

Planar Linkages Following a Prescribed Motion

**M. Gallet, C. Koutschan, Z. Li, G.
Regensburger, J. Schicho, N. Villamizar**

RICAM-Report 2015-03

PLANAR LINKAGES FOLLOWING A PRESCRIBED MOTION

MATTEO GALLET^{*,◦}, CHRISTOPH KOUTSCHAN^{*}, ZIJIA LI^{*},
GEORG REGENSBURGER[†], JOSEF SCHICHO, AND NELLY VILLAMIZAR

ABSTRACT. Designing mechanical devices, called linkages, that draw a given plane curve has been a topic that interested engineers and mathematicians for hundreds of years, and recently also computer scientists. Already in 1876, Kempe proposed a procedure for solving the problem in full generality, but his constructions tend to be extremely complicated. We provide a novel algorithm that produces much simpler linkages, but works only for parametric curves. Our approach is to transform the problem into a factorization task over some noncommutative algebra. We show how to compute such a factorization, and how to use it to construct a linkage tracing a given curve.

1. INTRODUCTION

Kempe’s Universality Theorem [14], stating that any plane algebraic curve can be drawn by a mechanical linkage with only rotational joints, surprised his contemporaries: his work represented a major breakthrough in a topic investigated by mathematicians of the 19th century. In the middle of the last century, it appeared in textbooks, for example, the ones by Lebesgue [20] and Blaschke [2]. Recently, there was a revived interest in the problem among mathematicians and computer scientists as we outline in the following. We would also like to mention [22] for a historical overview and further references.

In modern terms, the procedure proposed by Kempe is a parsing algorithm. It takes the defining polynomial of a plane curve as input and realizes arithmetic operations via certain elementary linkages. In this work, we approach the question from a different perspective. Instead of a polynomial, we start with a rational parametrization of a curve. From this, we obtain a parametrized family of elements of the group SE_2 of direct isometries of the plane, namely a motion, whose action on a point traces the given curve. In order to realize such a motion by a linkage, we decompose it into a series of revolutions. By encoding motions via polynomials over a noncommutative algebra, we reduce this task to a factorization problem. Eventually, we design a linkage whose rotational joints move according to the previously obtained revolutions, yielding a device drawing the desired curve.

Requiring a rational parametrization, it is clear that our approach is less general than Kempe’s because it cannot be applied to planar curves of positive genus. On the other hand, the parametric setup has advantages in design processes: in the language of robotics/kinematics, we are not just prescribing the position of the end effector but also its orientation, in a similar way as in [25]; moreover, when one has

* Supported by the Austrian Science Fund (FWF): W1214.

◦ Supported by the Austrian Science Fund (FWF): P26607 - “Algebraic Methods in Kinematics: Motion Factorisation and Bond Theory”.

† Supported by the Austrian Science Fund (FWF): P27229.

to interpolate prescribed poses of the end effector, we can specify a tracing order and have control over the speed of the motion.

Linkages obtained via Kempe’s procedure do not, in general, trace only the curve they are designed for. More precisely, the curve is only drawn by a component of the configuration space (also called workspace), that is, the set of all positions reachable by the linkage. This is due to the fact that already the elementary linkages may admit degenerate configurations, allowing the device to flip into an unwanted component. Several solutions to this problem have been suggested [1, 6, 12, 13, 15]. The linkages produced by our method present the same issue, however, this can be treated using the techniques presented in [1, 6]; see Remark 6.7.

Generation of linkages drawing an arbitrary curve may involve many links and joints. This was already observed by Kempe in his foundational paper [14]: “... *this method would not be practically useful on account of the complexity of the link work employed* ...”. As summarized in [6], the problem of finding upper bounds for the number of joints needed to trace a given planar curve of degree d was first addressed by Gao and Zhu [7, 8]. Their upper bound $O(d^4)$ was later improved by Abbott and Barton to $O(d^2)$. Moreover, they show that this bound is optimal in the worst case; see [1]. In this paper, we provide an algorithm that applies to rational planar curves. Given a rational parametrization with denominator of degree d without real roots, our algorithm produces a linkage with $4d$ links and $6d - 2$ joints (Proposition 6.8). For example, in the first part of Section 8, we apply our technique to the parametrization of an ellipse, obtaining a linkage with 8 links and 10 joints. In contrast, an unoptimized linkage returned by Kempe’s procedure has 158 links and 235 joints; we are grateful to Alexander Kobel, who gave a full implementation [16] of Kempe’s procedure using the *Cinderella* interactive geometry system, for assisting us in the computation of these numbers.

In Section 2, we develop a mathematical model for linkages with only rotational joints. In particular, we define their configuration space in terms of isometries sending a fixed initial configuration to a reachable one. This differs from the commonly used models [5, 12, 13, 15, 24].

In Section 3, we start by recalling an embedding of SE_2 as an open subset of a real projective space; see for example [11]. This allows us to introduce a non-commutative algebra \mathbb{K} whose multiplication corresponds to the group operation in SE_2 , hence mimicking the role played by dual quaternions with respect to SE_3 . More precisely, \mathbb{K} is the even subalgebra of the Clifford algebra for SE_2 ; see for example [26, Section 9.2]. A polynomial with coefficients in \mathbb{K} therefore describes a family of direct isometries, which we call a rational motion. Consequently, we refer to such polynomials as motion polynomials; they are the two-dimensional analogue of the motion polynomials introduced in [10], and they are the key for turning our geometric problem into an algebraic one. In fact, we show that linear motion polynomials represent motions constrained by a revolute or prismatic joint. Using the properties of the algebra \mathbb{K} , we then deduce that a factorization of a motion polynomial into linear factors gives rise to a decomposition of the corresponding rational motion into simpler ones, constrained by revolute or prismatic joints.

In Section 4, we give necessary and sufficient criteria for the existence of a factorization of a motion polynomial. Not every motion polynomial admits such a factorization. However, the correspondence between motion polynomials and rational motions is not one-to-one: for every rational motion, there is a whole equivalence

class of motion polynomials. Moreover, the motions we are dealing with are special, since they admit bounded orbits. Once we restrict to such bounded rational motions, we can prove that the equivalence class of every bounded rational motion contains a motion polynomial admitting a factorization into linear polynomials of revolute type. We provide an algorithm for computing the smallest factorizable polynomial in an equivalence class and one of its factorizations; see Theorem 4.15. In this context, we would like to refer to the preprint [21], where this result is used to show the existence of factorizations, but for motion polynomials corresponding to rational motions in SE_3 .

At the beginning of Section 6, we construct from a factorization a linkage—an open chain—such that the given rational motion can be realized as the relative motion of the last link of the chain with respect to the first one. Open chains have high mobility, so we have to constrain our linkage such that it performs only the motion we are interested in. The technique we employ, called flip procedure, is introduced in Section 5. In the remaining part of Section 6, we exploit the properties of flips and propose our main algorithm; see Theorem 6.6.

In Section 7, we address the problem of self-collisions. For arbitrary linkages, this is a challenging problem [5, Section 9.3]. Here, for the first time in the paper, we take into account how linkages are physically realized. We show that self-collisions can be efficiently detected for linkages obtained by our algorithm when links are realized by bars. If, instead, we allow links of different shapes, we describe a construction showing that it is possible to realize these linkages without collisions. This addresses the open problems raised in [6, Open Problem 3.2] and [23, Section 2.3].

A popular formulation of Kempe’s Theorem states that “*There is a linkage that signs your name*”. Inspired by this, in Section 8, we give an example of a linkage drawing a calligraphic letter.

All algorithms described in the paper have been implemented by the second-named author in the computer algebra system *Mathematica*. The source code, an expository notebook, and animations for the main examples are available for free as electronic supplementary material [17].

Here is an explanation for the fact that this contribution has six authors: we discussed the problem of constructing linkages following a planar rational motion in our research seminar at RICAM. After some time, the discussions and partial solutions became the main topic of the seminar. The final result is a product of the symbolic computation group, where each participant contributed according to his/her own background and special skills (algebraic geometry, combinatorics, kinematics, etc.).

2. LINKAGES

In this section, we define a mathematical model for kinematic objects known as *linkages*. A linkage is a device constituted by rigid bodies, called *links*, connected by *joints*, which restrict the relative position of two neighboring links. In this paper we focus on *planar* linkages with *revolute joints*, namely linkages for which all links move in parallel planes and whose joints allow only rotations around a point. In our model we will not be concerned with the shape of the links—the only exception will be Section 7—and we will suppose that joints are the only constraints for the motion of the links. Keeping this in mind, we can represent a linkage by a graph, whose vertices correspond to links, and where two vertices are connected by an edge

if and only if the corresponding links are connected by a joint (for an example, see Figure 1). We call two links *neighboring* if they are connected by a joint, and they correspond to pairs of vertices connected by an edge.

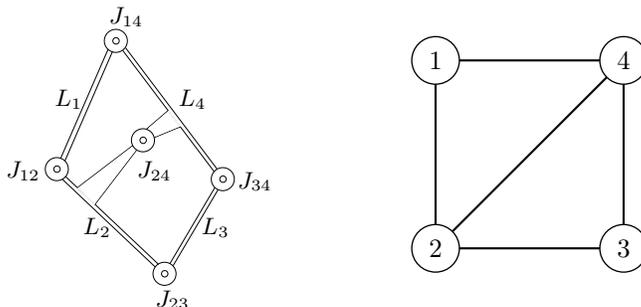


FIGURE 1. A picture of a linkage (on the left) and the corresponding graph (on the right): the links L_1, \dots, L_4 correspond to vertices, while the joints J_{12}, \dots, J_{24} correspond to edges.

Since the graph does not encode how the joints constrain the motion of the links, we need to add some extra information in order to faithfully represent a linkage.

We precede introducing this extra data with a discussion on how to describe the configuration of a linkage, namely, the positions of all the links. There are essentially two ways to indicate the position of a link: we can give either its *absolute position* with respect to some frame of reference, or its *relative position* with respect to some other link. We describe these (absolute/relative) positions by means of isometries.

Notation. We denote by SE_2 the group of *direct isometries* of \mathbb{R}^2 , i.e., maps that preserve distances and orientation (of the standard basis of \mathbb{R}^2):

$$SE_2 = \{ \sigma: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \sigma \text{ is an isometry, } \det \sigma = 1 \}.$$

We fix an *initial configuration* of the linkage. Then there is a unique isometry in SE_2 that takes a certain link from its initial position to the actual one, and that determines the absolute position of the link (see Figure 2). The relative position of a link L_2 with respect to a link L_1 is defined in terms of absolute positions: if σ_1 and σ_2 are the isometries giving the absolute position of L_1 and L_2 respectively, then the relative position of L_2 with respect to L_1 is given by $\sigma_2 \circ \sigma_1^{-1}$. In this way, we have the relation (where \circ denotes composition):

$$\text{rel. pos. of } L_3 \text{ w.r.t. } L_1 = (\text{rel. pos. of } L_3 \text{ w.r.t. } L_2) \circ (\text{rel. pos. of } L_2 \text{ w.r.t. } L_1).$$

If L_1 and L_2 are neighboring links, i.e., if they are connected by a joint, then the relative position of L_2 w.r.t. L_1 is a rotation around a fixed point P_{12} . The point P_{12} stays always the same for any configuration of the linkage, and depends only on the joint J_{12} that connects L_1 and L_2 . Hence relative positions of two neighboring links are elements of a subgroup of SE_2 that is completely specified by a single point, the center of the rotation. Together with the previous considerations, this justifies the following definition for a linkage.

Definition 2.1. A *linkage* with revolute joints is a connected undirected graph $G = (V, E)$, together with a map $\rho: E \longrightarrow \mathbb{R}^2$, such that G does not have self-loops, i.e., all edges connect different vertices. The elements of V are called *links*,



FIGURE 2. The initial position of the link L_2 (on the left) and an arbitrary position (on the right): the isometry representing the absolute position of L_2 on the right is the one taking it from its initial (dashed) position to the actual one.

while the elements of E are called *joints*. For a joint $e \in E$, the point $\rho(e)$ is called the *center of rotation* of e . In the following, we will always assume that V is of the form $\{1, \dots, n\}$ and that elements of E are given by unordered pairs $\{i, j\}$ of elements $i, j \in V$.

Remark 2.2. An implementation of a linkage, where links are realized by line segments between the centers of rotation, also looks like a graph. Note that this graph is not the same as the one in Definition 2.1, rather it is its dual.

Following our previous discussion, we model relative positions of two neighboring links by rotations around a fixed point. We start by defining “virtual” relative positions: the name indicates that they do not take into account all the constraints imposed by the linkage.

Definition 2.3. Let $L = (G, \rho)$ be a linkage, and let $i, j \in V$ be links connected by a joint $e \in E$. The *set of virtual relative positions* $\text{VRP}(i, j)$ of the link i with respect to the link j is the subgroup of SE_2 of rotations around the point $\rho(e)$. Notice that $\text{VRP}(i, j) = \text{VRP}(j, i)$.

We define a *configuration* of a linkage L as a collection of virtual relative positions of links, satisfying the following conditions: if $(i, h_1), (h_1, h_2), \dots, (h_s, i)$ is a *directed cycle* in G , namely a sequence of pairs of links connected by a joint, starting and ending at the same link, then the composition $\sigma_{i, h_1} \circ \dots \circ \sigma_{h_s, i}$, which gives the relative position of i with respect to itself, should be the identity.

Definition 2.4. The *set of configurations* of a linkage L is defined to be

$$\text{Conf}(L) = \left\{ (\sigma_{k,l}) \in \prod_{\{i,j\} \in E} \text{VRP}(i,j) \times \text{VRP}(j,i) : \text{for every directed cycle } (i, h_1), (h_1, h_2), \dots, (h_s, i) \text{ in } G, \text{ we have } \sigma_{i, h_1} \circ \dots \circ \sigma_{h_s, i} = \text{id} \right\},$$

where \prod denotes the Cartesian product so that $(\sigma_{k,l})$ is a tuple of size $2|E|$.

This definition of configuration space is the main difference between our model and the ones appearing in the literature [5, 12, 13, 15, 24].

Remark 2.5. The cycle conditions imposed in the definition of $\text{Conf}(L)$ have the following consequence: suppose that $i, j \in V$ are two links connected by a joint. Then $(i, j), (j, i)$ is a directed cycle from i to i . Therefore if $\Sigma = (\sigma_{k,l})$ is a

configuration of L , the cycle condition imposes that $\sigma_{i,j} \circ \sigma_{j,i} = \text{id}$, implying that $\sigma_{i,j} = \sigma_{j,i}^{-1}$. Hence for every linkage $L = (G, \rho)$, the projection

$$\prod_{\{i,j\} \in E} \text{VRP}(i,j) \times \text{VRP}(j,i) \longrightarrow \prod_{\substack{\{i,j\} \in E \\ i < j}} \text{VRP}(i,j)$$

is a bijection when restricted to $\text{Conf}(L)$ (and an isomorphism if we consider the projective structure on $\text{Conf}(L)$ we will define soon), and similarly for every projection that forgets exactly one among each pair (i,j) and (j,i) . Still, we chose the above definition for $\text{Conf}(L)$ because it does not fix an orientation of the edges, allowing us to deal with arbitrary directed paths (see Definition 2.7).

