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APPROXIMATE CONTROLLABILITY FOR EQUATIONS OF FLUID MECHANICS WITH A FEW BODY CONTROLS

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Abstract. The approximate controllability by means of a few controls is proven for 2D Navier–Stokes system in an infinite channel under generalized Navier boundary conditions in the finite direction and the periodicity assumption in the infinite direction. The set of controls is given explicitly, and do not depend on the viscosity of the fluid.

The case of 1D Burgers system in a bounded interval under Dirichlet boundary conditions, where the controls are supported in an arbitrary small nonempty open subset, is also discussed.

Contents

1. Introduction

Let D be a two-dimensional cylinder $D = \left(\frac{a}{2\pi}\mathbb{S}^1\right) \times (0,\, b) \sim (0,\, a) \times (0,\, b)$, with boundary $\partial D = (0,\, a) \times \{0,\, b\}$. This corresponds to the case we consider the system in an infinite 2D channel and assume that in the infinite direction the velocity of the fluid is periodic, with period a.

The Navier–Stokes system, in $(0,\, T)\times D$, under generalized Navier boundary conditions and controlled by a body forcing η, reads

$$\begin{aligned} &\partial_t u + \langle u\cdot\nabla\rangle u - \nu\Delta u + \nabla p_u + h = \eta, \quad \operatorname{div} u = 0,\\ &u\cdot \mathbf{n}\big|_{\partial D} = 0, \quad \operatorname{curl} u\,\big|_{\partial D} = \beta u \wedge \mathbf{n}\big|_{\partial D}\,,\\ &u(0,\, x) = u_0(x), \end{aligned} \tag{1}$$

where η is a control at our disposal. Furthermore, as usual $u = (u_1,\, u_2)$ and p_u, defined for $(t,\, x) = (t,\, x_1,\, x_2) \in I \times D$, are the unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity; the operators ∇ and Δ are respectively the well known gradient and Laplacean in the space variables $(x_1,\, x_2)$; $\langle u\cdot\nabla\rangle v$ stands for $(u\cdot\nabla v_1,\, u\cdot\nabla v_2)$, $\operatorname{div} u := \partial_{x_1} u_1 + \partial_{x_2} u_2$, $\operatorname{curl} u := -\partial_{x_2} u_1 + \partial_{x_1} u_2$, $\mathbf{n}$ is the unit outward normal vector to ∂D, $u\wedge\mathbf{n} := u_1\mathbf{n}_2 - u_2\mathbf{n}_1$; and h and β are fixed functions.

We consider also the controlled Burgers system in an interval $I = (0,\, L) \subset \mathbb{R}$, $L > 0$:

$$\begin{aligned} &\partial_t u + u\partial_x u - \nu\partial_{xx} u + h = \zeta, \quad u|_{\partial I} = 0,\\ &u(0,\, x) = u_0(x), \end{aligned} \tag{2}$$

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Here ζ is a control taking values in the space of square-integrable functions in I, whose support, in x, is contained in a given open subset $\mathcal{O} \subseteq I$, u stands again for the unknown velocity of the fluid, $\nu > 0$ for the viscosity, h for a fixed function, and ∂I for the boundary $\{0, L\}$ of I.

The aim of this work is to give a finite number of suitable functions (i.e., controllers) $\mathcal{C} = \{\Phi_i \mid i \in \{1, 2, \ldots, M\}\}$ which enable us to drive the systems (1) and (2) from $u(0) := u(0, x) = u_0$ at time $t = 0$, to $u(T) := u(T, x) \approx u_1$ at time T, for any a priori given u_1, with controls taking values in $\operatorname{span} \mathcal{C}$. Further, for each system, the set $\mathcal{C}$ should be independent of (u_0, u_1, T, ν). We underline, in particular, the independence of $\mathcal{C}$ on ν.

To find $\mathcal{C}$ independent of (u_0, u_1, T, ν), we follow a procedure introduced in [AS05] to prove the analogous result for 2D Navier–Stokes equation under periodic boundary conditions. The same procedure (or slight variations) has been used in other works in cases of different equations, different domains (e.g., manifolds), and different boundary conditions, and also to prove some other type of controllability properties. We refer the reader to [AS05, AS06, AS08, AKSS07, Sar12, Shi06, Shi07b, Shi07a, Shi08a, Shi08b, Shi10, Rod08, Rod06, Rod05, Rod07a, Rod07b, Ner10, Ner11, CMSB09].

