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On the dimension of spline spaces on triangulations

Nelly Villamizar and Bernard Mourrain

Abstract. We consider, from an algebro-geometric perspective, the problem of determining the dimension of the space of bivariate and trivariate piecewise polynomial functions (or *splines*) defined on triangular and tetrahedral partitions. Classical splines on planar rectangular grids play an important role in Computer Aided Geometric Design, and splines spaces over arbitrary subdivisions of planar domains are now considered for isogeometric analysis applications. Using the homological approach introduced by L. J. Billera, we establish upper and lower bounds on the dimension of the spline spaces; these formulas include terms that take into account the geometry of the faces surrounding the interior faces of the partition and, having no restriction on the orderings of the faces, these bounds yield more accurate approximations to the dimension than previous methods.

Keywords. Splines, dimension, bounds, tetrahedral partitions, Hilbert function, Fröberg's conjecture, Ideals of powers of linear forms.

1. Introduction

For a polyhedral partition Δ embedded in $\mathbb{R}^d$, a polynomial *spline* is a piecewise polynomial function defined on Δ with a specified order of global smoothness.

It is commonly accepted that the first mathematical reference to splines is in the article from 1946 by Schoenberg [27], which is probably the first place where the word "spline" is used in connection with smooth, piecewise polynomial approximation. However, the idea that polynomials were the most convenient functions for approximation and interpolation has its roots in the aircraft and shipbuilding industries. On an irregular domain, such as an airplane or a human skull, the approximation functions are needed to be defined and adaptable to satisfy boundary conditions on domains of any reasonable shape. This idea developed into what is now one of the most powerful tools to solve partial differential equations, the finite element method [11]. Splines are nowadays important not only in numerical analysis and approximation theory, they are very useful for modeling surfaces of arbitrary topology and are a widely recognized tool in isogeometric analysis [9], image analysis and free-form representation in Computer Aided Design (CAD) and Computer Aided Geometric Design (CAGD) [10].

To be useful in computations, the space of spline functions must have a basis, and it in turn makes essential to study the dimension of these spaces. For the space of piecewise polynomials on a given triangulation, or on a simplicial partition in $\mathbb{R}^d$, the problem of finding the dimension and a suitable basis for it was first formally formulated by Strang [30, 31]. He conjectured a formula for the dimension of the spline space on a general triangulation [30]. However, serious difficulties already begin to arise in the planar case, and the actual lower bound on the dimension of the space is usually larger than the formula in the conjecture depending on the embedding of the triangulation in $\mathbb{R}^2$ [19, 20, 28]. The methods to compute the dimension include the construction of nodal bases and the Bernstein–Bézier approach, see [16] and the references therein.

In 1988, Billera introduced the use of homological algebra and some algebraic machinery to study the spaces of splines on triangulations in any dimension [4]. By means of this approach, complicated linear algebra can be presented in a more organized way, and he was able to find the dimension for the space of C^1 bivariate splines for triangulations whose edges are in sufficiently general position, for any fixed polynomial degree. See also [5, 7] for more results concerning the algebraic structure of the spline space.

The homological construction was continued by Schenck and Stillman in [25], and studied in [13, 24–26]. We start the next section by recalling this construction, which we latter use to prove a formula for an upper bound on the dimension of bivariate spline spaces, and new lower and upper bounds for trivariate spline spaces.

These bounds improve previous results [16, 28] and the approach leads to connections of the dimension problem on spline spaces and classical problems in algebraic geometry.

The formula for the upper bound in the bivariate case applies to any ordering established on the interior vertices of the partition. Having no restriction on the ordering makes it possible to obtain accurate approximations to the dimension and even exact values in many cases, for instance it leads to a simple proof for a dimension formula of the C^r spline space when the degree of the polynomials is at least $4r + 1$ [21].

The results for the trivariate case in the literature do not take into account the exact geometry of faces in the partition [1, 2, 17]. The bounds we prove include terms that take into account the geometry of the faces surrounding the interior edges and vertices of the partition. For our results we explore connections between spline functions and ideals generated by powers of linear forms, ideals of fat points, the Fröberg's conjecture and the weak Lefschetz property, giving so an insight into ways of improving these bounds by using results from algebraic geometry [22].

The structure of this chapter is as follows. In Section 2 we present in detail the construction of the chain complex introduced by Schenck and Stillman [25] in general settings i.e., for any finite d–dimensional simplicial complex. We recall some properties of the homology modules, which leads to a formula for the dimension with terms corresponding to the dimension of low homology modules. By bounding these terms we get lower and upper bounds for two dimensional simplicial complexes in Section 3, and the three dimensional case is considered in Section 4. At the end of each section we present some examples and some final remarks.

2. Construction of the chain complex

We introduce the notation and some definitions from [24] that will be used throughout this chapter.

We denote by Δ a connected, finite d-dimensional simplicial complex, supported on $|\Delta| \subset \mathbb{R}^d$, such that Δ and all its links are pseudomanifolds [4]. We could think of Δ as the triangulation of a (topological) d-ball.

For integers r and k, with $r \geq 0$, $k \geq r$, denote by $C^r_k(\Delta)$ the space of polynomial splines defined on Δ, of degree at most k, and continuously differentiable of order r.

The problem of finding the dimension of $C^r_k(\Delta)$ can be reduced to the case in which each maximal face of Δ contains the origin in $\mathbb{R}^d$, in the following way [6]. We embed Δ in the hyperplane $\{x_{d+1} = 1\} \subset \mathbb{R}^{d+1}$ and form the cone $\hat{\Delta}$ with vertex at the origin. Let $C^r(\hat{\Delta})$ be the set of splines defined on $\hat{\Delta}$ and continuously differentiable of order r. For a fixed k, let $C^r(\hat{\Delta})_k$ denote the subset of splines on $\hat{\Delta}$ of degree exactly k. The elements of $C^r(\hat{\Delta})_k$ are precisely the homogenization of the elements of $C^r_k(\Delta)$, and

$$C^r(\hat{\Delta})_k \cong C^r_k(\Delta)$$

as $\mathbb{R}$-vector spaces [6]; in particular

$$\dim C^r_k(\Delta) = \dim C^r(\hat{\Delta})_k. \tag{2.1}$$

Thus, we turn the problem of finding $\dim C^r_k(\Delta)$ (for different values of k) into the problem of finding the Hilbert function of a graded algebra, namely $C^r(\hat{\Delta})$, and we may apply the tools of commutative and homological algebra to solve the problem.

Let us denote by Δ^0 the set of interior faces of Δ. For $i = 0, \dots, d-1$ let Δ_i^0 be the set of i-dimensional interior faces of Δ whose support is not contained in the boundary $\partial\Delta$ of $|\Delta|$, and let Δ_d^0 be the set of all maximal d-faces of Δ. We denote by f_i^0 the cardinality of these sets, for $i = 0, \dots, d$.

Let us denote by $R := \mathbb{R}[x_1, \dots, x_{d+1}]$ the polynomial ring in $d+1$ variables. Let $\mathcal{R}$ be the constant (chain) complex on Δ i.e., $\mathcal{R}(\beta) = R$ for every $\beta \in \Delta^0$. For $i = 0, \dots, d$ we have $\mathcal{R}_i = R^{f_i^0}$. The maps ∂_i in the complex $\mathcal{R}$ are induced by the usual simplicial boundary maps $\bar{\partial}_i$ used to compute the relative homology of $(\Delta, \partial\Delta)$ with coefficients in R (see [4] and [23] for more general details about relative homology).