We notice that for every $i, j \in V$ connected by a joint, the subgroup $\text{VRP}(i,j)$ can be set-theoretically identified with the real projective line $\mathbb{P}_{\mathbb{R}}^1$. Under this identification, every cycle condition imposed in Definition 2.4 becomes a closed condition in the Zariski topology, since it is given by multihomogeneous polynomials. In this way $\text{Conf}(L)$ acquires the structure of a projective subvariety of $(\mathbb{P}_{\mathbb{R}}^1)^{2|E|}$.

Definition 2.6. We define the *mobility* of a linkage L to be the dimension of the configuration space $\text{Conf}(L)$ as a projective subvariety of $(\mathbb{P}_{\mathbb{R}}^1)^{2|E|}$.

So far we only took into account the relative position of two neighboring links. For our purposes, namely to construct a linkage which follows a prescribed motion, we need to take one link (the “base”) as fixed, and consider the relative positions of all the other links with respect to the base.

Definition 2.7. Let $L = (G, \rho)$ be a linkage. Let $\Sigma \in \text{Conf}(L)$ and let $i, j \in V$ be links. Let $(i, h_1), (h_1, h_2), \dots, (h_s, j)$ be a directed path in G from i to j —which exists, since by Definition 2.1 the graph G is connected. Then we define the *relative position of j with respect to i in the configuration $\Sigma = (\sigma_{k,l})$* as

$$\text{RP}(i, j, \Sigma) = \sigma_{i, h_1} \circ \dots \circ \sigma_{h_s, j} \in \text{SE}_2.$$

Notice that, because of the cycle condition, this definition is independent of the chosen path. We define the *set of relative positions of j with respect to i* to be

$$\text{RP}(i, j) = \{\text{RP}(i, j, \Sigma) : \Sigma \in \text{Conf}(L)\} \subseteq \text{SE}_2.$$

3. MOTION POLYNOMIALS

As mentioned in Section 1, our goal is to reduce the main problem to an algebraic one. We first introduce an algebraic setting for manipulating isometries, consisting in a noncommutative \mathbb{R} -algebra \mathbb{K} whose multiplication corresponds to the group operation in SE_2 . This is an instance of a general construction that allows to associate a Clifford algebra to each group of isometries SE_n ; see for example [26, Section 9.2]. Then we define the main object of our work, motion polynomials, as polynomials over \mathbb{K} .

We start by embedding SE_2 in the real projective space $\mathbb{P}_{\mathbb{R}}^3$, with coordinates x_1, x_2, y_1, y_2 , as the open subset

$$\mathcal{U} = \mathbb{P}_{\mathbb{R}}^3 \setminus \{x_1^2 + x_2^2 = 0\}.$$

Hence \mathcal{U} is the complement of the line $x_1 = x_2 = 0$. We do not report the explicit equations giving the embedding, but instead describe the action of an element

$(x_1 : x_2 : y_1 : y_2) \in \mathcal{U}$ on a point $(x, y) \in \mathbb{R}^2$:

$$(3.1) \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{x_1^2 + x_2^2} \left[\begin{pmatrix} x_1^2 - x_2^2 & -2x_1x_2 \\ 2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1y_1 - x_2y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix} \right].$$

Moreover, the group operation in SE_2 becomes a bilinear map given by:

$$(3.2) \quad \begin{aligned} (x_1 : x_2 : y_1 : y_2) \cdot (x'_1 : x'_2 : y'_1 : y'_2) &= (x_1x'_1 - x_2x'_2 : x_1x'_2 + x_2x'_1 : \\ & x_1y'_1 + x_2y'_2 + y_1x'_1 - y_2x'_2 : \\ & x_1y'_2 - x_2y'_1 + y_1x'_2 + y_2x'_1), \end{aligned}$$

where $(x_1 : x_2 : y_1 : y_2)$ and $(x'_1 : x'_2 : y'_1 : y'_2)$ represent two direct isometries $\sigma, \sigma' \in \text{SE}_2$, respectively.

For an easier handling of the multiplication in this embedding of SE_2 , we introduce the following notation: we write a representative $(x_1 : x_2 : y_1 : y_2)$ of an element of SE_2 as a pair $(z, w) \in \mathbb{C}^2$, where $z = x_1 + \iota x_2$ and $w = y_1 + \iota y_2$ (here ι is the imaginary unit). Then Equation (3.2) can be rewritten concisely as

$$(3.3) \quad (z, w) \cdot (z', w') = (z z', \bar{z} w' + z' w)$$

where z, z', w and w' are multiplied as complex numbers, and the bar $(\bar{\cdot})$ is complex conjugation. We can go further, by writing a pair (z, w) in the form $z + \eta w$, and by postulating that η satisfies the two relations:

$$z \eta = \eta \bar{z} \quad \text{for all } z \in \mathbb{C} \quad \text{and} \quad \eta^2 = 0.$$

Then the multiplication we obtain is exactly the one described in Equation (3.3):

$$(z + \eta w) \cdot (z' + \eta w') = z z' + \eta (\bar{z} w' + z' w).$$

Definition 3.1. We define the \mathbb{R} -algebra

$$\mathbb{K} = \mathbb{C}[\eta]/(\eta^2, \iota\eta + \eta\iota).$$

Based on the previous discussions, we can identify elements of \mathbb{K} with elements of SE_2 . In this way the projective space $\mathbb{P}_{\mathbb{R}}^3$ in which we embed SE_2 can be thought as the projectivization $\mathbb{P}(\mathbb{K})$ of \mathbb{K} , considered as an \mathbb{R} -vector space.

Notice that the algebra \mathbb{K} is constructed in such a way that its multiplication is a lift of the group operation of SE_2 : this means that if two isometries $\sigma_1, \sigma_2 \in \text{SE}_2$ are represented by $k_1, k_2 \in \mathbb{K}$, then $\sigma_1 \circ \sigma_2$ is represented by $k_1 \cdot k_2$. More precisely, one can prove that \mathbb{K} is isomorphic to the even subalgebra $\mathcal{C}^+(0, 2, 1)$ of the Clifford algebra $\mathcal{C}(0, 2, 1)$ of SE_2 ; see [26, Section 9.2].

Remark 3.2. Looking at the construction of the algebra \mathbb{K} , we notice that the condition $x_1^2 + x_2^2 \neq 0$ for points in $\mathbb{P}_{\mathbb{R}}^3$ representing isometries in SE_2 becomes $z \neq 0$ when we consider elements of \mathbb{K} . Moreover, the identity isometry is represented in $\mathbb{P}_{\mathbb{R}}^3$ by the point $(1 : 0 : 0 : 0)$, hence by any purely real element of \mathbb{K} , namely elements of the form $z + \eta w$ with $z \in \mathbb{R}$ and $w = 0$. From this we see that, given $k = z + \eta w \in \mathbb{K}$ representing an isometry $\sigma \in \text{SE}_2$, then $k' = \bar{z} - \eta w$ represents σ^{-1} , since the product $k k'$ equals $|z|^2$, which is purely real.

The following lemma characterizes simple subgroups of SE_2 in a geometric fashion: it shows that both translational and revolute motions correspond to lines in $\mathbb{P}_{\mathbb{R}}^3$.

Lemma 3.3. *Let $\ell \subseteq \mathbb{P}_{\mathbb{R}}^3$ be a projective line passing through the point $(1 : 0 : 0 : 0)$, and define $\ell_{\mathcal{U}} = \ell \cap \mathcal{U}$. Let $X = \ell \cap \{x_1 = x_2 = 0\}$. Then:*

- (1) if X has cardinality 1, then $\ell_{\mathcal{U}}$ corresponds to a subgroup of SE_2 that consists of all translations along a fixed common direction;
- (2) if X is empty, then $\ell_{\mathcal{U}}$ corresponds to a subgroup of SE_2 that consists of all rotations around a fixed common point.

Proof. We analyze the two cases separately.

Ad (1): By hypothesis we have that $(1 : 0 : 0 : 0) \in \ell$ and $(0 : 0 : a : b) \in X \subset \ell$ for some $a, b \in \mathbb{R}$, not both zero. Hence the line ℓ admits the parametrization:

$$(\lambda : 0 : a\mu : b\mu), \quad \text{for } (\lambda : \mu) \in \mathbb{P}_{\mathbb{R}}^1.$$

Plugging this parametrization into Equation (3.1), we see that the elements of $\ell_{\mathcal{U}}$ are translations by the vector $\mu/\lambda(a, b)$.

Ad (2): Again by hypothesis we have that $(1 : 0 : 0 : 0) \in \ell$; moreover there exists a point in ℓ which is of the form $(0 : a : b : c)$. Since X is assumed to be empty, we have $a \neq 0$, and hence the line ℓ admits the parametrization:

$$(\lambda : a\mu : b\mu : c\mu), \quad \text{for } (\lambda : \mu) \in \mathbb{P}_{\mathbb{R}}^1.$$

Equation (3.1) is now used to compute the fixed points of an arbitrary element of ℓ . A direct calculation reveals that we always obtain the point $(-c/2a, b/2a)$, which is independent of λ and μ . Thus every element of $\ell_{\mathcal{U}}$ represents a rotation around that point. \square

The description of revolutions around a point given by Lemma 3.3 enables us to compute effectively the configuration curve of a linkage, as shown in Example 3.4.

Example 3.4. Let us consider the linkage L whose graph is depicted in Figure 3. The linkage L is given by 4 links and 4 joints, and the map ρ is determined by:

$$\begin{aligned} \rho(1, 2) &= u_1 = (0, -3/2), \\ \rho(2, 3) &= u_3 = (-3/4, 9/2), \\ \rho(3, 4) &= u_4 = (-1, 13/2), \\ \rho(4, 1) &= u_2 = (-1/4, 1/2). \end{aligned}$$

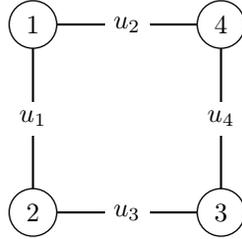


FIGURE 3. Graph of a linkage constituted by 4 links and 4 joints, a so-called closed $4R$ -linkage. The edges are labeled by points u_i in \mathbb{R}^2 that are the images of the edges under the map ρ .

We compute the configuration set of L and its dimension as a projective variety. Since we have 4 joints, it will be a projective subvariety of $(\mathbb{P}_{\mathbb{R}}^1)^4$ (here, rather than $\text{Conf}(L)$, we are considering one of its projections, but as noticed in Remark 2.5 we get an isomorphic object). In order to do so, we use the structure of projective lines of the subgroups of SE_2 of rotations around a fixed point ensured by Lemma 3.3.

One can explicitly compute a parametrization for each line $\ell_i \subseteq \mathbb{P}_{\mathbb{R}}^3$ corresponding to rotations around u_i for all $i \in \{1, \dots, 4\}$, obtaining:

center	parametrization of line
$(0, -3/2)$	$(\lambda_1 : \mu_1 : -3\mu_1 : 0)$
$(-3/4, 9/2)$	$(\lambda_3 : 2\mu_3 : 18\mu_3 : 3\mu_3)$
$(-1, 13/2)$	$(\lambda_4 : \mu_4 : 13\mu_4 : 2\mu_4)$
$(-1/4, 1/2)$	$(\lambda_2 : 2\mu_2 : 2\mu_2 : \mu_2)$

where $(\lambda_i : \mu_i) \in \mathbb{P}_{\mathbb{R}}^1$ for all $i \in \{1, \dots, 4\}$. Notice that, in accordance with Lemma 3.3, none of the lines ℓ_i intersects the line $\{x_1 = x_2 = 0\}$. Recall from Definition 2.4 that, in this case, a configuration for L is a 4-tuple $(\sigma_{1,2}, \sigma_{2,3}, \sigma_{3,4}, \sigma_{4,1})$ of direct isometries that satisfies the cycle condition $\sigma_{1,2} \circ \sigma_{2,3} \circ \sigma_{3,4} \circ \sigma_{4,1} = \text{id}$. Each of the isometries $\sigma_{k,l}$ gives a point on one line ℓ_i , and in our projective model of SE_2 composition corresponds to multiplication according to Equation (3.2). If we perform these computations, we see that the composition $\sigma_{1,2} \circ \sigma_{2,3} \circ \sigma_{3,4} \circ \sigma_{4,1}$ corresponds to a point $(F_1 : F_2 : F_3 : F_4) \in \mathbb{P}_{\mathbb{R}}^3$, where all F_i are multihomogeneous polynomials in the variables $(\lambda_1 : \mu_1), \dots, (\lambda_4 : \mu_4)$ of multidegree $(1, 1, 1, 1)$. Noticing that the identity element of SE_2 is represented by the point $(1 : 0 : 0 : 0) \in \mathbb{P}_{\mathbb{R}}^3$, one realizes that the cycle condition is equivalent to

$$\text{rk} \begin{pmatrix} F_1 & F_2 & F_3 & F_4 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 1.$$

Hence, as a subvariety of $(\mathbb{P}_{\mathbb{R}}^1)^4$, the configuration set $\text{Conf}(L)$ is given by the zero set of the polynomials F_2, F_3, F_4 . A computer algebra computation shows that this is a one-dimensional variety with two components.

We introduce now one of the main concepts of this paper, namely the notion of *motion polynomial*. Intuitively, a motion can be described as a curve in the space of direct isometries. In our case, we want such a curve to be defined by a rational parametrization.

Definition 3.5. A *rational motion* is a map $\phi: \mathbb{R} \rightarrow \mathbb{P}_{\mathbb{R}}^3$ given by four polynomials $X_1, X_2, Y_1, Y_2 \in \mathbb{R}[t]$ such that $X_1^2 + X_2^2$ is not identically zero.

Notice that, as a function between algebraic varieties, a rational motion is in particular a rational map, not defined on the set of common zeros of the polynomials X_1, X_2, Y_1 and Y_2 . Moreover, taking into account the projective model of SE_2 we introduced, the condition on $X_1^2 + X_2^2$ ensures that for all but finitely many $t \in \mathbb{R}$ we have $\phi(t) \in \text{SE}_2$.

Definition 3.6. Let ϕ be a rational motion given by (X_1, X_2, Y_1, Y_2) . The polynomial $P(t) = Z(t) + \eta W(t) \in \mathbb{K}[t]$, where $Z = X_1 + \iota X_2$ and $W = Y_1 + \iota Y_2$, is called a *motion polynomial* encoding the motion ϕ . The polynomials Z and W are respectively called the *primal* and *secondary part* of P , denoted by $\text{pp}(P)$ resp. $\text{sp}(P)$.