Concerning the Navier boundary conditions we would like to say that they have been addressed by many authors in the last years, either because in some situations they may be more realistic than the no-slip boundary conditions, $u|_{\partial D} = 0$, or because they are more appropriate in finding a solution for the Euler system as a limit of solutions for the Navier–Stoles system as ν goes to zero (cf. [XX07, WXZ12], [Kel06, section 8]), or even because of the possibility to recover the solution under no-slip boundary conditions as a limit of solutions under Navier boundary conditions (cf. [JM01], and conversely (cf. [Kel06, section 9]). See also [IP06, FŇ05, CCG10, AS11].

Concerning the 1D Burgers system, we would like to say that the question on whether the procedure in [AS05] could be useful in the case of localized internal controls has been raised in [Agr, Section VII] for the case of the Navier–Stokes equations; here we aim to reinforce the idea that that procedure could lead to internal controllability results by means of a few body controls.

Notation. We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$. Given an open interval $I \subseteq \mathbb{R}$ and two Banach spaces X and Y, we write $W(I, X, Y) := \{f \in L^2(I, X) \mid \partial_t f \in L^2(I, Y)\}$. In the case $X = Y$ we write $H^1(I, X) := W(I, X, X)$. The dual space of X will be denoted X'. Further C_i, $i \in \mathbb{N}$, stand for unessential positive constants.

2. Preliminaries

We can, and wish to, deal with systems (1) and (2) in a similar way. For that from now Ω will stand for either $D \subset \mathbb{R}^2$ or $I \subset \mathbb{R}$, the domains containing the fluid. Recall that the systems can be rewritten as an evolutionary system

$$\partial_t u + Au + B(u, u) + Cu + h = \eta, \quad u(0, x) = u_0(x), \tag{3}$$

in a suitable subspace $H \subseteq L^2(\Omega, \mathbb{R}^d)$, with $d = 2$ for (1) and $d = 1$ for (2). We consider H as a pivot space: $H = H'$. For regular enough vector fields we have $(A + C)u = -\nu\Pi\Delta u$ and $B(u, v) = \Pi\langle u\cdot\nabla\rangle v$ for system (1), and $Au = -\nu\Pi\partial_{xx}$, $Cu = 0$, and $B(u, v) = \Pi u \partial_x v$ for system (2), where in both cases Π is the orthogonal projection in $L^2(\Omega, \mathbb{R}^d)$ onto H (for system (2), Π is just the identity operator). A and C can be chosen so that for a suitable space $V \subseteq H \cap H^1(\Omega, \mathbb{R}^d)$ and $\mathrm{D}(A) := \{u \in V \cap H^2(\Omega, \mathbb{R}^d) \mid Au \in H\}$ we have that $(u, v)_H := (u, v)_{L^2(\Omega, \mathbb{R}^d)}$, $(u, v)_V := \langle Au, v\rangle_{V', V}$, and $(u, v)_{\mathrm{D}(A)} := (Au, Av)_H$ are scalar products that induce norms in H, V, and $\mathrm{D}(A)$ equivalent to those inherited from the usual norms of $L^2(\Omega, \mathbb{R})$, $H^1(\Omega, \mathbb{R}^d)$, and $H^2(\Omega, \mathbb{R}^d)$, respectively. Further C

is self-adjoint, $\langle Cu, v\rangle_{V',V} = \langle Cv, u\rangle_{V',V}$, and $\langle Cu, v\rangle_{V',V} \le C_0|u|_H|v|_V$. Furthermore the operator A is a compact and self-adjoint isomorphism from V onto V' and from its domain $\mathrm{D}(A)$ onto H. The inclusions $\mathrm{D}(A) \subset V \subset H$ are dense and compact. For further details concerning the spaces H, V, and $\mathrm{D}(A)$, we refer to [Rod08, Section 4.2] for the case of system (1), and to [KR14] for the case of system (2).

We will take controls in H, so $\eta = \Pi\eta$, and suppose that $h = \Pi h$ (otherwise we have just to take Πh in (3) instead).