For $\tau \in \Delta_{d-1}^0$, let ℓ_τ be the linear form that vanishes on $\hat{\tau}$ (it is just the homogenization of the linear polynomial vanishing on τ). For every interior face $\beta \in \Delta^0$ define

$$\mathcal{J}(\beta) := \langle \ell_\tau^{r+1} \rangle_{\tau \ni \beta}.$$

Let us denote by $\mathcal{J}$ the complex of ideals defined in this way, with the restriction of the maps ∂_i defined on $\mathcal{R}_i$ to $\mathcal{J}_i := \bigoplus_{\beta \in \Delta_i^0} \mathcal{J}(\beta)$, for $i = 0, \dots, d$.

We consider the chain complex $\mathcal{R}/\mathcal{J}$ defined as the quotient of $\mathcal{R}$ by $\mathcal{J}$:

$$0 \xrightarrow{\partial_{d+1}} \bigoplus_{\sigma \in \Delta_d^0} \mathcal{R} \xrightarrow{\partial_d} \bigoplus_{\tau \in \Delta_{d-1}^0} \mathcal{R}/\mathcal{J}(\tau) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} \bigoplus_{\beta \in \Delta_0^0} \mathcal{R}/\mathcal{J}(\gamma) \xrightarrow{\partial_0} 0$$

where ∂_i are the induced relative (module $\partial\Delta$) simplicial boundary maps.

In [4] it was proved that $C^r(\hat{\Delta})$ is isomorphic to the top homology module of $\mathcal{R}/\mathcal{J}$, i.e.,

$$H_d(\mathcal{R}/\mathcal{J}) := \ker(\partial_d) \cong C^r(\hat{\Delta}).$$

This together with (2.1) and the Euler characteristic equation [29, p. 172]

$$\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J}),$$

implies that, for any fixed $k \geq 1$

$$\dim C_k^r(\Delta) = \sum_{i=0}^{d} (-1)^i \sum_{\beta \in \Delta_{d-i}^0} \dim \mathcal{R}/\mathcal{J}(\beta)_k - \sum_{i=1}^{d} (-1)^i \dim\, H_{d-i}(\mathcal{R}/\mathcal{J})_k. \tag{2.2}$$

The subindex k indicates that we are considering the k-the part of the graded module.

The aim is to determine a formula for all the modules in the latter equation in terms of known information about Δ.

One first thing that we can use is the short exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{R} \longrightarrow \mathcal{R}/\mathcal{J} \longrightarrow 0.$$

It gives rise to the long exact sequence of homology modules

$$\cdots \to H_{i+1}(\mathcal{R}/\mathcal{J}) \to H_i(\mathcal{J}) \to H_i(\mathcal{R}) \to H_i(\mathcal{R}/\mathcal{J}) \to H_{i-1}(\mathcal{J}) \to \cdots \tag{2.3}$$

When Δ is supported on a topological d-ball (as is the case for our simplicial complexes), then $H_i(\mathcal{R}) = 0$ for every $i \neq d$ and $H_d(\mathcal{R}) = R$ [23, p. 181]. Then from the long exact sequence (2.3) it follows that $H_0(\mathcal{R}/\mathcal{J}) = 0$, and for all $i \leq d-1$

$$H_i(\mathcal{R}/\mathcal{J}) \cong H_{i-1}(\mathcal{J}). \tag{2.4}$$

In particular, we get the short sequence

$$0 \longrightarrow H_d(\mathcal{R}) \longrightarrow H_d(\mathcal{R}/\mathcal{J}) \longrightarrow H_{d-1}(\mathcal{J}) \longrightarrow 0,$$

and it follows

$$C^r(\hat{\Delta}) \cong R \oplus H_{d-1}(\mathcal{J}), \tag{2.5}$$

therefore, the study of the spline space reduces to the study of the homology module $H_{d-1}(\mathcal{J})$.

Let us recall that by definition, for $i = 0, \ldots, d$

$$H_i(\mathcal{J}) := \ker(\partial_i)/\operatorname{Im}\partial_{i+1} \tag{2.6}$$

and we have

$$\bigoplus_{\beta\in\Delta_i^0} \mathcal{J}(\beta) = \ker\partial_i \oplus \operatorname{Im}\partial_i, \tag{2.7}$$

where ∂_i are the maps in the chain complex $\mathcal{J}$.

The previous construction is valid in any dimension d, in particular for $d = 2$ and $d = 3$ that are the cases we want to explore here. The importance of the study of ideals of powers of linear forms is easily detectable.

Since $\mathcal{J}(\beta) = 0$ for all maximal faces β of Δ, then

$$\bigoplus_{\beta\in\Delta_d^0} \mathcal{R}/\mathcal{J}(\beta)_k = \bigoplus_{\beta\in\Delta_d^0} \mathcal{R}_k \quad \text{and hence} \quad \dim \bigoplus_{\beta\in\Delta_d^0} \mathcal{R}_k = f_d^0 \cdot \binom{k+d}{d}. \tag{2.8}$$

Also by definition $\mathcal{J}(\beta) = \langle \ell_\beta^{r+1}\rangle$, the ideal generated by the power $r+1$ of the linear form that vanishes on $\hat{\beta}$, for all $\beta \in \Delta_{d-1}^0$, thus

$$\dim \bigoplus_{\beta\in\Delta_{d-1}^0} \mathcal{R}/\mathcal{J}(\beta)_k = f_{d-1}^0 \cdot \left[\binom{k+d}{d} - \binom{k+d-(r+1)}{d}\right]. \tag{2.9}$$

Here and throughout the chapter, we adopt the convention that for $m, u \in \mathbb{Z}$ the binomial coefficient $\binom{m}{u} = 0$ if $m < u$.

Let us consider $\beta \in \Delta_i^0$ for some $0 \le i < d-1$. Observe that for a specific face β, we may make an affine change of coordinates and assume that the linear forms in $\mathcal{J}(\beta)$ involve only the variables $x_1, \ldots, x_{d-i}$. Hence

$$\mathcal{R}/\mathcal{J}(\beta) \cong \mathbb{R}[x_{d+1-i}, \ldots, x_{d+1}] \otimes_{\mathbb{R}} \mathbb{R}[x_1, \ldots, x_{d-i}]/\mathcal{J}(\beta).$$

Then in order to obtain the dimension of a spline space we need to analyze ideals generated by powers of linear forms in two, three,… and d-variables.

For instance, for a triangulation of a region in the plane, the ideals associated to the vertices ($\beta \in \Delta_0^0$, $d = 2$) are generated by linear forms in two variables; similarly the ideals corresponding to edges in a 3-dimensional partition ($\beta \in \Delta_1^0$, $d = 3$).

Schenck and Stillman [25] proved the following free resolution of ideals in two variables generated by powers of homogeneous linear forms (in [13] Geramita and Schenck extended this result by using *inverse systems of fat points*, they gave a completely characterization of the possible free resolutions for these kind of ideals allowing mixed powers).