Next we want to connect motion polynomials with rational curves. For this purpose it is helpful to rephrase the action of SE_2 on \mathbb{R}^2 shown in Equation (3.1) in the following way: we identify a point $(x, y) \in \mathbb{R}^2$ with the element $u = x + \iota y \in \mathbb{C}$; if $\sigma \in \text{SE}_2$ is represented by an element $z + \eta w \in \mathbb{K}$, then the action of σ on u is given by

$$u^{z+\eta w} = \frac{u z^2 + z w}{|z|^2}.$$

Using this formulation of the action, it is easy to prove the following result.

Proposition 3.7. *Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}^2$ be a rational parametrization of a real curve, which means that φ is of the form*

$$\varphi(t) = \left(\frac{f(t)}{h(t)}, \frac{g(t)}{h(t)} \right)$$

for some real polynomials f, g and h . Then the orbit of the origin under the motion given by the motion polynomial $P = h + \eta(f + ig)$ is exactly the image of φ .

Let P_1 and P_2 be two motion polynomials. Because of the algebraic properties of \mathbb{K} , the polynomial $P = P_1 P_2$ defines a motion that is the composition of the motions determined by P_1 and P_2 . More precisely, for every $t \in \mathbb{R}$, the isometry $P(t)$ is the composition of the isometries $P_1(t)$ and $P_2(t)$. This shows how important the factorization of a motion polynomial is in our framework: it provides a decomposition of a motion into simpler ones.

We saw in Lemma 3.3 that rational motions whose image is a line in $\mathbb{P}_{\mathbb{R}}^3$ are translational motions or revolutions. Hence linear motion polynomials encode this kind of motions. It follows that, if we are able to factor a motion polynomial into linear ones, then we can decompose a rational motion into revolutions and translational motions. Later we will consider a suitable subclass of motion polynomials, so that only the first situation occurs.

Remark 3.8. Let $\phi: \mathbb{R} \rightarrow \mathbb{P}_{\mathbb{R}}^3$ be a rational motion and let $P(t) \in \mathbb{K}[t]$ be the corresponding motion polynomial, so $P = Z + \eta W$ with $Z, W \in \mathbb{C}[t]$. We notice that, although Z and W have coefficients in \mathbb{C} , the polynomial P gives by definition a real curve in $\mathbb{P}_{\mathbb{R}}^3$, namely a curve described by real equations. Therefore, if $R \in \mathbb{R}[t]$ is a nonzero real polynomial, then one can check that $RP \in \mathbb{K}[t]$ provides the same curve in $\mathbb{P}_{\mathbb{R}}^3$ as P .

Example 3.9. The rational motion given by the polynomial $t + \eta$ is a vertical translational motion. Indeed, its image in $\mathbb{P}_{\mathbb{R}}^3$ is parametrized by $(\lambda : 0 : \mu : 0)$, so that we are in case 1 of Lemma 3.3. In contrast, the motion polynomial $P(t) = t + \iota$ gives a revolution around the origin $(0, 0)$. This is an instance of case 2 of Lemma 3.3, using the parametrization $(\lambda : \mu : 0 : 0)$.

Example 3.10. Consider the following product of two linear motion polynomials:

$$\underbrace{(t + \iota)}_{\substack{\text{rotation around} \\ \text{a point}}} \cdot \underbrace{(t - \iota + \eta)}_{\substack{\text{rotation in the opposite} \\ \text{sense, around another point}}} = \underbrace{(t^2 + 1) + \eta(t - \iota)}_{\substack{\text{this is a translation, since} \\ \text{there is no imaginary part in } t^2 + 1}}$$

In this case the translational vector is given by $\frac{1}{t^2+1}(t, -1)$. This vector changes with t and describes a circle. Hence we get a *circular translation* (see Figure 4).

In Example 3.10, we see that the factorization of a motion polynomial into linear polynomials provides a decomposition of the described motion into revolutions. However, in Example 3.11 we see that this is not always possible.

Example 3.11. Let us consider an *elliptic translational motion*:

$$P(t) = (t^2 + 1) + \eta(at - bt), \quad a, b \in \mathbb{R}.$$

We try to factor P into two linear polynomials $P_1, P_2 \in \mathbb{K}[t]$. First notice that the primal parts of P_1 and P_2 should be factors of $(t^2 + 1)$, so they have to be of the

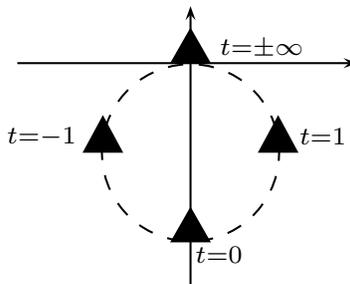


FIGURE 4. The trace of the point $(0, 0)$ under the circular translation given by the motion polynomial $(t^2 + 1) + \eta(t - \iota)$ is depicted via a dashed line. The action of the motion on the black triangle highlights the fact that we have a purely translational motion.

form $t \pm \iota$. Thus we have only two possibilities:

$$\begin{cases} \text{pp}(P_1) = t + \iota \\ \text{pp}(P_2) = t - \iota \end{cases} \quad \begin{cases} \text{pp}(P_1) = t - \iota \\ \text{pp}(P_2) = t + \iota \end{cases}$$

By a direct computation one can prove that none of those two choices gives a factorization if $a \neq b$, thus P cannot be factored into linear polynomials.

On the other hand, we show now that if we multiply P by a real polynomial R we can achieve a factorization. Recall that, by Remark 3.8, the polynomials P and RP describe the same motion. In this case we take $R = t^2 + 1$. Hence we need four linear polynomials P_1, \dots, P_4 to factorize RP , and again $\text{pp}(P_i) = t \pm \iota$. We make the following ansatz:

$$(3.4) \quad \begin{aligned} P_1 &= (t - \iota) + \eta w_1, & P_3 &= (t + \iota) + \eta w_3, \\ P_2 &= (t + \iota) + \eta w_2, & P_4 &= (t - \iota) + \eta w_4. \end{aligned}$$

Imposing $P_1 \cdots P_4 = RP$ gives a linear system in the w_i , namely:

$$(3.5) \quad \begin{cases} w_1 + w_2 + w_3 + w_4 = a, \\ w_1 + w_2 - w_3 - w_4 = b. \end{cases}$$

Each solution of the system 3.5 gives rise to a factorization of the polynomial RP . If, instead, we take the following ansatz:

$$\begin{aligned} P_1 &= (t + \iota) + \eta w_1, & P_3 &= (t + \iota) + \eta w_3, \\ P_2 &= (t - \iota) + \eta w_2, & P_4 &= (t - \iota) + \eta w_4. \end{aligned}$$

one can check that the corresponding linear system does not admit any solution.

In the next section, we consider the factorization problem in a more systematic way. We will see that, once we restrict to a certain class of motion polynomials, all situations can be treated as we did in Example 3.11: in general a motion polynomial P cannot be factored into linear polynomials in $\mathbb{K}[t]$, but it is always possible to find a real polynomial R such that RP can be factored.

We end this section by defining precisely what we mean by saying that a linkage realizes a given rational motion. Recall from Definition 2.7 that, given two links i and j of a linkage L , we can define the set $\text{RP}(i, j)$ of relative positions of the link j with respect to the link i , and this is a subset of SE_2 . Furthermore, recall that the

set $\text{Conf}(L)$ of configurations of L has the structure of a real projective variety. We are going to use some concepts and results coming from real algebraic geometry:

- (i) every projective space $\mathbb{P}_{\mathbb{R}}^n$ is a real affine variety, namely there exists an embedding $\xi: \mathbb{P}_{\mathbb{R}}^n \hookrightarrow \mathbb{R}^k$ for some $k \in \mathbb{N}$, and $\xi(\mathbb{P}_{\mathbb{R}}^n)$ is the zero set of a collection of real polynomials; see for example [3, Theorem 3.4.4];
- (ii) a subset $X \subseteq \mathbb{R}^k$ is called *semialgebraic* if it can be described as the set of points satisfying a disjunction of conjunctions of polynomial equalities and inequalities; see for example [3, Definition 2.1.4];
- (iii) a subset $X \subseteq \mathbb{P}_{\mathbb{R}}^n$ is called semialgebraic if its image under any embedding $\xi: \mathbb{P}_{\mathbb{R}}^n \hookrightarrow \mathbb{R}^k$ as in (i) is a semialgebraic set in the sense of (ii).

Lemma 3.12. *Let L be a linkage and let i, j be links of L . Then, under the projective embedding of SE_2 , the set $\text{RP}(i, j)$ becomes a semialgebraic subset of SE_2 .*

Proof. Let $(i, h_1), \dots, (h_s, j)$ be any directed path from i to j . Consider the map $F: \text{Conf}(L) \rightarrow \text{SE}_2$ sending a configuration $\Sigma = (\sigma_{k,l})$ to $\sigma_{i,h_1} \circ \dots \circ \sigma_{h_s,j}$. One can check that $\text{RP}(i, j)$ coincides with $F(\text{Conf}(L))$. Moreover, if we write the map F in terms of the projective coordinates of $\text{Conf}(L)$ and SE_2 , then F is given by real polynomials. Hence, because of a general result in real algebraic geometry (see for example [3, Theorem 1.4.2]), the set $\text{RP}(i, j)$ is semialgebraic. \square

In the following definition, and later in this paper, the word ‘‘component’’ will always stand for ‘‘irreducible component’’ with respect to the Zariski topology.

Definition 3.13. Let L be a linkage and let $\phi: \mathbb{R} \rightarrow \mathbb{P}_{\mathbb{R}}^3$ be a rational motion. We say that

- (1) L *weakly realizes* the motion ϕ if there exist links i and j of L such that $\phi(\mathbb{R}) \subseteq \text{RP}(i, j)$.
- (2) L *strongly realizes* the motion ϕ if there exist links i and j of L such that $\phi(\mathbb{R}) \subseteq \text{RP}(i, j)$ and $\overline{\phi(\mathbb{R})}$ is a component of $\text{RP}(i, j)$, where $\overline{(\cdot)}$ denotes the Zariski closure in $\text{RP}(i, j)$.

Remark 3.14. Notice that in case 2 of Definition 3.13 we allow the given variety of relative positions to have several components, and we ask that only one of them coincides with the Zariski closure of the image of the given rational motion, which is irreducible by construction. This situation occurs in the example given on the right of Figure 5, since the variety of relative positions between base and end effector has two components (see Lemma 5.5).

4. FACTORIZATION OF MOTION POLYNOMIALS

Motivated by the considerations we made in the previous section, we address the problem of factorizing a motion polynomial $P = Z + \eta W \in \mathbb{K}[t]$ into linear factors. We restrict our attention to monic polynomials, i.e., to polynomials P whose leading coefficient is 1, which implies that $\deg W < \deg Z$.

Given a monic polynomial $P \in \mathbb{K}[t]$, our ultimate goal is to write it in the form $P = P_1 \cdots P_n$, where $n = \deg P$ and $\deg P_i = 1$ for all i . If $P = Z + \eta W$ and $P_i = Z_i + \eta W_i$, then $Z = Z_1 \cdots Z_n$. Hence the primal part of each P_i has to be one of the linear factors of the factorization of Z over $\mathbb{C}[t]$. We start by a few definitions and results characterizing monic motion polynomials that split into linear factors.

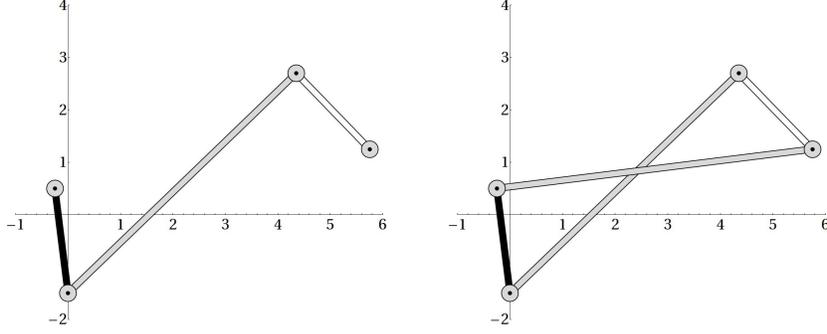


FIGURE 5. Two linkages that both realize the motion given by $(t + \iota - 3\eta) \cdot (t + 2\iota + \eta(18 + 3\iota))$, discussed in Example 3.4. The base is painted in black and the end effector in white. The depicted position is reached at $t = -2$. The linkage on the left has two degrees of freedom and thus weakly realizes the given motion as the relative position between base and end effector. The linkage on the right has mobility one and provides a strong realization of this motion.

Definition 4.1. Let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$. We define the following polynomials:

$$Q_i(\mathbf{z}) = (t - \bar{z}_1) \cdots (t - \bar{z}_{i-1})(t - z_{i+1}) \cdots (t - z_n) \quad \text{for } i \in \{1, \dots, n\}.$$

Hence we have that $\deg Q_i(\mathbf{z}) = n - 1$ for all i .

Remark 4.2. Let $Z \in \mathbb{C}[t]$ and let $\mathbf{z} = (z_1, \dots, z_n)$ be a fixed permutation of the roots of Z . Then the polynomials $Q_i(\mathbf{z})$ have a non-trivial gcd if and only if the polynomial Z admits a pair of complex-conjugate roots α and $\bar{\alpha}$ (this includes the case of multiple real roots).

Lemma 4.3. Let $P = Z + \eta W \in \mathbb{K}[t]$ be a monic motion polynomial, and let $\mathbf{z} = (z_1, \dots, z_n)$ be a fixed permutation of the roots of Z over \mathbb{C} . Then P admits a factorization $P = P_1 \cdots P_n$, where $P_i(t) = (t - z_i) + \eta w_i$ with $w_i \in \mathbb{C}$, if and only if W lies in $\langle Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z}) \rangle_{\mathbb{C}}$, i.e., in the \mathbb{C} -linear span of the polynomials $Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z})$.

Proof. Suppose that $P \in \mathbb{K}[t]$ admits a factorization into linear factors:

$$P(t) = \prod_{i=1}^n (t - z_i + \eta w_i) = \prod_{i=1}^n (t - z_i) + \eta \underbrace{\sum_{k=1}^n \left(\prod_{j=1}^{k-1} (t - \bar{z}_j) \right) \left(\prod_{j=k+1}^n (t - z_j) \right)}_{= Q_k(\mathbf{z})} w_k.$$

Such a factorization exists if and only if we can choose $w_1, \dots, w_n \in \mathbb{C}$ such that $\sum_{k=1}^n w_k Q_k(\mathbf{z})$ matches the prescribed η -part of $P(t) = Z(t) + \eta W(t)$, namely if and only if $W \in \langle Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z}) \rangle_{\mathbb{C}}$. \square

Lemma 4.4. Let $P \in \mathbb{K}[t]$ be a monic motion polynomial and suppose that $Z = \text{pp}(P)$ has no pair of complex-conjugate roots. Then for every permutation $\mathbf{z} = (z_1, \dots, z_n)$ of the roots of Z , the polynomial P admits a factorization $P = P_1 \cdots P_n$ with $\text{pp}(P_i) = t - z_i$ for all i .