3. H-approximate controllability

For simplicity we denote $\mathcal{B}(u)v := B(u, v) + B(v, u)$.

Definition 3.1. A finite subset $\mathcal{C} = \{W_k \mid k \in \{1, 2, \dots, M\}\} \subset V$ is said V-saturating if for the following sequence of subspaces of V, defined recursively by

$$\mathcal{G}^0 := \operatorname{span}\mathcal{C},$$
$$\mathcal{G}^{j+1} := \mathcal{G}^j + \operatorname{span}\{\mathcal{B}(u)v \mid u \in \mathcal{C},\, v \in \mathcal{G}^j\} \cap V,$$

we have that the union $\bigcup_{j\in\mathbb{N}} \mathcal{G}^j$ is dense in H.

For a given open subset $\mathcal{O} \subseteq \Omega$ we define the space $V_\mathcal{O} := \{u \in V \mid \operatorname{supp}(u) \subseteq \overline{\mathcal{O}}\}$.

Definition 3.2. Let us be given an open nonempty subset $\mathcal{O} \subseteq \Omega$. A finite subset $\mathcal{C} = \{W_k \mid k \in \{1, 2, \dots, M\}\} \subset V_\mathcal{O}$ is said $V_\mathcal{O}$-saturating if for the following sequence of subspaces of $V_\mathcal{O} \subset V$, defined recursively by

$$\mathcal{G}^0 := \operatorname{span}\mathcal{C},$$
$$\mathcal{G}^{j+1} := \mathcal{G}^j + \operatorname{span}\{\mathcal{B}(u)v \mid u \in \mathcal{C},\, v \in \mathcal{G}^j\} \cap V_\mathcal{O},$$

we have that the union $\bigcup_{j\in\mathbb{N}} \mathcal{G}^j|_\mathcal{O}$ is dense in $L^2(\mathcal{O}, \mathbb{R}^d)$.

Definition 3.3. Let $i \in \mathbb{N}$. System (3) is said $(\mathcal{G}^i, H)$-approximately controllable at time $T > 0$, if for any given triple $(u_0, u_1, \varepsilon) \in H \times H \times (0, +\infty)$, there exists $\eta \in L^2((0, T), \mathcal{G}^i)$ such that the corresponding (weak) solution $u \in W((0, T), V, V')$ satisfies $|u(T) - u_1|_H \le \varepsilon$.

Theorem 3.4. *Let $\mathcal{C}$ be a V-saturating set and let us be given $T > 0$ and a function $h \in L^2((0, T), H)$. Then system (3) is $(\mathcal{G}^1, H)$-approximately controllable at time T.*

Proof. We prove the Theorem into three main steps. Notice that we can suppose that $u_0 \in V$, because we can apply zero control up to time $t_0 < T$, arriving to $u(t_0) \in V$, and then work in the smaller time interval $[t_0, T]$.

- Step 1: *the system is $(\mathcal{G}^j, H)$-approximately controllable at time T, for big enough $j \in \mathbb{N}$:*

Let us be given $(u_0, u_1, \rho) \in V \times H \times (0, +\infty)$. If we apply zero control in system (3), for time $t \in (0, T)$, we arrive at time T to a vector field $u_T \in V$. Since $h \in L^2((0, T), H)$ the solution is in $C([0, T], V)$. Now let u_δ satisfy $|u_\delta - u(T)|_V \le \delta$. Let $T_\delta \in (0, T)$ and consider system (3) in the interval of time (T_δ, T) with initial condition $u(T_\delta) = u_\delta$. Then apply the constant control $\eta_c := \frac{\tilde{u}_1 - u_\delta}{T - T_\delta}$, where $\tilde{u}_1 \in V$ satisfies $|\tilde{u}_1 - u_1|_H \le \frac{\epsilon}{3}$, for a suitable $\epsilon > 0$. Notice that $z := u - u_\delta - (t - T_\delta)\eta_c$ solves

$$\begin{aligned} \partial_t z &= -A(z + y) - B(z + y, z + y) \\ &\quad -C(z + y) - h, \\ z(T_\delta) &= 0. \end{aligned} \tag{4}$$