Let $\mathcal{J}(\beta)$ be an ideal generated by $\ell_1^{r+1}, \ldots, \ell_t^{r+1}$ where ℓ_j for $j = 1, \ldots, t$ are linearly independent homogeneous linear forms in $\mathbb{R}[x_1, x_2]$. A free resolution of $\mathcal{R}/\mathcal{J}(\beta)$ is given by

$$0 \to \mathcal{R}(-\Omega-1)^a \oplus \mathcal{R}(-\Omega)^b \to \oplus_{j=1}^t \mathcal{R}(-r-1) \to \mathcal{R} \to \mathcal{R}/\mathcal{J}(\beta) \to 0 \tag{2.10}$$

where $\Omega - 1$ is the socle degree of $\mathcal{R}/\mathcal{J}(\beta)$; Ω and the multiplicities a and b are given by

$$\Omega = \left\lfloor \frac{t\,r}{t-1} \right\rfloor + 1, \quad a = t\,(r+1) + (1-t)\,\Omega, \quad b = t - 1 - a. \tag{2.11}$$

Thus, for a fixed polynomial degree k:

$$\dim \mathcal{R}/\mathcal{J}(\beta)_k = \binom{k+d}{d} - t\binom{k+d-(r+1)}{d} + b\binom{k+d-\Omega}{d} + a\binom{k+d-(\Omega+1)}{d}. \tag{2.12}$$

Considering each face β at a the time, the dimension of $\mathcal{R}/\mathcal{J}(\beta)_k$ is given by the previous formula. Summing them up we get the a dimension formula for the corresponding module in (2.2).

For ideals generated by powers of linear forms in three or more variables there is no resolution known. There is a formula conjectured by Fröberg [12] on the expected Hilbert series of an ideal generated by a generic set of forms (not necessarily powers of linear forms) in a polynomials ring in n-variables. Fröberg's conjecture has been proved true in some cases, for instance for $n = 2$, that is the generic situation of the module in (2.12), for $n = 3$ [3]. For other values of n, it has been proved for some particular cases that depend on the number of generators, see [15] for a detailed list. By using Fröberg's conjecture and its equivalent form known as the *maximal rank property*, one might find approximations for the dimension of the spline space, see the discussion in [22] where the case of tetrahedral partitions in $\mathbb{R}^3$ is studied. In Section 4 below, with $n = 3$, we use a special case for which Fröberg's sequence gives the exact dimension of the ideal and prove a lower bound on the dimension of the spline space $C^r_k(\varDelta)$.

As the dimension of the simplicial complexes increases, the homology modules become quite complicated. In Section 3 we consider splines spaces defined on triangulations embedded in $\mathbb{R}^2$ ($d = 2$), and in Section 4 spline spaces on simplicial complexes embedded in $\mathbb{R}^3$ ($d = 3$). We find lower and upper bounds for the homology modules and manipulating the formula (2.2) we deduce lower and upper bounds for the dimension of the spline space $C^r_k(\varDelta)$.

3. Dimension of bivariate triangular spline spaces

Let $\varDelta$ be a connected, finite two-dimensional simplicial complex supported on $|\varDelta| \subset \mathbb{R}^2$ which is homotopy equivalent to a disk. Applying (2.4), the fomula in (2.2) reduces to

$$\dim C^r_k(\varDelta) = \sum_{i=0}^{2}(-1)^i \sum_{\beta\in\varDelta^0_{2-i}} \dim \mathcal{R}/\mathcal{J}(\beta)_k + \dim H_0(\mathcal{J})_k. \tag{3.1}$$

Let us recall that in this setting the ring R is the polynomial ring in three variables $R = \mathbb{R}[x_1, x_2, x_3]$, corresponding to the homogenization $\hat{\varDelta}$ of $\varDelta$.

Theorem 3.1. *The dimension of $C^r_k(\varDelta)$ is bounded below by*

$$\begin{aligned}\dim C^r_k(\varDelta) \geq &\binom{k+2}{2} + f^0_1\binom{k+1-r}{2}\\ &-\sum_{i=1}^{f^0_0}\left[t_i\binom{k+1-r}{2} - b_i\binom{k+2-\Omega_i}{2} - a_i\binom{k+1-\Omega_i}{2}\right],\end{aligned}$$

where t_i is the number of different slopes of the edges containing the vertex γ_i, and

$$\Omega_i = \left\lfloor \frac{t_i\, r}{t_i - 1}\right\rfloor + 1, \quad a_i = t_i\,(r+1) + (1-t_i)\,\Omega_i, \quad b_i = t_i - 1 - a_i.$$

Proof. Since $\dim H_0(\mathcal{J})_k \geq 0$, and (2.8), (2.9) and (2.12) give us formulas for the remaining terms of (3.1), we get the lower bound given in the theorem. □

In contrast with the formula of the lower bound, the theorem below provides an upper bound on $\dim C^r_k(\varDelta)$. The ordering on the vertices plays an important role in the formula. Different orderings can give different upper bounds, and since the theorem can be applied to any ordering on the vertices in $\varDelta^0_0$, it leads to find the exact dimension in many cases [21], see Example 1 below.

Let $\gamma_1, \ldots, \gamma_{f^0_0}$ be an ordering on $\varDelta^0_0$. For each vertex γ_i, denote by $N(\gamma_i)$ the set of edges that contain γ_i, and define $\tilde{t}_i$ as the number of different slopes of the edges connecting γ_i to a vertex on the boundary or to one of the first $i-1$ vertices in the list.

Theorem 3.2. *The dimension of* $C_k^r(\Delta)$ *is bounded above by*

$$\dim C_k^r(\Delta) \leq \binom{k+2}{2} + f_1^0 \binom{k+1-r}{2}$$
$$- \sum_{i=1}^{f_0^0} \left[\tilde{t}_i \binom{k+1-r}{2} - \tilde{b}_i \binom{k+2-\tilde{\Omega}_i}{2} - \tilde{a}_i \binom{k+1-\tilde{\Omega}_i}{2} \right]$$

with $\tilde{t}_i$ *as we have defined above and*

$$\tilde{\Omega}_i = \left\lfloor \frac{\tilde{t}_i\, r}{\tilde{t}_i - 1} \right\rfloor + 1, \quad \tilde{a}_i = \tilde{t}_i\,(r+1) + (1-\tilde{t}_i)\,\tilde{\Omega}_i, \quad \tilde{b}_i = \tilde{t}_i - 1 - \tilde{a}_i.$$

If $\tilde{t}_i = 1$ *or* 0*, then* $\tilde{a}_i = \tilde{b}_i = \tilde{\Omega}_i = 0$.

Proof. In the case $d = 2$, the isomorphism in (2.5) gives

$$\dim C_k^r(\Delta) = \dim R_k + \dim H_1(\mathcal{J})_k.$$

We know that $H_1(\mathcal{J}) = \ker \partial_1$ by (2.6), and $\bigoplus_{\tau \in \Delta_1^0} \mathcal{J}(\tau) = \ker(\partial_1) \oplus \operatorname{Im}(\partial_1)$ by (2.7). Hence we can write

$$\dim C_k^r(\Delta) = \dim R_k + \sum_{\tau \in \Delta_1^0} \dim \mathcal{J}(\tau)_k - \dim(\operatorname{Im} \partial_1)_k$$

where ∂_1 is the map in the chain complex $\mathcal{J}$. Therefore, to find an upper bound on $\dim C_k^r(\Delta)$ it is enough to find a lower bound on the dimension of $\operatorname{Im} \partial_1$ in degree k. We define the maps δ, φ and π by R-linear extensions as follows.

