning a direction to each pipe and demand branch, we obtain an orientation of $\mathcal{G}_{\mathcal{N}}$, allowing to speak of a positive or negative mass flow: If the flow on an edge agrees with the assigned direction, the flow is called positive, else it is called negative. Note that the given orientation is arbitrary and only serves as a reference condition, it must not be related with the true or expected direction of the fluid flow. Represented as an oriented graph, the structure of the network $\mathcal{N}$ is described by the incidence matrix $A_{\mathcal{N}}$ associated with $\mathcal{G}_{\mathcal{N}}$. According to the numbering of $\mathcal{V}, \mathcal{E}$, we partition the incidence matrix as

$$A_{\mathcal{N}} = \begin{bmatrix} A_{\mathrm{Jc,Pi}} & A_{\mathrm{Jc,De}} \\ A_{\mathrm{Re,Pi}} & A_{\mathrm{Re,De}} \end{bmatrix} = \begin{bmatrix} A_{\mathrm{Jc}} \\ A_{\mathrm{Re}} \end{bmatrix}.$$

Furthermore, we summarize the flows, pressures and pressure differences as

$$q = \begin{bmatrix} q_{\mathrm{Pi}} \\ q_{\mathrm{De}} \end{bmatrix}, \quad p = \begin{bmatrix} p_{\mathrm{Jc}} \\ p_{\mathrm{Re}} \end{bmatrix}, \quad \Delta p = \begin{bmatrix} \Delta p_{\mathrm{Pi}} \\ \Delta p_{\mathrm{De}} \end{bmatrix}.$$

Setting $C_1 = \mathrm{diag}\,(c_{1,j})_j$, $C_2 = \mathrm{diag}\,(c_{2,j})_j$, $C_3 = [c_{3,j}]_j$ for $j = 1, ...n_{\mathrm{Pi}_j}$ and noting that $A_{\mathcal{N}}^T p = \Delta p$, we set

$$f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}) := C_1(A_{\mathrm{Jc,Pi}}^T p_{\mathrm{Jc}} + A_{\mathrm{Re,Pi}}^T p_{\mathrm{Re}}) + C_2\,\mathrm{diag}\,(|q_{\mathrm{Pi},j}|)\,q_{\mathrm{Pi}} + C_3,$$

and the differential equation for the pipe flow q_{Pi} is given by

$$\dot{q}_{\mathrm{Pi}} = f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}). \tag{8a}$$

Noting that $\sum_{j\in\hat{J}_i} q_{i_j} = e_i^T A_{\mathcal{N}} q$ for a junction Jc_i, where e_i denotes the i-th standard canonical basis vector, such that the equations for the junctions can be summarized as

$$A_{\mathrm{Jc,Pi}} q_{\mathrm{Pi}} + A_{\mathrm{Jc,De}} q_{\mathrm{De}} = 0. \tag{8b}$$

For the demand branches and reservoirs, we obtain the simple relations

$$q_{\mathrm{De}} = \bar{q}_{\mathrm{De}}, \tag{8c}$$
$$p_{\mathrm{Re}} = \bar{p}_{\mathrm{Re}}. \tag{8d}$$

In conclusion, the dynamics in a network consisting of pipes, reservoirs and demand vertices, are modeled by the DAE (8).
To analyze the solvability of (8), we formally define the DAE function

$$F(t, q, p, \dot{q}, \dot{p}) := \begin{bmatrix} \dot{q}_{\mathrm{Pi}} - f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}) \\ A_{\mathrm{Jc,Pi}} q_{\mathrm{Pi}} + A_{\mathrm{Jc,De}} q_{\mathrm{De}} \\ q_{\mathrm{De}} - \bar{q}_{\mathrm{De}} \\ p_{\mathrm{Re}} - \bar{p}_{\mathrm{Re}} \end{bmatrix}. \tag{9}$$

On an interval $\mathcal{I} \subset \mathbb{R}$, where the flow doesn't change its direction, the smoothness of F depends on the smoothness of the input functions $\bar{q}_{\mathrm{De}}, \bar{p}_{\mathrm{Re}}$.

Lemma 3.1. *For a network $\mathcal{N}$, consider the function F defined in (9). Let $\bar{q}_{De} \in C^k(\mathcal{I}, \mathbb{R}^{n_{De}})$, $\bar{p}_{Re} \in C^k(\mathcal{I}, \mathbb{R}^{n_{Re}})$. Then, $F \in C^k(\mathbb{D}, \mathbb{R}^n)$ for $\mathbb{D} := \mathcal{I} \times \Omega_x \times \Omega_{\dot{x}}$ with $\Omega_x := \mathbb{R}^{n_\mathcal{E}} \times \mathbb{R}_+^{n_\mathcal{E}}$ and $\Omega_{\dot{x}} := \mathbb{R}^{n_\mathcal{E}} \times \mathbb{R}^{n_\mathcal{E}}$.*

Proof. If $q_{\mathrm{Pi}} \in \mathbb{R}_+^{n_\mathcal{E}}$, then $\mathrm{sgn}(q_{\mathrm{Pi},j}) \geq 0$ for every $j = 1, ..., n_\mathcal{E}$ and $f_{\mathrm{Pi}}(q_{\mathrm{Pi}}, p_{\mathrm{Jc}}, p_{\mathrm{Re}}) = C_1(A_{\mathrm{Jc,Pi}}^T p_{\mathrm{Jc}} + A_{\mathrm{Re,Pi}}^T p_{\mathrm{Re}}) + C_2 \operatorname{diag}(q_{\mathrm{Pi},j})\, q_{\mathrm{Pi}} + C_3$. Hence, $f_{\mathrm{Pi}} \in C^k(\mathbb{R}_+^{n_\mathcal{E}} \times \mathbb{R}^{n_\mathcal{V}}, \mathbb{R}^{n_{\mathrm{Pi}}})$. With $\bar{q}_{\mathrm{De}} \in C^k(\mathcal{I}, \mathbb{R}^{n_{\mathrm{De}}})$, $\bar{p}_{\mathrm{Re}} \in C^k(\mathcal{I}, \mathbb{R}^{n_{\mathrm{Re}}})$, it follows that F satisfies the asserted smoothness property. □

Furthermore, we impose the following assumptions on the network.

Assumption 3.1. *Let $\mathcal{N}$ be a network of pipes, junctions, reservoir and demand branches that satisfies the following assertions.*

(i) Every pair of adjacent vertices, junctions or reservoirs, is connected at most by one pipe.

(ii) Every pair of vertices, junctions or reservoirs, can be reached by a sequence of pipes.

(iii) Each pipe has a fixed but arbitrary direction.

(iv) Every junction is adjacent to at most one demand branch and every reservoir is connected to at most one branch element.

Under Assumption 3.1, the graph $\mathcal{G}_\mathcal{N}$ associated with $\mathcal{N}$ is simple, connected and directed. The reservoirs $\mathrm{Re}_1, ..., \mathrm{Re}_{n_{\mathrm{Re}}}$ are end vertices, i.e., $d(\mathrm{Re}_i) = 1$ for $i = 1, ..., n_{\mathrm{Re}}$.

3.2 Analysis of the hydraulic flow network

To analyze the solvability of the DAE (8), we exploit the structure of the network $\mathcal{N}$ and the associated graph $\mathcal{G}_\mathcal{N}$.

Lemma 3.2. *Let $\mathcal{N}$ be a network of pipes, junctions, reservoirs and demand branches that satisfies Assumption 3.1. Let $A_\mathcal{N}$ be the incidence matrix of the associated graph $\mathcal{G}_\mathcal{N}$.*
If $n_{Re} > 0$, then $\operatorname{rank}(A_{Jc,Pi}) = n_{Jc}$ and there exists a permutation $\Pi = [\Pi_1, \Pi_2]$ such that $A_{Jc,Pi}\Pi_1$ is nonsingular. Furthermore, $A_{Jc,Pi} C_1 A_{Jc,Pi}^T$ is nonsingular.

Proof. Neglecting the demand branches $\mathrm{De}_1, ..., \mathrm{De}_{n_{\mathrm{De}}}$ from $\mathcal{G}_{\mathcal{N}}$ leads to the subgraph $\mathcal{G}_{\mathcal{N}\setminus\mathrm{De}} := \{(\mathrm{Jc}, \mathrm{Re}), \mathrm{Pi}\}$. The associated incidence matrix is given by $A_{\mathrm{Pi}} = [A^T_{\mathrm{Jc,Pi}}, A^T_{\mathrm{Re,Pi}}]^T$. Since there is at least one reservoir in the network, the submatrix $A_{\mathrm{Jc,Pi}}$ is a *reduced* incidence matrix, implying that $\mathrm{rank}(A_{\mathrm{Jc,Pi}}) = n_{\mathrm{Jc}}$, cp. [10]. Consequently, there exists a permutation $\Pi = [\Pi_1, \Pi_2]$ such that $A_{\mathrm{Jc,Pi}}\Pi_1$ is nonsingular, while $A_{\mathrm{Jc,Pi}}\Pi_2$ contains the linear dependent columns.
With $c_{1,j} > 0$, $j = 1, ...n_{\mathrm{Pi}_j}$, the matrix $C_1 = \mathrm{diag}\,(c_{1,j})_{j=1,...n_{\mathrm{Pi}_j}}$ is positive definite, allowing to factorize the considered product according to $A_{\mathrm{Jc,Pi}} C_1 A^T_{\mathrm{Jc,Pi}} = (A_{\mathrm{Jc,Pi}}\sqrt{C_1})(A_{\mathrm{Jc,Pi}}\sqrt{C_1})^T$, where $\sqrt{C_1} = \mathrm{diag}\left(\sqrt{c_{1,j}}\right)_{j=1,...n_{\mathrm{Pi}_j}}$ is nonsingular. Then,

$$\mathrm{rank}(A_{\mathrm{Jc,Pi}} C_1 A^T_{\mathrm{Jc,Pi}}) = \mathrm{rank}(A_{\mathrm{Jc,Pi}}\sqrt{C_1}) = \mathrm{rank}(A_{\mathrm{Jc,Pi}}) = n_{\mathrm{Jc}}, \tag{10}$$

implying that $A_{\mathrm{Jc,Pi}} C_1 A^T_{\mathrm{Jc,Pi}}$ is pointwise nonsingular. □

Remark 3.1. *Under the assertions of Lemma 3.2, the permutation $\Pi = [\Pi_1, \Pi_2]$ partitions the graph $\mathcal{G}_{\mathcal{N}}$ into a spanning tree and chord set [6, p. 141, Thm 7-3]. More exactly, if $\Pi_1 = [e_j]_{j\in\mathcal{I}_1}$ and $\Pi_2 = [e_j]_{j\in\mathcal{I}_2}$, where $e_1, ..., e_{n_{Pi}}$ denotes the standard canonical basis, then $\{Pi_j | j \in \mathcal{I}_1\}$ and the adjacent vertices define a spanning tree of $\mathcal{G}_{\mathcal{N}}$ while $\{Pi_j | j \in \mathcal{I}_2\}$ denote the corresponding chord set.*
The matrix $A_{Jc,Pi}\sqrt{C_1}$ corresponds to the incidence matrix of the graph $\{(Jc, Re), Pi\}$ whose edges are weighted with $\sqrt{c_{1,1}}, ..., \sqrt{c_{1,n_{Pi}}}$. The product $A_{Jc,Pi} C_1 A^T_{Jc,Pi}$ corresponds to the adjacency matrix of this weighted graph $\{(Jc, Re), Pi\}$ up to addition of the degree matrix $D := \mathrm{diag}\,(d(Jc_i))_{i=1,...,Jc_i}$, cp. [8, p. 24].

Using Lemma 3.2, we give conditions when the DAE (8) is uniquely solvable and reformulate (8) as s-free system.

Theorem 3.1. *Let $\mathcal{N}$ be a network of pipes, junctions, reservoirs and demand branches that satisfies Assumption 3.1. Let $F \in C^2(\mathbb{D}, \mathbb{R}^n)$ be the DAE function associated with $\mathcal{N}$. If $n_{Re} > 0$, then the DAE (8) has regular s-index 1 and is uniquely solvable for every $(t_0, q_0, p_0) \in \mathcal{C}_2(F)$. The solution is given by $q \in C^1(\mathcal{I}, \mathbb{R}^{n_\mathcal{E}})$, $p \in C^1(\mathcal{I}, \mathbb{R}^{n_\mathcal{V}})$. The s-free model of $\mathcal{N}$ is given by*

$$\dot{q}_2 - \Pi_2^T f_{Pi}(q_{Pi}, p_{Jc}, p_{Re}) = 0 \tag{11a}$$
$$A_{Jc,Pi}\Pi_1\Pi_1^T q_{Pi} + A_{Jc,Pi}\Pi_2\Pi_2^T q_{Pi} + A_{Jc,De} q_{De} = 0 \tag{11b}$$
$$A_{Jc,Pi} f_{Pi}(q_{Pi}, p_{Jc}, p_{Re}) = -A_{Jc,De}\dot{\bar{q}}_{De} \tag{11c}$$
$$q_{De} = \bar{q}_{De} \tag{11d}$$
$$p_{Re} = \bar{p}_{Re}, \tag{11e}$$

where $[\Pi_1\, \Pi_2]$ is a permutation such that $A_{Jc,Pi}\Pi_1$ is nonsingular. The number of differential and algebraic equations is given by $d = n_{Pi} - n_{Jc}$ and $a = 2n_{Jc} + n_{De} + n_{Re}$, respectively.

Proof. We prove the assertion by verifying the individual items of Hypothesis 2.1, such that we can apply Theorem 2.1. First, we show that (8) satisfies Hypothesis 2.1 with values $\mu = 1$, $d = n_{\mathrm{Pi}} - n_{\mathrm{Jc}}$ and $a = 2n_{\mathrm{Jc}} + n_{\mathrm{De}} + n_{\mathrm{Re}}$. We consider the derivative array $\mathcal{F}_{F,1} := [F^T, \frac{d}{dt}F^T]$ with F given as in (9) and

$$\frac{d}{dt}F = \begin{bmatrix} \ddot{q}_{\mathrm{Pi}} - \mathrm{D}_1 f_{\mathrm{Pi}}\dot{q}_{\mathrm{Pi}} - C_1(A^T_{\mathrm{Jc,Pi}}\dot{p}_{\mathrm{Jc}} + A^T_{\mathrm{Re,Pi}}\dot{p}_{\mathrm{Re}}) \\ A_{\mathrm{Jc,Pi}}\dot{q}_{\mathrm{Pi}} + A_{\mathrm{Jc,De}}\dot{q}_{\mathrm{De}} \\ \dot{q}_{\mathrm{De}} - \dot{\bar{q}}_{\mathrm{De}} \\ \dot{p}_{\mathrm{Re}} - \dot{\bar{p}}_{\mathrm{Re}} \end{bmatrix}.$$

The Jacobians $M_1 := \partial_{\dot{q},\dot{p},\ddot{q},\ddot{p}} \mathcal{F}_{F,1}$ and $N_1 := \partial_{q,p} \mathcal{F}_{F,1}$ are given by

$$M_1 = \begin{array}{c} \\ \text{Pi} \\ \text{Jc} \\ \text{De} \\ \text{Re} \\ \dot{\text{Pi}} \\ \dot{\text{Jc}} \\ \dot{\text{De}} \\ \dot{\text{Re}} \end{array} \begin{array}{c} \begin{array}{cccccccc} \dot{q}_{\text{Pi}} & \dot{q}_{\text{De}} & \dot{p}_{\text{Jc}} & \dot{p}_{\text{Re}} & \ddot{q}_{\text{Pi}} & \ddot{q}_{\text{De}} & \ddot{p}_{\text{Jc}} & \ddot{p}_{\text{Re}} \end{array} \\ \left[\begin{array}{cccccccc} I_{\text{Pi}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{D}_1 f_{\text{Pi}} & 0 & -C_1 A_{\text{Jc},\text{Pi}}^T & -C_1 A_{\text{Re},\text{Pi}}^T & I_{\text{Pi}} & 0 & 0 & 0 \\ A_{\text{Jc},\text{Pi}} & A_{\text{Jc},\text{De}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{\text{De}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\text{Re}} & 0 & 0 & 0 & 0 \end{array} \right] \end{array}, \tag{12}$$

$$N_1 = \begin{array}{c} \\ \text{Pi} \\ \text{Jc} \\ \text{De} \\ \text{Re} \\ \dot{\text{Pi}} \\ \dot{\text{Jc}} \\ \dot{\text{De}} \\ \dot{\text{Re}} \end{array} \begin{array}{c} \begin{array}{cccc} q_{\text{Pi}} & q_{\text{De}} & p_{\text{Jc}} & p_{\text{Re}} \end{array} \\ \left[\begin{array}{cccc} -\mathrm{D}_1 f_{\text{Pi}} & 0 & -C_1 A_{\text{Jc},\text{Pi}}^T & -C_1 A_{\text{Re},\text{Pi}}^T \\ A_{\text{Jc},\text{Pi}} & A_{\text{Jc},\text{De}} & 0 & 0 \\ 0 & I_{\text{De}} & 0 & 0 \\ 0 & 0 & 0 & I_{\text{Re}} \\ -\partial_{q_{\text{Pi}}} \dot{f}_{\text{Pi}} & 0 & -\partial_{p_{\text{Jc}}} \dot{f}_{\text{Pi}} & -\partial_{p_{\text{Re}}} \dot{f}_{\text{Pi}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}. \tag{13}$$

To check Hypothesis 2.1, (i), we permute M_1 using permutations $\bar{\Pi}_{M_1}, \Pi_{M_1}$, such that $\bar{\Pi}_{M_1}^T M_1 \Pi_{M_1} = [\tilde{M}_{ij}]_{i,j=1,2}$, with

$$\tilde{M}_{11} = \begin{array}{c} \\ \dot{\text{De}} \\ \dot{\text{Re}} \\ \text{Pi} \\ \dot{\text{Pi}} \end{array} \begin{array}{c} \begin{array}{cccc} \dot{q}_{\text{De}} & \dot{p}_{\text{Re}} & \dot{q}_{\text{Pi}} & \ddot{q}_{\text{Pi}} \end{array} \\ \left[\begin{array}{cccc} I_{\text{De}} & 0 & 0 & 0 \\ 0 & I_{\text{Re}} & 0 & 0 \\ 0 & 0 & I_{\text{Pi}} & 0 \\ 0 & -C_1 A_{\text{Re},\text{Pi}}^T & -\mathrm{D}_1 f_{\text{Pi}} & I_{\text{Pi}} \end{array} \right] \end{array}, \quad \tilde{M}_{12} = \begin{array}{c} \\ \dot{\text{De}} \\ \dot{\text{Re}} \\ \text{Pi} \\ \dot{\text{Pi}} \end{array} \begin{array}{c} \begin{array}{ccc} \dot{p}_{\text{Jc}} & \ddot{p}_{\text{Jc}} & \ddot{p}_{\text{Re}} \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -C_1 A_{\text{Jc},\text{Pi}}^T & 0 & 0 \end{array} \right] \end{array},$$

$$\tilde{M}_{21} = \begin{array}{c} \\ \dot{\text{Jc}} \\ \text{Jc} \\ \text{De} \\ \text{Re} \end{array} \begin{array}{c} \begin{array}{cccc} \dot{q}_{\text{De}} & \dot{p}_{\text{Re}} & \dot{q}_{\text{Pi}} & \ddot{q}_{\text{Pi}} \end{array} \\ \left[\begin{array}{cccc} A_{\text{Jc},\text{De}} & 0 & A_{\text{Jc},\text{Pi}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}, \qquad \tilde{M}_{22} = 0.$$