with $y := u_\delta + (t - T_\delta)\eta_c$. Since $\langle B(z+y,\, z+y),\, z\rangle_{V',V} = \langle B(z+y,\, y),\, z\rangle_{V',V} + \langle B(y,\, z),\, z\rangle_{V',V}$, because $\langle B(z,\, z),\, z\rangle_{V',V} = 0$, we can see that $|\langle B(z+y,\, z+y),\, z\rangle_{V',V}|_\mathbb{R}$ is bounded by $C_1|z|_H|z|_V|y|_V + C_1|y|_H|y|_V|z|_V$ and, from (4),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z|_H^2 \le C_2^2\left(|y|_V^2 + |z|_H^2|y|_V^2 + |y|_H^2|y|_V^2 + |h|_{V'}^2\right).$$

Observe that $|y|_{L^\infty((T_\delta,T),\,V)} \le |u_\delta|_V + |\tilde{u}_1 - u_\delta|_V \le 2|u(T)|_V + 2\delta + |\tilde{u}_1|_V =: D_\delta$, and thus $|y|^2_{L^2((T_\delta,\,T),\,V)} \le D_\delta^2(T - T_\delta)$. From the Gronwall lemma it follows that

$$|z(T)|_H^2 \le \mathrm{e}^{2C_2^2D_\delta^2(T-T_\delta)}2C_2^2\Big(D_\delta^2(T-T_\delta) + C_3D_\delta^4(T-T_\delta)^2 + |h|^2_{L^2((T_\delta,\,T),\,V')}\Big)$$

which implies that if we set T_δ close enough to T, then $|z(T)|_H \le \frac{\epsilon}{3}$. This means that if we apply zero control in system (3) up to time $T = T_\delta$ and after apply the constant control $\frac{\tilde{u}_1 - u(T_\delta)}{T - T_\delta}$, we arrive at a vector field $u(T)$ with $|u(T) - \tilde{u}_1|_H = |z(T)|_H \le \frac{\epsilon}{3}$. Now, if P^j is the orthogonal projection in H onto $\mathcal{G}^j$, then by a continuity argument, we can set j big enough such that $|(1 - P^j)(\tilde{u}_1 - u_\delta)|_H$ is so small that if we replace the control η_c by $P^j\eta_c$ then the corresponding solution u^j satisfies $|u^j(T) - u(T)|_H \le \frac{\epsilon}{3}$. In conclusion, setting T_δ close enough to T, and then setting j big enough, the control $\eta = \begin{cases} 0, & \text{if } t \in (0,\, T_\delta) \\ \frac{P^j(\tilde{u}_1 - u_\delta)}{T - T_\delta}, & \text{if } t \in (T_\delta,\, T)\end{cases}$, takes its values in $\mathcal{G}^j$, and drives the system to a vector field $u^j(T)$ such that $|u^j(T) - u_1|_H \le |u^j(T) - u(T)|_H + |u(T) - \tilde{u}_1|_H + |\tilde{u}_1 - u_1|_H \le \epsilon$.

• Step 2: *Imitation*: Here we prove the following:
Let $u_0 \in V$, $\eta_0 \in L^2((0,\,T),\,H)$, and $\zeta \in H^1((0,\,T),\,\mathbb{R})$. Let also u^1 and u^2 be the solutions corresponding to the controls η^1 and η^2 given by

$$\eta^1 := \eta_0 + \zeta\mathcal{B}(W_k)v,$$
$$\eta^2 := \eta_0 + \zeta(t)^2K^2B(W_k,\, W_k) + \partial_t\phi$$

with $\phi(t) := \sqrt{2}\sin(\frac{N\pi t}{T})(\zeta(t)KW_k - K^{-1}v)$. Then given $\rho > 0$ we have that $|u^1(T) - u^2(T)|_H \le \rho$, if $W_k \in \mathcal{C}$, $v \in V$, and if K and N are big enough.