Proof. We fix a permutation \mathbf{z} of the roots of Z , and we denote $Q_k = Q_k(\mathbf{z})$ for all k . From Lemma 4.3, we know that $P = Z + \eta W$ admits a factorization if and only if $W \in \langle Q_1, \dots, Q_n \rangle_{\mathbb{C}}$. Clearly, this is always possible (for arbitrary W) if the following matrix $M_n \in \mathbb{C}^{n \times n}$ is non-singular:

$$M_n = \begin{pmatrix} \langle t^0 \rangle Q_1 & \cdots & \langle t^0 \rangle Q_n \\ \langle t^1 \rangle Q_1 & \cdots & \langle t^1 \rangle Q_n \\ \vdots & & \vdots \\ \langle t^{n-1} \rangle Q_1 & \cdots & \langle t^{n-1} \rangle Q_n \end{pmatrix}.$$

Here $\langle t^i \rangle Q_k$ denotes the coefficient of t^i in the polynomial Q_k . Notice that the matrix entries are, up to sign, elementary symmetric polynomials in the z_i and \bar{z}_i . We now exhibit that the determinant of M_n is nonzero under the assumptions on the roots of Z . Indeed, we find that

$$(4.1) \quad \det(M_n) = \prod_{1 \leq i < j \leq n} (\bar{z}_i - z_j),$$

which follows, as a special case, from Lemma 3 in [18] (by substituting $A_k = -z_k$, $B_k = -\bar{z}_{k-1}$, and by extracting the coefficient of $\prod_{i=1}^n X_i^{i-1}$). A similar determinant evaluation is given in [19] where the z_i appear without conjugation. Since $\det(M_n)$ is very much reminiscent of the Vandermonde determinant, it is no surprise that (4.1) can also be proved in an analogous fashion. \square

Corollary 4.5. *Let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ such that $\gcd(Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z})) = 1$. Then*

$$\langle Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z}) \rangle_{\mathbb{C}} = \{W \in \mathbb{C}[t] : \deg(W) < n\}.$$

Proof. The gcd condition implies that \mathbf{z} has no pair of complex-conjugate roots, as noticed in Remark 4.2. The statement is then a direct consequence of Lemma 4.3 and Lemma 4.4. \square

Since the gcd of the polynomials $Q_i(\mathbf{z})$ introduced in Definition 4.1 plays an important role for the factorization of motion polynomials, we make some effort to describe it precisely (Lemma 4.6, Corollary 4.7, Proposition 4.10); for this purpose the combinatorial notion of *matching* will be introduced. At the same time we aim at a nicer characterization of those secondary parts in Lemma 4.3 that ensure the existence of a factorization; the result is presented in Proposition 4.12. To simplify the proofs of Propositions 4.10 and 4.12, we formulate the somewhat technical Lemma 4.9.

Lemma 4.6. *Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta \neq \bar{\alpha}$, and let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $\tilde{\mathbf{z}} = (z_1, \dots, z_{j-1}, \beta, z_j, \dots, z_n) \in \mathbb{C}^{n+1}$ for some $j \in \{1, \dots, n+1\}$. Then the multiplicities of the roots α and $\bar{\alpha}$ in $\gcd(Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z}))$ and in $\gcd(Q_1(\tilde{\mathbf{z}}), \dots, Q_{n+1}(\tilde{\mathbf{z}}))$ are the same.*

Proof. First observe that $Q_k(\mathbf{z})$ and $Q_\ell(\tilde{\mathbf{z}})$ have the same multiplicities of α and $\bar{\alpha}$, where $\ell = k$ if $k < j$ and $\ell = k+1$ if $k \geq j$. On the other hand, the polynomial $Q_j(\tilde{\mathbf{z}})$ has no such counterpart among the $Q_k(\mathbf{z})$, but its multiplicities of α and $\bar{\alpha}$ are greater than or equal to those in its (one or two) neighbors $Q_{j-1}(\tilde{\mathbf{z}})$ and $Q_{j+1}(\tilde{\mathbf{z}})$. The claim follows. \square

Corollary 4.7. *Let $\mathbf{x} \in \mathbb{C}^m$ and $\mathbf{y} \in \mathbb{C}^n$ with $x_i \neq y_j \neq \bar{x}_i$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. If $\mathbf{z} \in \mathbb{C}^{m+n}$ is an arbitrary interlacement of \mathbf{x} and \mathbf{y} , i.e., there are $1 \leq i_1 \leq \dots \leq i_m \leq m+n$ and $1 \leq j_1 \leq \dots \leq j_n \leq m+n$ with $\{i_1, \dots, i_m\} \cap \{j_1, \dots, j_n\} = \emptyset$ such that $z_{i_k} = x_k$, $1 \leq k \leq m$, and $z_{j_k} = y_k$, $1 \leq k \leq n$, then*

$$\gcd(Q_1(\mathbf{z}), \dots, Q_{m+n}(\mathbf{z})) = \gcd(Q_1(\mathbf{x}), \dots, Q_m(\mathbf{x})) \cdot \gcd(Q_1(\mathbf{y}), \dots, Q_n(\mathbf{y})).$$

Proof. This is a direct consequence of Lemma 4.6. \square

Definition 4.8. Let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$. A set

$$M \subseteq \{(i, j) : 1 \leq i < j \leq n \text{ and } z_i = \bar{z}_j\}$$

is called a *matching* of \mathbf{z} if for all $(i_1, j_1), (i_2, j_2) \in M$ we have $i_1 \neq i_2$ and $j_1 \neq j_2$. (Note that for each $\mathbf{z} \in \mathbb{C}^n$ there is a bipartite graph with at most $2n - 2$ vertices such that the matchings in that graph and the matchings of \mathbf{z} are in bijection.)

Lemma 4.9. *Let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, let $G = \gcd(Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z}))$, and let $g = \deg G$. For a matching M of \mathbf{z} define $H(M) := \prod_{(i,j) \in M} (t - z_j)$. Then the following statements hold:*

- (1) *For any matching M of \mathbf{z} we have $H(M) \mid G$.*
- (2) *There is a matching M of \mathbf{z} such that $H(M) = G$.*
- (3) *There are indices $1 \leq i_1 \leq \dots \leq i_{n-g} \leq n$ such that, setting $\tilde{\mathbf{z}} = (z_{i_1}, \dots, z_{i_{n-g}})$, we have $Q_k(\tilde{\mathbf{z}}) = Q_{i_k}(\mathbf{z})/G$ for all $1 \leq k \leq n-g$ and $\gcd(Q_1(\tilde{\mathbf{z}}), \dots, Q_{n-g}(\tilde{\mathbf{z}})) = 1$.*

Proof. For (1) we show that $H(M)$ divides $Q_k(\mathbf{z})$ for any matching M of \mathbf{z} and for any $k \in \{1, \dots, n\}$. Indeed, by rewriting

$$H(M) = \prod_{(i,j) \in M} (t - z_j) = \left(\prod_{\substack{(i,j) \in M \\ j > k}} (t - z_j) \right) \left(\prod_{\substack{(i,j) \in M \\ j \leq k}} (t - \bar{z}_i) \right)$$

the claim follows, since $Q_k(\mathbf{z})$ has precisely the roots $\bar{z}_1, \dots, \bar{z}_{k-1}, z_{k+1}, \dots, z_n$.

For (2) and (3) note that the statements trivially hold for $G = 1$, so we assume now that $G \neq 1$. Using Corollary 4.7 the proof is reduced to the case where $\mathbf{z} \in \{\alpha, \bar{\alpha}\}^n$ for some $\alpha \in \mathbb{C}$. If $\alpha \in \mathbb{R}$ then we have $Q_1(\mathbf{z}) = \dots = Q_n(\mathbf{z}) = (t - \alpha)^{n-1} = G$ and we can choose $M = \{(1, 2), (2, 3), \dots, (n-1, n)\}$ and $i_k = k$ for $1 \leq k < n$. Hence from now on we assume $\alpha \neq \bar{\alpha}$.

Both assertions (2) and (3) are proved in parallel by an inductive argument on the degree of G , so in the following we show how to reduce the given scenario to one with smaller g . The main difficulty here is to pick an index j such that deleting z_j from \mathbf{z} yields a gcd of smaller degree. We distinguish three cases and assume w.l.o.g. that $z_n = \alpha$ (case A) resp. $z_1 = \alpha$ (cases B and C).

Case A: There exists an index j such that $\#\alpha(z_j, \dots, z_n) \leq \#\bar{\alpha}(z_j, \dots, z_n)$ where $\#\alpha$ denotes the number of occurrences of α . In this situation we set $\tilde{\mathbf{z}} = (z_1, \dots, z_{n-1})$ and $\tilde{G} = \gcd(Q_1(\tilde{\mathbf{z}}), \dots, Q_{n-1}(\tilde{\mathbf{z}}))$. By induction hypothesis there exists a matching \tilde{M} for $\tilde{\mathbf{z}}$ such that $H(\tilde{M}) = \tilde{G}$. Because of the assumption on the number of α 's and $\bar{\alpha}$'s, there must be an index $i \in \{j, \dots, n-1\}$ with $z_i = \bar{\alpha}$ that does not occur as the first entry of any pair in \tilde{M} . Hence $M := \tilde{M} \cup \{(i, n)\}$ is a matching for \mathbf{z} and we have $H(M) = (t - \alpha)H(\tilde{M})$. Since by (1) we have

$H(M) \mid G$, we obtain $(t - \alpha)\tilde{G} \mid G$. Next observe that $Q_k(\tilde{\mathbf{z}}) = Q_k(\mathbf{z})/(t - z_n)$ for all $1 \leq k < n$ and hence

$$\tilde{G} = \gcd(Q_1(\tilde{\mathbf{z}}), \dots, Q_{n-1}(\tilde{\mathbf{z}})) = \frac{1}{t - \alpha} \gcd(Q_1(\mathbf{z}), \dots, Q_{n-1}(\mathbf{z})).$$

From the definition of G it follows that $G \mid (t - \alpha)\tilde{G}$, and thus $G = (t - \alpha)\tilde{G} = H(M)$.

Case B: There exists an index j such $\#_\alpha(z_1, \dots, z_j) \leq \#\bar{\alpha}(z_1, \dots, z_j)$. The reasoning in this case is completely analogous to case A, setting $\tilde{\mathbf{z}} = (z_2, \dots, z_n)$.

Case C: Neither case A nor case B applies, which means that for all indices j we have $\#_\alpha(z_1, \dots, z_j) > \#\bar{\alpha}(z_1, \dots, z_j)$ and $\#_\alpha(z_j, \dots, z_n) > \#\bar{\alpha}(z_j, \dots, z_n)$. Note that in this situation we automatically have $z_1 = z_n$, which w.l.o.g. we assumed to be α . Now there exists no entry in \mathbf{z} whose removal would yield a gcd of degree $\deg G - 1$. We circumvent this problem by setting $\tilde{\mathbf{z}} = (z_1, \dots, z_{j-1}, z_{j+2}, \dots, z_n)$ where j is the largest index such that $z_j = \bar{\alpha}$; in particular $j < n$. The rest is analogous to the previous reasoning. As before, we may assume that there exists a matching \tilde{M} of $\tilde{\mathbf{z}}$ with $H(\tilde{M}) = \tilde{G} = \gcd(Q_1(\tilde{\mathbf{z}}), \dots, Q_{n-2}(\tilde{\mathbf{z}}))$. Since $\#_\alpha(z_1, \dots, z_j) > \#\bar{\alpha}(z_1, \dots, z_j)$ there exists an index $i < j$ with $z_i = \alpha$ such that i does not appear as the first entry of any pair in \tilde{M} . In order to extend \tilde{M} to a matching for \mathbf{z} we need some relabeling; for this purpose we define $\tau(a) = a$ if $a < j$ and $\tau(a) = a + 2$ otherwise. Then

$$M := \{(\tau(a), \tau(b)) \mid (a, b) \in \tilde{M}\} \cup \{(i, j), (j, j + 1)\}$$

is a matching for \mathbf{z} and $H(M) = (t - \alpha)(t - \bar{\alpha})H(\tilde{M})$; hence $(t - \alpha)(t - \bar{\alpha})\tilde{G} \mid G$. A simple calculation shows that $Q_k(\tilde{\mathbf{z}}) = Q_k(\mathbf{z})/((t - \alpha)(t - \bar{\alpha}))$ for $k < j$ and that $Q_k(\tilde{\mathbf{z}}) = Q_{k+2}(\mathbf{z})/((t - \alpha)(t - \bar{\alpha}))$ for $k \geq j$. Thus we get

$$\tilde{G} = \frac{1}{(t - \alpha)(t - \bar{\alpha})} \gcd(Q_1(\mathbf{z}), \dots, Q_{j-1}(\mathbf{z}), Q_{j+2}(\mathbf{z}), \dots, Q_n(\mathbf{z})).$$

Again, this is sufficient to conclude that $G = (t - \alpha)(t - \bar{\alpha})\tilde{G} = H(M)$, which concludes the proof. \square

Proposition 4.10. *Let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ and let M be a matching of \mathbf{z} of maximal size, and let $Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z})$ be as in Definition 4.1. Then we have*

$$G := \gcd(Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z})) = \prod_{(i,j) \in M} (t - z_j)$$

(where the gcd is assumed to be a monic polynomial).