On $\mathbb{D}$, the diagonal block $\tilde{M}_{11}$ is non-singular and the associated Schur complement vanishes, i.e., $\tilde{M}_{21} \tilde{M}_{11}^{-1} \tilde{M}_{12} = 0$. Then

$$\operatorname{rank}(M_1) = \operatorname{rank}(\tilde{M}_{11}) + \operatorname{rank}(\tilde{M}_{21} \tilde{M}_{11}^{-1} \tilde{M}_{12}) = 2n_{\text{Pi}} + n_{\text{De}} + n_{\text{Re}}$$

on $\mathbb{D}$ by Lemma 2.1, implying that $a = \operatorname{corank}(M_1) = 2n_{\text{Jc}} + n_{\text{De}} + n_{\text{Re}}$ and $d = n_{\mathcal{V}} + n_{\mathcal{E}} - a = n_{\text{Pi}} - n_{\text{Jc}}$. Furthermore, there exists a basis $Z_2 \in \mathbb{R}^{2n \times a}$ of $\operatorname{corange}(M_1)$ that, e.g., is given by

$$Z_2^T = \begin{array}{c} \begin{array}{cccccccc} \dot{\text{De}} & \dot{\text{Re}} & \text{Pi} & \dot{\text{Pi}} & \dot{\text{Jc}} & \text{Jc} & \text{De} & \text{Re} \end{array} \\ \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & I_{\text{Jc}} & 0 & 0 \\ -A_{\text{Jc},\text{De}} & 0 & -A_{\text{Jc},\text{Pi}} & 0 & I_{\text{Jc}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\text{De}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\text{Re}} \end{array} \right] \end{array} \bar{\Pi}_{M_1}^T.$$

In order to check Hypothesis 2.1, *(ii)*, we apply the matrix Z_2 to the Jacobian N_1 and find that

$$Z_2^T N_1 = \begin{array}{c} \begin{matrix} q_{\text{Pi}} & q_{\text{De}} & p_{\text{Jc}} & p_{\text{Re}} \end{matrix} \\ \begin{bmatrix} A_{\text{Jc,Pi}} & A_{\text{Jc,De}} & 0 & 0 \\ A_{\text{Jc,Pi}} \text{D}_1 f_{\text{Pi}} & 0 & A_{\text{Jc,Pi}} C_1 A_{\text{Jc,Pi}}^T & A_{\text{Jc,Pi}} C_1 A_{\text{Re,Pi}}^T \\ 0 & I_{\text{De}} & 0 & 0 \\ 0 & 0 & 0 & I_{\text{Re}} \end{bmatrix} \end{array}. \tag{14}$$

As $n_{\text{Re}} > 0$, we have that $\text{rank}(A_{\text{Jc,Pi}}) = n_{\text{Jc}}$ and there exists a permutation $[\Pi_1, \Pi_2]$ such that $A_{\text{Jc,Pi},1}\Pi_1 := A_{\text{Jc,Pi}}\Pi_1$ is nonsingular, cp. Lemma 3.2. The linear depending columns are given by $A_{\text{Jc,Pi},2} := A_{\text{Jc,Pi}}\Pi_2$. Choosing suitable permutations $\bar{\Pi}_{N_1}, \Pi_{N_1}$, we thus find that $Z_2^T N_1 \Pi_{N_1} = [\tilde{N}_1, \tilde{N}_2]$ with

$$\tilde{N}_1 = \begin{bmatrix} A_{\text{Jc,Pi}} C_1 A_{\text{Jc,Pi}}^T & A_{\text{Jc,Pi}} \text{D}_1 f_{\text{Pi}} \Pi_1 & A_{\text{Jc,Pi}} C_1 A_{\text{Re,Pi}}^T & 0 \\ 0 & A_{\text{Jc,Pi},1}\Pi_1 & 0 & A_{\text{Jc,De}} \\ 0 & 0 & I_{\text{Re}} & 0 \\ 0 & 0 & 0 & I_{\text{De}} \end{bmatrix}, \quad \tilde{N}_2 = \begin{bmatrix} A_{\text{Jc,Pi}} \text{D}_1 f_{\text{Pi}} \Pi_2 \\ A_{\text{Jc,Pi},2} \\ 0 \\ 0 \end{bmatrix}.$$

As the Jacobian C_1 is pointwise positive definite on $\mathbb{R}_+^{n\varepsilon} \times \mathbb{R}^{n\nu}$, the product $A_{\text{Jc,Pi}} C_1 A_{\text{Jc,Pi}}^T$ is pointwise nonsingular on $\mathbb{R}_+^{n\varepsilon} \times \mathbb{R}^{n\nu}$, cp. Lemma 3.2. The matrix $A_{\text{Jc,Pi},1}\Pi_1$ is nonsingular due to the choice of Π_1. Hence, $\tilde{N}_1$ is pointwise nonsingular on $\mathbb{R}_+^{n\varepsilon} \times \mathbb{R}^{n\nu}$, implying that $\text{rank}(Z_2^T N_1) = 2n_{\text{Jc}} + n_{\text{De}} + n_{\text{Re}} = a$ on on $\mathbb{R}_+^{n\varepsilon} \times \mathbb{R}^{n\nu}$. In particular, on $\mathbb{R}_+^{n\varepsilon} \times \mathbb{R}^{n\nu}$, there exists a basis T_2 of $\ker(Z_2^T N_1)$ that, e.g., is given by

$$T_2 = \Pi_{N_1} \begin{bmatrix} -(A_{\text{Jc,Pi}} C_1 A_{\text{Jc,Pi}}^T)^{-1} A_{\text{Jc,Pi}} \text{D}_1 f_{\text{Pi}} (\Pi_2 - \Pi_1 A_{\text{Jc,Pi},1} \Pi_1^{-1} A_{\text{Jc,Pi},2}) \\ -A_{\text{Jc,Pi},1} \Pi_1^{-1} A_{\text{Jc,Pi},2} \\ 0 \\ I_{n_{\text{Pi}} - n_{\text{Jc}}} \end{bmatrix}.$$

Finally, to check Hypothesis 2.1, *(iii)*, we consider the product

$$F_{\dot{q},\dot{p}} T_2 = \begin{bmatrix} -\Pi_1 A_{\text{Jc,Pi},1} \Pi_1^{-1} A_{\text{Jc,Pi},2} + \Pi_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

As Π_1, Π_2 are linearly independent, we have that

$$\text{rank}(\Pi_2 - \Pi_1 (U_2^T A_{\text{Jc,Pi}} \Pi_1)^{-1} U_2^T A_{\text{Jc,Pi}} \Pi_2) = n_{\text{Pi}} - n_{\text{Jc}}$$

and it follows that $\text{rank}(F_{\dot{q},\dot{p}} T_2) = d$ on $\mathbb{D}$. Considering the matrix

$$Z_1^T = \begin{bmatrix} 0 & \Pi_2^T & 0 & 0 \end{bmatrix},$$

we find that $\text{rank}(Z_1^T F_{\dot{q},\dot{p}} T_2) = d$ on $\mathbb{D}$. In conclusion, we have shown that the network model (8) satisfies the assertions of Hypothesis 2.1 for $\mu = 1$ and $d = n_{\text{Pi}} - n_{\text{Jc}}$ and $a = 2n_{\text{Jc}} + n_{\text{De}} + n_{\text{Re}}$, i.e., the network model (8) has s-index $\mu = 1$.

To prove that the network model is regular, we show that (8) satisfies Hypothesis 2.1 with values $\mu = 2$, $d = n_{\text{Pi}} - n_{\text{Jc}}$, $a = 2n_{\text{Jc}} + n_{\text{De}} + n_{\text{Re}}$. Therefore, we consider $\mathcal{F}_{F,2} := [F^T, \frac{d}{dt}F^T, \frac{d^2}{dt^2}F^T]^T$ with

$$\frac{d^2}{dt^2} F(t, q, p, \dot{q}, \dot{p}) = \begin{bmatrix} \dddot{q}_{\text{Pi}} - \text{D}_1 f_{\text{Pi}} \ddot{q}_{\text{Pi}} - C_1 (A_{\text{Jc,Pi}}^T \ddot{p}_{\text{Jc}} + A_{\text{Re,Pi}}^T \ddot{p}_{\text{Re}}) + R(q, p, \dot{q}, \dot{p}) \\ A_{\text{Jc,Pi}} \ddot{q}_{\text{Pi}} + A_{\text{Jc,De}} \ddot{q}_{\text{De}} \\ \ddot{q}_{\text{De}} - \dddot{q}_{\text{De}} \\ \ddot{p}_{\text{Re}} - \dddot{p}_{\text{Re}} \end{bmatrix},$$

where $R(q,p,\dot q,\dot p)$ contains the mixed derivatives in the pipe function. The Jacobians $M_2 := \partial_{\dot q,\dot p,\ddot q,\ddot p}\mathcal{F}_{F,2}$ and $N_2 := \partial_{q,p}\mathcal{F}_{F,2}$ are given by

$$M_2 = \begin{bmatrix} M_1 & 0 \\ M_{21} & M_{22} \end{bmatrix}, \quad N_2 = \begin{bmatrix} N_1 \\ N_{21} \end{bmatrix},$$

with M_1, N_1 given by (12), (13) and

$$M_{21} = \begin{array}{c} \\ \ddot{\mathrm{P}}\mathrm{i} \\ \ddot{\mathrm{J}}\mathrm{c} \\ \ddot{\mathrm{D}}\mathrm{e} \\ \ddot{\mathrm{R}}\mathrm{e} \end{array} \begin{array}{c} \begin{array}{cccccccc} \dot q_{\mathrm{Pi}} & \dot q_{\mathrm{De}} & \dot p_{\mathrm{Jc}} & \dot p_{\mathrm{Re}} & \ddot q_{\mathrm{Pi}} & \ddot q_{\mathrm{De}} & \ddot p_{\mathrm{Jc}} & \ddot p_{\mathrm{Re}} \end{array} \\ \begin{bmatrix} \partial_{\dot q_{\mathrm{Pi}}}\ddot F_2 & 0 & \partial_{\dot p_{\mathrm{Jc}}}\ddot F_2 & \partial_{\dot p_{\mathrm{Re}}}\ddot F_2 & -\mathrm{D}_1 f_{\mathrm{Pi}} & 0 & -C_1 A_{\mathrm{Jc,Pi}}^T & -C_1 A_{\mathrm{Re,Pi}}^T \\ 0 & 0 & 0 & 0 & A_{\mathrm{Jc,Pi}} & A_{\mathrm{Jc,De}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\mathrm{De}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathrm{Re}} \end{bmatrix} \end{array},$$

$$M_{22} = \begin{array}{c} \\ \ddot{\mathrm{P}}\mathrm{i} \\ \ddot{\mathrm{J}}\mathrm{c} \\ \ddot{\mathrm{D}}\mathrm{e} \\ \ddot{\mathrm{R}}\mathrm{e} \end{array} \begin{array}{c} \begin{array}{cccc} \dddot q_{\mathrm{Pi}} & \dddot q_{\mathrm{De}} & \dddot p_{\mathrm{Jc}} & \dddot p_{\mathrm{Re}} \end{array} \\ \begin{bmatrix} I_{\mathrm{Pi}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad N_{21} = \begin{array}{c} \\ \ddot{\mathrm{P}}\mathrm{i} \\ \ddot{\mathrm{J}}\mathrm{c} \\ \ddot{\mathrm{D}}\mathrm{e} \\ \ddot{\mathrm{R}}\mathrm{e} \end{array} \begin{array}{c} \begin{array}{cccc} q_{\mathrm{Pi}} & q_{\mathrm{De}} & p_{\mathrm{Jc}} & p_{\mathrm{Re}} \end{array} \\ \begin{bmatrix} \partial_q R & 0 & \partial_{p_{\mathrm{Jc}}} R & \partial_{p_{\mathrm{Re}}} R \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

We permute M_2 using permutations $\bar\Pi_{M_2}, \Pi_{M_2}$, such that

$$\bar\Pi_{M_2}^T M_2 \Pi_{M_2} = \begin{bmatrix} \tilde M_{11} & \tilde M_{12} & 0 \\ \tilde M_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$\tilde M_{11} = \begin{array}{c} \\ \dot{\mathrm{D}}\mathrm{e} \\ \ddot{\mathrm{D}}\mathrm{e} \\ \dot{\mathrm{R}}\mathrm{e} \\ \mathrm{Pi} \\ \dot{\mathrm{P}}\mathrm{i} \\ \ddot{\mathrm{R}}\mathrm{e} \\ \ddot{\mathrm{P}}\mathrm{i} \end{array} \begin{array}{c} \begin{array}{ccccccc} \dot q_{\mathrm{De}} & \ddot q_{\mathrm{De}} & \dot p_{\mathrm{Re}} & \dot q_{\mathrm{Pi}} & \ddot q_{\mathrm{Pi}} & \ddot p_{\mathrm{Re}} & \dddot q_{\mathrm{Pi}} \end{array} \\ \begin{bmatrix} I_{\mathrm{De}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{\mathrm{De}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{\mathrm{Re}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\mathrm{Pi}} & 0 & 0 & 0 \\ 0 & 0 & -C_1 A_{\mathrm{Re,Pi}}^T & -\mathrm{D}_1 f_{\mathrm{Pi}} & I_{\mathrm{Pi}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\mathrm{Re}} & 0 \\ 0 & 0 & \partial_{\dot p_{\mathrm{Re}}} R & \partial_{\dot q_{\mathrm{Pi}}} R & -\mathrm{D}_1 f_{\mathrm{Pi}} & -C_1 A_{\mathrm{Re,Pi}}^T & I_{\mathrm{Pi}} \end{bmatrix} \end{array},$$

$$\tilde M_{12} = \begin{array}{c} \\ \dot{\mathrm{D}}\mathrm{e} \\ \ddot{\mathrm{D}}\mathrm{e} \\ \dot{\mathrm{R}}\mathrm{e} \\ \mathrm{Pi} \\ \dot{\mathrm{P}}\mathrm{i} \\ \ddot{\mathrm{R}}\mathrm{e} \\ \ddot{\mathrm{P}}\mathrm{i} \end{array} \begin{array}{c} \begin{array}{cc} \dot p_{\mathrm{Jc}} & \ddot p_{\mathrm{Jc}} \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -C_1 A_{\mathrm{Jc,Pi}}^T & 0 \\ 0 & 0 \\ \partial_{\dot p_{\mathrm{Jc}}} R & -C_1 A_{\mathrm{Jc,Pi}}^T \end{bmatrix} \end{array},$$

$$\tilde M_{21} = \begin{array}{c} \\ \dot{\mathrm{J}}\mathrm{c} \\ \ddot{\mathrm{J}}\mathrm{c} \end{array} \begin{array}{c} \begin{array}{ccccccc} \dot q_{\mathrm{De}} & \ddot q_{\mathrm{De}} & \dot p_{\mathrm{Re}} & \dot q_{\mathrm{Pi}} & \ddot q_{\mathrm{Pi}} & \ddot p_{\mathrm{Re}} & \dddot q_{\mathrm{Pi}} \end{array} \\ \begin{bmatrix} A_{\mathrm{Jc,De}} & 0 & 0 & A_{\mathrm{Jc,Pi}} & 0 & 0 & 0 \\ 0 & A_{\mathrm{Jc,De}} & 0 & 0 & A_{\mathrm{Jc,Pi}} & 0 & 0 \end{bmatrix} \end{array}.$$

The matrix $\tilde{M}_{11}$ is nonsingular and noting that

$$\tilde{M}_{21}\tilde{M}_{11}^{-1}\tilde{M}_{12} = \begin{bmatrix} 0 & 0 \\ -A_{\mathrm{Jc,Pi}}C_1A_{\mathrm{Jc,Pi}}^T & 0 \end{bmatrix},$$

we find that

$$\mathrm{rank}(M_2) = \mathrm{rank}(\tilde{M}_{11}) + \mathrm{rank}(\tilde{M}_{21}\tilde{M}_{11}^{-1}\tilde{M}_{12}) = 2n_{\mathrm{De}} + 2n_{\mathrm{Re}} + 3n_{\mathrm{Pi}} + n_{\mathrm{Jc}}.$$

Hence, with $\mathrm{corank}(M_2) = n_{\mathrm{De}} + n_{\mathrm{Re}} + 2n_{\mathrm{Jc}} = a$ and $d = n_{\mathcal{V}} + n_{\mathcal{E}} - a = n_{\mathrm{Pi}} - n_{\mathrm{Jc}}$, we obtain the same characteristic values d, a. A basis $Z_2 \in \mathbb{R}^{2n\times a}$ of $\mathrm{corange}(M_1)$ is given by

$$Z_2^T = \begin{array}{c} \\ \left[\begin{array}{cccccccccccc} \dot{\mathrm{De}} & \ddot{\mathrm{De}} & \dot{\mathrm{Re}} & \mathrm{Pi} & \dot{\mathrm{Pi}} & \ddot{\mathrm{Re}} & \ddot{\mathrm{Pi}} & \dot{\mathrm{Jc}} & \ddot{\mathrm{Jc}} & \mathrm{Jc} & \mathrm{De} & \mathrm{Re} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathrm{Jc}} & 0 & 0 \\ -A_{\mathrm{Jc,De}} & -A_{\mathrm{Jc,Pi}} & 0 & 0 & 0 & 0 & 0 & I_{\mathrm{Jc}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathrm{De}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathrm{Re}} \end{array}\right] \end{array} \bar{\Pi}_{M_2}^T$$

and we find that

$$Z_2^T N_1 = \left[\begin{array}{cccc} q_{\mathrm{Pi}} & q_{\mathrm{De}} & p_{\mathrm{Jc}} & p_{\mathrm{Re}} \\ A_{\mathrm{Jc,Pi}} & A_{\mathrm{Jc,De}} & 0 & 0 \\ A_{\mathrm{Jc,Pi}}\mathrm{D}_1 f_{\mathrm{Pi}} & 0 & A_{\mathrm{Jc,Pi}}C_1A_{\mathrm{Jc,Pi}}^T & A_{\mathrm{Jc,Pi}}C_1A_{\mathrm{Re,Pi}}^T \\ 0 & I_{\mathrm{De}} & 0 & 0 \\ 0 & 0 & 0 & I_{\mathrm{Re}} \end{array}\right].$$

Comparing $Z_2^TN_1$ and $Z_1^TN_1$, cp. (29), we can verify the assertions $(ii), (iii)$ of Hypothesis 2.1 using the same matrices Z_1, T_2. In conclusion, the network model (8) is regular.
Applying the matrices Z_1, Z_2 to the derivative array $\mathcal{F}_{F,1}$, we obtain the s-free remodeling (11). □

The solvability condition of Theorem 3.1 agrees with the result given in [15].