We can choose K big enough so that η^1 will be close to

$$\eta^3 := \eta_0 + \zeta\mathcal{B}(W_k)v - K^{-2}B(v,\, v)$$

in $L^2((0,\,T),\,V')$. For simplicity, let us denote $\Psi(t) = \zeta(t)KW_k - K^{-1}v$. Then, from the density of $\mathrm{D}(A)$ in V we can set $\tilde{W}_k$ and $\tilde{v}$ in $\mathrm{D}(A)$ and close enough to W_k and v so that Ψ is close to $\widetilde{\Psi} := \zeta(t)K\widetilde{W}_k - K^{-1}\widetilde{v}$ in $L^2((0,\,T),\,V)$ and so η^3 will be close to

$$\widetilde{\eta}^3 := \eta_0 + \zeta\mathcal{B}(\widetilde{W}_k)\widetilde{v} - K^{-2}B(\widetilde{v},\, \widetilde{v})$$

in $L^2((0,\,T),\,V')$. That is to say, we may choose K, $\widetilde{W}_k$, and $\widetilde{v}$ so that the solution $\widetilde{u}^3$, corresponding to $\widetilde{\eta}^3$, satisfies $|\widetilde{u}^3(T) - u^1(T)|_H \le \frac{\rho}{3}$.

Now we set the control

$$\widetilde{\eta}^2 := \eta_0 + \zeta(t)^2K^2B(\widetilde{W}_k,\, \widetilde{W}_k) + \partial_t\left(\sqrt{2}\sin(\tfrac{N\pi t}{T})\widetilde{\Psi}(t)\right)$$

and denote by $\widetilde{u}^2$ the corresponding solution. Notice that $\phi(t) = \sqrt{2}\sin(\frac{N\pi t}{T})\Psi(t)$; set also $\widetilde{\phi}(t) := \sqrt{2}\sin(\frac{N\pi t}{T})\widetilde{\Psi}(t)$, $y := u^2 - \phi$, and $\widetilde{y} := \widetilde{u}^2 - \widetilde{\phi}$. Then y and $\widetilde{y}$ solve

$$\begin{array}{rcl}\partial_t\bar{y} &=& -A(\bar{y}+\varphi) - B(\bar{y}+\varphi,\, \bar{y}+\varphi) - C(\bar{y}+\varphi) \\ && -h + \eta_0 + \zeta(t)^2K^2B(\beta,\, \beta), \\ \bar{y}(0) &=& u_0, \qquad \bar{y}(T) \;=\; \bar{y}_T.\end{array} \tag{5}$$

with $(\varphi,\, \beta,\, \bar{y}_T) = (\phi,\, W_k,\, u^2(T))$ and $(\varphi,\, \beta,\, \bar{y}_T) = (\widetilde{\phi},\, \widetilde{W_k},\, \widetilde{u}^2(T))$, respectively. Since $\phi(t)$ is close to $\widetilde{\phi}(t)$ in $C([0,\, T], V)$, and $\widetilde{W_k}$ is close to W_k in V, from [Rod08, Proposition 3.2.2], we can conclude that $u^2(T) = y(T)$ is close to $\widetilde{u}^2(T) = \widetilde{y}(T)$ in H. That is, for suitable K, $\widetilde{W_k}$, and $\widetilde{v}$ we have $|u^2(T) - \widetilde{u}^2(T)|_H \leq \frac{\varrho}{3}$.

Hence from $|u^2(T)-u^1(T)|_H = |u^2(T)-\widetilde{u}^2(T)|_H+|\widetilde{u}^2(T)-\widetilde{u}^3(T)|_H+|\widetilde{u}^3(T)-u^1(T)|_H \leq \frac{2\varrho}{3} + |\widetilde{u}^2(T) - \widetilde{u}^3(T)|_H = \frac{2\varrho}{3} + |\widetilde{y}(T) - \widetilde{u}^3(T)|_H$, it remains to prove that we can set N big enough so that $|\widetilde{y}(T) - \widetilde{u}^3(T)|_H \leq \frac{\varrho}{3}$. Now, $d \coloneqq \widetilde{u}^3 - \widetilde{y}$ solves

$$\begin{aligned} \partial_t d &= -Ad - \mathcal{B}(\widetilde{y})d - B(d,\, d) - Cd \\ &\quad + A\widetilde{\phi} + \mathcal{B}(\widetilde{y})\widetilde{\phi} + C\widetilde{\phi} + B(\widetilde{\phi},\, \widetilde{\phi}) - B(\widetilde{\Psi},\, \widetilde{\Psi}), \\ d(0) &= 0, \end{aligned}$$