Proof. In Lemma 4.9 part (1) it was shown that the product on the right-hand side, which we denoted by $H(M)$, divides G for any matching M . For the other direction we have to argue that the maximality of M implies that $G \mid H(M)$, or equivalently, that $G \nmid H(M)$ implies that M is not maximal. From Lemma 4.9 part (2) it follows that there exists a matching M' of \mathbf{z} such that $H(M') = G$. Note that $|M'| = \deg(G)$. Thus, if $G \nmid H(M)$, which means that $\deg(H(M)) < \deg(G)$, then M is not maximal because $|M| < |M'|$. \square

Example 4.11. Let $Z = (t - \alpha)^r(t - \bar{\alpha})^{r+1}$ for some $\alpha \in \mathbb{C} \setminus \mathbb{R}$. In the following table we consider different permutations $\mathbf{z} \in \{\alpha, \bar{\alpha}\}^{2r+1}$ of the roots of Z ; to each permutation we give the gcd $G = \gcd(Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z}))$ and a matching M of maximal size that witnesses G :

permutation	G	M
$(\alpha, \dots, \alpha, \bar{\alpha}, \dots, \bar{\alpha})$	$(t - \bar{\alpha})^r$	$\{(1, r+1), (2, r+2), \dots, (r, 2r)\}$
$(\bar{\alpha}, \dots, \bar{\alpha}, \alpha, \dots, \alpha)$	$(t - \alpha)^r$	$\{(1, r+2), (2, r+3), \dots, (r, 2r+1)\}$
$(\bar{\alpha}, \alpha, \bar{\alpha}, \alpha, \dots, \alpha, \bar{\alpha})$	$(t - \alpha)^r (t - \bar{\alpha})^r$	$\{(1, 2), (2, 3), \dots, (2r, 2r+1)\}$

The cases displayed above are the extreme ones: indeed, it is easy to see that $r \leq \deg(G) \leq 2r$. Moreover, for any $G = (t - \alpha)^i (t - \bar{\alpha})^j$ with $0 \leq i, j \leq r$ and $i + j \geq r$ there exists a permutation \mathbf{z} that produces exactly this gcd G . These considerations lead to interesting combinatorial questions—e.g., how many permutations of a given set of roots are there that produce a prescribed gcd—which, however, are irrelevant for our purposes.

Proposition 4.12. *Let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $W \in \mathbb{C}[t]$ with $\deg W < n$. Then*

$$W \in \langle Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z}) \rangle_{\mathbb{C}} \iff W \in (Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z})) \cdot \mathbb{C}[t],$$

where $(Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z})) \cdot \mathbb{C}[t]$ is the ideal of $\mathbb{C}[t]$ generated by $Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z})$.

Proof. The claim is equivalent to the following statement:

$$(4.2) \quad \langle Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z}) \rangle_{\mathbb{C}} = (Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z})) \cdot \mathbb{C}[t] \cap \mathbb{C}[t]_{<n},$$

where $\mathbb{C}[t]_{<n}$ denotes the set of complex polynomials of degree less than n . Note that the containment “ \subseteq ” is trivial. Let $G = \gcd(Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z}))$. If $G = 1$, then by Corollary 4.5 we have $\langle Q_1(\mathbf{z}), \dots, Q_n(\mathbf{z}) \rangle_{\mathbb{C}} = \mathbb{C}[t]_{<n}$, which implies Equation (4.2). On the other hand, if G is non-trivial, Equation (4.2) is equivalent to

$$(4.3) \quad \langle \tilde{Q}_1, \dots, \tilde{Q}_n \rangle_{\mathbb{C}} = (\tilde{Q}_1, \dots, \tilde{Q}_n) \cdot \mathbb{C}[t] \cap \mathbb{C}[t]_{<n-g},$$

where $\tilde{Q}_i = Q_i(\mathbf{z})/G$ and $g = \deg G$. By Lemma 4.9 part (3), there exists $\tilde{\mathbf{z}} = (z_{i_1}, \dots, z_{i_{n-g}})$ such that $\gcd(Q_1(\tilde{\mathbf{z}}), \dots, Q_{n-g}(\tilde{\mathbf{z}})) = 1$, and $Q_k(\tilde{\mathbf{z}}) = \tilde{Q}_{i_k}$ for all k . Hence, by Corollary 4.5 we have $\langle Q_1(\tilde{\mathbf{z}}), \dots, Q_{n-g}(\tilde{\mathbf{z}}) \rangle_{\mathbb{C}} = \mathbb{C}[t]_{<n-g}$. This implies Equation (4.3), since $\langle Q_1(\tilde{\mathbf{z}}), \dots, Q_{n-g}(\tilde{\mathbf{z}}) \rangle_{\mathbb{C}} \subseteq \langle \tilde{Q}_1, \dots, \tilde{Q}_n \rangle_{\mathbb{C}}$. \square

Recall that our goal is to decompose a given motion into a sequence of revolutions, by factorizing a corresponding motion polynomial. For this purpose we have to restrict the domain of motion polynomials we are working with: for example, there is no hope to write a motion whose orbits are unbounded as the composition of revolutions, since the orbits of revolutions are always bounded. In particular, if we consider a rational bounded curve, then it can be given by a parametrization $\varphi = (f/h, g/h)$ for some real polynomials f, g and h with $\deg h \geq \max\{\deg f, \deg g\}$. The boundedness of the curve implies that h has no real roots. The limit of $\varphi(t)$ for $t \rightarrow \infty$ gives a point in \mathbb{R}^2 , which can be translated to the origin. Then we even have $\deg h > \max\{\deg f, \deg g\}$. If we further assume that h is monic, then also the motion polynomial $P = h + \eta(f + ig)$ is monic, and by Proposition 3.7 it traces the image of φ . This motivates the following definition.

Definition 4.13. Let $P = Z + \eta W$ be a motion polynomial in $\mathbb{K}[t]$. We say that P is *bounded* if it is monic and if Z does not have real roots. Notice that the motion given by a bounded motion polynomial is a map defined for all $t \in \mathbb{R}$.

Let $P(t) = t - k$ be a bounded linear motion polynomial. Then the motion described by P admits exactly one fixed point Q (independent of t), since P represents a revolution around Q (see Lemma 3.3). We start by determining the fixed point (x, y) of an isometry $z + \eta w \in \mathbb{K}$, employing the notation $u = x + iy$ used before:

$$u = u^{z+\eta w} \iff u = \frac{uz^2 + zw}{|z|^2} \iff u = \frac{zw}{|z|^2 - z^2} = \frac{w}{\bar{z} - z}.$$

It follows that for every $t \in \mathbb{R}$ the isometry $t - k$ has the same fixed point as k (in accordance with Lemma 3.3).

Definition 4.14. Let $k = z + \eta w \in \mathbb{K}$ with $z \in \mathbb{C} \setminus \mathbb{R}$, then we denote the fixed point of the isometry represented by k by

$$\chi(k) = \frac{w}{\bar{z} - z}.$$

We are now ready to state and prove the main result of this section:

Theorem 4.15. *Let $P \in \mathbb{K}[t]$ be a bounded motion polynomial. Then there exists a real polynomial $R \in \mathbb{R}[t]$ such that RP can be factored into linear polynomials.*

Proof. If P is of the form $P = S\tilde{P}$ with $S \in \mathbb{R}[t] \setminus \mathbb{R}$ and $\tilde{P} \in \mathbb{K}[t]$, then we can apply the theorem to \tilde{P} obtaining $R \in \mathbb{R}[t]$ such that $R\tilde{P}$ factors; thus also $RS\tilde{P} = RP$ factors. Hence we may assume that P does not have non-constant real factors.

Let $P(t) = Z(t) + \eta W(t)$ and let $\tilde{R} = \gcd(Z, \bar{Z})$. The desired polynomial (of minimal degree) is obtained by $R = \tilde{R} / \gcd(\tilde{R}, W\bar{W})$. Using Lemma 4.3 and Proposition 4.12, the proof is reduced to showing that there exists a permutation of the n roots of RZ (counted with multiplicities) such that the corresponding $G = \gcd(Q_1, \dots, Q_n)$ divides RW . To this end write

$$Z = \prod_{i=1}^h \left((t - \alpha_i)^{r_i} (t - \bar{\alpha}_i)^{s_i} \right), \quad r_i \geq s_i \geq 0, \quad r_i > 0,$$

where $\alpha_i \in \mathbb{C}$ are pairwise conjugate-distinct, i.e., $\alpha_i \neq \alpha_j$ and $\alpha_i \neq \bar{\alpha}_j$ for $i \neq j$. Note that in the special case $s_1 = \dots = s_h = 0$ we get $R = 1$, in agreement with the fact that in this case already P itself factors by Lemma 4.4. On the other hand, we have always $\deg(R) \leq \deg(P)$, and equality is attained when $r_i = s_i$ for all i and $\gcd(Z, W) = 1$.

Define $R_i := (t - \alpha_i)(t - \bar{\alpha}_i)$ and write $\tilde{R} = \prod_{i=1}^h R_i^{s_i}$. Next let u_i and v_i be the multiplicities of α_i and $\bar{\alpha}_i$ in W , i.e.,

$$W = \tilde{W} \cdot \prod_{i=1}^h \left((t - \alpha_i)^{u_i} (t - \bar{\alpha}_i)^{v_i} \right)$$

for some polynomial $\tilde{W} \in \mathbb{C}[t]$ such that $\gcd(\tilde{W}, R_1 \cdots R_h) = 1$. By introducing the quantity $m_i := \min\{s_i, u_i + v_i\}$, the polynomial R can be written as

$$R = \frac{\tilde{R}}{\gcd(\tilde{R}, W\bar{W})} = \prod_{i=1}^h R_i^{s_i - m_i},$$

and hence

$$RZ = \prod_{i=1}^h \left((t - \alpha_i)^{r_i + s_i - m_i} (t - \bar{\alpha}_i)^{2s_i - m_i} \right),$$

$$RW = \widetilde{W} \cdot \prod_{i=1}^h \left((t - \alpha_i)^{u_i + s_i - m_i} (t - \bar{\alpha}_i)^{v_i + s_i - m_i} \right).$$

In order to construct an admissible permutation, the roots $\alpha_i, \bar{\alpha}_i$ are arranged as follows:

$$\left(\underbrace{\bar{\alpha}_i, \dots, \bar{\alpha}_i}_{s_i - \min\{s_i, v_i\}}, \underbrace{\alpha_i, \dots, \alpha_i}_{r_i + s_i - m_i}, \underbrace{\bar{\alpha}_i, \dots, \bar{\alpha}_i}_{s_i - \min\{s_i, u_i\}} \right).$$

The assumption on P ensures that if $s_i > 0$ then $u_i = 0$ or $v_i = 0$; indeed, if s_i, u_i, v_i were all strictly positive then R_i would be a common real factor of Z and W . Then $m_i = \min\{s_i, u_i\} + \min\{s_i, v_i\}$, and hence the previous multiplicities agree with those in RZ .

The final permutation is obtained as an arbitrary interlacement of all arrangements for $1 \leq i \leq h$. Proposition 4.10 then implies that

$$G = \prod_{i=1}^h \left((t - \alpha_i)^{s_i - \min\{s_i, v_i\}} (t - \bar{\alpha}_i)^{s_i - \min\{s_i, u_i\}} \right).$$

Since $s_i - \min\{s_i, v_i\} \leq u_i + s_i - m_i$ and $s_i - \min\{s_i, u_i\} \leq v_i + s_i - m_i$, we conclude that G divides RW . \square

Algorithm 1 FactorMotionPolynomial

Input: $P = Z + \eta W \in \mathbb{K}[t]$ a bounded motion polynomial such that Z and W have no common factor in $\mathbb{R}[t] \setminus \mathbb{R}$.

Output: a polynomial $R \in \mathbb{R}[t]$ and a tuple (k_1, \dots, k_n) of elements of \mathbb{K} such that $(t - k_1) \cdots (t - k_n) = R(t) \cdot P(t)$.

- 1: **Factor** $Z(t)$ over \mathbb{C} , obtaining $Z = \prod_{i=1}^h (t - \alpha_i)^{r_i} (t - \bar{\alpha}_i)^{s_i}$ with $r_i \geq s_i \geq 0$.
 - 2: **Initialize** $q = \text{empty}$ (the empty tuple).
 - 3: **For** $i = 1, \dots, h$ **Do**
 - 4: **Set** $u_i = \max_j((t - \alpha_i)^j \mid W)$ and $v_i = \max_j((t - \bar{\alpha}_i)^j \mid W)$.
 - 5: **Set** $m_i = \min\{s_i, u_i + v_i\}$
 - 6: **Set** $\omega = \left(\underbrace{\bar{\alpha}_i, \dots, \bar{\alpha}_i}_{s_i - \min\{s_i, v_i\}}, \underbrace{\alpha_i, \dots, \alpha_i}_{r_i + s_i - m_i}, \underbrace{\bar{\alpha}_i, \dots, \bar{\alpha}_i}_{s_i - \min\{s_i, u_i\}} \right)$.
 - 7: **Set** $q = \text{concatenate}(q, \omega)$.
 - 8: **End For**
 - 9: **Set** $R = \prod_{i=1}^h ((t - \alpha_i)(t - \bar{\alpha}_i))^{s_i - m_i}$.
 - 10: **Set** $n = \text{length}(q)$.
 - 11: **Set** $Q_j = \prod_{l=1}^{j-1} (t - q_l) \prod_{l=j+1}^n (t - \bar{q}_l)$ for all $j \in \{1, \dots, n\}$.
 - 12: **Compute** $\{w_j\}_{j=1}^n$ such that $RW = \sum_{j=1}^n w_j Q_j$ using linear algebra.
 - 13: **Set** $k_j = q_j - \eta w_j$ for all $j \in \{1, \dots, n\}$.
 - 14: **Return** $(R, (k_1, \dots, k_n))$.
-

The proof of Theorem 4.15 immediately gives rise to a factorization algorithm for motion polynomials, called `FactorMotionPolynomial`. It produces one single factorization for a given polynomial P , although there could be many for the following reasons: first, there may be several admissible permutations of the roots of RZ , and second, the choice of the coefficients $\{w_j\}$ in Step 12 may not be unique, since the polynomials $\{Q_j\}$ need not be \mathbb{C} -linearly independent.

5. THE FLIP PROCEDURE

The procedure described in this section will be crucial for the construction of linkages with mobility one. It is inspired by similar techniques—involving the interchange of factors in a factorization of a quadratic motion polynomial—used in [9] and [10].

Lemma 5.1. *Let $k_1, k_2 \in \mathbb{K}$ be such that $\text{pp}(k_1) \neq \overline{\text{pp}(k_2)}$. Then there exists a unique pair $(k_3, k_4) \in \mathbb{K}^2$ such that:*

- (1) $\text{pp}(k_3) = \text{pp}(k_2)$ and $\text{pp}(k_4) = \text{pp}(k_1)$;
- (2) $(t - k_1)(t - k_2) = (t - k_3)(t - k_4)$ as polynomials in $\mathbb{K}[t]$.

Proof. Let us suppose that $k_i = z_i + \eta w_i$ for $i \in \{1, 2\}$. We make the ansatz

$$k_3 = z_2 + \eta w_3 \quad \text{and} \quad k_4 = z_1 + \eta w_4,$$

where w_3 and w_4 are elements of \mathbb{C} to be determined. The condition

$$(t - k_1)(t - k_2) = (t - k_3)(t - k_4)$$

is equivalent to the linear system

$$\begin{pmatrix} 1 & 1 \\ z_1 & \bar{z}_2 \end{pmatrix} \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_1 + w_2 \\ \bar{z}_1 w_2 + z_2 w_1 \end{pmatrix},$$

which has a unique solution, since by hypothesis the determinant of the matrix is different from zero. \square

Definition 5.2. Let $k_1, k_2 \in \mathbb{K}$ be such that $\text{pp}(k_1) \neq \overline{\text{pp}(k_2)}$. Then we define $\text{Flip}(k_1, k_2) = (k_3, k_4)$, where $(k_3, k_4) \in \mathbb{K}^2$ is the pair from Lemma 5.1.