The s-free formulation (11) can be reformulated as semi-explicit system, allowing to use semi-explicit DAE solvers and to specify the set of consistent initial values.

Lemma 3.3. *Let $\mathcal{N}$ be a network of pipes, junctions, reservoirs and demand branches that satisfies Assumption 3.1. Let $F \in C^2(\mathbb{D}, \mathbb{R}^n)$ be the DAE function associated with $\mathcal{N}$. If $n_{Re} > 0$, then functions $q \in C^1(\mathcal{I}, \mathbb{R}^{n_\mathcal{E}})$, $p \in C^1(\mathcal{I}, \mathbb{R}^{n_\mathcal{V}})$ solve the DAE (8) if and only if they solve the semi-explicit system*

$$\dot{q}_2 = K_{Tr,Ch}\Sigma(q_2)\big(L_{q_2}q_2 + L_{q_{De}}\bar{q}_{De}\big) + K_{Tr,De}\dot{\bar{q}}_{De} + K_{Tr,Re}\bar{p}_{Re} + C_5, \tag{15a}$$
$$p_{Jc} = K_{Jc,Pi}\Sigma(q_2)\big(L_{q_2}q_2 + L_{q_{De}}\bar{q}_{De}\big) + K_{Jc,De}\dot{\bar{q}}_{De} + K_{Jc,Re}\bar{p}_{Re} + C_4, \tag{15b}$$
$$q_1 = -(A_{Jc,Pi}\Pi_1)^{-1}A_{Jc,Pi,2}q_2 + (A_{Jc,Pi}\Pi_1)^{-1}A_{Jc,De}\bar{q}_{De}, \tag{15c}$$
$$q_{De} = \bar{q}_{De}, \tag{15d}$$
$$p_{Re} = \bar{p}_{Re}, \tag{15e}$$

where $q_i = \Pi_i^T q_{Pi}$, $i = 1, 2$, for a permutation $[\Pi_1\,\Pi_2]$ such that $A_{Jc,Pi}\Pi_1$ is nonsingular and

$$L_{q_2} := \Pi_2 - \Pi_1(A_{Jc,Pi}\Pi_1)^{-1}A_{Jc,Pi}\Pi_2, \qquad L_{q_{De}} := -(A_{Jc,Pi}\Pi_1)^{-1}A_{Jc,De},$$
$$K_{Tr,Ch} := \Pi_2^T G C_2 \qquad K_{Tr,De} := \Pi_2^T C_1 A_{Jc,Pi}^T K_{Jc,De}, \qquad K_{Tr,Re} := \Pi_2^T G^T A_{Re,Pi}^T,$$
$$K_{Jc,Pi} := -B_{Jc}^{-1}A_{Jc,Pi}C_2, \qquad K_{Jc,De} := -B_{Jc}^{-1}A_{Jc,De}, \qquad K_{Jc,Re} := -B_{Jc}^{-1}A_{Jc,Pi}C_1A_{Re,Pi}^T,$$
$$C_5 := C_1A_{Jc,Pi}^TC_4 + C_3, \qquad C_4 := -B_{Jc}^{-1}C_3,$$

with $B_{Jc} := A_{Jc,Pi}C_1A_{Jc,Pi}^T$, $G = (I - C_1A_{Jc,Pi}^TB_{Jc}^{-1}A_{Jc,Pi})$ and $\Sigma(q_2) := \mathrm{diag}\,(|[L_{q_2}q_2 + L_{q_{De}}\bar{q}_{De}]_j|)$.

Proof. Under the given assertions, the DAE (8) is uniquely solvable for a consistent initial value $(t_0, q_0, p_0) \in \mathcal{C}_2(F)$ and the solution $q \in C^1(\mathcal{I}, \mathbb{R}^{n_\mathcal{E}})$, $p \in C^1(\mathcal{I}, \mathbb{R}^{n_\mathcal{V}})$ is also the unique solution of the s-free formulation (11) cp. Theorem 3.1. Choosing a permutation $[\Pi_1, \Pi_2]$, such that $A_{\mathrm{Jc,Pi}}\Pi_1$ is nonsingular, cp. Lemma 3.2, we use the variable partitioning $q_1 := \Pi_1^T q_{\mathrm{Pi}}$, $q_2 := \Pi_2^T q_{\mathrm{Pi}}$ to solve the is-free formulation (11) and derive the semi-explicit formulation (15).
First, however, we exploit the boundary conditions (11d), (11e) to eliminate the controlled states $p_{\mathrm{Re}}, q_{\mathrm{De}}$ in the equations (11a) - (11c). Considering the variables q_1, q_2, then we solve the junction equation (11b) for q_1 and get

$$q_1 = -(A_{\mathrm{Jc,Pi}}\Pi_1)^{-1} A_{\mathrm{Jc,Pi},2} q_2 + (A_{\mathrm{Jc,Pi}}\Pi_1)^{-1} A_{\mathrm{Jc,De}} \bar{q}_{\mathrm{De}}. \tag{16}$$

Noting that $q = \Pi_1 q_1 + \Pi_2 q_2$, the mass flow q can be expressed as affine linear transformation of the flows q_2 on the chord set, i.e.,

$$q_{\mathrm{Pi}}(q_2) = L_{q_2} q_2 + L_{q_{\mathrm{De}}} \bar{q}_{\mathrm{De}}, \tag{17}$$

where $L_{q_2} := \Pi_2 - \Pi_1 (A_{\mathrm{Jc,Pi}}\Pi_1)^{-1} A_{\mathrm{Jc,Pi}}\Pi_2$ and $L_{q_{\mathrm{De}}} := -(A_{\mathrm{Jc,Pi}}\Pi_1)^{-1} A_{\mathrm{Jc,De}}$. Inserting the transformation (17) into equation (11c), we get that

$$0 = A_{\mathrm{Jc,Pi}}\big(C_1 A_{\mathrm{Jc,Pi}}^T p_{\mathrm{Jc}} + C_1 A_{\mathrm{Re,Pi}}^T \bar{p}_{\mathrm{Re}} + C_2 \operatorname{diag}\left(|q_{\mathrm{Pi},j}|\right) q_{\mathrm{Pi}}(q_2) + C_3\big) + A_{\mathrm{Jc,De}} \dot{\bar{q}}_{\mathrm{De}}.$$

As $B_{\mathrm{Jc}} := A_{\mathrm{Jc,Pi}} C_1 A_{\mathrm{Jc,Pi}}^T$ is nonsingular, we can solve this equation for p_{Jc} as a function of q_2, i.e.,

$$\begin{aligned} p_{\mathrm{Jc}}(q_2) &= -B_{\mathrm{Jc}}^{-1}\big(A_{\mathrm{Jc,Pi}} C_1 A_{\mathrm{Re,Pi}}^T \bar{p}_{\mathrm{Re}} + A_{\mathrm{Jc,Pi}} C_2 \operatorname{diag}\left(|q_{\mathrm{Pi},j}|\right) q_{\mathrm{Pi}}(q_2) + A_{\mathrm{Jc,Pi}} C_3 + A_{\mathrm{Jc,De}} \dot{\bar{q}}_{\mathrm{De}}\big) \\ &= K_{\mathrm{Jc,Pi}} \Sigma(q_2)\big(L_{q_2} q_2 + L_{q_{\mathrm{De}}} \bar{q}_{\mathrm{De}}\big) + K_{\mathrm{Jc,De}} \dot{\bar{q}}_{\mathrm{De}} + K_{\mathrm{Jc,Re}} \bar{p}_{\mathrm{Re}} + C_4, \end{aligned} \tag{18}$$

where $K_{\mathrm{Jc,Pi}} := -B_{\mathrm{Jc}}^{-1} A_{\mathrm{Jc,Pi}} C_2 \Sigma(q_2)$, $K_{\mathrm{Jc,De}} := -B_{\mathrm{Jc}}^{-1} A_{\mathrm{Jc,De}}$, $K_{\mathrm{Jc,Re}} := -B_{\mathrm{Jc}}^{-1} A_{\mathrm{Jc,Pi}} C_1 A_{\mathrm{Re,Pi}}^T$, $C_4 := -B_{\mathrm{Jc}}^{-1} A_{\mathrm{Jc,Pi}} C_3$ and $\Sigma(q_2) := \operatorname{diag}\left(|[L_{q_2} q_2 + L_{q_{\mathrm{De}}} \bar{q}_{\mathrm{De}}]_j|\right)$. Using (18), we can reformulate (11a) as an ODE depending only on q_2, i.e.,

$$\dot{q}_2 = \Pi_2^T\big(C_1 (A_{\mathrm{Jc,Pi}}^T p_{\mathrm{Jc}} + A_{\mathrm{Re,Pi}}^T \bar{p}_{\mathrm{Re}}) + C_2 \Sigma(q_2) q_{\mathrm{Pi}}(q_2) + C_3\big).$$

Noting that

$$\begin{aligned} C_1 A_{\mathrm{Jc,Pi}}^T K_{\mathrm{Jc,Pi}}(q_2) + C_2 \Sigma(q_2) &= (I - C_1 A_{\mathrm{Jc,Pi}}^T B_{\mathrm{Jc}}^{-1} A_{\mathrm{Jc,Pi}}) C_2 \Sigma(q_2), \\ A_{\mathrm{Jc,Pi}}^T K_{\mathrm{Jc,Re}} + A_{\mathrm{Re,Pi}}^T &= (I - A_{\mathrm{Jc,Pi}}^T B_{\mathrm{Jc}}^{-1} A_{\mathrm{Jc,Pi}} C_1) A_{\mathrm{Re,Pi}}^T \end{aligned}$$

we set $G = (I - C_1 A_{\mathrm{Jc,Pi}}^T B_{\mathrm{Jc}}^{-1} A_{\mathrm{Jc,Pi}})$ and get that

$$\dot{q}_2 = K_{\mathrm{Tr,Ch}} \Sigma(q_2)\big(L_{q_2} q_2 + L_{q_{\mathrm{De}}} \bar{q}_{\mathrm{De}}\big) + K_{\mathrm{Tr,De}} \dot{\bar{q}}_{\mathrm{De}} + K_{\mathrm{Tr,Re}} \bar{p}_{\mathrm{Re}} + C_5. \tag{19}$$

where $K_{\mathrm{Tr,Ch}} := \Pi_2^T G C_2$, $K_{\mathrm{Tr,De}} := \Pi_2^T C_1 A_{\mathrm{Jc,Pi}}^T K_{\mathrm{Jc,De}}$, $K_{\mathrm{Tr,Re}} := \Pi_2^T G^T A_{\mathrm{Re,Pi}}^T$ and $C_5 := \Pi_2^T (C_3 + C_1 A_{\mathrm{Jc,Pi}}^T C_4)$. In conclusion, (11d), (11e), (17), (18), (19) establish the semi-explicit reformulation (15). □

Remark 3.2. *Considering the semi-explicit system* (15)*, we observe the following for a network $\mathcal{N}$ satisfying the solvability condition of Theorem 3.1: The mass flows q_1 lying on a spanning tree of the graph $\mathcal{G}$ of $\mathcal{N}$ are specified by Kirchoff's current law* (8b)*, which ensures the conservation of mass within the system. Only the mass flows q_2 lying on a chord set, i.e., on pipes closing a loop, are determined by the transient momentum equation* (8a)*. The pressures in the junctions are obtained by the derivative of Kirchoff's current law* (8b)*, using the transient momentum equation* (8a) *to express the change of the mass flows.*

As the algebraic constraints arising from the conservation of mass are linear, we can use the the semi-explicit system (15) to specify the set of consistent initial values. Therefore, we consider the evaluation of (15) in $t_0 \in \mathcal{I}$, i.e.,

$$\dot{q}_{2,0} = K_{\mathrm{Tr,Ch}} \Sigma(q_2)\big(L_{q_2} q_{2,0} + L_{q_{\mathrm{De}}} \bar{q}_{\mathrm{De}}(t_0)\big) + K_{\mathrm{Tr,De}} \dot{\bar{q}}_{\mathrm{De}}(t_0) + K_{\mathrm{Tr,Re}} \bar{p}_{\mathrm{Re}}(t_0) + C_5 \tag{20a}$$

$$p_{\mathrm{Jc}} = K_{\mathrm{Jc,Pi}} \Sigma(q_2)\big(L_{q_2} q_{2,0} + L_{q_{\mathrm{De}}} \bar{q}_{\mathrm{De}}\big) + K_{\mathrm{Jc,De}} \dot{\bar{q}}_{\mathrm{De}}(t_0) + K_{\mathrm{Jc,Re}} \bar{p}_{\mathrm{Re}}(t_0) + C_4 \tag{20b}$$

$$q_1 = -(A_{\mathrm{Jc,Pi}} \Pi_1)^{-1} A_{\mathrm{Jc,Pi,2}} q_{2,0} + (A_{\mathrm{Jc,Pi}} \Pi_1)^{-1} A_{\mathrm{Jc,De}} \bar{q}_{\mathrm{De}}(t_0) \tag{20c}$$

$$q_{\mathrm{De},0} = \bar{q}_{\mathrm{De}}(t_0) \tag{20d}$$

$$p_{\mathrm{Re},0} = \bar{p}_{\mathrm{Re}}(t_0), \tag{20e}$$

and, noting that $\frac{d}{dt}\Sigma(q_2) = \Sigma(\dot{q}_2)$ as $\mathrm{sgn}(q_{\mathrm{Pi},j}(t)) = const$ on $\mathcal{I}$, the evaluation of the time derivative of (15) in $t_0 \in \mathcal{I}$, i.e.,

$$\begin{aligned}\ddot{q}_{2,0} = K_{\mathrm{Tr,Ch}} \big(&\Sigma(\dot{q}_2)\big(L_{q_2} q_{2,0} + L_{q_{\mathrm{De}}} \bar{q}_{\mathrm{De}}\big) + \Sigma(q_2)\big(L_{q_2} \dot{q}_{2,0} + L_{q_{\mathrm{De}}} \dot{\bar{q}}_{\mathrm{De}}\big) + K_{\mathrm{Tr,De}} \ddot{\bar{q}}_{\mathrm{De}}(t_0)\\ &+ K_{\mathrm{Tr,Re}} \dot{\bar{p}}_{\mathrm{Re}}(t_0)\end{aligned} \tag{21a}$$

$$\begin{aligned}\dot{p}_{\mathrm{Jc}} = K_{\mathrm{Jc,Pi}} \big(&\Sigma(\dot{q}_2)\big(L_{q_2} q_{2,0} + L_{q_{\mathrm{De}}} \bar{q}_{\mathrm{De}}\big) + \Sigma(q_2)\big(L_{q_2} \dot{q}_{2,0} + L_{q_{\mathrm{De}}} \dot{\bar{q}}_{\mathrm{De}}\big) + K_{\mathrm{Jc,De}} \ddot{\bar{q}}_{\mathrm{De}}(t_0)\\ &+ K_{\mathrm{Jc,Re}} \dot{\bar{p}}_{\mathrm{Re}}(t_0)\end{aligned} \tag{21b}$$

$$\dot{q}_{1,0} = -(A_{\mathrm{Jc,Pi}} \Pi_1)^{-1} A_{\mathrm{Jc,Pi}} \Pi_2 \dot{q}_{2,0} + (A_{\mathrm{Jc,Pi}} \Pi_1)^{-1} \dot{\bar{q}}_{\mathrm{De}}(t_0) \tag{21c}$$

$$\dot{q}_{\mathrm{De},0} = \dot{\bar{q}}_{\mathrm{De}}(t_0) \tag{21d}$$

$$\dot{p}_{\mathrm{Re},0} = \dot{\bar{p}}_{\mathrm{Re}}(t_0). \tag{21e}$$

Lemma 3.4. *Let $\mathcal{N}$ be a network of pipes, junctions, reservoirs and demand branches that satisfies Assumption 3.1. Let $F \in C^2(\mathbb{D}, \mathbb{R}^n)$ be the DAE function associated with $\mathcal{N}$. If $n_{Re} > 0$, then the set of consistent initial values and initializations is given by*

$$\mathcal{C}_2(F) = \big\{(t_0, q_0, p_0) \in \mathcal{I} \times \Omega_x \,|\, (t_0, q_0, p_0) \text{ solves } (20) \text{ with } q_{2,0} = \Pi_2^T q_0 \;\big\},$$

$$\mathcal{L}_2(F) = \big\{(t_0, q_0, p_0, \dot{q}_0, \dot{p}_0) \in \mathbb{D} \,|\, (t_0, q_0, p_0, \dot{q}_0, \dot{p}_0) \text{ solves } (20) \text{ and } (21) \text{ with } q_{2,0} = \Pi_2^T q_0 \;\big\}.$$

Proof. As the subspace $\mathrm{coker}(M_2)$ is constant, the s-free formulation (11) of the DAE (8) is defined globally on $\mathbb{D}$, independent of the considered intial value. Hence, the semi-explicit formulation (15) is defined globally on $\mathbb{D}$, implying that every solution of (15) also solves the DAE (8). In particular, this is true in $t = t_0$, implying that $(t_0, q_0, p_0) \in \mathcal{C}_1(F)$ if (t_0, q_0, p_0) solves (20) with $q_{2,0} = \Pi_2^T q_0$. The time derivative of the algebraic components $q_1, p_{\mathrm{Jc}}, q_{\mathrm{De}}, p_{\mathrm{Re}}$ can be obtained by differentiating (20), so considering (21), we obtain the consistent initializations $\mathcal{L}_1(F)$. Similarly, we can construct a vector $(t_0, q_0, p_0, \dot{q}_0, \dot{p}_0, \ddot{q}_0, \ddot{p}_0)$ solving $\mathcal{F}_2(F)$, implying that $\mathcal{C}_2(F) = \mathcal{C}_1(F)$ and $\mathcal{L}_2(F) = \mathcal{L}_1(F)$. ☐

Remark 3.3. *Numerically, we can either directly solve the semi-explicit formulation* (15) *by applying an implicit solver to* (15a) *for q_2 and evaluating* (15b), (30f), (15d) *and* (15e) *using these values, or, if the implicit function p_{Jc} is analytically available, by solving the resolved system* (15) *by applying an ODE solver to the ODE* (15a) *and evaluating* (15b) *to* (15e).