Let us consider

$$\delta : \bigoplus_{\tau=(\gamma,\gamma')\in\Delta_1^0} \mathcal{J}(\tau)\,[\tau] \to \bigoplus_{\gamma\in\Delta_0^0} \bigoplus_{\tau\in N(\gamma)} R\,[\tau|\gamma]$$

such that $\delta([\tau]) = [\tau|\gamma] - [\tau|\gamma']$ for $\tau = (\gamma, \gamma') \in \Delta_1^0$, and the map

$$\varphi : \bigoplus_{\gamma\in\Delta_0^0} \bigoplus_{\tau\in N(\gamma)} R\,[\tau|\gamma] \to \bigoplus_{\gamma\in\Delta_0^0} R\,[\gamma]$$

defined by

$$\varphi([\tau|\gamma]) = \begin{cases} [\gamma] & \text{if } \gamma \in \Delta_0^0, \\ 0 & \text{if } \gamma \notin \Delta_0^0. \end{cases}$$

Then, we have $\partial_1 = \varphi \circ \delta$. We consider now the map

$$\pi : \bigoplus_{\gamma\in\Delta_0^0} \bigoplus_{\tau\in N(\gamma)} R\,[\tau|\gamma] \to \bigoplus_{\gamma\in\Delta_0^0} \bigoplus_{\tau\in N(\gamma)} R\,[\tau|\gamma]$$

such that $\pi([\tau|\gamma]) = 0$ if γ is the end point of biggest index of τ, and $\pi([\tau|\gamma]) = [\tau|\gamma]$ otherwise. Denote by $\tilde{\partial}_1 = \varphi \circ \pi \circ \delta$.

For every $\gamma \in \Delta_0^0$, let us define $\tilde{N}(\gamma)$ as the set of interior edges τ connecting γ to another vertex which is not of bigger index. Let us consider the ideal

$$\widetilde{\mathcal{J}}(\gamma) = \sum_{\tau\in\tilde{N}(\gamma)} R\,\ell_\tau^{r+1} \subseteq \mathcal{J}(\gamma).$$

By construction, we have

$$\operatorname{Im} \tilde{\partial}_1 = \bigoplus_{\gamma\in\Delta_0^0} \widetilde{\mathcal{J}}(\gamma)\,[\gamma]$$

and $\dim(\operatorname{Im} \partial_1)_k \geq \dim(\operatorname{Im} \tilde{\partial}_1)_k$.

Since for each edge $\tau \in \Delta_1^0$ its correspondent linear form ℓ_τ appears among the generators of the ideal $\widetilde{\mathcal{J}}(\gamma)$ for (only) one interior vertex, then

$$\dim \operatorname{Im} \tilde{\partial}_1 = \sum_{i=1}^{f_0^0} \left[\tilde{t}_i \binom{k+2-(r+1)}{2} - \tilde{b}_i \binom{k+2-\tilde{\Omega}_i}{2} - \tilde{a}_i \binom{k+2-(\tilde{\Omega}_i+1)}{2} \right]$$

with $\tilde{t}_i = |\tilde{N}(\gamma_i)|$ and $\tilde{\Omega}_i$, $\tilde{a}_i$, $\tilde{b}_i$ as defined in (2.11). This gives a lower bound on $\dim \operatorname{Im} \partial_1$ and proves the theorem. □

Example 1. Effect of the ordering of the vertices on the upper bound.

Let Δ be the triangulated polygon in Fig. 1, and consider three different numberings of the interior vertices as shown in (1), (2) and (3).

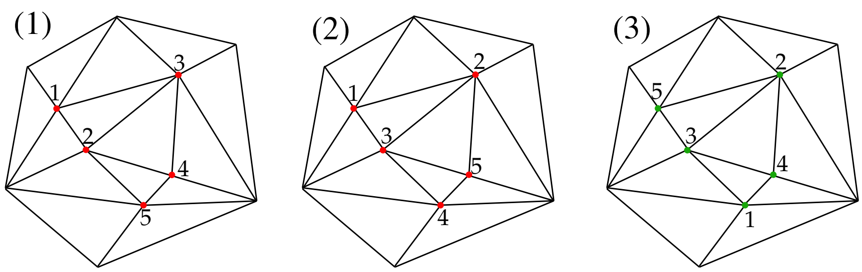

FIGURE 1. Effect of the numbering on the upper bound.

By Theorem 3.1, $\dim C_2^1(\Delta) \geq 9$. By Theorem 3.2 for the numbering (1), the upper bound on $\dim C_2^1(\Delta)$ is 11, for the numbering (2) the upper bound is 10, and in fact there is a numbering that give us 9 as upper bound, namely the one shown in (3). Thus, we can compute the exact dimension by combining these two bounds, $\dim C_2^1(\Delta) = 9$.

Example 2. Let us consider the triangulation in Fig. 2. It is easy to see that the lower bound on $\dim C_k^r(\Delta)$ always equals the upper bound, for any k and r.

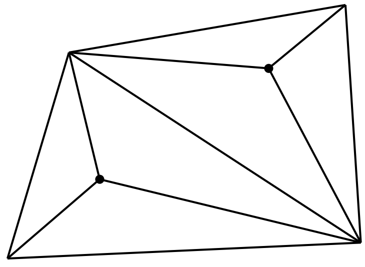

FIGURE 2. Triangulation with two no connected interior vertices.

Thus, the dimension of the spline space is given by the formula

$$\dim C_k^r(\Delta) = \binom{k+2}{2} + \binom{k+1-r}{2} + 2\binom{k-1}{2}.$$

4. Dimension of trivariate splines

Let Δ be a connected, finite 3-dimensional simplicial complex, supported on $|\Delta| \subset \mathbb{R}^3$, such that $|\Delta|$ is homotopy equivalent to a 3-dimensional ball.

In this setting the formula in (2.2) takes the form

$$\dim C_k^r(\Delta) = \sum_{i=0}^{3}(-1)^i \sum_{\beta\in\Delta_{3-i}^0} \dim \mathcal{R}/\mathcal{J}(\beta)_k + \dim H_1(\mathcal{J})_k - \dim H_0(\mathcal{J})_k \tag{4.1}$$

and by the equalities (2.6) and (2.7)

$$= \dim R_k + \dim \bigoplus_{\sigma\in\Delta_2} \mathcal{J}(\sigma)_k - \dim \operatorname{Im}(\partial_2)_k, \tag{4.2}$$

where ∂_2 is the corresponding map in the chain complex $\mathcal{J}$:

$$0 \longrightarrow \bigoplus_{\sigma\in\Delta_2^0} \mathcal{J}(\sigma) \xrightarrow{\partial_2} \bigoplus_{\tau\in\Delta_1^0} \mathcal{J}(\tau) \xrightarrow{\partial_1} \bigoplus_{\gamma\in\Delta_0^0} \mathcal{J}(\gamma) \longrightarrow 0 \tag{4.3}$$

The ring R is in this case the polynomial ring in four variables $R = \mathbb{R}[x_1, x_2, x_3, x_4]$, corresponding to the homogenization $\hat{\Delta}$ of Δ.