Definition 5.3. Let $k_1, k_2 \in \mathbb{K}$ be such that $\text{pp}(k_1) \neq \overline{\text{pp}(k_2)}$. Starting from this data, we define a linkage $\text{FlipLinkage}(k_1, k_2) = (G, \rho)$ via the algorithm `FlipLinkage`. The situation is summarized in Figure 6.

Algorithm 2 `FlipLinkage`

Input: $k_1, k_2 \in \mathbb{K}$ such that $\text{pp}(k_1) \neq \overline{\text{pp}(k_2)}$.

Output: `FlipLinkage`(k_1, k_2).

1: **Define** $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (2, 3), (3, 4), (1, 4)\}$ and set $G = (V, E)$.

2: **Define** $(k_3, k_4) = \text{Flip}(k_1, k_2)$.

3: **Define** $u_i = \chi(k_i)$ for all $i \in \{1, \dots, 4\}$.

4: **Define**

$$\begin{aligned} \rho(1, 3) &= (\Re(u_1), \Im(u_1)) & \rho(2, 1) &= (\Re(u_2), \Im(u_2)) \\ \rho(3, 4) &= (\Re(u_3), \Im(u_3)) & \rho(4, 2) &= (\Re(u_4), \Im(u_4)) \end{aligned}$$

5: **Return** (G, ρ) .

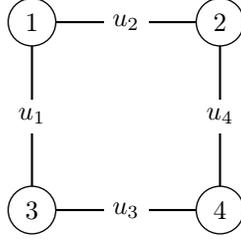


FIGURE 6. The graph of the linkage $\text{FlipLinkage}(k_1, k_2)$, where $k_1, k_2 \in \mathbb{K}$ satisfy $\overline{\text{pp}(k_1)} \neq \text{pp}(k_2)$. The edges of the graph are labeled by the images of the map ρ .

Lemma 5.4. *Let $k_1, k_2 \in \mathbb{K}$ with $\overline{\text{pp}(k_1)} \neq \text{pp}(k_2)$. Let $(k_3, k_4) = \text{Flip}(k_1, k_2)$. Then*

$$|\chi(k_1) - \chi(k_2)| = |\chi(k_3) - \chi(k_4)|.$$

If, in addition, $z_1, z_2 \notin \mathbb{R}$ and $\chi(k_1) \neq \chi(k_2)$, then the following holds:

$$|\chi(k_1) - \chi(k_2)| \neq |\chi(k_1) - \chi(k_3)|.$$

Proof. Let $k_1 = z_1 + \eta w_1$ and $k_2 = z_2 + \eta w_2$. Then the condition $(t - k_1)(t - k_2) = (t - k_3)(t - k_4)$ implies that $k_3 = z_2 + \eta w_3$ and $k_4 = z_1 + \eta w_4$ with

$$w_3 = \frac{(z_2 - \bar{z}_2)w_1 + (\bar{z}_1 - \bar{z}_2)w_2}{z_1 - \bar{z}_2} \quad \text{and} \quad w_4 = \frac{(z_1 - z_2)w_1 + (z_1 - \bar{z}_1)w_2}{z_1 - \bar{z}_2}.$$

A direct calculation shows that

$$|\chi(k_3) - \chi(k_4)| = \left| \frac{w_3}{\bar{z}_2 - z_2} - \frac{w_4}{\bar{z}_1 - z_1} \right| = \left| \frac{w_1}{\bar{z}_1 - z_1} - \frac{w_2}{\bar{z}_2 - z_2} \right| = |\chi(k_1) - \chi(k_2)|$$

as claimed. The second claim is also obtained via a direct computation. \square

Lemma 5.4 implies that the linkages produced by the algorithm FlipLinkage have the shape of a (possibly degenerated) antiparallelogram; see the right part of Figure 5 for an example. We now introduce the notion of *flip mobility* to exclude some degenerated cases, namely the ones where the antiparallelogram is in fact a square.

Notation. Let $k_1 = z_1 + \eta w_1$ and $k_2 = z_2 + \eta w_2$ be two elements of \mathbb{K} . We say that the condition $\text{FM}(k_1, k_2)$ (which stands for *flip mobility*) holds if and only if

$$z_1, z_2 \notin \mathbb{R}, \quad z_1 \neq z_2, \quad z_1 \neq \bar{z}_2, \quad \chi(k_1) \neq \chi(k_2).$$

Lemma 5.5. *Let $k_1, k_2 \in \mathbb{K}$ such that $\text{FM}(k_1, k_2)$ holds. Then the linkage L obtained by $\text{FlipLinkage}(k_1, k_2)$ has mobility one and the configuration curve has two components C_1 and C_2 . Moreover, the natural maps $C_1 \rightarrow \text{VRP}(i, j)$ and $C_2 \rightarrow \text{VRP}(i, j)$ are isomorphisms for every two neighboring links i and j .*

Proof. Since the linkages obtained via FlipLinkage are constituted by four bars, this is a well-known result; see for example [4, Table on p. 426]. One can also prove the statement using the notions we introduced in Section 2 as follows. We set $L = \text{FlipLinkage}(k_1, k_2)$, and denote $(k_3, k_4) = \text{Flip}(k_1, k_2)$. Recalling Definition 2.6, our goal is to prove that $\text{Conf}(L)$ is a one-dimensional variety. One notices that

$$\text{Conf}(L) \cong \{(\lambda_1 : \mu_1), \dots, (\lambda_4 : \mu_4) : f_{12}(\lambda_1, \mu_1, \lambda_2, \mu_2) = f_{34}(\lambda_3, \mu_3, \lambda_4, \mu_4)\},$$

where

$$f_{12}: \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \longrightarrow \mathbb{P}(\mathbb{K})$$

$$(\lambda_1 : \mu_1), (\lambda_1 : \mu_1) \mapsto (\lambda_1 - \mu_1 k_1)(\lambda_2 - \mu_2 k_2)$$

and analogously for f_{34} . Under our hypotheses, the images of f_{12} and f_{34} are two smooth quadrics whose intersection is a curve with two components. This proves the first part of the statement. For the second, one proves that natural projections $\text{Conf}(L) \rightarrow \text{VRP}(i, j)$ become isomorphisms once restricted to each of the two components. \square

Remark 5.6. Results in Example 3.4 are particular instances of Lemma 5.5. One can in fact check that in that case $L = \text{FlipLinkage}(k_1, k_2)$, where $k_1 = \iota - \eta \cdot 3$ and $k_2 = -2\iota - \eta(2 + \iota)$.

6. CONSTRUCTION OF LINKAGES

In this section, we show how to construct a linkage with mobility one that traces an algebraic curve described by a rational parametrization. Based on the results of Section 4, we first describe an algorithm which takes a bounded motion polynomial P and constructs a linkage weakly realizing P (see Definition 3.13).

Algorithm 3 ConstructWeakLinkage

Input: $P \in \mathbb{K}[t]$ a bounded motion polynomial.

Output: (G, ρ) a linkage weakly realizing P .

- 1: **Compute** $S \in \mathbb{R}[t]$ of maximal degree, monic, such that S divides P .
 - 2: **Compute** $R, (k_1, \dots, k_n) = \text{FactorMotionPolynomial}(P/S)$.
 - 3: **Set** $u_i = \chi(k_i) = \frac{w_i}{z_i - z_i}$, where $k_i = z_i + \eta w_i$, for every $i \in \{1, \dots, n\}$.
 - 4: **Set** $V = \{1, \dots, n + 1\}$ and $E = \{(1, 2), (2, 3), \dots, (n, n + 1)\}$.
 - 5: **Set** $G = (V, E)$.
 - 6: **Set** $\rho(i, i + 1) = (\Re(u_i), \Im(u_i)) \in \mathbb{R}^2$ for every $i \in \{1, \dots, n\}$.
 - 7: **Return** (G, ρ) .
-

Proposition 6.1. *Algorithm ConstructWeakLinkage is correct.*

Proof. The linkage L constructed by the algorithm `ConstructWeakLinkage` is an open chain, so its configuration space is isomorphic to $(\mathbb{P}_{\mathbb{C}}^1)^{n-1}$, where $n - 1$ is the number of joints of L . Let $\phi: \mathbb{R} \rightarrow \mathbb{P}_{\mathbb{C}}^3$ be the motion corresponding to RP . We consider links 1 and n , and we prove that $\phi(\mathbb{R}) \subseteq \text{RP}(1, n)$, namely that L weakly realizes RP according to Definition 3.13. In our situation we have that

$$\text{RP}(1, n) = \{\sigma_{1,2} \circ \sigma_{2,3} \circ \dots \circ \sigma_{n-1,n} : \sigma_{i,i+1} \in \mathbb{P}_{\mathbb{C}}^1\}.$$

Now we fix an arbitrary $t \in \mathbb{R}$ and we take $\sigma_{i,i+1}$ to be the isometry given by $t - k_i$ for all $i \in \{1, \dots, n - 1\}$. Then we see that $\phi(t) \in \text{RP}(1, n)$, so the statement is proven. \square

Suppose that we are given a rational parametrization of a bounded curve. As we saw in Proposition 3.7, starting from it we can construct a bounded polynomial P such that the curve is the orbit of the origin in \mathbb{R}^2 under the motion described by P . Then the algorithm `ConstructWeakLinkage` returns a linkage L that weakly realizes P . However, this is rather unsatisfactory, since the configuration space

of L is, in general, a variety of high dimension. Our goal is to obtain a linkage with mobility one, providing a strong realization of P . The main idea is to “rigidify” L , which by construction is an open chain, by introducing additional links and joints, forming antiparallelograms. At the level of graphs, this corresponds to extending the linear graph of L to a $2 \times n$ ladder graph (see Figure 7). To do so, we pick a linear motion polynomial $t - l$, which will constitute the first “step” of the ladder, and then apply the flip procedure iteratively to complete the ladder. In the following, we investigate how $l \in \mathbb{K}$ has to be chosen so that the resulting linkage has mobility one.

Definition 6.2. Let $\mathbf{k} = (k_1, \dots, k_m)$ be a tuple of elements in \mathbb{K} , and let $l \in \mathbb{K}$. Define

$$l_1 := l, \quad (\tilde{k}_i, l_{i+1}) = \text{Flip}(l_i, k_i) \quad \text{for all } i \in \{1, \dots, m\}.$$

Then we say that the condition $\text{IFM}(l, \mathbf{k})$ (which stands for *iterated flip mobility*) holds if and only if $\text{FM}(l_i, k_i)$ holds for all $i \in \{1, \dots, m\}$.

In order to prove that the algorithm we propose works, we have to show that it is always possible to find an element $l \in \mathbb{K}$ such that the iterated flip mobility condition holds. For this we use a property of the flip construction.

Definition 6.3. Given an element $k = z + \eta w \in \mathbb{K}$ with $z \neq 0$, we define $\text{inv}(k)$ to be the element $\bar{z} - \eta w \in \mathbb{K}$. Recalling Remark 3.2, we have that k represents an isometry σ , and $\text{inv}(k)$ its inverse σ^{-1} . Moreover, if $z \notin \mathbb{R}$, then $t - k$ represents an isometry for every $t \in \mathbb{R}$, and $t - \text{inv}(k)$ the inverse isometry.

Lemma 6.4. Let $k_1, k_2 \in \mathbb{K}$ be such that $\text{pp}(k_1) \neq \overline{\text{pp}(k_2)}$ and $\text{pp}(k_2) \notin \mathbb{R}$. Let $(k_3, k_4) = \text{Flip}(k_1, k_2)$. Then we have that

$$(\text{inv}(k_3), k_1) = \text{Flip}(k_4, \text{inv}(k_2)).$$

Proof. Notice that the fact that $\text{pp}(k_2) \notin \mathbb{R}$ implies that the element $(t - k_2)$ represents an isometry for all $t \in \mathbb{R}$. From Definition 6.3 one has that the inverse isometry is represented by $(t - \text{inv}(k_2))$. The two situations are depicted below:

$$\begin{array}{ccc} \begin{array}{ccc} \bullet & \xrightarrow{t-k_2} & \bullet \\ \uparrow t-k_1 & & \uparrow t-k_4 \\ \bullet & \xrightarrow{t-k_3} & \bullet \end{array} & & \begin{array}{ccc} \bullet & \xleftarrow{t-\text{inv}(k_2)} & \bullet \\ \uparrow t-k_1 & & \uparrow t-k_4 \\ \bullet & \xleftarrow{t-\text{inv}(k_3)} & \bullet \end{array} \\ (k_3, k_4) = \text{Flip}(k_1, k_2) & & (\text{inv}(k_3), k_1) = \text{Flip}(k_4, \text{inv}(k_2)) \end{array}$$

A direct computation shows the desired result. \square

Lemma 6.5. Let $\mathbf{k} = (k_1, \dots, k_m)$ be a tuple of elements in \mathbb{K} such that $\text{pp}(k_i) \notin \mathbb{R}$ for all $i \in \{1, \dots, m\}$. Then there exists $l \in \mathbb{K}$ with $\text{pp}(l) \notin \mathbb{R}$ such that $\text{IFM}(l, \mathbf{k})$ holds.

Proof. Following the notation of Definition 6.2 we have that $\text{pp}(l_i) = \text{pp}(l)$ for all $i \in \{1, \dots, m\}$. Hence, in order to ensure that $\text{IFM}(l, \mathbf{k})$ holds, we need to choose l such that $\text{pp}(l) \neq \text{pp}(k_i)$ and $\text{pp}(l) \neq \overline{\text{pp}(k_i)}$ for all $i \in \{1, \dots, m\}$. From now on, we fix such a value for $\text{pp}(l)$. The other situations we should keep away from are the ones where $\chi(l_i) = \chi(k_i)$. Since $\text{pp}(l_i)$ is now fixed, the condition $\chi(l_i) = \chi(k_i)$

becomes a linear equation for $\text{sp}(l_i)$, which always admits a unique solution $\widehat{w}_i \in \mathbb{C}$. Let us define $\widehat{l}_i = \text{pp}(l) + \eta \widehat{w}_i$. Using Lemma 6.4 repeatedly (namely $i - 1$ times) starting from the pair $(\widehat{l}_i, \text{inv}(k_{i-1}))$, we obtain an element $h_i \in \mathbb{K}$ such that

$$\chi(l_i) \neq \chi(k_i) \iff l \neq h_i.$$

Summing up, for every fixed value of $\text{pp}(l)$ different from all $\{\text{pp}(k_i)\}_{i=1}^m$ and all $\{\overline{\text{pp}(k_i)}\}_{i=1}^m$, there are finitely many values for $\text{sp}(l)$ that should be avoided for $\text{IFM}(l, \mathbf{k})$ to hold. Thus we can always find an $l \in \mathbb{K}$ with the claimed properties. \square

We are ready to present the algorithm **ConstructStrongLinkage**. Figure 7 shows the labeled graph produced by the algorithm.