and from the identities $\sin^2(\tau) = \frac{1-\cos(2\tau)}{2}$ and $B(\widetilde{\phi},\, \widetilde{\phi}) = 2\sin^2(\frac{N\pi t}{T})B(\widetilde{\Psi},\, \widetilde{\Psi})$, it follows

$$\begin{aligned} \partial_t d = -Ad - \mathcal{B}(\widetilde{y})d - B(d,\, d) - Cd \\ + A\widetilde{\phi} + \mathcal{B}(\widetilde{y})\widetilde{\phi} + C\widetilde{\phi} - \cos(\tfrac{2N\pi t}{T})B(\widetilde{\Psi},\, \widetilde{\Psi}), \end{aligned}$$

from which, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|d|_H^2 \leq C_1|\widetilde{y}|_V^2|d|_H^2 + C_1|d|_H^2 + \langle \Phi,\, d\rangle_{V',\,V}$$

with $\Phi = 2(A+C)\widetilde{\phi} + 2\mathcal{B}(\widetilde{y})\widetilde{\phi} - 2\cos(\frac{2N\pi t}{T})B(\widetilde{\Psi},\, \widetilde{\Psi})$ and, by the Gronwall lemma, we have for $s \in [0,\, T]$

$$|d(s)|_H^2 \leq \int_0^s \langle \Phi(t),\, d(t)\rangle_{V',\,V} \mathrm{e}^{C_1 \int_t^s |\widetilde{y}(\tau)|_V^2 + 1\, \mathrm{d}\tau}\, \mathrm{d}t.$$

From [Rod08, Proposition 3.2.2] a bound for $|\widetilde{y}|^2_{L^2((0,\,T),\,V)}$ can be taken independent of N, on the other hand, by standard estimates we can prove that $\widetilde{u}^2$ (and so also $\widetilde{y}$) and $\widetilde{u}^3$ are strong solutions in $W((0,\, T),\, \mathrm{D}(A),\, H)$, because $u_0 \in V$, which allow us to conclude that the integrand $\langle \Phi,\, d\rangle_{V',\,V}\mathrm{e}^{C_1\int_t^s |y|_V^2+1\,\mathrm{d}\tau}$ takes the form $\sin(\frac{N\pi t}{T})\Theta_s^{\varsigma} + \cos(\frac{2N\pi t}{T})\Theta_s^{\varkappa}$, with Θ_s^{ς} and Θ_s^{ς} bounded in $W^{1,1}((0,\, s),\, \mathbb{R}) \subset L^\infty((0,\, s),\, \mathbb{R})$ by a constant independent of N and $s \in (0,\, T)$ (to show this we may proceed as in [Rod08, section 3.2.5]). Thus for some $C_3 \geq 0$ independent of N and $s \in (0,\, T)$,

$$\begin{aligned} |d(s)|_H^2 \leq \frac{s}{N\pi}|\Theta_s^{\varsigma}|_{L^\infty((0,\,s),\,\mathbb{R})} + \frac{s}{2N\pi}|\Theta_s^{\varkappa}|_{L^\infty((0,\,s),\,\mathbb{R})} \\ + \frac{s}{N\pi}\int_0^s |\partial_t \Phi_s^{\varsigma}|_{\mathbb{R}} + \frac{1}{2}|\partial_t \Phi_s^{\varkappa}|_{\mathbb{R}}\, \mathrm{d}t \leq \frac{C_3 T}{N\pi}. \end{aligned}$$

Thus, if N is big enough, $|\widetilde{u}^3(T) - \widetilde{y}(T)|_H = |d(T)|_H \leq \frac{\varrho}{3}$.

- Step 3: *Iteration and conclusion*:

Let us be given $(u_0,\, u_1,\, \varepsilon,\, T) \in V \times H \times (0,\, +\infty)^2$. From Step 1 there is a piecewise constant control $\eta \in L^2((0,\, T),\, \mathcal{G}^j)$ driving (3) from u_0 at time $t = 0$ to $u(T)$ at time $t = T$, with $|u(T)-u_1|_H \leq \frac{\epsilon}{4}$; we can approximate that control in $L^2((0,\, T),\, H)$-norm by a control $\tilde{\eta} \in H^1((0,\, T),\, \mathcal{G}^j)$, so that the corresponding solution $\tilde{u}$ satisfies $|\tilde{u}(T)-u_1|_H \leq \frac{\epsilon}{2}$. That control can be written as