4 A quasi-stationary network model for incompressible thermal flow

In this section, a DAE model for incompressible thermal flow networks is derived by adding energy balance laws to the model analyzed in Section 3. Performing a topology index analysis, we again derive solvability conditions that can be reinterpreted in terms of conditions on the network.

4.1 Model

Again, we consider a network $\mathcal{N}$ of pipes as introduced in Subsection 3.1. In addition to the pressure distribution across the system, we are also interested in the temperature distribution. For the thermal part, the state variable of our choice is the specific enthalpy h, which changes with respect to enthalpy fluxes H. In order to capture the thermal inertia of the liquid within the network, each junction is equipped with a volume $V \geq 0$. Junctions with zero volume are interpreted as *virtual* connection points. Those virtual connections point have a certain importance in the design of system simulation software, since they allow to connect standardized sub-components without introducing additional volumes (and as a consequence additional thermal inertia).

In order to set up the network model, we assume that the junctions Jc_i are numbered such that $V_i > 0,\ i = 1, ..., n_{\mathrm{Jc}_V}$ and $V_i = 0,\ i = n_{\mathrm{Jc}_V} + 1, ..., n_{\mathrm{Jc}_V} + n_{\mathrm{Jc}_0}$ where $n_{\mathrm{Jc}} = n_{\mathrm{Jc}_V} + n_{\mathrm{Jc}_0}$. We partition the incidence matrix and the pressure vector accordingly as

$$A_{\mathcal{N}} = \begin{bmatrix} A_{\mathrm{Jc}_V,\mathrm{Pi}} & A_{\mathrm{Jc}_V,\mathrm{De}} \\ A_{\mathrm{Jc}_0,\mathrm{Pi}} & A_{\mathrm{Jc}_0,\mathrm{De}} \\ A_{\mathrm{Re},\mathrm{Pi}} & A_{\mathrm{Re},\mathrm{De}} \end{bmatrix}, \quad p = \begin{bmatrix} p_{\mathrm{Jc}_V} \\ p_{\mathrm{Jc}_0} \\ p_{\mathrm{Re}} \end{bmatrix}$$

and define the volume matrix $V_{\mathrm{Jc}} := \mathrm{diag}\left(\hat{V}, 0\right)$, $\hat{V} = \mathrm{diag}\,(V_i)_{i=1,\ldots,n_{\mathrm{Jc}_V}}$. To set up the governing equations for the network, we consider the characteristic relation that every element imposes on the enthalpy flux and specific enthalpy. In a pipe Pi_j, the enthalpy flux H_j agrees with the product of the mass flow q_j and the specific enthalpy h_i in the originating vertex v_i. Hence, if the pipe Pi_j is directed from v_{j_1} to v_{j_2}, we have that

$$H_{\mathrm{Pi},j} = f_{H_{\mathrm{Pi},j}}(q_{\mathrm{Pi},j}, h_{v_{j_1}}, h_{v_{j_2}}),$$

where

$$f_{H_{\mathrm{Pi},j}}(q_{\mathrm{Pi},j}, h_{v_{j_1}}, h_{v_{j_2}}) := \left((sgn(q_{\mathrm{Pi},j}) + 1)h_{v_{j_1}} - (sgn(q_{\mathrm{Pi},j}) - 1)h_{v_{j_2}}\right)\frac{q_{\mathrm{Pi},j}}{2}.$$

Due to energy conservation, in a junction Jc_i, the sum of all enthalpy fluxes $H_{\mathrm{Pi},j}$ entering or leaving Jc_i equals the product of the volume V_i and the change of the specific enthalpy $h_{\mathrm{Jc},i}$, i.e.,

$$\sum_{j \in \hat{J}_{\mathrm{Jc}_i}} H_{\mathrm{Pi},j} = V_i \dot{h}_{\mathrm{Jc},i}.$$

In a demand branch De_i, we assume that with the mass flow $\bar{q}_{\mathrm{De},j}$ also the enthalpy flux $\bar{H}_{\mathrm{De},j}$ that is taken from or filled into the network is known, i.e., $H_{\mathrm{De},j} = \bar{H}_{\mathrm{De},j}$. Similarly, in a reservoir Re_i, we assume that with the pressure also the specific enthalpy h_i is kept at a fixed value $\bar{h}_{\mathrm{Re},i}$, i.e., $h_{\mathrm{Re},i} = \bar{h}_{\mathrm{Re},i}$. To lift the element equations to the network, we summarize the enthalpy fluxes and specific enthalpies as

$$H = \begin{bmatrix} H_{\mathrm{Pi}} \\ H_{\mathrm{De}} \end{bmatrix}, \quad h = \begin{bmatrix} h_{\mathrm{Jc}_V} \\ h_{\mathrm{Jc}_0} \\ h_{\mathrm{Re}} \end{bmatrix},$$

where $h_{\mathrm{Jc}_V}, h_{\mathrm{Jc}_0}$ refer to the enthalpies associated with junctions of positive and zero volume, respectively. Furthermore, we consider the matrix

$$|A_{\mathcal{N}}| = \left[|A_{\mathcal{N},ij}|\right]_{(i,j)\in\mathcal{V}\times\mathcal{E}}$$

containing the element-wise absolute values of incidence matrix $A_{\mathcal{N}}$. Setting

$$D_\star(q_{\mathrm{Pi}}) = \frac{1}{2}\,\mathrm{diag}\,(q_{\mathrm{Pi}}(t))\left(\mathrm{diag}\,(\mathrm{sgn}(q_{\mathrm{Pi}}(t)))\,A^T_{\star,\mathrm{Pi}} + |A^T_{\star,\mathrm{Pi}}|\right),$$

for $\star = \mathrm{Jc}_0, \mathrm{Jc}_V, \mathrm{Re}$, we define the function

$$f_{\bar{\mathrm{Pi}}}(q_{\mathrm{Pi}}, h_{\mathrm{Jc}_V}, h_{\mathrm{Jc}_0}, h_{\mathrm{Re}}) := D_{\mathrm{Jc}_0}(q_{\mathrm{Pi}})h_{\mathrm{Jc}_0} + D_{\mathrm{Jc}_V}(q_{\mathrm{Pi}})h_{\mathrm{Jc}_V} + D_{\mathrm{Re}}(q_{\mathrm{Pi}})h_{\mathrm{Re}} \quad (22)$$

and the enthalpy fluxes $H_{\rm Pi}$ for the pipes $\rm Pi_1, ..., Pi_{n_{Pi}}$ are given by

$$H_{\rm Pi} = f_{\rm \bar{Pi}}(q_{\rm Pi}, h). \tag{23a}$$

For the junctions, we obtain that

$$A_{\rm Jc,Pi} H_{\rm Pi} + A_{\rm Jc,De} H_{\rm De} = V_{\rm Jc} \dot{h}_{\rm Jc}. \tag{23b}$$

For the demand branches and reservoirs, we obtain the simple relations

$$H_{\rm De} = \bar{H}_{\rm De}, \tag{23c}$$

$$h_{\rm Re} = \bar{h}_{\rm Re}. \tag{23d}$$

Including thermal effects, the function $f_{{\rm Pi},j}$ relating the mass flow q_j with the pressure difference Δp_j now depends on the specific enthalpy of the originating vertex as physical properties like the density typically depend on the temperature or specific enthalpy, respectively. More exactly, if the pipe Pi_j is directed from v_{j_1} to v_{j_2}, then $c_{2,j} = c_{2,j}(h_{j_1})$ and equation (6) reads

$$\dot{q}_{\rm Pi} = f_{\rm Pi}(q_{\rm Pi}, A_{\rm Jc,Pi}^T p_{\rm Jc} + A_{\rm Re,Pi}^T p_{\rm Re}, h). \tag{6a*}$$

where

$$f_{\rm Pi}(q_{\rm Pi}, p_{\rm Jc}, p_{\rm Re}, h_{\rm Jc}, h_{\rm Re}) := C_1(A_{\rm Jc,Pi}^T p_{\rm Jc} + A_{\rm Re,Pi}^T p_{\rm Re}) + C_2(h)\,\mathrm{diag}\,(sgn(q_{{\rm Pi},j}) q_{{\rm Pi},j})\, q_{\rm Pi} + C_3.$$

In conclusion, the dynamics in a network consisting of pipes, reservoirs, junctions and demand vertices, are modeled by the DAE system (8) and (23), where (8a) is replaced by (6a*).
Formally, we define the DAE function

$$\tilde{F}(t, q, p, H, h, \dot{q}, \dot{p}, \dot{H}, \dot{h}) = \begin{bmatrix} \dot{q}_{\rm Pi} - f_{\rm Pi}(q_{\rm Pi}, p_{\rm Jc}, p_{\rm Re}, h_{\rm Jc}, h_{\rm Re}) \\ V_{\rm Jc}\dot{h}_{\rm Jc} - A_{\rm Jc,Pi} H_{\rm Pi} - A_{\rm Jc,De} H_{\rm De} \\ A_{\rm Jc,Pi} q_{\rm Pi} + A_{\rm Jc,De} q_{\rm De} \\ H_{\rm Pi} - f_{\rm \bar{Pi}}(q_{\rm Pi}, h_{\rm Jc}, h_{\rm Re}) \\ q_{\rm De} - \bar{q}_{\rm De} \\ p_{\rm Re} - \bar{p}_{\rm Re} \\ H_{\rm De} - \bar{H}_{\rm De} \\ h_{\rm Re} - \bar{h}_{\rm Re}. \end{bmatrix}. \tag{24}$$

Again, on an interval $\mathcal{I} \subset \mathbb{R}$, where the flow doesn't change its direction, the smoothness of $\tilde{F}$ depends on the smoothness of the input functions $\bar{q}_{\rm De}, \bar{p}_{\rm Re}, \bar{H}_{\rm De}, \bar{h}_{\rm Re}$.

Lemma 4.1. *For a network $\mathcal{N}$, consider the function $\tilde{F}$ defined in* (24)*. Let $\bar{q}_{De}, \bar{H}_{De} \in C^k(\mathcal{I}, \mathbb{R}^{n_{De}})$, $\bar{p}_{Re}, \bar{h}_{Re} \in C^k(\mathcal{I}, \mathbb{R}^{n_{Re}})$. Then, $\tilde{F} \in C^k(\mathbb{D}, \mathbb{R}^n)$ for $\mathbb{D} := \mathcal{I} \times \Omega_{x,ext} \times \Omega_{\dot{x},ext}$ where $\Omega_{x,ext} := (\mathbb{R}^{n_\mathcal{E}} \times \mathbb{R}_+^{n_\mathcal{E}})^2$ and $\Omega_{\dot{x},ext} := (\mathbb{R}^{n_\mathcal{E}} \times \mathbb{R}^{n_\mathcal{E}})^2$.*

Proof. If $q_{\rm Pi} \in \mathbb{R}_{++}^{n_\mathcal{E}}$, then $sgn(q_{{\rm Pi},j}) \geq 0$ for every $j = 1, ..., n_\mathcal{E}$ and $f_{\rm Pi}(q_{\rm Pi}, p_{\rm Jc}, p_{\rm Re}) = C_1(A_{\rm Jc,Pi}^T p_{\rm Jc} + A_{\rm Re,Pi}^T p_{\rm Re}) + C_2\,\mathrm{diag}\,(q_{{\rm Pi},j})\, q_{\rm Pi} + C_3$ and $f_{\rm \bar{Pi}}(q_{\rm Pi}, h_{\rm Jc}, h_{\rm Re}) = \frac{1}{2}\,\mathrm{diag}\,(q_{\rm Pi})\,\big((A_{\rm Jc,Pi}^T + |A_{\rm Jc,Pi}^T|) h_{\rm Jc} + (A_{\rm Re,Pi}^T + |A_{\rm Re,Pi}^T|) h_{\rm Re}\big)$. Hence, $f_{\rm Pi}, f_{\rm \bar{Pi}} \in C^k(\mathbb{R}_+^{n_\mathcal{E}} \times \mathbb{R}^{n_\mathcal{V}} \times \mathbb{R}^{n_\mathcal{V}}, \mathbb{R}^{n_{\rm Pi}})$. With $\bar{q}_{\rm De}, \bar{H}_{\rm De} \in C^k(\mathcal{I}, \mathbb{R}^{n_{\rm De}})$, $\bar{p}_{\rm Re}, \bar{h}_{\rm Re} \in C^k(\mathcal{I}, \mathbb{R}^{n_{\rm Re}})$, it follows that $\tilde{F}$ satisfies the asserted smoothness property. □

4.2 Analysis of the hydraulic thermal flow network

To analyze the solvability of the coupled DAE (8) and (23), we again exploit the structure of the network $\mathcal{N}$ and the associated graph $\mathcal{G}_\mathcal{N}$. For $t \in \mathcal{I}$, we consider the matrix

$$B_{\rm \bar{J}c_0}(t) = A_{\rm Jc_0,Pi} D_{\rm \bar{J}c_0}(q_{\rm Pi}(t)), \tag{25}$$

and give conditions on its associated graph $\mathcal{G}(B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)) = \big\{\{v_1, ..., v_n\}, \{(v_i, v_j)|B_{\bar{\mathrm{J}}\mathrm{c}_0,ij} \neq 0\}\big\}$, cp. [21, p. 528], that guarantee the invertibility of $B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)$. Recall that $\hat{J}_i = \{\ell \in \mathcal{I}_\mathcal{E} \mid \exists v_k \colon e_\ell = (v_i, v_k)\}$ is the index sets of edges $e_\ell \in \mathcal{E}_\mathcal{N}$ having v_i as originating vertex. As the mass flow q has an a direction of flow independent of the direction of the graph, we partition $\hat{J}_i$ according to $\hat{J}_i = \hat{J}_{i,s} \cup \hat{J}_{i,e}$ where

$$\hat{J}_{i,s}(t) := \{j \in \hat{J}_i | q_j(t) \text{ starts in } v_i\}, \quad \hat{J}_{i,e}(t) := \{j \in \hat{J}_i | q_j(t) \text{ ends in } v_i\}$$

are time-dependent index sets.

Lemma 4.2. *Let $\mathcal{N}$ be a network of pipes, reservoirs, junctions and demand branches that satisfies Assumption 3.1. For $t \in \mathcal{I}$, consider the matrix $B_{\bar{J}c_0}$ defined in (25). If $\sum_{j \in \hat{J}_{i,s}(t)} |q_j(t)| > 0$ for $i = 1, ..., n_{Jc_0}$, then the following assertions hold.*

1. *The graph $\mathcal{G}(B_{\bar{J}c_0}(t))$ is given by $\mathcal{G}(B_{\bar{J}c_0}(t)) = \{\mathcal{V}_{Jc_0}, \mathcal{E}_{B_{\bar{J}c_0}}\}$, where $\mathcal{V}_{Jc_0} = \big\{Jc_{0,1}, ..., Jc_{0,n_{Jc_0}}\big\}$ and $\mathcal{E}_{B_{\bar{J}c_0}} := \{Pi_j \mid \hat{J}_{i,e}(t) \cap \hat{J}_k(t),\, i \neq k \in I_{Jc_0}\} \cup \{e_j \mid e_j = (Jc_{0,i}, Jc_{0,i}), i \in I_{Jc_0}\}$.*
2. *Let $\mathrm{Con}_K = \{\mathcal{V}_K, \mathcal{E}_K\}$, $K = 1, ..., C_{\mathcal{G}(B_{\bar{J}c_0}(t))}$ be the strongly connected components in $\mathcal{G}(B_{\bar{J}c_0}(t))$. Then, $B_{\bar{J}c_0}(t)$ is nonsingular if*

$$\sum_{j \in \hat{J}_{i,e}(t) \cap (J_\mathcal{G} \setminus J_{\mathrm{Con}_K})} |q_j| \geq \sum_{j \in \hat{J}_{i,s}(t) \cap J_{De}} |q_j|, \qquad i_1, ..., i_{n_{Jc_0}} \in I_{Jc_0}, \tag{26a}$$

$$\sum_{j \in \hat{J}_{\hat{\imath},e}(t) \cap (J_\mathcal{G} \setminus J_{\mathrm{Con}_K})} |q_j| > \sum_{j \in \hat{J}_{\hat{\imath},s}(t) \cap J_{De}} |q_j|, \qquad \hat{\imath} \in I_{Jc_0}. \tag{26b}$$

Proof. 1. By the definition of the incidence matrix $A_\mathcal{N}$, its entries satisfy

$$A_{ij} A_{kj} = \begin{cases} 1, & j \in \hat{J}_i,\, i = k, \\ -1, & j \in \hat{J}_i \cap \hat{J}_k,\, i \neq k, \\ 0, & \text{else}, \end{cases}$$

$$A_{ij} |A_{kj}| = \begin{cases} A_{ij}, & j \in \hat{J}_i \cap J_{\mathrm{Pi}},\, i = k \text{ or } j \in \hat{J}_i \cap \hat{J}_k,\, i \neq k, \\ 0, & \text{else} \end{cases}$$

for $i, k \in I_\mathcal{G}$. Including the direction of the flow, we further get that

$$A_{ij} \operatorname{sgn}(q_j(t)) = \begin{cases} 1, & j \in \hat{J}_{i,s}(t), \\ -1, & j \in \hat{J}_{i,e}(t), \\ 0, & \text{else}. \end{cases}$$

Then, noting that

$$B_{\bar{\mathrm{J}}\mathrm{c}_0,ik}(t) = \sum_{j=1}^{n_{\mathrm{Pi}}} |q_j(t)| (A_{\mathrm{Jc}_0,\mathrm{Pi}})_{ij} \big((A_{\mathrm{Jc}_0,\mathrm{Pi}})_{kj} + |(A_{\mathrm{Jc}_0,\mathrm{Pi}})_{kj}| \operatorname{sgn}(q_j(t))\big),$$

the entries of $B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)$ are given by

$$B_{\bar{\mathrm{J}}\mathrm{c}_0,ik}(t) = \begin{cases} 2 \sum_{j \in \hat{J}_{i,s}(t) \cap J_{\mathrm{Pi}}} |q_j|, & i = k, \\ -2|q_j(t)|, & i \neq k,\, j \in \hat{J}_{i,e} \cap \hat{J}_k \cap J_{\mathrm{Pi}}, \\ 0, & \text{else}. \end{cases}$$

As $\sum_{j \in \hat{J}_{i,s}} |q_j| > 0$ for $i \in I_{\mathrm{Jc}_0}$, it follows that $B_{\bar{\mathrm{J}}\mathrm{c}_0}$ is a Z-matrix with strictly positive diagonal entries. Furthermore, we verify the proposed structure of $\mathcal{G}(B_{\bar{\mathrm{J}}\mathrm{c}_0}(t))$.