Algorithm 4 ConstructStrongLinkage

Input: $P \in \mathbb{K}[t]$ a bounded motion polynomial.

Output: (G, ρ) a linkage strongly realizing the motion induced by P .

- 1: **Compute** $S \in \mathbb{R}[t]$ of maximal degree, monic, such that S divides P .
 - 2: **Compute** $R, (k_1, \dots, k_n) = \text{FactorMotionPolynomial}(P/S)$.
 - 3: **Choose** $l \in \mathbb{K}$ such that $\text{IFM}(l, \bar{k})$ holds.
 - 4: **Set** $l_1 = l$ and $(\widetilde{k}_i, l_{i+1}) = \text{Flip}(l_i, k_i)$ for all $i \in \{1, \dots, n\}$.
 - 5: **Set**

$$\left. \begin{array}{l} u_i = \chi(k_i) \\ \widetilde{u}_i = \chi(\widetilde{k}_i) \\ v_j = \chi(l_j) \end{array} \right\} \begin{array}{l} \text{for every } i \in \{1, \dots, n\} \\ \text{for every } j \in \{1, \dots, n+1\} \end{array}$$
 - 6: **Set** $V = \{1, \dots, 2n+2\}$.
 - 7: **Set** $E = \left\{ (i, i+1), (n+1+i, n+2+i), (j, n+1+j) \text{ for all } i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, n+1\} \right\}$.
 - 8: **Set** $G = (V, E)$.
 - 9: **Set**

$$\left. \begin{array}{l} \rho(i, i+1) = (\mathfrak{R}(u_i), \mathfrak{I}(u_i)) \\ \rho(n+1+i, n+2+i) = (\mathfrak{R}(\widetilde{u}_i), \mathfrak{I}(\widetilde{u}_i)) \\ \rho(j, n+1+j) = (\mathfrak{R}(v_j), \mathfrak{I}(v_j)) \end{array} \right\} \begin{array}{l} \text{for every } i \in \{1, \dots, n\} \\ \text{for every } j \in \{1, \dots, n+1\} \end{array}$$
 - 10: **Return** (G, ρ) .
-

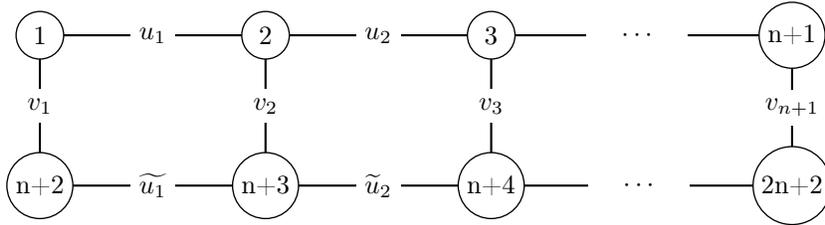
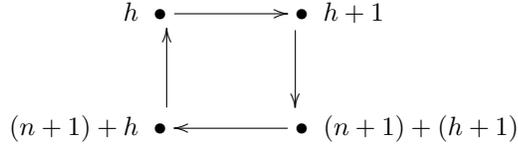


FIGURE 7. The labeled graph of the linkage returned by the algorithm **ConstructStrongLinkage**.

Theorem 6.6. *Let P be a bounded polynomial in $\mathbb{K}[t]$ and let $\phi: \mathbb{R} \rightarrow \mathbb{P}_{\mathbb{R}}^3$ be the rational motion induced by P . Let us denote by L the linkage obtained by applying *ConstructStrongLinkage* to P . Then L strongly realizes the motion ϕ .*

Proof. Let K be the configuration space of $L = (G, \rho)$. As in the statement of Lemma 5.5, we are going to prove that for any two neighboring edges i and j of G , every irreducible component of K is isomorphic to $\text{RP}(i, j)$. This will prove that K has dimension one, which means that L has mobility one.

Recall from Definition 2.4 that K is the collection of configurations $\Sigma = (\sigma_{k,l})$ satisfying the equations imposed by the directed cycles of G . First of all, notice that in the definition of K one can consider only those equations imposed by “square cycles”, namely directed cycles of the form:



In fact, one can easily see that all other equations belong to the ideal generated by these ones. Let us now introduce the following notation: for $h \in \{1, \dots, n\}$ we denote by $\text{sq}(h)$ the set

$$\{(h, h + 1), (h + 1, (n + 1) + (h + 1)), ((n + 1) + (h + 1), (n + 1) + h), ((n + 1) + h, h)\}.$$

We define $\pi_{\text{sq}(h)}$ to be the projection

$$\pi_{\text{sq}(h)}: K \longrightarrow \prod_{(r,s) \in \text{sq}(h)} \text{VRP}(r, s).$$

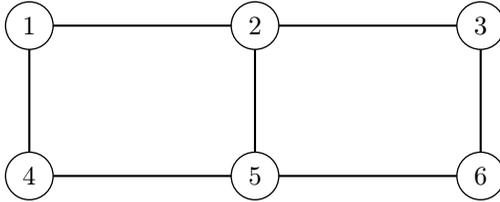
Then the image of $\pi_{\text{sq}(h)}$ is contained in the configuration space C_h of a flip linkage, which by Lemma 5.5 has two components, so $C_h = C_h^1 \cup C_h^2$.

For every sequence $b \in \{1, 2\}^n$, we set

$$K_b = \pi_{\text{sq}(1)}^{-1} \left(C_1^{b(1)} \right) \cap \dots \cap \pi_{\text{sq}(n)}^{-1} \left(C_n^{b(n)} \right).$$

Then by construction we have $K = \bigcup_{b \in \{1, 2\}^n} K_b$. We are going to prove that for every sequence b and for any neighboring links i and j in G , we have $K_b \cong \text{VRP}(i, j)$.

For the sake of simplicity, we start by proving this result when the graph G consists only of two squares (i.e. $n = 2$):



In this case we only have two flip linkages with configuration spaces C_1 and C_2 . Let us fix $b \in \{1, 2\}^2$, and consider the diagram:

$$(6.1) \quad \begin{array}{ccc} C_1^{b(1)} & & C_2^{b(2)} \\ & \searrow \cong & \swarrow \cong \\ & \pi_{2,5} & \pi_{2,5} \\ & & \text{VRP}(2, 5) \end{array}$$

We know that the maps $\pi_{2,5}$ are isomorphisms, because of Lemma 5.5. We consider the set

$$\{(x, y) : x \in C_1^{b(1)}, y \in C_2^{b(2)} \text{ and } \pi_{2,5}(x) = \pi_{2,5}(y)\},$$

which is nothing but the pullback of Diagram (6.1). One can check that this set equals K_b since, as we observed before, the elements of K are only subject to the equations coming from the cycles in C_1 and C_2 . Because of the property of isomorphisms of being stable under pullbacks, we obtain the following commutative diagram of isomorphisms:

$$\begin{array}{ccc} & K_b & \\ \pi_{\text{sq}(1)} \swarrow \cong & & \searrow \cong \pi_{\text{sq}(2)} \\ C_1^{b(1)} & & C_2^{b(2)} \\ \pi_{2,5} \searrow \cong & & \swarrow \cong \pi_{2,5} \\ & \text{VRP}(2, 5) & \end{array}$$

The composition $\pi_{2,5} \circ \pi_{\text{sq}(1)}$ equals the projection $\pi_{2,5}: K_b \rightarrow \text{VRP}(2, 5)$, and the same holds for every projection $\pi_{i,j}: K_b \rightarrow \text{VRP}(i, j)$, where i and j are neighboring links. Hence each of these maps is an isomorphism, and this proves our claim.

If now G is constituted by more than two squares, then K_b is obtained via an iteration of several pullbacks, and eventually we get a diagram of isomorphisms. Thus the claim holds also in this case.

Since K is covered by finitely many varieties of dimension one, then it has dimension one. Moreover, each of the varieties K_b is an irreducible component of K .

We are left to prove that L realizes the motion ϕ induced by P . Similarly as we did in the proof of Proposition 6.1, one can show that $\phi(\mathbb{R}) \subseteq \text{RP}(1, n+1)$. Since K is one-dimensional, then also $\text{RP}(1, n+1)$ is so; since $\phi(\mathbb{R})$ is one-dimensional, then $\overline{\phi(\mathbb{R})}$ is a component of $\text{RP}(1, n+1)$. \square

Remark 6.7. As we mentioned in Section 1, the linkages constructed via our algorithms present the same issues as the ones produced by Kempe's procedure, namely the devices do not realize only the motion they are designed for, but also other ones. This is due to the fact that the elementary linkages produced by the flip procedure, commonly known as antiparallelograms, admit configuration spaces with more than one component (see Proposition 5.5). On the other hand, since antiparallelograms are among the linkages used also by Kempe's procedure, the techniques developed in [1] and [6, Section 3.2.2] can be applied also in our case to prevent this unwanted behavior.

We now state explicitly how many links and joints the linkages created by our algorithm have, when applied to a parametrization of a curve.

Proposition 6.8. *Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}^2$ the parametrization of a real bounded planar curve. Without loss of generality, suppose that*

$$\varphi(t) = \left(\frac{f(t)}{h(t)}, \frac{g(t)}{h(t)} \right)$$

with $f, g, h \in \mathbb{R}[t]$, and h monic, and $d = \deg h > \max\{\deg f, \deg g\}$. Then, there exists a linkage with at most $4d$ links and $6d - 2$ joints drawing the curve given by φ .

Proof. As discussed before Definition 4.13, the motion polynomial $P = h + \eta(f + \iota g)$ is bounded, and by Proposition 3.7 the orbit of the origin in \mathbb{R}^2 under the motion that it determines is exactly the image of φ . Let us denote by L the linkage obtained by applying `ConstructStrongLinkage` to P . Then L is constituted by at most $4d + 2$ links and $6d + 1$ joints: in fact, as noticed in the proof of Theorem 4.15, the real polynomial R returned by `FactorMotionPolynomial` has at most degree d . Since here we are only interested in the curve traced by the linkage, and not in the orientation of the end effector, we can remove the leftmost square from the graph of the linkage L , obtaining a linkage with $4d$ links and $6d - 2$ joints (for an example, see Section 8). \square

7. SELF-COLLISIONS

A natural question that arises in the construction of a linkage is whether it can be physically realized such that no collisions between components of the linkage occur. More precisely, one asks whether there exists an assignment of $1, 2, \dots, n$ to the links such that for any $i < k < j$ for which i and j are neighboring links, the joint connecting i and j never overlaps with the link k . For general linkages it is a difficult problem to detect such collisions; see for example [5, Section 9.3 and Theorem 9.5.5]. However, it turns out that the same problem has a straightforward solution for linkages constructed by our algorithm, and the reason is that all joints in our case follow rational curves whose explicit parametrization is a direct byproduct of the algorithm `ConstructStrongLinkage`. Assuming that a link is realized as a collection of line segments connecting the (two or three) joints attached to it, a collision is described as follows: for some $i < k < j$ as above there exists a $t \in \mathbb{R} \cup \{\infty\}$ such that the position $(x_1(t), y_1(t))$ of the joint (i, j) lies on one of the line segments of link k ; denote its endpoints by $(x_2(t), y_2(t))$ and $(x_3(t), y_3(t))$. Then a collision happens if the system

$$(7.1) \quad \begin{cases} x_1(t) = s \cdot x_2(t) + (1 - s) \cdot x_3(t) \\ y_1(t) = s \cdot y_2(t) + (1 - s) \cdot y_3(t) \end{cases}$$

has a solution subject to $0 \leq s \leq 1$ and $t \in \mathbb{R} \cup \{\infty\}$. In our construction, the coordinates x_i, y_i are given by rational functions in t , so that the system (7.1) is equivalent to a bivariate polynomial system that can be easily solved. Note that for each admissible configuration (i.e., joint plus line segment) we have to solve a system of the form (7.1). This way we can decide whether collisions occur or not, and if so, we obtain the precise description when and where they happen.

At the same time, we want to mention a remarkable method for preventing collisions for planar linkages with a link diagram of the form shown in Figure 7. This method is based on designing the shape of some links and joints: we introduce three types of links (F-link, U-link, Z-link), and two types of joints (T-joint, Z-joint).

As before, all links are arranged in different layers, and w.l.o.g. we assume that these layers correspond to integer numbers. An F-link is located in a single layer, in other words, we associate one integer for each F-link. On the other hand, U-links and Z-links stretch across two, not necessarily neighboring, layers. Therefore, to such links we associate a pair of integers (a, b) with $a < b$, which means that one part of the link is located in layer a and the other part in layer b ; the two parts are rigidly connected by a vertical rod. For any two different U-links with layers (a_1, b_1) and (a_2, b_2) , we prohibit the situations $a_1 < a_2 < b_1 < b_2$ and $a_2 < a_1 < b_2 < b_1$, which could yield a collision between these two U-links. For a Z-link on layers (a, b) we impose the condition $b - a = 2$, and the link located on layer $a + 1$ (in our case this will always be some part of a U-link) is connected with this Z-link by a revolute joint (which we call Z-joint) around its vertical rod. In contrast, a T-joint joins two links located on neighboring layers.

Using the above design, one can check that collisions can only happen between links, and not between links and joints. We now argue that even these collisions can be avoided. First, an F-link can not collide with another F-link or a Z-link. Second, a Z-link can not collide with another Z-link. Third, if we move the vertical connections of all U-links sufficiently far away, the F-links and Z-links do not collide with the U-links. Fourth, the above conditions on the layers of U-links imply that two U-links could collide only if they occur in a nested way (e.g., $a_1 < a_2 < b_2 < b_1$). Again, this can be avoided by moving the vertical connection of the outer U-link far away enough. Thus we can always manage to make a collision-free design, by manipulating the shape of the U-links.