From (2.8) and (2.9), we have explicit formulas for the first two terms in (4.2), and thus, in order to find an upper bound on $\dim C_k^r(\Delta)$ we need to find a lower bound on the dimension of $\operatorname{Im}(\partial_2)$ in degree k. We proceed in an analogous way as in the proof of Theorem 3.2 in the previous section.

To find an upper bound, let us consider a numbering on the interior edges τ_i of Δ, say $\tau_1, \ldots, \tau_{f_1^0}$. For each $i = 1, \ldots, f_1^0$, let s_i be the number of different linear forms defining the hyperplanes incident to τ_i, and define $\tilde{s}_i$ as the number of different linear forms defining the hyperplanes incident to τ_i, which correspond to triangles whose other two edges are either on $\partial\Delta$, or have index smaller than i. See Example 3.

Example 3. Numbering on the interior edges.

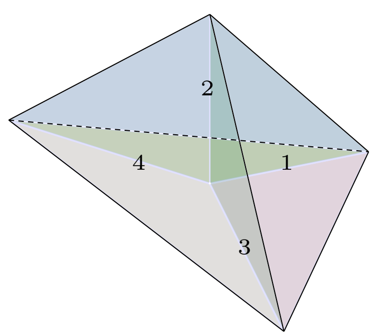

FIGURE 3. Clough–Tocher split.

Let Δ be the Clough–Tocher split consisting of a tetrahedron which has been split about an interior point into four subtetrahedra, see Fig. 3.

Let us consider the numbering on the edges as in the figure. In this case, three different planes meet at each interior edges of the partition, then $s_i = 3$ for $i = 1, \ldots, 4$. On the other hand, following the counting and the definition above, $\tilde{s_1} = 0$, $\tilde{s_2} = 1$, $\tilde{s_3} = 2$, and $\tilde{s_4} = 3$.

For each $\tau_i \in \Delta_1^0$, denote by $\widetilde{\mathcal{J}}(\tau_i)$ the ideal generated by the $r + 1$ powers of the linear forms defining the $\tilde{s}_i$ hyperplanes.

Let us notice that by construction, for the edges $\tau_i \in \Delta_i^0$ of each triangle σ in Δ_2^0, the linear form ℓ_σ is among the generators of $\widetilde{\mathcal{J}}(\tau_i)$ for only one τ_i, namely that with highest index.

Theorem 4.1. *The dimension of $C_k^r(\Delta)$ is bounded above by*

$$\dim C_k^r(\Delta) \leq \binom{k+3}{3} + f_2^0 \binom{k+2-r}{3} - \sum_{i=1}^{f_1^0} \left[\tilde{s}_i \binom{k+2-r}{3} - \tilde{b}_i \binom{k+3-\tilde{\Omega}_i}{3} - \tilde{a}_i \binom{k+2-\tilde{\Omega}_i}{3} \right]$$

with $\tilde{s}_i$ as we have defined above and

$$\tilde{\Omega}_i = \left\lfloor \frac{\tilde{s}_i\, r}{\tilde{s}_i - 1} \right\rfloor + 1, \quad \tilde{a}_i = \tilde{s}_i\,(r+1) + (1-\tilde{s}_i)\,\tilde{\Omega}_i, \quad \tilde{b}_i = \tilde{s}_i - 1 - \tilde{a}_i.$$

If $\tilde{s}_i = 1$ or 0 then $\tilde{a}_i = \tilde{b}_i = \tilde{\Omega}_i = 0$.

Proof. We define the maps δ, φ and π by R-linear extensions as follows. Let δ be the map

$$\delta : \bigoplus_{\sigma=(\tau,\tau',\tau'')} \mathcal{J}(\sigma)[\sigma] \to \bigoplus_{\tau_i \in \Delta_1^0} \bigoplus_{\sigma \in N(\tau_i)} \mathcal{R}[\sigma|\tau_i]$$

where, for each i, $N(\tau_i)$ denotes the set of triangles that contain the edge τ_i. The map δ is induced by the boundary map ∂_2. Thus, $\delta([\sigma]) = [\sigma|\tau] - [\sigma|\tau'] + [\sigma|\tau'']$ for $\sigma = (\tau, \tau', \tau'') \in \Delta_2^0$, see Fig. 4.

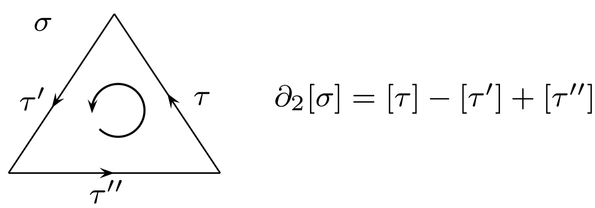

FIGURE 4. Orientation of a triangle $\sigma \in \Delta_2^0$.

Let

$$\varphi : \bigoplus_{\tau_i \in \Delta_1^0} \bigoplus_{\sigma \in N(\tau_i)} \mathcal{R}[\sigma|\tau_i] \to \bigoplus_{\tau_i \in \Delta_1^0} \mathcal{R}[\tau_i]$$

with

$$\varphi([\sigma|\tau_i]) = \begin{cases} [\tau_i] & \text{if } \tau_i \in \Delta_1^0 \\ 0 & \text{if } \tau_i \notin \Delta_1^0. \end{cases}$$

Then $\partial_2 = \varphi \circ \delta$. We consider the map

$$\pi : \bigoplus_{\tau_i \in \Delta_1^0} \bigoplus_{\sigma \in N(\tau_i)} \mathcal{R}[\sigma|\tau_i] \to \bigoplus_{\tau_i \in \Delta_1^0} \bigoplus_{\sigma \in N(\tau_i)} \mathcal{R}[\sigma|\tau_i]$$

defined by $\pi([\sigma|\tau]) = [\sigma|\tau]$, $\pi([\sigma|\tau']) = \pi([\sigma|\tau'']) = 0$ if $\sigma = (\tau, \tau', \tau'')$ and either τ' or τ'' are on the boundary of Δ or the index assigned to them is bigger than the assigned to τ. Denote by $\tilde{\partial}_2 = \varphi \circ \pi \circ \delta$.

For $\tau_i \in \Delta_1^0$, let $\tilde{N}(\tau_i)$ be the set of triangles $\sigma \in \Delta_2^0$ that contain τ as an edge and whose other two edges do not have bigger index than the index of τ. Then we have,

$$\widetilde{\mathcal{J}}(\tau_i) = \sum_{\sigma \in \tilde{N}(\tau_i)} R\ell_\sigma^{r+1} \subseteq \mathcal{J}(\tau_i).$$

By construction,

$$\operatorname{Im} \tilde{\partial}_2 = \bigoplus_{\tau_i \in \Delta_1^0} \widetilde{\mathcal{J}}(\tau_i)[\tau_i]$$

and $\dim \mathrm{Im}\,(\partial_2)_k \geq \dim \mathrm{Im}\,(\tilde{\partial}_2)_k$. Thus, from the formula for $\dim C^r_k(\Delta)$ given in (4.2), it follows

$$\dim C^r_k(\Delta) \leq \dim \mathcal{R}_k + \dim \bigoplus_{\sigma\in\Delta_2} \mathcal{J}(\sigma)_k - \dim \mathrm{Im}\,(\tilde{\partial}_2).$$

By an affine change of coordinates we may assume the edge τ_i to be along one of the coordinate edges, and thus the linear forms in $\widetilde{\mathcal{J}}(\tau_i)$ only involve two variables. Then $\dim \mathrm{Im}\,(\tilde{\partial}_2)_k$ is given by (2.12) with $d = 3$, $\tilde{s}_i = |\tilde{N}(\tau_i)|$, and $\tilde{\Omega}_i$, $\tilde{a}_i$ and $\tilde{b}_i$ as given by the formulas (2.11) with $\tilde{s}_i$ instead of t_i. This together with (2.8) and (2.9) proves the theorem. □

Our aim now is to provide a lower bound on the dimension of $C^r_k(\Delta)$ to complement the upper bound in Theorem 4.1.