To prove that $B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)$ is nonsingular, we show that $B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)$ is an M-matrix. For $i \in I_{\mathrm{Jc}_0}$, the i-th row sum of $B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)$ is given by

$$\sum_{k\in I_{\mathrm{Jc}_0}} |B_{\bar{\mathrm{J}}\mathrm{c}_0,ik}(t)| = 2\left(\sum_{j\in \hat{J}_{i,s}(t)\cap J_{\mathrm{Pi}}} |q_j| - \sum_{\substack{j\in \hat{J}_{i,e}(t)\cap \hat{J}_k(t)\cap J_{\mathrm{Pi}} \\ k\neq i}} |q_j| \right).$$

Considering the subgraph $\mathcal{G}_0 := \{\mathcal{V}_{\mathrm{Jc}_0}, \mathcal{E}_{\mathcal{G}_0}\}$, where $\mathcal{V}_{\mathrm{Jc}_0} = \left\{\mathrm{Jc}_{0,1}, ..., \mathrm{Jc}_{0,n_{\mathrm{Jc}_0}}\right\}$ and $\mathcal{E}_{\mathcal{G}_0} := \{\mathrm{Pi}_j \,|\, \mathrm{Pi}_j = (\mathrm{Jc}_{0,i}, \mathrm{Jc}_{0,k})\}$, we find that

$$\bigcup_{k\neq i\in I_{\mathrm{Jc}_0}} \left(\hat{J}_i(t) \cap \hat{J}_k(t) \cap J_{\mathrm{Pi}}\right) = \hat{J}_i(t) \cap J_{\mathcal{G}_0},$$

where $J_{\mathcal{G}_0}$ contains the indices of $\mathrm{Pi}_j \in \mathcal{E}_{\mathcal{G}_0}$. Furthermore, using that $\sum_{j\in \hat{J}_{i,s}(t)} |q_j| = \sum_{j\in \hat{J}_{i,e}(t)} |q_j|$ due to $\sum_{j\in \hat{J}_i} A_{ij} q_j(t) = 0$, we have that

$$\sum_{j\in \hat{J}_{i,s}(t)\cap J_{\mathrm{Pi}}} |q_j| = -\sum_{j\in \hat{J}_{i,s}(t)\cap J_{\mathrm{De}}} |q_j| + \sum_{j\in \hat{J}_{i,e}(t)\cap J_{\mathrm{De}}} |q_j| + \sum_{j\in \hat{J}_{i,e}(t)\cap J_{\mathrm{Pi}}} |q_j|.$$

Then, it follows that

$$|B_{\bar{\mathrm{J}}\mathrm{c}_0,ik}(t)| = 2\left(-\sum_{j\in \hat{J}_{i,s}(t)\cap J_{\mathrm{De}}} |q_j| + \sum_{j\in \hat{J}_{i,e}(t)\cap J_{\mathrm{De}}} |q_j| + \sum_{j\in \hat{J}_{i,e}(t)\cap J_{\mathrm{Pi}}} |q_j| - \sum_{j\in \hat{J}_{i,e}(t)\cap \mathcal{E}_{\mathcal{G}_0}} |q_j| \right),$$

and using that $J_{\mathrm{De}} \cup (J_{\mathrm{Pi}} \setminus J_{\mathcal{G}_0} = J_{\mathcal{G}} \setminus J_{\mathcal{G}_0}$, this reads

$$|B_{\bar{\mathrm{J}}\mathrm{c}_0,ik}(t)| = 2\left(-\sum_{j\in \hat{J}_{i,s}(t)\cap (J_{\mathrm{De}})} |q_j| + \sum_{j\in \hat{J}_{i,e}(t)\cap (J_{\mathcal{G}}\setminus J_{\mathcal{G}_0})} |q_j| \right).$$

Hence, if

$$\begin{aligned} \sum_{j\in \hat{J}_{i,e}(t)\cap (J_{\mathcal{G}}\setminus J_{\mathcal{G}_0})} |q_j| &\geq \sum_{j\in \hat{J}_{i,s}(t)\cap J_{\mathrm{De}}} |q_j|, & i_1, ..., i_{n_{\mathrm{Jc}_0}} \in I_{\mathrm{Jc}_0}, \\ \sum_{j\in \hat{J}_{\hat{\imath},e}(t)\cap (J_{\mathcal{G}}\setminus J_{\mathcal{G}_0})} |q_j| &> \sum_{j\in \hat{J}_{\hat{\imath},s}(t)\cap J_{\mathrm{De}}} |q_j|, & \hat{\imath} \in I_{\mathrm{Jc}_0}. \end{aligned}$$

is satisfied, then $B_{\bar{\mathrm{J}}\mathrm{c}_0}$ is an M-matrix, [21, p. 531].
If $B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)$ is reducible, then $B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)$ is congruent to a block upper triangular matrix with irreducible diagonal blocks, cp. [21]. Then, there exists a permutation Π, such that $\Pi^T B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)\Pi = [\tilde{B}_{\bar{\mathrm{J}}\mathrm{c}_0,KL}(t)]_{KL}$ with $\tilde{B}_{\bar{\mathrm{J}}\mathrm{c}_0,KL}(t) = 0$, $K > L$ and the diagonal blocks $\tilde{B}_{\bar{\mathrm{J}}\mathrm{c}_0,KK}(t)$ are irreducible. Hence, we can apply the the same arguments that we have used in the case $B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)$ is irreducible to find conditions when the diagonal blocks $\tilde{B}_{KK}(t)$ are M-matrices. Due to the triagonal structure, then $B_{\bar{\mathrm{J}}\mathrm{c}_0}(t)$ is nonsingular. Noting that the diagonal blocks $\tilde{B}_{KK}(t)$ correspond to the strongly connected components of $\mathcal{G}(B_{\bar{\mathrm{J}}\mathrm{c}_0}(t))$, cp. [21, p. 529], we arrive at condition (26). □

Remark 4.1. *The graph $\mathcal{G}(B_{\bar{J}c_0}(t))$ agrees with the directed subgraph $\mathcal{G}_0 = \{\mathcal{V}_{Jc_0}, \mathcal{E}_{\mathcal{G}_0}\}$ spanned by the zero volume junctions $\mathcal{V}_{Jc_0} = \left\{Jc_{0,1}, ..., Jc_{0,n_{Jc_0}}\right\}$ and pipes connecting these junctions, i.e., $\mathcal{E}_{\mathcal{G}_0} := \{Pi_j \,|\, Pi_j = (Jc_{0,i}, Jc_{0,k})\}$. As $\sum_{j\in \hat{J}_{i,s}} |q_j(t)| > 0$, $\mathcal{G}(B_{\bar{J}c_0}(t))$ has a self loop at every vertex and the direction of the edges in $\mathcal{E}_{\mathcal{G}_0}$ is reciprocal to the direction of the mass flows q_{Pi}. This, however, is a consequence of the convention that $A_{ij} = 1$ if $e_j = (v_i, v_k)$.*

The consequence important for our analysis is, that condition (26) ensures that in the strongly connected component of this subgraph $\mathcal{G}_0 = \{\mathcal{V}_{Jc_0}, \mathcal{E}_{\mathcal{G}_0}\}$, there is at least on zero volume junction $Jc_{0,\hat{i}}$ that receives a mass flow from outside the strongly connected component - either from another strongly connected component or from a reservoir. Considering the temperature distribution across the network, this ensures that every zero junction node receives information from the thermal reference node.

Lemma 4.2 gives conditions when equation (22) is solvable for the enthalpies h_{Jc_0} associated with the zero volume junctions. Combined with the solvability conditions of Theorem 3.1, we can give conditions when the coupled DAE (8) and (23) is uniquely solvable and derive a s-free remodeling.

Theorem 4.1. *Let $\mathcal{N}$ be a network of pipes, junctions, reservoirs and demand branches that satisfies Assumption 3.1. Let $\tilde{F} \in C^2(\mathbb{D}, \mathbb{R}^{2n})$ be the DAE function associated with $\mathcal{N}$. If $n_{Re} > 0$ and condition (26) is satisfied, then the DAE (8) and (23) has regular s-index 1 and is uniquely solvable for every $(t_0, q_0, p_0, h_0, H_0) \in \mathcal{C}_2(\tilde{F})$. The solution is given by $q, H \in C^1(\mathcal{I}, \mathbb{R}^{n_\mathcal{E}})$, $p, h \in C^1(\mathcal{I}, \mathbb{R}^{n_\mathcal{V}})$. The s-free model of $\mathcal{N}$ is given by*

$$\dot{q}_2 - \Pi_2^T f_{Pi}(q, A_\mathcal{N}^T p, h) = 0 \tag{27a}$$
$$A_{Jc,Pi}\Pi_1 q_1 + A_{Jc,Pi}\Pi_2 q_2 + A_{Jc,De} q_{De} = 0 \tag{27b}$$
$$A_{Jc,Pi} f_{Pi}(q, A_\mathcal{N}^T p, h) + A_{Jc,De}\dot{q}_{De} = 0 \tag{27c}$$
$$\hat{V}\dot{h}_{Jc_V} - A_{Jc_V,Pi} H_{Pi} - A_{Jc_V,De} H_{De} = 0 \tag{27d}$$
$$A_{Jc_0,Pi} H_{Pi} + A_{Jc_0,De} H_{De} = 0 \tag{27e}$$
$$H_{Pi} - f_{\bar{P}i}(q_{Pi}, h) = 0 \tag{27f}$$
$$p_{Re} = \bar{p}_{Re} \tag{27g}$$
$$h_{Re} = \bar{h}_{Re} \tag{27h}$$
$$q_{De} = \bar{q}_{De} \tag{27i}$$
$$H_{De} = \bar{H}_{De}, \tag{27j}$$

where the number of differential and algebraic equations is given by $d = n_{Pi} - n_{Jc_0}$ and $a = 2(n_{Jc} + n_{De} + n_{Re}) + n_{Pi} + n_{Jc_0}$, respectively.

Proof. To prove the assertion, we consider the derivative array $\mathcal{F}_{F_\mathcal{N}^{ext},1} := [(F_\mathcal{N}^{ext})^T, (\frac{d}{dt}F_\mathcal{N}^{ext})^T]^T$ of size $\mu = 1$ with $F_\mathcal{N}$ given as in (24) and

$$\frac{d}{dt}F_\mathcal{N} = \begin{bmatrix} \ddot{q}_{\mathrm{Pi}} - \mathrm{D}_1 f_{\mathrm{Pi}}\dot{q}_{\mathrm{Pi}} - \mathrm{D}_2 f_{\mathrm{Pi}}(A_{\mathrm{Jc,Pi}}^T \dot{p}_{\mathrm{Jc}} + A_{\mathrm{Re,Pi}}^T \dot{p}_{\mathrm{Re}}) - \mathrm{D}_3 f_{\mathrm{Pi}}\dot{h} \\ V_{\mathrm{Jc}}\ddot{h}_{\mathrm{Jc}} - A_{\mathrm{Jc,Pi}}\dot{H}_{\mathrm{Pi}} - A_{\mathrm{Jc,De}}\dot{H}_{\mathrm{De}} \\ A_{\mathrm{Jc,Pi}}\dot{q}_{\mathrm{Pi}} + A_{\mathrm{Jc,De}}\dot{q}_{\mathrm{De}} \\ -A_{\mathrm{Jc,Pi}}\dot{H}_{\mathrm{Pi}} - A_{\mathrm{Jc,De}}\dot{H}_{\mathrm{De}} \\ \dot{H}_{\mathrm{Pi}} - \mathrm{D}_1 f_{\bar{\mathrm{P}}\mathrm{i}}\dot{q}_{\mathrm{Pi}} - \mathrm{D}_2 f_{\bar{\mathrm{P}}\mathrm{i}}\dot{h} \\ \dot{q}_{\mathrm{De}} - \dot{\bar{q}}_{\mathrm{De}} \\ \dot{p}_{\mathrm{Re}} - \dot{\bar{p}}_{\mathrm{Re}} \\ \dot{H}_{\mathrm{De}} - \dot{\bar{H}}_{\mathrm{De}} \\ \dot{h}_{\mathrm{Re}} - \dot{\bar{h}}_{\mathrm{Re}} \end{bmatrix}.$$

To check Hypothesis 2.1, (i), we transform the Jacobian $M_{\mu=1} := \partial_{\dot{q},\dot{p},\ddot{q},\ddot{p}}\mathcal{F}_{F_\mathcal{N},\mu=1}$ using nonsingular transformations $\bar{\Pi}_{M_{\mu=1}}, \Pi_{M_{\mu=1}}$, such that

$$\bar{\Pi}_{M_{\mu=1}}^T M_{\mu=1}\Pi_{M_{\mu=1}} = \begin{bmatrix} I_{n_{\mathrm{Pi}}+n_{\mathrm{Jc}_V}+2n_{\mathrm{De}}+2n_{\mathrm{Re}}} & 0 & 0 & 0 \\ * & \tilde{M}_{22} & \tilde{M}_{24} & 0 \\ * & \tilde{M}_{42} & 0 & 0 \\ \tilde{M}_{61} & 0 & 0 & 0 \end{bmatrix},$$

where

$$
\tilde{M}_{22} = \begin{array}{c} \\ \dot{\mathrm{Pi}} \\ \dot{\bar{\mathrm{Pi}}} \\ \dot{\bar{\mathrm{Jc}}}_V \end{array}
\begin{array}{c} \begin{array}{cccc} \dot{p}_{\mathrm{Jc}}U_1 & \ddot{q}_{\mathrm{Pi}} & \dot{H}_{\mathrm{Pi}} & \ddot{h}_{\mathrm{Jc},V} \end{array} \\
\begin{bmatrix} -\mathrm{D}_2\, f_{\mathrm{Pi}} A_{\mathrm{Jc,Pi}}^T U_1 & I_{\mathrm{Pi}} & 0 & 0 \\ 0 & 0 & I_{\mathrm{Pi}} & 0 \\ 0 & 0 & -A_{\mathrm{Jc}_V,\mathrm{Pi}} & V_{\mathrm{Jc}} \end{bmatrix} \end{array}, \tag{28a}
$$

$$
\tilde{M}_{24} = \begin{array}{c} \\ \dot{\mathrm{Pi}} \\ \dot{\bar{\mathrm{Pi}}} \\ \dot{\bar{\mathrm{Jc}}}_V \end{array}
\begin{array}{c} \begin{array}{cccc} \dot{h}_{\mathrm{Jc},0} & \ddot{h}_{\mathrm{Jc},0} & \dot{p}_{\mathrm{Jc}} & \ddot{p}_{\mathrm{Jc}} \end{array} \\
\begin{bmatrix} -\partial_{\dot{h}_{\mathrm{Jc},0}} f_{\mathrm{Pi}} & 0 & \mathrm{D}_2\, f_{\mathrm{Pi}} A_{\mathrm{Jc,Pi}}^T & 0 \\ -D_{h_{\mathrm{Jc},0}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{28b}
$$

$$
\tilde{M}_{42} = \begin{array}{c} \\ \dot{\bar{\mathrm{Jc}}}_0 \end{array}
\begin{array}{c} \begin{array}{ccc} \ddot{q}_{\mathrm{Pi}} & \dot{H}_{\mathrm{Pi}} & \ddot{h}_{\mathrm{Jc},V} \end{array} \\
\begin{bmatrix} 0 & -A_{\mathrm{Jc}_0,\mathrm{Pi}} & 0 \end{bmatrix} \end{array}, \tag{28c}
$$

$$
\tilde{M}_{61} = \begin{array}{c} \\ \dot{\mathrm{Jc}} \\ \mathrm{De}, \bar{\mathrm{De}} \\ \mathrm{Re}, \bar{\mathrm{Re}} \\ \mathrm{Pu}, \bar{\mathrm{Pu}} \\ \bar{\mathrm{Pi}} \\ \mathrm{Jc}, \bar{\mathrm{Jc}}_0 \end{array}
\begin{array}{c} \begin{array}{cccccc} \dot{q}_{\mathrm{De}} & \dot{H}_{\mathrm{De}} & \dot{p}_{\mathrm{Re}} & \dot{h}_{\mathrm{Re}} & \dot{q}_{\mathrm{Pi}} & \dot{h}_{\mathrm{Jc}_V} \end{array} \\
\begin{bmatrix} A_{\mathrm{Jc}_0,\mathrm{De}} & 0 & 0 & 0 & A_{\mathrm{Jc}_0,\mathrm{Pi}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}. \tag{28d}
$$

The reason for the block numbering becomes clear when we prove that the DAE is regular. By the choice of U_1, V_1 and the assumption on $\mathrm{D}f_{\mathrm{Pu}}$, the diagonal block $\tilde{M}_{22}$ is nonsingular, implying that

$$
\operatorname{rank}(M_{\mu=1}) = n_{\mathrm{Pi}} + n_{\mathrm{Jc}_V} + 2n_{\mathrm{De}} + 2n_{\mathrm{Re}} + \operatorname{rank}(\tilde{M}_{22}) + \operatorname{rank}(\mathcal{S}_{11}(\bar{\Pi}_{M_{\mu=1}}^T M_{\mu=1} \Pi_{M_{\mu=1}})),
$$

where the Schur complement is given by

$$
\mathcal{S}_{11}(\bar{\Pi}_{M_{\mu=1}}^T M_{\mu=1} \Pi_{M_{\mu=1}}) = -\tilde{M}_{42}\tilde{M}_{22}^{-1}\tilde{M}_{24} = \begin{array}{c} \\ \dot{\bar{\mathrm{Jc}}}_0 \end{array}
\begin{array}{c} \begin{array}{cccc} \dot{h}_{\mathrm{Jc},0} & \ddot{h}_{\mathrm{Jc},0} & \dot{p}_{\mathrm{Jc}} & \ddot{p}_{\mathrm{Jc}} \end{array} \\
\begin{bmatrix} A_{\mathrm{Jc}_0,\mathrm{Pi}} D_{h_{\mathrm{Jc},0}} & 0 & 0 & 0 \end{bmatrix} \end{array}.
$$