$$\tilde{\eta} = \hat{\eta} + \sum_{\substack{1\leq m\leq M \\ 1\leq n\leq N_j}} \zeta_{(m,\,n)}\mathcal{B}(W_m)v_n \tag{6}$$

with $\hat{\eta} \in H^1((0,\,T),\,\mathcal{G}^{j-1})$, M is the cardinality of $\mathcal{C}$, $N_j \in \mathbb{N}_0$ and $W_m \in \mathcal{C}$ and $v_n \in \mathcal{G}^{j-1}$. Setting

$$\eta_0 = \hat{\eta} + \sum_{\substack{1\le m\le M\\ 1\le n\le N_j\\ (m,\,n)\neq(1,\,1)}} \zeta_{(m,\,n)}\mathcal{B}(W_m)v_n,$$

from Step 2 we can imitate $\eta^1 := \tilde{\eta} = \eta_0 + \zeta_{(1,\,1)}\mathcal{B}(W_1)v_1$ by a control of the form $\eta^2 = \eta_0 + \zeta^2_{(1,\,1)}K^2B(W_1,\,W_1) + \partial_t\phi$, which we can rewrite as

$$\eta^2 = \hat{\eta} + \bar{\eta}_1 + \bar{\eta}_{j-1} + \sum_{\substack{1\le m\le M\\ 1\le n\le N_j\\ (m,\,n)\neq(1,\,1)}} \tilde{\zeta}_{(m,\,n)}\mathcal{B}(W_m)v_n$$

with $\bar{\eta}_1 \in L^2((0,\,T),\,\mathcal{G}^1)$ and $\bar{\eta}_{j-1} \in H^1((0,\,T),\,\mathcal{G}^{j-1})$, that is, the control takes again the form (6).

It is now clear that, applying Step 2 (at most) MN_j times, we arrive to a control of the form $\eta^3 = \hat{\eta}^3 + \bar{\eta}^3_1$ with $\hat{\eta}^3 \in H^1((0,\,T),\,\mathcal{G}^{j-1})$ and $\bar{\eta}^3_1 \in L^2((0,\,T),\,\mathcal{G}^1)$. If $j-1 = 1$, the proof is finished, otherwise we rewrite

$$\eta^3 = \bar{\eta}^3_1 + \hat{\eta}^4 + \sum_{\substack{1\le m\le M\\ 1\le n\le N_{j-1}}} \tilde{\zeta}_{(m,\,n)}\mathcal{B}(W_m)v_n$$

with $\hat{\eta}^4 \in H^1((0,\,T),\,\mathcal{G}^{j-2})$, $N_{j-1} \in \mathbb{N}_0$ and $v_n \in \mathcal{G}^{j-2}$, which takes the form (6).

We can conclude that repeating the procedure above we are able to arrive to a control $\eta^5 \in L^2((0,\,T),\,\mathcal{G}^1)$ driving system (3) from $u(0) = u_0$ at time $t = 0$, to $u(T)$ close to u_1 at time $t = T$, say $|u(T) - u_1|_H \le \epsilon$. □

Corollary 3.5. *Let $\mathcal{C}$ be a V-saturating set and let us be given $T > 0$ and a function $h \in L^2((0,\,T),\,H)$. If $B(W,\,W) = 0$ for all $W \in \mathcal{C}$, then system (3) is $(\mathcal{G}^0,\,H)$-approximately controllable at time T.*

Proof. From Theorem 3.4 there exists a control

$$\eta = \hat{\eta} + \sum_{\substack{1\le m\le M\\ 1\le n\le N_1\\ (m,\,n)\neq(1,\,1)}} \tilde{\zeta}_{(m,\,n)}\mathcal{B}(W_m)v_n + \tilde{\zeta}_{(1,\,1)}\mathcal{B}(W_1)v_1$$

with $\hat{\eta} \in L^2((0,\,T),\,\mathcal{G}^0)$ and $v_n \in \mathcal{G}^0$, driving the system from $u(0) = u_0$ at time $t = 0$, to $u(T)$ close to u_1 at time $t = T$, say with $|u(T) - u_1|_H \le \frac{\epsilon}{2}$. From Step 2 in the proof of Theorem 3.4 this control can be imitated by a control of the form $\eta^2 = \eta_0 + \zeta^2_{(1,\,1)}K^2B(W_1,\,W_1) + \partial_t\phi = \eta_0 + \partial_t\phi$, with $\partial_t\phi \in L^2((0,\,T),\,\mathcal{G}^0)$. Actually, η^2 takes the form