Let us take zero as a lower bound for $\dim H_1(\mathcal{J})_k$, then from (4.1), for any $k \geq 0$:

$$\dim C^r_k(\Delta) \geq \dim R_k + \sum_{i=1}^{2}(-1)^i \dim \bigoplus_{\beta\in\Delta^0_{3-i}} \mathcal{J}(\beta)_k + \dim \mathrm{Im}\,(\partial_1)_k \tag{4.4}$$

Since we have explicit formulas for all the terms in (4.4) except for the last one. We can establish a lower bound on $\dim C^r_k(\Delta)$ by finding a lower bound on $\dim \mathrm{Im}\,(\partial_1)$.

We proceed analogously as before. This time we give an ordering to the vertices in the interior of Δ. For each vertex γ_i, we denote by $M(\gamma_i)$ the set of edges τ in Δ^0_1 that contain the vertex γ_i, and by $\tilde{M}(\gamma_i)$ be the set of interior edges connecting γ_i to one of the first $i-1$ vertices in the list, or to a vertex in the boundary.

For each $\gamma_i \in \Delta^0_0$, let t_i be defined as before, the number of generators of $\mathcal{J}(\gamma_i)$. Define the ideal $\widetilde{\mathcal{J}}(\gamma_i)$ as

$$\widetilde{\mathcal{J}}(\gamma_i) = \langle \ell^{r+1}_\sigma \rangle_{\sigma\ni\tau} \quad \text{for } \tau \in \tilde{M}(\gamma_i),$$

and let $\tilde{t}_i$ be the number of generators of $\widetilde{\mathcal{J}}(\gamma_i)$. An analogous argument to the one we used in the proofs above leads to the following lemma.

Lemma 4.2. *For a three dimensional simplicial complex Δ, the dimension of the spline space $C^r_k(\Delta)$ is bounded by*

$$\dim C^r_k(\Delta) \geq \dim R_k + \sum_{i=1}^{2}(-1)^i \dim \bigoplus_{\beta\in\Delta^0_{3-i}} \mathcal{J}(\beta)_k + \dim \bigoplus_{i=1}^{f^0_0} \widetilde{\mathcal{J}}(\gamma_i)_k. \tag{4.5}$$

Proof. Follows from (4.4), and the argument as before. □

We have explicit formulas for the terms in (4.5), except for $\dim \bigoplus_{i=1}^{f^0_0} \widetilde{\mathcal{J}}(\gamma_i)_k$. This term involves ideals generated by powers of linear forms in three variables, which correspond to equations of planes going through a common point.

As we mentioned above, there is a formula $F(t, r+1, n)$ conjectured by Fröberg [12] concerning the dimension of ideals generated by a generic set of forms (not necessarily powers of linear forms) in a polynomial ring in n variables. This conjecture, in particular, was proved for the case $n = 3$ of forms in three variables [3]. But since the conjectured dimension is not true in general when the generators are powers of linear forms [22], such formula only gives us a lower bound on the dimension of $\mathcal{R}/\mathcal{J}(\gamma)_k$ for every vertex γ, namely

$$\dim \mathcal{R}/\mathcal{J}(\gamma)_k \geq \sum_{j=0}^{k} F(t, r+1, 3)_j, \tag{4.6}$$

where t is the number of linear forms that corresponds to the different planes that contain the vertex γ in Δ. Equality holds in (4.6) when $t \leq 3$ for each vertex $\gamma \in \Delta^0_0$ [15].

The formula $F(t, r+1, n)$ associated to the Hilbert function of an ideal generated by t forms of degree $r+1$ in a polynomial ring of $n=3$ variables over $\mathbb{R}$ (or any field of characteristic zero) is be defined as follows, for $j \geq 0$

$$F(t, r+1, 3)_j = \begin{cases} F'(t, r+1, 3)_j, & \text{if } F'(t, r+1, 3)_u > 0 \text{ for all } u \leq j, \\ 0 & \text{otherwise;} \end{cases} \tag{4.7}$$

where $F'(t, r+1, 3)_j$ is given by

$$F'(t, r+1, 3)_j = \dim R_j + \sum_{v=1}^{3} (-1)^v \dim R_{j-(r+1)v} \binom{t}{v}.$$

It is a particular case of [15, Theorem 1.6].

Theorem 4.3. *The dimension* $\dim C_k^r(\varDelta)$ *is bounded below by*

$$\begin{aligned} \dim C_k^r(\varDelta) \geq \binom{k+3}{3} + \Bigg[f_2^0 \binom{k+2-r}{3} \\ - \sum_{i=1}^{f_1^0} \left[s_i \binom{k+2-r}{3} - b_i \binom{k+3-\Omega_i}{3} - a_i \binom{k+2-\Omega_i}{3} \right] \\ + f_0^0 \binom{k+3}{3} - \sum_{i=1}^{f_0^0} \left(\sum_{j=0}^{k} F(\zeta_i, r+1, 3)_j \right) \Bigg]_+ \end{aligned} \tag{4.8}$$

with $\zeta_i = \min(3, \tilde{t}_i)$, s_i *as defined above, and*

$$\Omega_i = \left\lfloor \frac{s_i\, r}{s_i - 1} \right\rfloor + 1, \quad a_i = s_i\,(r+1) + (1 - s_i)\,\Omega_i, \quad b_i = s_i - 1 - a_i.$$

Proof. It is clear from Lemma 4.2 and the previous remarks. Since the dimension of the spline space is at least the number of polynomials in tree variables of degree less than or equal to k, then we take the positive part of the additional terms in (4.8). See [22] for the detailed proof. □

Remark 4.4. The lower bound on $C_k^r(\varDelta)$ in the previous theorem can be improved if the linear forms defining the ideals $\mathcal{J}(\gamma_i)$ are generic, or if the Hilbert function of ideals generated by powers of $\tilde{t}_i \geq 4$ linear forms in three variables is known, in which case one might avoid the step of taking $\zeta_i = \min(3, \tilde{t}_i)$. In consequence, the results in algebraic geometry about the Hilbert function of ideals of powers of linear forms and the related ideal of fat points would significantly improve results concerning the dimension of spline spaces.

For the central configurations that we will consider in this section, it is easy to see that $H_0(\mathcal{J})$ is always zero, see [22] for more details.

Example 4. Let $\varDelta$ be a octahedron subdivided into eight tetrahedra by placing a symmetric central vertex, see Fig. 5.