As $B_{\bar{\mathrm{Jc}}_0} = A_{\mathrm{Jc}_0,\mathrm{Pi}} D_{h_{\mathrm{Jc},0}}$ is nonsingular with $\operatorname{rank}(B_{\bar{\mathrm{Jc}}_0}) = n_{\mathrm{Jc}_0}$, it follows that

$$
\operatorname{rank}(M_{\mu=1}) = 3n_{\mathrm{Pi}} + 2n_{\mathrm{Jc}_V} + n_{\mathrm{Jc}_0} + 2n_{\mathrm{De}} + 2n_{\mathrm{Re}} + \tilde{c}_{\mathrm{Jc,Pu}},
$$

and with $a = \operatorname{corank}(M_{\mu=1})$ and $d = n_{\mathcal{V}} + n_{\mathcal{E}} - a$ we verify the values of d, a.
Exploiting the structure of $\bar{\Pi}_{M_{\mu=1}}^T M_{\mu=1} \Pi_{M_{\mu=1}}$, a basis of $\operatorname{corange}(M_{\mu=1})$ is given by

$$
Z_{2,\mu=1}^T = \begin{bmatrix} -\tilde{M}_{61} & 0 & 0 & I_a \end{bmatrix} \bar{\Pi}_{M_{\mu=1}}^T \in \mathbb{R}^{2n \times a}.
$$

To check Hypothesis 2.1, (ii), we apply the matrix $Z_{2,\mu=1}$ to the Jacobian $N_{\mu=1} := \partial_{q,p}\mathcal{F}_{F_N,\mu=1}$. Transforming $Z_{2,\mu=1}^T N_{\mu=1}$ by nonsingular matrices $\bar{\Pi}_N, \Pi_N$ build from the matrices U, V, Π, we get that

$$
\bar{\Pi}_N^T Z_{2,\mu=1}^T N_{\mu=1} \Pi_N = \begin{bmatrix} I_{2n_{\mathrm{De}}+2n_{\mathrm{Re}}} & 0 & 0 \\ * & \tilde{N}_{22} & \tilde{N}_{23} \\ * & \tilde{N}_{32} & 0 \end{bmatrix}, \tag{29}
$$

where

$$\tilde{N}_{22} = \begin{array}{c} \\ \text{Jc} \\ \star \\ \bar{\text{Pi}} \end{array} \begin{array}{c} \begin{array}{cccc} q_{\text{Pi}}\Pi_1 & p_{\text{Jc}} & H_{\text{Pi}} & \end{array} \\ \begin{bmatrix} A_{\text{Jc},\text{Pi},1} & 0 & 0 & 0 \\ U_2^T A_{\text{Jc},\text{Pi}} \,\mathrm{D}_1 f_{\text{Pi}}\Pi_1 & \tilde{\mathrm{D}}_2 f_{\text{Pi}} & \hat{\mathrm{D}}_2 f_{\text{Pi}} & 0 \\ -\mathrm{D}_1 f_{\bar{\text{Pi}}}\Pi_1 & 0 & 0 & I_{\text{Pi}} \end{bmatrix} \end{array},$$

$$\tilde{N}_{23} = \begin{array}{c} \\ \text{Jc} \\ \star \\ \bar{\text{Pi}} \end{array} \begin{array}{c} \begin{array}{ccc} q_{\text{Pi}}\Pi_2 & h_{\text{Jc},V} & h_{\text{Jc},0} \end{array} \\ \begin{bmatrix} A_{\text{Jc},\text{Pi}}\Pi_2 & 0 & 0 \\ U_2^T A_{\text{Jc},\text{Pi}} \,\mathrm{D}_1 f_{\text{Pi}}\Pi_2 & U_2^T A_{\text{Jc},\text{Pi}}\partial_{\dot{h}_{\text{Jc},V}} f_{\text{Pi}} & U_2^T A_{\text{Jc},\text{Pi}}\partial_{\dot{h}_{\text{Jc},0}} f_{\text{Pi}} \\ -\mathrm{D}_1 f_{\bar{\text{Pi}}}\Pi_2 & -\partial_{\dot{h}_{\text{Jc},V}} f_{\bar{\text{Pi}}} & -D_{\bar{\text{Jc}}_0} \end{bmatrix} \end{array},$$

$$\tilde{N}_{32} = \begin{array}{c} \\ \bar{\text{Jc}}_0 \end{array} \begin{array}{c} \begin{array}{cccc} q_{\text{Pi}}\Pi_1 & p_{\text{Jc}} & H_{\text{Pu}} & H_{\text{Pi}} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & -A_{\text{Jc}_0,\text{Pi}} \end{bmatrix} \end{array}.$$

with $\star := -A_{\text{Jc},\text{De}} \cdot \text{De} - U_2^T A_{\text{Jc},\text{Pi}} \cdot \text{Pi} + \dot{\text{Pi}}$. By the assumptions on $\mathrm{D}_1 f_{\text{Pu}}$ and $\mathrm{D}_2 f_{\text{Pi}}$, cp. (10), the diagonal block $\tilde{N}_{22}$ is nonsingular. Hence,

$$\operatorname{rank}(Z_{2,\mu=1}^T N_{\mu=1}) = 2n_{\text{De}} + 2n_{\text{Re}} + \operatorname{rank}(\tilde{N}_{22}) + \operatorname{rank}(\tilde{N}_{32}\tilde{N}_{22}^{-1}\tilde{N}_{23}),$$

where

$$\tilde{N}_{32}\tilde{N}_{22}^{-1}\tilde{N}_{23} = \begin{bmatrix} A_{\text{Jc}_0,\text{Pi}} \,\mathrm{D}_1 f_{\bar{\text{Pi}}}\big(\Pi_1(A_{\text{Jc},\text{Pi}}\Pi_1)^{-1}A_{\text{Jc},\text{Pi}}\Pi_2 - \Pi_2\big) \\ A_{\text{Jc}_0,\text{Pi}}\partial_{\dot{h}_{\text{Jc},V}} f_{\bar{\text{Pi}}} \\ B_{\bar{\text{Jc}}_0} \end{bmatrix}^T.$$

Again, as $B_{\bar{\text{Jc}}_0} = A_{\text{Jc}_0,\text{Pi}} D_{\bar{\text{Jc}}_0}$ is nonsingular with $\operatorname{rank}(B_{\bar{\text{Jc}}_0}) = n_{\text{Jc}_0}$, it follows that

$$\operatorname{rank}(Z_{2,\mu=1}^T N_{\mu=1}) = n_{\text{Pi}} + 2n_{\text{Jc}_V} + 3n_{\text{Jc}_0} + 2n_{\text{De}} + 2n_{\text{Re}} = a.$$

Exploiting the structure of $Z_{2,\mu=1}^T N_{\mu=1}\Pi_{N_{\mu=1}}$, a basis of $\ker(Z_{2,\mu=1}^T N_{\mu=1})$ is given by

$$T_{2,\mu=1} = \Pi_{N_{\mu=1}} \begin{bmatrix} 0 & -\tilde{N}_{22}^{-1}\tilde{N}_{23} & I \end{bmatrix}^T X_3 \in C(\mathbb{D}, \mathbb{R}^{n\times d}),$$

where $\operatorname{span}(X_3) = \ker(\tilde{N}_{32}\tilde{N}_{22}^{-1}\tilde{N}_{23})$. The first block row of $\tilde{N}_{22}^{-1}\tilde{N}_{23}$ corresponds to the pipe flows $q_{\text{Pi},1}$ while the identity $I_{n_{\text{Pi}\Pi_2}}$ is associated with the flows $q_{\text{Pi},2}$. With

$$F_{\dot{q},\dot{p}} = \begin{array}{c} \\ \text{Pi} \\ \bar{\text{Jc}}_V \\ \star \end{array} \begin{array}{c} \begin{array}{ccc} \dot{q}_{\text{Pi}} & \dot{h}_{\text{Jc}_V} & \diamond \end{array} \\ \begin{bmatrix} I_{n_{\text{Pi}}} & 0 & 0 \\ 0 & V_{\text{Jc}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array},$$

where $\diamond = \text{Jc}, \text{Jc}_0, \bar{\text{Pi}}, \text{De}, \bar{\text{De}}, \text{Re}, \bar{\text{Re}}$ and $\star = p_{\text{Jc}}, h_{\text{Jc}_0}, H_{\text{Pi}}, q_{\text{De}}, H_{\text{De}}, p_{\text{Re}}, h_{\text{Re}}$, we find that

$$F_{\dot{q},\dot{p}} T_{2,\mu=1} = \begin{array}{c} \text{Pi} \\ \bar{\text{Jc}}_V \\ \star \end{array} \begin{bmatrix} \Pi_2 - \Pi_1(U_2^T A_{\text{Jc},\text{Pi}}\Pi_1)^{-1} U_2^T A_{\text{Jc},\text{Pi}}\Pi_2) & 0 \\ 0 & I_{n_{\text{Jc}_V}} \\ 0 & 0 \end{bmatrix}.$$

As Π_1, Π_2 are linearly independent, we have that

$$\operatorname{rank}(\Pi_2 - \Pi_1(U_2^T A_{\text{Jc},\text{Pi}}\Pi_1)^{-1} U_2^T A_{\text{Jc},\text{Pi}}\Pi_2) = n_{\text{Pi}} - (n_{\text{Jc}} - \tilde{c}),$$

and we find that $\operatorname{rank}(F_{\dot{q},\dot{p}}T_{2,\mu=1}) = n_{\mathrm{Pi}} - n_{\mathrm{Jc}_0} + \tilde{c} = d$. Setting

$$Z_{1,\mu=1}^T = \begin{array}{c} \\ \end{array}\begin{matrix} \mathrm{Pi} & \bar{\mathrm{J}}\mathrm{c} & \star \\ \end{matrix} \begin{bmatrix} \Pi_2^T & 0 & 0 \\ 0 & I_{n_{\mathrm{Jc}_V}} & 0 \end{bmatrix},$$

we verify that $\operatorname{rank}(Z_{1,\mu=1}^T F_{\dot{q},\dot{p}} T_{2,\mu=1}) = d$.
Hence, the network model (8) has has s-index $\mu = 1$.

To prove that the network model is regular, we show that (8) satisfies Hypothesis 2.1 for $\mu = 2$ and same values d, a. The proof follows the same line as for $\mu = 1$, but the derivative array under consideration is increased to order 2, i.e., we consider $\mathcal{F}_{F_{\mathcal{N}},2} := [F_{\mathcal{N}}^T, \frac{d}{dt}F_{\mathcal{N}}^T, \frac{d^2}{dt^2}F_{\mathcal{N}}^T]^T$ with

$$\ddot{F}_{\mathcal{N}} = \begin{bmatrix} \dddot{q}_{\mathrm{Pi}} - \mathrm{D}_1 f_{\mathrm{Pi}} \ddot{q}_{\mathrm{Pi}} - \mathrm{D}_2 f_{\mathrm{Pi}}(A_{\mathrm{Jc,Pi}}^T \ddot{p}_{\mathrm{Jc}} + A_{\mathrm{Re,Pi}}^T \ddot{p}_{\mathrm{Re}}) - \mathrm{D}_3 f_{\mathrm{Pi}} \ddot{h} + R_{\mathrm{Pi}}(q,p,h,\dot{q},\dot{p},\dot{h}) \\ V_{\mathrm{Jc}} \dddot{h}_{\mathrm{Jc}} - A_{\mathrm{Jc,Pi}} \ddot{H}_{\mathrm{Pi}} - A_{\mathrm{Jc,De}} \ddot{H}_{\mathrm{De}} \\ A_{\mathrm{Jc,Pi}} \ddot{q}_{\mathrm{Pi}} + A_{\mathrm{Jc,De}} \ddot{q}_{\mathrm{De}} \\ A_{\mathrm{Jc,Pi}} \ddot{q}_{\mathrm{Pi}} + A_{\mathrm{Jc,De}} \ddot{q}_{\mathrm{De}} \\ \ddot{H}_{\mathrm{Pi}} - \mathrm{D}_1 f_{\bar{\mathrm{P}}\mathrm{i}} \ddot{q}_{\mathrm{Pi}} - \mathrm{D}_2 f_{\bar{\mathrm{P}}\mathrm{i}} \ddot{h} \\ \ddot{q}_{\mathrm{De}} - \ddot{\bar{q}}_{\mathrm{De}} \\ \ddot{p}_{\mathrm{Re}} - \ddot{\bar{p}}_{\mathrm{Re}} \\ \ddot{H}_{\mathrm{De}} - \ddot{\bar{H}}_{\mathrm{De}} \\ \ddot{h}_{\mathrm{Re}} - \ddot{\bar{h}}_{\mathrm{Re}} \end{bmatrix}.$$

where R_{Pi} contains the mixed derivatives in $F_{\mathcal{N},1}$. To check Hypothesis 2.1, (i), we transform the Jacobian $M_{\mu=2} := \partial_{\dot{q},\dot{p},\ddot{q},\ddot{p}} \mathcal{F}_{F_{\mathcal{N}},2}$ using nonsingular transformations $\bar{\Pi}_{M_{\mu=2}}, \Pi_{M_{\mu=2}}$, such that

$$\bar{\Pi}_{M_{\mu=2}}^T M_{\mu=2} \Pi_{M_{\mu=2}} = \begin{bmatrix} I_{n_{\mathrm{Pi}}+n_{\mathrm{Jc}_V}+4n_{\mathrm{De}}+4n_{\mathrm{Re}}} & 0 & 0 & 0 & 0 \\ * & \tilde{M}_{22} & 0 & \tilde{M}_{24} & 0 \\ * & \tilde{M}_{32} & \tilde{M}_{33} & \tilde{M}_{34} & 0 \\ * & \tilde{M}_{42} & 0 & 0 & 0 \\ * & \tilde{M}_{52} & \tilde{M}_{53} & 0 & 0 \\ \tilde{M}_{61} & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\tilde{M}_{22}, \tilde{M}_{24}, \tilde{M}_{42}, \tilde{M}_{61}$ are given by (28) and

$$\tilde{M}_{32} = \begin{matrix} \\ \ddot{\mathrm{P}}\mathrm{i} \\ \ddot{\bar{\mathrm{P}}}\mathrm{i} \\ \ddot{\bar{\mathrm{J}}}\mathrm{c}_V \end{matrix} \begin{matrix} \ddot{q}_{\mathrm{Pi}} & \dot{H}_{\mathrm{Pi}} & \ddot{h}_{\mathrm{Jc},V} \\ -\mathrm{D}_1 f_{\mathrm{Pi}} & 0 & -\partial_{\ddot{h}_{\mathrm{Jc},V}} f_{\mathrm{Pi}} \\ -\mathrm{D}_1 f_{\bar{\mathrm{P}}\mathrm{i}} & 0 & -\partial_{\ddot{h}_{\mathrm{Jc},V}} f_{\bar{\mathrm{P}}\mathrm{i}} \\ 0 & 0 & 0 \end{matrix}, \qquad \tilde{M}_{33} = \begin{matrix} \\ \ddot{\mathrm{P}}\mathrm{i} \\ \ddot{\bar{\mathrm{P}}}\mathrm{i} \\ \ddot{\bar{\mathrm{J}}}\mathrm{c}_V \end{matrix} \begin{matrix} \dddot{q}_{\mathrm{Pi}} & \ddot{H}_{\mathrm{Pi}} & \dddot{h}_{\mathrm{Jc},V} \\ I_{\mathrm{Pi}} & 0 & 0 \\ 0 & I_{\mathrm{Pi}} & 0 \\ 0 & -A_{\mathrm{Jc}_V,\mathrm{Pi}} & V_{\mathrm{Jc}} \end{matrix},$$

$$\tilde{M}_{34} = \begin{matrix} \\ \ddot{\mathrm{P}}\mathrm{i} \\ \ddot{\bar{\mathrm{P}}}\mathrm{i} \\ \ddot{\bar{\mathrm{J}}}\mathrm{c}_V \end{matrix} \begin{matrix} \dot{h}_{\mathrm{Jc},0} & \ddot{h}_{\mathrm{Jc},0} & \dot{p}_{\mathrm{Jc}} & \ddot{p}_{\mathrm{Jc}} \\ 0 & -\partial_{\ddot{h}_{\mathrm{Jc},0}} f_{\mathrm{Pi}} & 0 & 0 \\ 0 & -\partial_{\ddot{h}_{\mathrm{Jc},0}} f_{\bar{\mathrm{P}}\mathrm{i}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix},$$

$$\tilde{M}_{52} = \begin{matrix} \\ \ddot{\bar{\mathrm{J}}}\mathrm{c}_0 \\ \ddot{\mathrm{J}}\mathrm{c} \end{matrix} \begin{matrix} \ddot{q}_{\mathrm{Pi}} & \dot{H}_{\mathrm{Pi}} & \ddot{h}_{\mathrm{Jc},V} \\ 0 & 0 & 0 \\ U_2^T A_{\mathrm{Jc,Pi}} & 0 & 0 \end{matrix}, \qquad \tilde{M}_{53} = \begin{matrix} \\ \ddot{\bar{\mathrm{J}}}\mathrm{c}_0 \\ U_2^T \ddot{\mathrm{J}}\mathrm{c} \end{matrix} \begin{matrix} \dddot{q}_{\mathrm{Pi}} & \ddot{H}_{\mathrm{Pi}} & \dddot{h}_{\mathrm{Jc},V} \\ 0 & -A_{\mathrm{Jc}_0,\mathrm{Pi}} & 0 \\ 0 & 0 & 0 \end{matrix}.$$