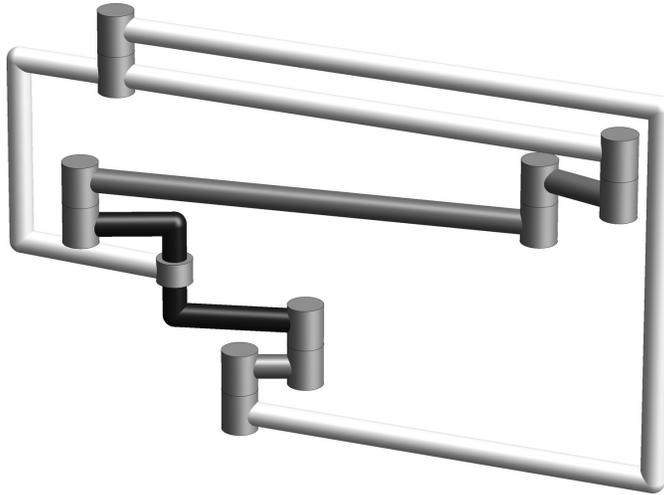


FIGURE 8. A linkage without self-collisions composed of three F-links (gray), two U-links (white), and one Z-link (black)

It remains to argue that for a general ladder-shaped link diagram, as returned by our algorithm `ConstructStrongLinkage` and as displayed in Figure 7, we can

find an assignment of link types and layers such that all the above conditions are fulfilled. One such assignment is depicted in Figure 9; it demonstrates that we can realize a motion—given by a factored motion polynomial of degree n —by a collision-free linkage using $4n + 1$ layers.

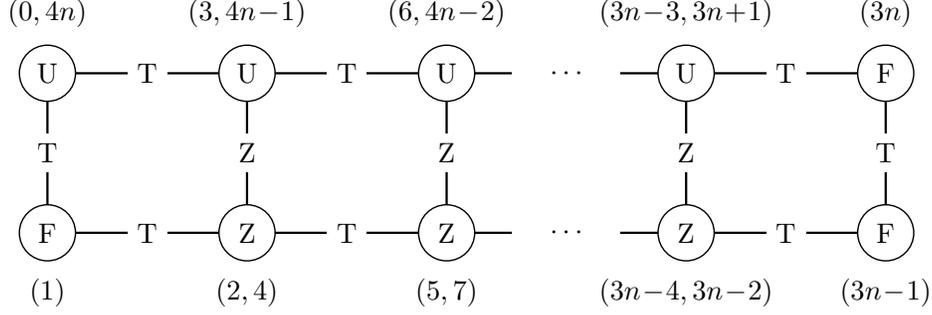


FIGURE 9. Assignment of joint types (T, Z), link types (F, U, Z) and layers (above and below the corresponding vertices) such that a linkage with ladder-shaped link graph can be realized without self-collisions.

8. FINAL EXAMPLES

We consider again the elliptic translational motion from Example 3.11, described by the motion polynomial $P(t) = (t^2+1)+\eta(at-bi)$, but now for the concrete choice $a = i$ and $b = -2i$. Setting $R = t^2 + 1$ and fixing the permutation $(i, -i, -i, i)$ of the roots of $\text{pp}(RP)$, as in (3.4), we obtain a two-dimensional family of factorizations into linear factors, i.e., the solutions of the linear system (3.5), out of which we choose the following:

$$(t + i - \eta i) \cdot (t - i + \frac{1}{2}\eta i) \cdot (t - i + \frac{3}{2}\eta i) \cdot (t + i).$$

Next we have to fix an $l \in \mathbb{K}$ such that the IFM condition holds; it turns out that $l = t + 9/5i$ is a good choice (which yields a linkage that is well suited for visualization). Applying the flip procedure iteratively to this input data, one obtains the following:

$$\begin{aligned} \tilde{k}_1 &= -i - \frac{2}{7}\eta i, & \tilde{k}_2 &= i - \frac{41}{28}\eta i, & \tilde{k}_3 &= i - \frac{3}{8}\eta i, & \tilde{k}_4 &= -i + \frac{45}{56}\eta i, \\ l_1 &= -\frac{9}{5}i + \frac{9}{7}\eta i, & l_2 &= -\frac{9}{5}i + \frac{9}{4}\eta i, & l_3 &= -\frac{9}{5}i + \frac{9}{8}\eta i, & l_4 &= -\frac{9}{5}i + \frac{9}{28}\eta i. \end{aligned}$$

Now we can use these quantities to construct a linkage that realizes an elliptic translation. If, on the other hand, we content ourselves with a linkage that draws the ellipse (i.e., the end effector is allowed to rotate arbitrarily), we can omit the “leftmost square” in the link graph so that we obtain the graph shown in Figure 10. Note that all linear motion polynomials appearing here are purely imaginary in their secondary parts. This implies that all their fixed points are of the form $(x, 0)$, so that all joints are located on the horizontal axis when the linkage is in its initial position ($t = \infty$); the same happens for $t = 0$. A realization of this linkage is shown in Figure 11. It produces two collisions with the ordering $(7, 2, 8, 3, 9, 10, 5, 4)$ of the links, but both happen at $t = \infty$. Hence in principle

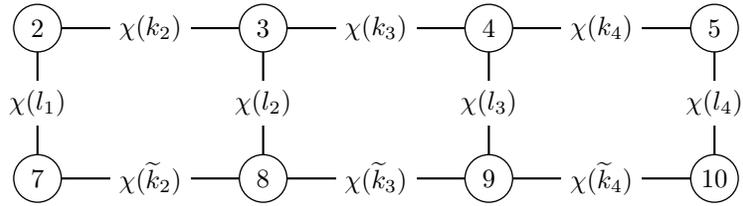
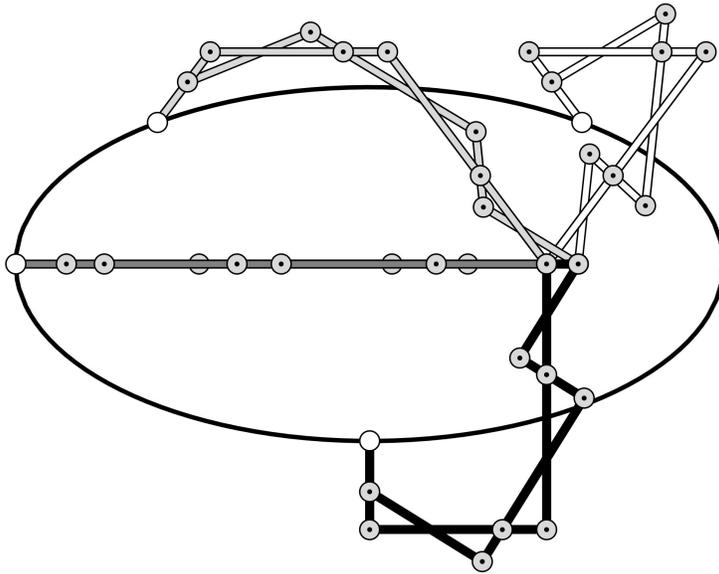


FIGURE 10. The link graph for the linkage drawing an ellipse.

FIGURE 11. The linkage that draws an ellipse. The same linkage is shown in different positions: $t = 2$ (white), $t = \frac{1}{2}$ (light gray), $t = 0$ (dark gray), and $t = -1$ (black).

we can trace, without disassembling the linkage, the full ellipse except a single point—see also the animations provided on our webpage [17].

We conclude with an example in connection to a popular formulation of Kempe’s Theorem, stating that *“There is a linkage that signs your name”*. King [15, Corollary 1.3] attributes this formulation to William Thurston. However, as remarked by O’Rourke [23], it is very implausible that a concrete “signing linkage” has ever been constructed due to the complexity, in terms of links and joints, of the linkages produced following Kempe’s procedure. As an example to support his claim, O’Rourke points out that already constructing a linkage drawing the “J” of John Hancock’s famous signature (see Figure 12) would be very difficult. We approximate the “J” by the rational curve given by the parametrization $(f/h, g/h)$, where

$$\begin{aligned} f(t) &= -321880t^5 - 436132t^4 - 237449t^3 - 64488t^2 - 8666t - 451, \\ g(t) &= -336018t^5 - 472949t^4 - 270569t^3 - 78158t^2 - 11325t - 651, \\ h(t) &= 170(7225t^6 + 13770t^5 + 11187t^4 + 4908t^3 + 1219t^2 + 162t + 9). \end{aligned}$$



FIGURE 12. John Hancock’s signature on the United States Declaration of Independence.

Our implementation of the algorithm `ConstructStrongLinkage`—provided on our webpage [17]—produces a linkage with 26 links and 37 joints realizing this curve, see Figure 13. When we let it draw the depicted “J”, we encounter seven collisions.

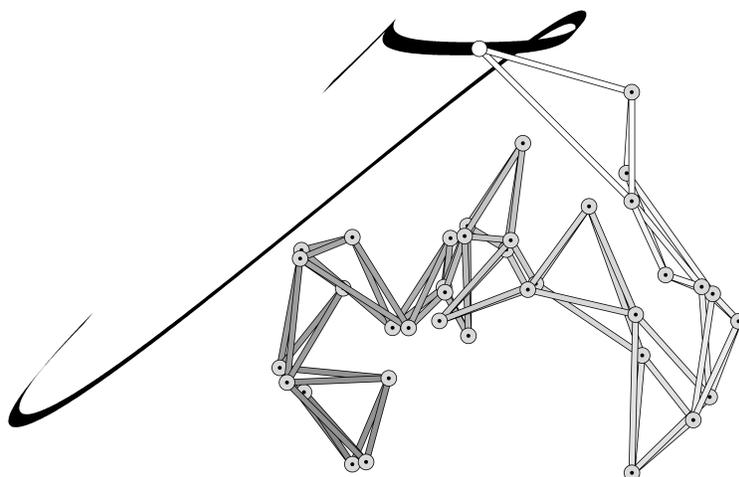


FIGURE 13. A rational curve approximating the “J” in John Hancock’s signature and a linkage drawing it. The corresponding motion is a translation, which we visualize by using a quill pen whose shape is a line segment in direction $(5, 6)$. The spatial arrangement of the links is indicated by hue: darker links lie below brighter ones.

REFERENCES

- [1] Timothy G. Abbott. Generalizations of Kempe’s universality theorem. Master’s thesis, Massachusetts Institute of Technology, 2008. Available at <http://web.mit.edu/tabbott/www/papers/mthesis.pdf>.
- [2] Wilhelm Blaschke and Hans R. Müller. *Ebene Kinematik*. Verlag von R. Oldenbourg, München, 1956.
- [3] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1998.
- [4] Oene Bottema and Bernard Roth. *Theoretical Kinematics*. North-Holland Publishing Company, 1979.
- [5] Robert Connelly and Erik D. Demaine. Geometry and topology of polygonal linkages. In Jacob E. Goodman and Joseph O’Rourke, editors, *Handbook of discrete and computational geometry*, pages 197–218. Chapman & Hall/CRC, Boca Raton, FL, second edition, 2004.

- [6] Erik D. Demaine and Joseph O’Rourke. *Geometric folding algorithms*. Cambridge University Press, Cambridge, 2007.
- [7] Xiao-Shan Gao and Chang-Cai Zhu. Automated generation of Kempe linkage and its complexity. *Journal of Computer Science and Technology*, 14(5):460–467, 1999.
- [8] Xiao-Shan Gao, Chang-Cai Zhu, Shang-Ching Chou, and Jian-Xin Ge. Automated generation of Kempe linkages for algebraic curves and surfaces. *Mechanism and Machine Theory*, 36(9):1019–1033, 2001.
- [9] Gábor Hegedüs, Josef Schicho, and Hans-Peter Schröcker. Construction of Overconstrained Linkages by Factorization of Rational Motions. In Jadran Lenarčič and Manfred Husty, editors, *Latest Advances in Robot Kinematics*, pages 213–220. Springer Netherlands, 2012.
- [10] Gábor Hegedüs, Josef Schicho, and Hans-Peter Schröcker. Factorization of Rational Curves in the Study Quadric. *Mechanism and Machine Theory*, 69:142–152, 2013.
- [11] Manfred L. Husty and Hans-Peter Schröcker. Kinematics and algebraic geometry. In J. Michael McCarthy, editor, *21st Century Kinematics*, pages 85–123. Springer London, 2013.
- [12] Denis Jordan and Marcel Steiner. Configuration spaces of mechanical linkages. *Discrete & Computational Geometry*, 22(2):297–315, 1999.
- [13] Michael Kapovich and John J. Millson. Universality theorems for configuration spaces of planar linkages. *Topology*, 41(6):1051–1107, 2002.
- [14] Alfred B. Kempe. On a General Method of describing Plane Curves of the nth degree by Linkwork. *Proceedings of the London Mathematical Society*, S1-7(1):213, 1876.
- [15] Henry C. King. Planar linkages and algebraic sets. *Turkish Journal of Mathematics*, 23(1):33–56, 1999.
- [16] Alexander Kobel. Automated generation of Kempe linkages for algebraic curves in a dynamic geometry system. Bachelor’s thesis, University of Saarbrücken, 2008. Available at <https://people.mpi-inf.mpg.de/~akobel/publications/Kobel08-kempe-linkages.pdf>.
- [17] Christoph Koutschan. Mathematica package PlanarLinkages.m and electronic supplementary material for the paper “Planar linkages following a prescribed motion”, 2015. Available at <http://www.koutschan.de/data/link/>.
- [18] Christian Krattenthaler. Advanced determinant calculus. *Séminaire Lotharingien de Combinatoire*, 42:1–67, 1999. Article B42q.
- [19] Alain Lascoux and Piotr Pragacz. Jacobians of symmetric polynomials. *Annals of Combinatorics*, 6(2):169–172, 2002.
- [20] Henri Lebesgue. *Leçons sur les Constructions Géométriques*. Gauthier-Villars, Paris, 1950.
- [21] Zijia Li, Josef Schicho, and Hans-Peter Schröcker. Factorization of motion polynomials. 2015. Preprint.
- [22] Joseph Malkevitch. Linkages: From fingers to robot arms. AMS Feature Column, September 2002. Available at <http://www.ams.org/samplings/feature-column/fcarc-linkages1>.
- [23] Joseph O’Rourke. *How to fold it*. Cambridge University Press, Cambridge, 2011.
- [24] John Owen and Stephen Power. Continuous curves from infinite Kempe linkages. *Bulletin of the London Mathematical Society*, 41(6):1105–1111, 2009.
- [25] Mark Plecnik, J. Michael McCarthy, and Charles W. Wampler. Kinematic Synthesis of a Watt I Six-Bar Linkage for Body Guidance. In *Advances in Robot Kinematics*, pages 317–325. Springer International Publishing, 2014.
- [26] Jon M. Selig. *Geometric fundamentals of robotics*. Monographs in Computer Science. Springer, New York, second edition, 2005.

(MG, CK, ZL, GR, NV) JOHANN RADON INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS (RICAM), AUSTRIAN ACADEMY OF SCIENCES, ALTENBERGER STRASSE 69, 4040 LINZ, AUSTRIA

(JS) RESEARCH INSTITUTE FOR SYMBOLIC COMPUTATION (RISC), JOHANNES KEPLER UNIVERSITY, ALTENBERGER STRASSE 69, 4040 LINZ, AUSTRIA

E-mail address: {matteo.gallet, christoph.koutschan, zijia.li, georg.regensburger, josef.schicho, nelly.villamizar}@ricam.oeaw.ac.at