Computations show that $H_1(\mathcal{J})$ is zero for all non-generic octahedra [24]. Since in this partition, there are exactly three different planes through the central vertex, then the Fröberg sequence gives us an explicit formula for the dimension of the ideal associated to the (unique) interior vertex. Hence

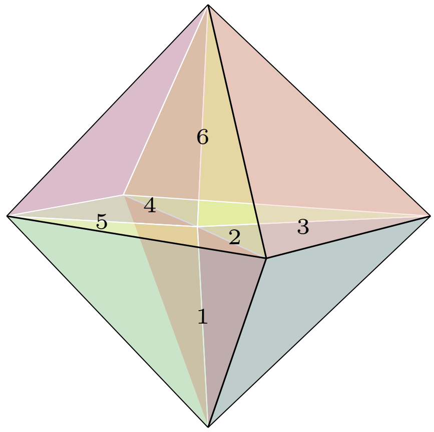

FIGURE 5. Regular octahedron.

the dimension $\dim C_k^r(\Delta)$ can be directly computed using (4.1) as follows,

$$\begin{aligned}\dim C_k^r(\Delta) =& \binom{k+3}{3} + 12\binom{k+3-(r+1)}{3} \\ & - \sum_{i=1}^{6}\left[2\binom{k+3-(r+1)}{3} - \binom{k+3-(2r+2)}{3}\right] \\ & + \binom{k+3}{3} - \sum_{j=0}^{k} F(3, r+1, 3)_j .\end{aligned}$$

From the definition of Fröberg's sequence (4.7),

$$F(3, r+1, 3)_j = \binom{j+2}{2} - 3\binom{j+1-r}{2} + 3\binom{j-2r}{2} - \binom{j-3r-1}{2}.$$

It is easy to check that $F(3, r+1, 3)_j > 0$ for every $0 \leq j < 3r+3$, and equal to zero otherwise. Hence, we can write

$$\sum_{j=0}^{k} F(3, r+1, 3)_j = \binom{k+3}{3} - 3\binom{k+2-r}{3} + 3\binom{k-2r+1}{3} - \binom{k-3r}{3} \tag{4.9}$$

and thus, the formula for the dimension of the spline space on the regular octahedron in Fig. 5 is given by the expression

$$\dim C_k^r(\Delta) = \binom{k+3}{3} + 3\binom{k+2-r}{3} + 3\binom{k+1-2r}{3} + \binom{k-3r}{3}.$$

Example 5. Let us consider the generic case of an octahedron subdivided into tetrahedra, where no set of four vertices of the octahedron is coplanar, Fig. 6. As we mentioned above, we have $H_0(\mathcal{J})$ equal to zero. But in contrast to the regular case, $H_1(\mathcal{J})$ is equal to zero when $r = 1$ but not for any other value of r [24].

For this partition Δ, we have $t = 12$ different planes corresponding to the triangles meeting at the central vertex. Then $\zeta = \min(3, t) = 3$, and using the formula (4.9) from the previous example for

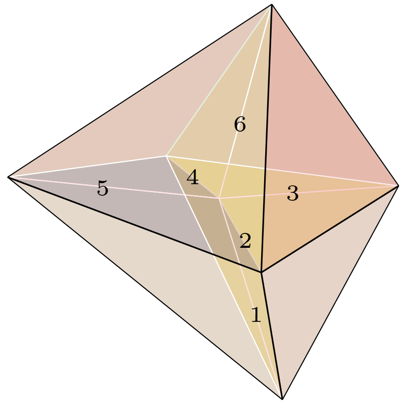

FIGURE 6. Generic octahedron.

the sum of the $F(3, r+1, 3)_j$ for $r = 1$, Theorem (4.3) gives us the following lower bound

$$\dim C_k^1(\Delta) \geq \binom{k+3}{3} + \Bigg[12\binom{k+1}{3} - 6\bigg[3\binom{k+1}{3} - 2\binom{k}{3}\bigg] + \binom{k+3}{3} - \sum_{j=0}^{k} F(3,2,3)_j\Bigg]_+$$
$$= \binom{k+3}{3} + \left[-3\binom{k+1}{3} + 12\binom{k}{3} - 3\binom{k-1}{3} + \binom{k-3}{3}\right]_+.$$

In order to find an upper bound, we apply Theorem 4.1 for some ordering on the interior edges of the partition. For instance, with the numbering on the edges as in Fig. 6, we have $\tilde{s}_1 = 0$, $\tilde{s}_2 = 1$, $\tilde{s}_3 = \tilde{s}_4 = 2$, $\tilde{s}_5 = 3$, and $\tilde{s}_6 = 4$, and so for any degree k:

$$\dim C_k^1(\Delta) \leq \binom{k+3}{3} + \binom{k+1}{3} + 4\binom{k}{3} + 2\binom{k-1}{3}.$$

Example 6. Let Δ be the Clough–Tocher split consisting of a tetrahedron which has been split about an interior point into four subtetrahedra, Fig. 3.

We consider $r = 1$ and $r = 2$. In these two cases the homology module $H_1(\mathcal{J})$ is zero.

(i) For $r = 1$, as in the previous example, we have

$$\sum_{j=0}^{k} F(3,2,3)_j = \binom{k+3}{3} - 3\binom{k+1}{3} + 3\binom{k-1}{3} - \binom{k-3}{3}$$

Then, the lower bound on the spline space proved in Theorem 4.3 is given by

$$\dim C_k^1(\Delta) \geq \binom{k+3}{3} + \left[-3\binom{k+1}{3} + 8\binom{k}{3} - 3\binom{k-1}{3} + \binom{k-3}{3}\right]_+.$$

The upper bound we obtained in this example, by applying Theorem 4.1 with the numbering of the edges as in Fig. 3 is the following:

$$\dim C_k^1(\Delta) \leq \binom{k+3}{3} + \binom{k-1}{3} + 2\binom{k}{3}.$$

Since for $r = 1$ the homology module $H_1(\mathcal{J}) = 0$, then we can apply (4.1) and the bound (4.6) with $t = 6$ in the sequence (4.7) for the (unique) interior vertex in Δ. This leads to the following upper bound

$$\dim C^1_k(\Delta) \leq \begin{cases} 1 & \text{for } k = 0 \\ 2\binom{k+3}{3} - 6\binom{k+1}{3} + 8\binom{k}{3} - 4 & \text{for } k \geq 1. \end{cases} \tag{4.10}$$

The formula (4.10) coincides with the generic dimension formula computed in [2] for this partition Δ. Although the formula in [2] holds only for $k \geq 8$ (and $r = 1$), it in turn coincides with the lower bound formula proved in [1] in every degree $k \geq 0$. In fact, in general, the dimension of the spline space of any nongeneric decomposition is always greater than or equal to the generic dimension, it is the smallest dimension encountered as one moves the vertices of the complex. Thus, since the lower bound formula proved in [1] coincides with the upper bound we proved above (4.10), we deduce the following result:
the exact dimension of the C^1 spline space over the Clough–Tocher split is

$$\dim C^1_k(\Delta) = \begin{cases} 1 & \textit{for } k = 0 \\ 2\binom{k+3}{3} - 6\binom{k+1}{3} + 8\binom{k}{3} - 4 & \textit{for } k \geq 1. \end{cases}$$