Note that $\tilde{M}_{33} = \tilde{M}_{22}$. Setting $\hat{M}_{22} = [\tilde{M}_{ij}]_{i,j=2,3}$, $\hat{M}_{23} = [\tilde{M}_{i4}]_{i=2,3}$, $\hat{M}_{32} = [\tilde{M}_{4j}]_{j=2,3}$, then the diagonal block $\hat{M}_{22}$ is nonsingular and we have that

$$\operatorname{rank}(M_{\mu=2}) = n_{\mathrm{Pi}} + n_{\mathrm{Jc}_V} + 4n_{\mathrm{De}} + 4n_{\mathrm{Re}} + \operatorname{rank}(\hat{M}_{22}) + \mathcal{S}_{11}(\bar{\Pi}^T_{M_{\mu=2}} M_{\mu=2} \Pi_{M_{\mu=2}})),$$

where the Schur complement is given by

$$\mathcal{S}_{11}(\bar{\Pi}^T_{M_{\mu=2}} M_{\mu=2} \Pi_{M_{\mu=2}}) = -\hat{M}_{32} \hat{M}_{22}^{-1} \hat{M}_{23}$$

$$= \begin{array}{c} \\ \dot{\bar{\mathrm{J}}}\mathrm{c}_0 \\ \ddot{\bar{\mathrm{J}}}\mathrm{c}_0 \\ \ddot{\mathrm{J}}\mathrm{c} \end{array} \begin{array}{c} \begin{array}{cccc} \dot{h}_{\mathrm{Jc},0} & \ddot{h}_{\mathrm{Jc},0} & \dot{p}_{\mathrm{Jc}} & \ddot{p}_{\mathrm{Jc}} \end{array} \\ \begin{bmatrix} A_{\mathrm{Jc}_0,\mathrm{Pi}} D_{\bar{\mathrm{J}}\mathrm{c}_0} & 0 & 0 & 0 \\ * & A_{\mathrm{Jc}_0,\mathrm{Pi}} \partial_{\ddot{h}_{\mathrm{Jc},0}} f_{\bar{\mathrm{P}}\mathrm{i}} & * & 0 \\ -U_1^T A_{\mathrm{Jc},\mathrm{Pi}} \partial_{\dot{h}_{\mathrm{Jc},0}} f_{\mathrm{Pi}} & 0 & \hat{\mathrm{D}}_2 f_{\mathrm{Pi}} & 0 \end{bmatrix} \end{array}.$$

As $A_{\mathrm{Jc}_0,\mathrm{Pi}} D_{\bar{\mathrm{J}}\mathrm{c}_0}$ and $A_{\mathrm{Jc},\mathrm{Pi}} C_1 A^T_{\mathrm{Jc},\mathrm{Pi}}$ are nonsingular, cp. Lemma 3.2 and Lemma 4.2, it follows that

$$\operatorname{rank}(M_{\mu=2}) = 5n_{\mathrm{Pi}} + 4n_{\mathrm{Jc}_V} + 3n_{\mathrm{Jc}_0} + 4n_{\mathrm{De}} + 4n_{\mathrm{Re}} + \tilde{c}.$$

Hence, the derivative array $\mathcal{F}_{F_\mathcal{N},2}$ yields the same values d, a.
Exploiting the structure of $\bar{\Pi}^T_{M_{\mu=2}} M_{\mu=2} \Pi_{M_{\mu=2}}$, a basis of corange$(M_{\mu=2})$ is given by

$$Z_{2,\mu=2} = \begin{bmatrix} -\tilde{M}_{61} & 0 & 0 & 0 & 0 & I_a \end{bmatrix} \bar{\Pi}^T_{M_{\mu=2}} \in \mathbb{R}^{3n \times a}.$$

Noting that $Z_{2,\mu=2}$ selects from $N_{2,\mu=2} := \partial_{q,p} \mathcal{F}_{F_\mathcal{N},\mu=2}$ the same block rows as $Z_{2,\mu=1}$ selects from $N_{2,\mu=1} := \partial_{q,p} \mathcal{F}_{F_\mathcal{N},\mu=1}$, we have that

$$Z^T_{2,\mu=2} N_{\mu=2} = Z^T_{2,\mu=1} N_{\mu=1}$$

and the remaining assertions of Hypothesis 2.1 are verified using the matrices $T_{2,\mu=2} := T_{2,\mu=1}$ and $Z_{1,\mu=2} := Z_{1,\mu=1}$. Hence the network model (8) is regular.

Applying the matrices $Z_{1,\mu=1}, Z_{1,\mu=2}$ to the derivative array $\mathcal{F}_{F_\mathcal{N},\mu=1}$, we obtain the s-free remodeling (27). □

Like for the standard flow model (8), the s-free formulation (27) of the thermal flow model can be reformulated as semi-explicit system.

Lemma 4.3. *Let $\mathcal{N}$ be a network of pipes, junctions, reservoirs and demand branches that satisfies Assumption 3.1. Let $F \in C^2(\mathbb{D}, \mathbb{R}^n)$ be the DAE function associated with $\mathcal{N}$. If $n_{Re} > 0$, then functions $q \in C^1(\mathcal{I}, \mathbb{R}^{n\varepsilon})$, $p \in C^1(\mathcal{I}, \mathbb{R}^{n\nu})$ solve the DAE (8) if and only if they solve the semi-explicit system*

$$\begin{aligned}
\dot{q}_2 &= K_{Ch,Ch}(q_2)\Sigma(q_2)\big(L_{q_2}q_2 + L_{q_{De}}\bar{q}_{De}\big) + K_{Ch,De}\dot{\bar{q}}_{De} + K_{Ch,Re}\bar{p}_{Re} + C_5 && \text{(30a)}\\
\hat{V}\dot{h}_{Jc_V} &= K_{Jc_V,Jc_V}h_{Jc_V} + K_{Jc_V,Re}\bar{h}_{Re} + K_{Jc_V,De}\bar{H}_{De} && \text{(30b)}\\
h_{Jc_0} &= K_{Jc_0,Jc_V}h_{Jc_V} + K_{Jc_0,Re}\bar{h}_{Re} + K_{Jc_0,De}\bar{H}_{De} && \text{(30c)}\\
H_{Pi} &= \bar{G}D_{\bar{J}c_V}h_{Jc_V} + \bar{G}D_{\bar{R}e}\bar{h}_{Re} - D_{\bar{J}c_0}B^{-1}_{\bar{J}c_0}(q_2)A_{Jc_0,De}\bar{H}_{De} && \text{(30d)}\\
p_{Jc} &= K_{Jc,Pi}(q_2)\Sigma(q_2)\big(L_{q_2}q_2 + L_{q_{De}}\bar{q}_{De}\big) + K_{Jc,De}\dot{\bar{q}}_{De} + K_{Jc,Re}\bar{p}_{Re} + C_4 && \text{(30e)}\\
q_1 &= -(A_{Jc,Pi}\Pi_1)^{-1}A_{Jc,Pi}\Pi_2 q_2 + (A_{Jc,Pi}\Pi_1)^{-1}\bar{q}_{De} && \text{(30f)}\\
q_{De} &= \bar{q}_{De} && \text{(30g)}\\
p_{Re} &= \bar{p}_{Re}, && \text{(30h)}
\end{aligned}$$

where $q_i = \Pi_i^T q_{Pi}$, $i = 1, 2$, for a permutation $[\Pi_1\,\Pi_2]$ such that $A_{Jc,Pi}\Pi_1$ is nonsingular and

$$L_{q_2} := -\Pi_1(A_{Jc,Pi}\Pi_1)^{-1}A_{Jc,Pi}\Pi_2 + \Pi_2, \qquad L_{q_{De}} := -(A_{Jc,Pi}\Pi_1)^{-1}A_{Jc,De},$$

and

$$\begin{aligned}
&K_{Ch,Ch}(q_2) := \Pi_2^T G C_2(h(q_2)), \quad K_{Ch,De} := \Pi_2^T C_1 A^T_{Jc,Pi}K_{Jc,De}, \quad K_{Ch,Re} := \Pi_2^T G^T A^T_{Re,Pi},\\
&K_{Jc_0,Jc_V} := -B^{-1}_{\bar{J}c_0}(q_2)A_{Jc_0,Pi}D_{\bar{J}c_V}, \quad K_{Jc_0,De} := -B^{-1}_{\bar{J}c_0}(q_2)A_{Jc_0,De}, \quad K_{Jc_0,Re} := -B^{-1}_{\bar{J}c_0}(q_2)A_{Jc_0,Pi}D_{\bar{R}e},\\
&C_4 := -(A_{Jc,Pi}C_1A_{Jc,Pi})^{-1}C_3, \quad C_5 := \Pi_2^T(C_3 + C_1A^T_{Jc,Pi}C_4),
\end{aligned}$$

with $G = (I - C_1A^T_{Jc,Pi}B^{-1}_{Jc}A_{Jc,Pi})$, $\bar{G} := I - D_{\bar{J}c_0}B^{-1}_{\bar{J}c_0}(q_2)A_{Jc_0,Pi}$, $\Sigma(q_2) := \operatorname{diag}(\operatorname{sgn}(q_{Pi,j}(q_2))q_{Pi,j}(q_2))$.

Proof. Under the given assertions, the DAE (8), (23) is uniquely solvable for a consistent initial value $(t_0, q_0, p_0) \in \mathcal{C}_2(\tilde{F})$ and the solution $q \in C^1(\mathcal{I}, \mathbb{R}^{n\varepsilon})$, $p \in C^1(\mathcal{I}, \mathbb{R}^{n\nu})$ is also the unique solution of the s-free formulation (27) cp. Theorem 4.1 To derive the semi-explicit formulation (15), we again solve the is-free formulation (27) using the variable partitioning $q_1 := \Pi_1^T q_{\mathrm{Pi}}$, $q_2 := \Pi_2^T q_{\mathrm{Pi}}$, where $[\Pi_1, \Pi_2]$ is a permutation, such that $A_{\mathrm{Jc,Pi},1}\Pi_1\Pi_1$ is nonsingular, cp. Lemma 3.2.

Again, we first use the boundary conditions (27g) - (27j) to eliminate the controlled states p_{Re}, h_{Re}, q_{De}, H_{De} in the equations (27a) - (27f). Using the variable transformation, we solve the junction equation (27b) for q_1 and obtain (30f). The flow q_{Pi} can again be represented as affine linear transformation of the flow q_2, cp. (17). For the enthalpy h_{Jc_0} associated with the zero volume junctions, we insert the identity $H_{\mathrm{Pi}} = f_{\bar{\mathrm{P}}\mathrm{i}}(q_{\mathrm{Pi}}, h)$, cp. (27f), into the mass balance at zero junctions (27e) and obtain that

$$0 = A_{\mathrm{Jc_0,Pi}}\big(D_{\bar{\mathrm{J}}\mathrm{c}_0}h_{\mathrm{Jc}_0} + D_{\bar{\mathrm{J}}\mathrm{c}_V}h_{\mathrm{Jc}_V} + D_{\bar{\mathrm{R}}\mathrm{e}}h_{\bar{\mathrm{R}}\mathrm{e}}\big) + A_{\mathrm{Jc_0,De}}\bar{H}_{\mathrm{De}}.$$

As $B_{\bar{\mathrm{J}}\mathrm{c}_0}(q_2) := A_{\mathrm{Jc_0,Pi}}D_{\bar{\mathrm{J}}\mathrm{c}_0}(q_{\mathrm{Pi}}(q_2))$ is nonsingular, this equation can be solved for h_{Jc_0} and we get that

$$\begin{aligned}
h_{\mathrm{Jc}_0} &= -B^{-1}_{\bar{\mathrm{J}}\mathrm{c}_0}(q_2)A_{\mathrm{Jc_0,Pi}}\big(D_{\bar{\mathrm{J}}\mathrm{c}_V}h_{\mathrm{Jc}_V} + D_{\bar{\mathrm{R}}\mathrm{e}}\bar{h}_{\mathrm{Re}}\big) - B^{-1}_{\bar{\mathrm{J}}\mathrm{c}_0}(q_2)A_{\mathrm{Jc_0,De}}\bar{H}_{\mathrm{De}}\\
&= K_{\mathrm{Jc_0,Jc}_V}h_{\mathrm{Jc}_V} + K_{\mathrm{Jc_0,Re}}\bar{h}_{\mathrm{Re}} + K_{\mathrm{Jc_0,De}}\bar{H}_{\mathrm{De}} && \text{(31)}
\end{aligned}$$

where $K_{\mathrm{Jc_0,Jc}_V} := -B^{-1}_{\bar{\mathrm{J}}\mathrm{c}_0}(q_2)A_{\mathrm{Jc_0,Pi}}D_{\bar{\mathrm{J}}\mathrm{c}_V}$, $K_{\mathrm{Jc_0,Re}} := -B^{-1}_{\bar{\mathrm{J}}\mathrm{c}_0}(q_2)A_{\mathrm{Jc_0,Pi}}D_{\bar{\mathrm{R}}\mathrm{e}}$, $K_{\mathrm{Jc_0,De}} := -B^{-1}_{\bar{\mathrm{J}}\mathrm{c}_0}(q_2)A_{\mathrm{Jc_0,De}}$. Inserting (31) back into the thermal pipe equation (27f), we get that

$$\begin{aligned}
H_{\mathrm{Pi}} &= \big(D_{\bar{\mathrm{J}}\mathrm{c}_V} - D_{\bar{\mathrm{J}}\mathrm{c}_0}B^{-1}_{\bar{\mathrm{J}}\mathrm{c}_0}(q_2)A_{\mathrm{Jc_0,Pi}}D_{\bar{\mathrm{J}}\mathrm{c}_V}\big)h_{\mathrm{Jc}_V} + \big(D_{\bar{\mathrm{R}}\mathrm{e}} - D_{\bar{\mathrm{J}}\mathrm{c}_0}B^{-1}_{\bar{\mathrm{J}}\mathrm{c}_0}(q_2)A_{\mathrm{Jc_0,Pi}}D_{\bar{\mathrm{R}}\mathrm{e}}\big)\bar{h}_{\mathrm{Re}}\\
&\quad - D_{\bar{\mathrm{J}}\mathrm{c}_0}B^{-1}_{\bar{\mathrm{J}}\mathrm{c}_0}(q_2)A_{\mathrm{Jc_0,De}}\bar{H}_{\mathrm{De}}\\
&= \bar{G}D_{\bar{\mathrm{J}}\mathrm{c}_V}h_{\mathrm{Jc}_V} + \bar{G}D_{\bar{\mathrm{R}}\mathrm{e}}\bar{h}_{\mathrm{Re}} - D_{\bar{\mathrm{J}}\mathrm{c}_0}B^{-1}_{\bar{\mathrm{J}}\mathrm{c}_0}(q_2)A_{\mathrm{Jc_0,De}}\bar{H}_{\mathrm{De}} && \text{(32)}
\end{aligned}$$

where $\bar{G} := I - D_{\bar{\text{Jc}}_0} B^{-1}_{\bar{\text{Jc}}_0}(q_2) A_{\text{Jc}_0,\text{Pi}}$. Using (31), we reformulate the mass balance at volume nodes(27d) as an ODE depending only on h_V, i.e.,

$$\begin{aligned}\hat{V}\dot{h}_{\text{Jc}_V} &= A_{\text{Jc}_V,\text{Pi}}\big(\bar{G}D_{\bar{\text{Jc}}_V} h_{\text{Jc}_V} + \bar{G}D_{\bar{\text{Re}}}\bar{h}_{\text{Re}} - D_{\bar{\text{Jc}}_0} B^{-1}_{\bar{\text{Jc}}_0}(q_2) A_{\text{Jc}_0,\text{De}}\bar{H}_{\text{De}}\big) + A_{\text{Jc}_V,\text{De}}\bar{H}_{\text{De}} \\ &= K_{\text{Jc}_V,\text{Jc}_V} h_{\text{Jc}_V} + K_{\text{Jc}_V,\text{Re}}\bar{h}_{\text{Re}} + K_{\text{Jc}_V,\text{De}}\bar{H}_{\text{De}} \end{aligned} \tag{33}$$

where $K_{\text{Jc}_V,\text{Jc}_V} := A_{\text{Jc}_V,\text{Pi}}\bar{G}D_{\bar{\text{Jc}}_V}$, $K_{\text{Jc}_V,\text{De}} := A_{\text{Jc}_V,\text{De}} - A_{\text{Jc}_V,\text{Pi}} D_{\bar{\text{Jc}}_0} B^{-1}_{\bar{\text{Jc}}_0}(q_2) A_{\text{Jc}_0,\text{De}}$ and $K_{\text{Jc}_V,\text{Re}} := A_{\text{Jc}_V,\text{Pi}}\bar{G}D_{\bar{\text{Re}}}$.
From (31) and (33) it follows that the enthalpy h is determined by the flow q_2, i.e.,

$$h = h(q_2). \tag{34}$$

For the pressure p_{Jc}, we get analogously to (15b) the relation

$$\begin{aligned} p_{\text{Jc}}(q_2) &= -B^{-1}_{\text{Jc}}\big(A_{\text{Jc},\text{Pi}} C_1 A^T_{\text{Re},\text{Pi}}\bar{p}_{\text{Re}} + A_{\text{Jc},\text{Pi}} C_2(h(q_2))\,\text{diag}\left(|q_{\text{Pi},j}|\right) q_{\text{Pi}}(q_2) + A_{\text{Jc},\text{Pi}} C_3 + A_{\text{Jc},\text{De}}\dot{\bar{q}}_{\text{De}}\big) \\ &= K_{\text{Jc},\text{Pi}}(q_2)\Sigma(q_2)\big(L_{q_2} q_2 + L_{q_{\text{De}}}\bar{q}_{\text{De}}\big) + K_{\text{Jc},\text{De}}\dot{\bar{q}}_{\text{De}} + K_{\text{Jc},\text{Re}}\bar{p}_{\text{Re}} + C_4, \end{aligned} \tag{35}$$

where $K_{\text{Jc},\text{Pi}}(q_2) := -B^{-1}_{\text{Jc}} A_{\text{Jc},\text{Pi}} C_2(h(q_2))\Sigma(q_2)$, $K_{\text{Jc},\text{Re}} := -B^{-1}_{\text{Jc}} A_{\text{Jc},\text{Pi}} C_1 A^T_{\text{Re},\text{Pi}}$ and $K_{\text{Jc},\text{De}} := -B^{-1}_{\text{Jc}} A_{\text{Jc},\text{De}}$, $C_4 := -B^{-1}_{\text{Jc}} A_{\text{Jc},\text{Pi}} C_3$ and $\Sigma(q_2) := \text{diag}\left(|[L_{q_2} q_2 + L_{q_{\text{De}}}\bar{q}_{\text{De}}]_j|\right)$.
Using (35), we can reformulate (27a) as an ODE depending only on q_2, i.e.,

$$\begin{aligned}\dot{q}_2 &= \Pi_2^T\big(C_1(A^T_{\text{Jc},\text{Pi}} p_{\text{Jc}}(q_2) + A^T_{\text{Re},\text{Pi}}\bar{p}_{\text{Re}}) + C_2(h(q_2))\Sigma(q_2) q_{\text{Pi}}(q_2) + C_3\big) \\ &= K_{\text{Ch},\text{Ch}}(q_2)\Sigma(q_2)\big(L_{q_2} q_2 + L_{q_{\text{De}}}\bar{q}_{\text{De}}\big) + K_{\text{Ch},\text{De}}\dot{\bar{q}}_{\text{De}} + K_{\text{Ch},\text{Re}}\bar{p}_{\text{Re}} + C_5. \end{aligned} \tag{36}$$

where $K_{\text{Ch},\text{Ch}}(q_2) := \Pi_2^T G C_2(h(q_2))$, $K_{\text{Ch},\text{De}} := \Pi_2^T C_1 A^T_{\text{Jc},\text{Pi}} K_{\text{Jc},\text{De}}$, $K_{\text{Ch},\text{Re}} := \Pi_2^T G^T A^T_{\text{Re},\text{Pi}}$ and $C_5 := \Pi_2^T(C_3 + C_1 A^T_{\text{Jc},\text{Pi}} C_4)$.
In conclusion, (11d), (11e), (17), (31), (32), (33), (35), (36) establish the semi-explicit reformulation (30). ☐

Remark 4.2. *In order to continue Remark 3.2, we observe that, considering the thermal distribution, the enthalpies associated with non-zero volume nodes are determined by a differential equation, while the enthalpies associated with zero volume nodes are specified algebraically. Including thermal effects in the pipe function for the flow q_{Pi}, the ODE for the flow q_2 becomes quasi-linear as the coefficient $K_{Ch,Ch}$ now depends on q_2 due to the its enthalpy dependance.*

The algebraic constraints are still linear, such that we can use the semi-explicit system (30) to specify the set of consistent initial values. Again, we evaluate (30) in $t_0 \in \mathcal{I}$, i.e.,

$$\dot{q}_{2,0} = K_{\text{Ch},\text{Ch}}(q_{2,0})\Sigma(q_{2,0})\big(L_{q_2} q_{2,0} + L_{q_{\text{De}}}\bar{q}_{\text{De}}(t_0)\big) + K_{\text{Ch},\text{De}}\dot{\bar{q}}_{\text{De}}(t_0) + K_{\text{Ch},\text{Re}}\bar{p}_{\text{Re}}(t_0) + C_5 \tag{37a}$$

$$\hat{V}\dot{h}_{\text{Jc}_V,0} = K_{\text{Jc}_V,\text{Jc}_V} h_{\text{Jc}_V,0} + K_{\text{Jc}_V,\text{Re}}\bar{h}_{\text{Re}}(t_0) + K_{\text{Jc}_V,\text{De}}\bar{H}_{\text{De}}(t_0) \tag{37b}$$

$$h_{\text{Jc}_0} = K_{\text{Jc}_0,\text{Jc}_V} h_{\text{Jc}_V,0} + K_{\text{Jc}_0,\text{Re}}\bar{h}_{\text{Re}}(t_0) + K_{\text{Jc}_0,\text{De}}\bar{H}_{\text{De}}(t_0) \tag{37c}$$

$$H_{\text{Pi},0} = \bar{G}D_{\bar{\text{Jc}}_V} h_{\text{Jc}_V,0} + \bar{G}D_{\bar{\text{Re}}}\bar{h}_{\text{Re}}(t_0) - D_{\bar{\text{Jc}}_0} B^{-1}_{\bar{\text{Jc}}_0}(q_{2,0}) A_{\text{Jc}_0,\text{De}}\bar{H}_{\text{De}}(t_0) \tag{37d}$$

$$p_{\text{Jc},0} = K_{q_2}(q_{2,0})\big(L_{q_2} q_{2,0} + L_{q_{\text{De}}}\bar{q}_{\text{De}}(t_0)\big) + K_{\dot{q}_{\text{De}}}\dot{\bar{q}}_{\text{De}}(t_0) + C_4 \tag{37e}$$

$$q_{1,0} = -(A_{\text{Jc},\text{Pi}}\Pi_1)^{-1} A_{\text{Jc},\text{Pi}}\Pi_2 q_{2,0} + (A_{\text{Jc},\text{Pi}}\Pi_1)^{-1}\bar{q}_{\text{De}}(t_0) \tag{37f}$$

$$q_{\text{De},0} = \bar{q}_{\text{De}}(t_0) \tag{37g}$$

$$p_{\text{Re},0} = \bar{p}_{\text{Re}}(t_0), \tag{37h}$$

as well as its time derivative

$$
\begin{aligned}
\ddot{q}_{2,0} &= K_{\mathrm{Ch,Ch}}(q_{2,0})\big(\Sigma(\dot{q}_{2,0})\big(L_{q_2}q_{2,0} + L_{q_{\mathrm{De}}}\bar{q}_{\mathrm{De}}(t_0)\big) + \Sigma(q_{2,0})\big(L_{q_2}\dot{q}_{2,0} + L_{q_{\mathrm{De}}}\dot{\bar{q}}_{\mathrm{De}}(t_0)\big)\big) \\
&\quad + \dot{K}_{\mathrm{Ch,Ch}}(q_{2,0})\Sigma(q_{2,0})\big(L_{q_2}q_{2,0} + L_{q_{\mathrm{De}}}\bar{q}_{\mathrm{De}}(t_0)\big) + K_{\mathrm{Ch,De}}\ddot{\bar{q}}_{\mathrm{De}}(t_0) + K_{\mathrm{Ch,Re}}\dot{\bar{p}}_{\mathrm{Re}}(t_0) \quad (38a) \\
\hat{V}\ddot{h}_{\mathrm{Jc}_V,0} &= K_{\mathrm{Jc}_V,\mathrm{Jc}_V}\dot{h}_{\mathrm{Jc}_V,0} + K_{\mathrm{Jc}_V,\mathrm{Re}}\dot{\bar{h}}_{\mathrm{Re}}(t_0) + K_{\mathrm{Jc}_V,\mathrm{De}}\dot{\bar{H}}_{\mathrm{De}}(t_0) \quad (38b) \\
\dot{h}_{\mathrm{Jc}_0} &= K_{\mathrm{Jc}_0,\mathrm{Jc}_V}\dot{h}_{\mathrm{Jc}_V,0} + K_{\mathrm{Jc}_0,\mathrm{Re}}\dot{\bar{h}}_{\mathrm{Re}}(t_0) + K_{\mathrm{Jc}_0,\mathrm{De}}\dot{\bar{H}}_{\mathrm{De}}(t_0) \quad (38c) \\
\dot{H}_{\mathrm{Pi},0} &= \bar{G}D_{\bar{\mathrm{J}}\mathrm{c}_V}\dot{h}_{\mathrm{Jc}_V,0} + \bar{G}D_{\bar{\mathrm{R}}\mathrm{e}}\dot{\bar{h}}_{\mathrm{Re}}(t_0) - D_{\bar{\mathrm{J}}\mathrm{c}_0}B^{-1}_{\bar{\mathrm{J}}\mathrm{c}_0}(q_{2,0})A_{\mathrm{Jc}_0,\mathrm{De}}\dot{\bar{H}}_{\mathrm{De}}(t_0) \quad (38d) \\
\dot{p}_{\mathrm{Jc},0} &= \dot{K}_{q_2}(q_{2,0})\big(L_{q_2}q_{2,0} + L_{q_{\mathrm{De}}}\bar{q}_{\mathrm{De}}(t_0)\big) + K_{q_2}(q_{2,0})\big(L_{q_2}\dot{q}_{2,0} + L_{q_{\mathrm{De}}}\dot{\bar{q}}_{\mathrm{De}}(t_0)\big) + K_{\dot{q}_{\mathrm{De}}}\ddot{\bar{q}}_{\mathrm{De}}(t_0) \quad (38e) \\
\dot{q}_{1,0} &= -(A_{\mathrm{Jc,Pi}}\Pi_1)^{-1}A_{\mathrm{Jc,Pi}}\Pi_2\dot{q}_{2,0} + (A_{\mathrm{Jc,Pi}}\Pi_1)^{-1}\dot{\bar{q}}_{\mathrm{De}}(t_0) \quad (38f) \\
\dot{q}_{\mathrm{De},0} &= \dot{\bar{q}}_{\mathrm{De}}(t_0) \quad (38g) \\
\dot{p}_{\mathrm{Re},0} &= \dot{\bar{p}}_{\mathrm{Re}}(t_0), \quad (38h)
\end{aligned}
$$

and make the following observation.

Lemma 4.4. *Let $\mathcal{N}$ be a network of pipes, junctions, reservoirs and demand branches that satisfies Assumption 3.1. Let $F \in C^2(\mathbb{D}, \mathbb{R}^n)$ be the DAE function associated with $\mathcal{N}$. If $n_{Re} > 0$, then the set of consistent initial values and initializations is given by*

$$
\begin{aligned}
\mathcal{C}_2(F) &= \big\{(t_0, q_0, p_0) \in \mathcal{I} \times \Omega_x \,|\, (t_0, q_0, p_0) \text{ solves } (37) \text{ with } q_{2,0} = \Pi_2^T q_0 \ \big\}, \\
\mathcal{L}_2(F) &= \big\{(t_0, q_0, p_0, \dot{q}_0, \dot{p}_0) \in \mathbb{D} \,|\, (t_0, q_0, p_0, \dot{q}_0, \dot{p}_0) \text{ solves } (37) \text{ and } (21) \text{ with } q_{2,0} = \Pi_2^T q_0 \ \big\}
\end{aligned}
$$

Proof. As the subspace $\operatorname{coker}(M_2)$ is constant, the s-free formulation (27) of the DAE (8) and (23) is defined globally on $\mathbb{D}$, independent of the considered intial value. Hence, the semi-explicit formulation (15) is defined globally on $\mathbb{D}$, implying that every solution of (30) also solves the DAE (8) and (23). In particular, this is true in $t = t_0$, implying that $(t_0, q_0, p_0) \in \mathcal{C}_1(F)$ if (t_0, q_0, p_0) solves (37) with $q_{2,0} = \Pi_2^T q_0$. The time derivative of the algebraic components $q_1, p_{\mathrm{Jc}}, q_{\mathrm{De}}, p_{\mathrm{Re}}$ can be obtained by differentiating (37), so considering (38), we obtain the consistent initializations $\mathcal{L}_1(F)$. Similarly, we can construct a vector $(t_0, q_0, p_0, \dot{q}_0, \dot{p}_0, \ddot{q}_0, \ddot{p}_0)$ solving $\mathcal{F}_2(F)$, implying that $\mathcal{C}_2(F) = \mathcal{C}_1(F)$ and $\mathcal{L}_2(F) = \mathcal{L}_1(F)$. □

4.3 Implementation issues

While the main criteria imposed on the underlying graph can be handled by standard graph algorithms, the conditions imposed in Lemma 4.2 require further discussion.

1. Lemma 4.2, condition $\sum_{j \in \hat{J}_{i,s}} |q_j| > 0$: Since B is a M-matrix, a matrix row criteria can be used in order to track the invertibility. E.g. if

$$
\sum_{j \in \hat{J}_{\mathrm{Jc}_i}} |q_j| = 0
$$

   in the vertex v_i, then the matrix is singular due to the corresponding row of this vertex. For practical applications, this means that there is neither inflow nor outflow in the vertex v_i, and therefore the pressure and enthalpy have to stay constant. Furthermore, $\sum_{j \in \hat{J}_{\mathrm{Jc}_i}} |q_j|$ is a measure for the *singularity* of the underlying system of equations, and therefore can be utilized for the choice (or parametrization) of the underlying solver.

2. Lemma 4.2, condition $\operatorname{sgn}(q_{\mathrm{Pi}}(t)) = \mathit{const}$ for $t \in \mathcal{I}$: This condition is only for theoretical interest in order to give a closed form for the Jacobians of the underlying system of equations. The general case is covered by providing an appropriate decomposition $\mathcal{I}_j$ of $\mathcal{I}$ and divide the full problem in subproblems restricted to the individual domains $\mathcal{I}_j$. In practice, if the sign of the massflow changes, then the system (especially the Jacobians) changes the structure, and therefore has to be reassembled.

The topology based index analysis provides a regular s-free formulation of the original problem, equipped with practical solvability conditions, which can be solved by combining standard algorithms of graph theory (e.g. The Boost Graph Library[7]) and numerical integration (e.g. SUNDIALS[8]) in a straight-forward fashion.

5 Conclusion and Outlook

This work provides a full analysis of a thermal fluid network, which is an extension of the well studied water networks. The first part confirms the results for water networks derived in [15], using a different index concept and proof strategy. Based on this, the new results are established in the second part, where a complete thermal model is added to the fluid network model. The analysis is based on a topological network approach, which allows to impose conditions on the underlying network structure, represented by a graph.
The provided topological solvability and index criteria in combination with efficient graph algorithm provide a powerful tool for the development of system simulation software. Parts of this work have been successfully established in the multi-disciplinary vehicle system simulation platform AVL CRUISE M[9] [9]. Anyhow, for the practical application it is important to extend those results to networks including pumps, valves and tanks, cp. the classification in [16], in order to be able to capture the whole cooling circuit. We mention, that further models for system simulation in automotive application (e.g. waste heat recovery, mobile air conditioning, lubrication systems), show up a similar network structure (with slightly modified equations). Therefore the presented analysis is representative for the latter mentioned.

5.1 Notation

Jc	junction nodes
Re	reservoir nodes
Pi	pipe edges
De	demand edges
Jc_0	junction nodes with $V = 0$
Jc_V	junction nodes with $V > 0$
$\mathcal{G}_{\mathrm{Jc}}$	subgraph of junction nodes and pipe edges
$\Pi_{V,1}$	matrix selecting junction nodes with $V > 0$
$\Pi_{V,2}$	matrix selecting junction nodes with $V = 0$
$A_{\mathrm{Jc}_0} = \Pi_{V,2}^T A_{\mathrm{Jc}_0,\mathrm{Pi}}$	incidence matrix of subgraph of $\mathcal{G}_{\mathrm{Jc}_0}$
n_N	number of nodes in $\mathcal{G}$
$\mathcal{I}_{n_N}$	index set of nodes in $\mathcal{G}$
$\mathcal{I}_{\mathrm{Jc}_0}$	index set of nodes in $\mathcal{G}_{\mathrm{Jc}_0}$
$E_1, ..., E_n \in \mathbb{R}^n$	standard canonical basis vectors in $\mathbb{R}^n$, where $E_i = [\delta_{ij}]_{j=1,...,n}$
D_i	partial derivative with respect to the i-th argument of a function

Acknowledgment

We would like to thank our colleagues from AVL List for fruitful and enlightening discussions and for proofreading the manuscript.

References

[1] K.E. Brenan, S.L. Campbell, and L.R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations.* SIAM Publications, Philadelphia, PA, 2nd edition, 1996.

[7] http://www.boost.org
[8] http://computation.llnl.gov/casc/sundials/main.html
[9] http://www.avl.com

[2] J. Burgschweiger, B. Gnädig, and M.C. Steinbach. Optimization models for operative planning in drinking water networks. *Optimization and Engineering*, 10(1):43–73, 2009.

[3] S.L. Campbell. One canonical form for higher index linear time varying singular systems. *Circuits Systems and Signal Processing*, 2:311–326, 1983.

[4] S.L. Campbell. A general form for solvable linear time varying singular systems of differential equations. *SIAM J. Math. Anal.*, 18:1101–1115, 1987.

[5] S.L. Campbell and C.W. Gear. The index of general nonlinear DAEs. *Numer. Math.*, 72:173–196, 1995.

[6] N. Deo. *Graph theory with applications to engineering and computer science.* Prentice-Hall, 1974.

[7] J.W. Deuerlein. Decomposition model of a general water supply network graph. *Journal of Hydraulic Engineering*, 134(6):822–832, 2008.

[8] R. Diestel. *Graduate Texts in Mathematics: Graph Theory.* Springer, Heidelberg, DE, 2000.

[9] AVL List GmbH. *Cruise M User Guide.* AVL Advanced Simulation Technologies, 2015.

[10] C. Godsil and G.F. Royle. *Algebraic Graph Theory.* Graduate Texts in Mathematics. Springer, New York, NY, 2013.

[11] E. Griepentrog and R. März. *Differential-Algebraic Equations and their numerical treatment.* Teubner-Verlag, Leipzig, DE, 1986.

[12] S. Grundel, L. Jansen, N. Hornung, T. Clees, C. Tischendorf, and P. Benner. In *Progress in Differential-Algebraic Equations*, pages 183–205. Springer, 2014.

[13] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, UK, 2012.

[14] C. Huck, L. Jansen, and C. Tischendorf. A topology based discretization of PDAEs describing water transportation networks. *PAMM*, 14(1):923–924, 2014.

[15] L. Jansen and J. Pade. Global unique solvability for a quasi-stationary water network model. *Preprint*, 11, 2013.

[16] L. Jansen and C. Tischendorf. A unified (P) DAE modeling approach for flow networks. In *Progress in Differential-Algebraic Equations*, pages 127–151. Springer, 2014.

[17] P. Kunkel and V. Mehrmann. Canonical forms for linear differential-algebraic equations with variable coefficients. *J. Comput. Appl. Math.*, 56:225–251, 1994.

[18] P. Kunkel and V. Mehrmann. Local and global invariants of linear differential algebraic equations and their relation. *Electr. Trans. Numer. Anal.*, 4:138–157, 1996.

[19] P. Kunkel and V. Mehrmann. Regular solutions of nonlinear differential-algebraic equations and their numerical determination. *Numer. Math.*, 79:581–600, 1998.

[20] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution.* EMS Publishing House, Zürich, CH, 2006.

[21] P. Lancaster and M. Tismenetsky. *The Theory of Matrices.* Academic Press, New York, NY, 1985.

[22] R. März. The index of linear differential algebraic equations with properly stated leading terms. *Results Math.*, 42:308–338, 2002.

[23] R. März. Solvability of linear differential algebraic equations with properly stated leading terms. *Results Math.*, 45(1-2):88–105, 2004.

[24] V. Mehrmann. Index concepts for differential-algebraic equations. Technical Report 2012-03, Institut für Mathematik, TU Berlin, DE, 2012.

[25] C.C. Pantelides. The consistent initialization of differential-algebraic systems. *SIAM J. Sci. Statist. Comput.*, 9(2):213–231, 1988.

[26] J. Pryce. A simple structural analysis method for DAEs. *BIT*, 41:364–394, 2001.

[27] R. Riaza and R. März. A simpler construction of the matrix chain defining the tractability index of linear DAEs. *Appl. Math. Letters*, 21:326–331, 2008.

[28] R. Sedgewick and K. Wayne. *Algorithms.* Addison-Wesley, Upper Saddle River, NJ, 4th edition, 2011.

[29] M.N. Spijker. Contractivity in the numerical solution of initial-value-problems. *Numer. Math.*, 42:271–290, 1983.

[30] M.C. Steinbach. *Topological Index Criteria in DAE for Water Networks.* Konrad-Zuse-Zentrum für Informationstechnik, 2005.

[31] K. Thulasiraman and M.N.S. Swamy. *Graphs: Theory and Algorithms.* JohnWileySons, New York, NY, 2011.

[32] C. Tischendorf. Topological index calculation of differential-algebraic equations in circuit simulation. *Surv. Math. Ind.*, 8:187–199, 1999.