(ii) Let us consider the case $r = 2$.
A lower bound is given by the formula

$$\dim C^2_k(\Delta) \geq \binom{k+3}{3} + \left[-3\binom{k}{3} + 4\binom{k-1}{3} + 4\binom{k-2}{3} - 3\binom{k-3}{3} + \binom{k-6}{3}\right]_+$$

Using that $H_1(\mathcal{J}) = 0$, and (4.1), (4.6) and (4.7) as before, the following is an upper bound for $k \geq 3$:

$$\dim C^2_k(\Delta) \leq 2\binom{k+3}{3} - 6\binom{k}{3} + 4\binom{k-1}{3} + 4\binom{k-2}{3} - 14$$

The values of the previous bounds on $\dim C^2_k(\Delta)$ for $k \leq 9$ are given in the following table. The first row shows the values obtained using the lower bound formula from [1].

k	1	2	3	4	5	6	7	8	9
Lower bound [1]	4	10	20	35	56	84	120	179	261
Lower bound	4	10	20	35	56	84	123	187	282
Upper bound	4	10	20	36	58	90	136	200	286

Remark 4.5. The examples above illustrate the improvement that our lower and upper bounds provide with respect to previous results in the literature. Furthermore, as we showed in the last example, the formulas we presented here might be combined with results obtained by using different techniques leading thus to sharper bounds, and in many cases to the exact dimension of the space.

Remark 4.6. The approaches we use in this work differ from the ones used before to find bounds on the dimension of a spline space defined on a simplicial complex, see [16] and the references therein. The results and examples we presented give an insight into ways of improving the bounds and finding the exact dimension formula under certain conditions. In [22] and [32], the reader can find a more extended discussion on the relationship between splines and fat points, and the connection of that theory with the Weak Lefschetz Property [14, 18], Hilbert series of ideals of powers of generic linear forms, and Fröberg's conjecture and its most recent versions [8].

References

[1] P. Alfeld and L. L. Schumaker, *Bounds on the dimensions of trivariate spline spaces*, Adv. Comput. Math. **29** (2008), no. 4, 315–335.

[2] P. Alfeld, L. L. Schumaker, and W. Whiteley, *The generic dimension of the space of C^1 splines of degree $d \geq 8$ on tetrahedral decompositions*, SIAM J. Numer. Anal. **30** (1993), no. 3, 889–920.

[3] D. J. Anick, *Thin algebras of embedding dimension three*, J. Algebra **100** (1986), no. 1, 235–259.

[4] L. J. Billera, *Homology of smooth splines: generic triangulations and a conjecture of Strang*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 325–340.

[5] ———, *The algebra of continuous piecewise polynomials*, Adv. Math. **76** (1989), no. 2, 170–183.

[6] L. J. Billera and L. L. Rose, *A dimension series for multivariate splines*, Discrete Comput. Geom. **6** (1991), no. 2, 107–128.

[7] ———, *Modules of piecewise polynomials and their freeness*, Math. Z. **209** (1992), no. 4, 485–497.

[8] C. Ciliberto, *Geometric aspects of polynomial interpolation in more variables and of Waring's problem*, European Congress of Mathematics, Vol. I (Barcelona, 2000). Progr., Math. 201, Birkhäuser, Basel (2001), 289–316.

[9] J. A. Cottrell, T. J. R. Hughes, and Y. Bazilevs, *Isogeometric analysis: toward integration of cad and fea*, John Wiley& sons, Ltd, 2009.

[10] G. Farin, *Curves and surfaces for computer aided geometric design*, 3rd ed., Computer Science and Scientific Computing, Academic Press Inc., 1993.

[11] G. Fix and G. Strang, *An analysis of the finite element method*, Wellesley-Cambridge Press, 2008.

[12] R. Fröberg, *An inequality for Hilbert series of graded algebras*, Math. Scand. **56** (1985), no. 2, 117–144.

[13] A. V. Geramita and H. Schenck, *Fat points, inverse systems, and piecewise polynomial functions*, J. Algebra **204** (1998), no. 1, 116–128.

[14] B. Harbourne, H. Schenck, and A. Seceleanu, *Inverse systems, Gelfand-Tsetlin patterns and the weak Lefschetz property*, J. Lond. Math. Soc. (2) **84** (2011), no. 3, 712–730.

[15] A. Iarrobino, *Inverse system of a symbolic power. III. Thin algebras and fat points*, Compositio Math. **108** (1997), no. 3, 319–356.

[16] M-J. Lai and L. L. Schumaker, *Spline functions on triangulations*, Encyclopedia of Mathematics and its Applications, vol. 110, Cambridge University Press, 2007.

[17] W. Lau, *A lower bound for the dimension of trivariate spline spaces*, Constr. Approx. **23** (2006), no. 1, 23–31.

[18] J. Migliore, R. Miró-Roig, and U. Nagel, *On the weak Lefschetz property for powers of linear forms*, Algebra & Number Theory **6** (2012), no. 3, 487–526.

[19] J. Morgan and R. Scott, *The dimension of the space of C^1 piecewise polynomials* (1975). Unpublished manuscript, avalible at: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.42.4635.

[20] ———, *A nodal basis for C^1 piecewise polynomials of degree $n \geq 5$*, Math. Comput. **29** (1975), 736–740.

[21] B. Mourrain and N. Villamizar, *Homological techniques for the analysis of the dimension of triangular spline spaces*, J. Symbolic Comput. **50** (2013), 564–577.

[22] ———, *Bounds on the dimension of trivariate spline spaces: a homological approach*, Math. Comput. Sci. **8** (2014), no. 2, 157–174.

[23] J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[24] H. Schenck, *A spectral sequence for splines*, Adv. in Appl. Math. **19** (1997), no. 2, 183–199.

[25] H. Schenck and M. Stillman, *A family of ideals of minimal regularity and the Hilbert series of $C^r(\hat{\Delta})$*, Adv. in Appl. Math. **19** (1997), no. 2, 169–182.

[26] ———, *Local cohomology of bivariate splines*, J. Pure Appl. Algebra **117/118** (1997), 535–548. Algorithms for algebra (Eindhoven, 1996).

[27] I. J. Schoenberg, *Contributions to the problem of approximation of equidistant data by analytic functions. Part A. On the problem of smoothing or graduation. A first class of analytic approximation formulae. Part B. On the problem of osculatory interpolation. A second class of analytic approximation formulae*, Quart. Appl. Math. **4** (1946), 45–99, 112–141.

[28] L. L. Schumaker, *Bounds on the dimension of spaces of multivariate piecewise polynomials*, Rocky Mountain J. Math. **14** (1984), no. 1, 251–264. Surfaces (Stanford, Calif., 1982).

[29] E. H. Spanier, *Algebraic topology*, McGraw-Hill Book Co., 1966.

[30] G. Strang, *Piecewise polynomials and the finite element method*, Bull. Amer. Math. Soc. **79** (1973), 1128–1137.

[31] ———, *The dimension of piecewise polynomial spaces, and one-sided approximation*, Conference on the Numerical Solution of Differential Equations (Univ. Dundee, Dundee, 1973), 1974, pp. 144–152. Lecture Notes in Math., Vol. 363.

[32] N Villamizar, *Algebraic Geometry for Splines*, Doctoral dissertation, University of Oslo, Norway http://urn.nb.no/URN:NBN:no-38634 , 2012.

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