$$\eta^2 = \hat{\eta} + \partial_t\phi + \sum_{\substack{1\le m\le M\\ 1\le n\le N_1\\ (m,\,n)\neq(1,\,1)}} \tilde{\zeta}_{(m,\,n)}\mathcal{B}(W_m)v_n$$

and by repeating the procedure we can arrive to a control $\eta^3 \in L^2((0,\,T),\,G^0)$ driving the system from $u(0) = u_0$ at time $t = 0$, to $u(T)$ close to u_1 at time $t = T$, say with $|u(T) - u_1|_H \le \epsilon$. □

Theorem 3.6. *Let $\mathcal{C}$ be a $V_{\mathcal{O}}$-saturating set and let us be given $T > 0$ and a function $h \in L^2((0,\,T),\,H)$. Then if for some $j \in \mathbb{N}$ system (3) is $(\mathcal{G}^j,\,H)$-approximately controllable at time T, it is also $(\mathcal{G}^1,\,H)$-approximately controllable at time T.*

Proof. The assumption is just the result of Step 1 in the proof of Theorem 3.4, thus we have just to repeat Steps 2 and 3 from that proof. Notice that $V_{\mathcal{O}} \subseteq V$. □

Remark 3.7. Notice N in the proof of Theorem 3.4 will depend on $|\widetilde{W}_k|_{\mathrm{D}(A)}$ and $|\widetilde{v}|_{\mathrm{D}(A)}$, which will increase as $(\widetilde{W}_k,\, \widetilde{v})$ approach $(W_k,\, v)$ in V, if $(W_k,\, v) \notin \mathrm{D}(A)^2$. That is, it is not clear whether η_2 and $\widetilde{\eta}_2$ are close in $L^2((0,\, T),\, V')$, and so whether u_2 and $\widetilde{u}_2$ are close. This was (probably) wrongly claimed in [Rod07b] (in step "Coming back to H" at the end of the Section 7.1), and later (partially) fixed by Proposition 3.2.2 in [Rod08].

4. Approximate controllability in a cylinder

We will show that in the case the Navier–Stokes system is considered in a cylinder Ω, a V-saturating set does exist that consists of a few eigenfuntions of Stokes operator A_L under Lions boundary conditions. By Lions boundary conditions, in two dimensions, we understand taking $\beta = 0$ in (1), that is the curl $\operatorname{curl} u$ of u (i.e., the vorticity $\nabla^\perp \cdot u$) vanishes at the boundary. The reason of the terminology (also adopted in [Kel06, Rod06]) is the work done in [Lio69, section 6.9].

The set of eigenfuntions of the Stokes operator $A_L := -\nu\Pi\Delta$ under Lions boundary conditions is given by

$$\mathcal{E} := \{Y_k^{\varsigma},\, Y_r^{\varkappa} \mid k \in \mathbb{N}_0^2,\, r \in \mathbb{N} \times \mathbb{N}_0\},$$

with

$$\begin{aligned} Y_k^{\varsigma} = Y_{(k_1,k_2)}^{\varsigma} &= \begin{pmatrix} -\frac{k_2\pi}{b} \sin\left(\frac{2k_1\pi x_1}{a}\right) \cos\left(\frac{k_2\pi x_2}{b}\right) \\ \frac{2k_1\pi}{a} \cos\left(\frac{2k_1\pi x_1}{a}\right) \sin\left(\frac{k_2\pi x_2}{b}\right) \end{pmatrix}, \\ Y_k^{\varkappa} = Y_{(k_1,k_2)}^{\varkappa} &= \begin{pmatrix} -\frac{k_2\pi}{b} \cos\left(\frac{2k_1\pi x_1}{a}\right) \cos\left(\frac{k_2\pi x_2}{b}\right) \\ -\frac{2k_1\pi}{a} \sin\left(\frac{2k_1\pi x_1}{a}\right) \sin\left(\frac{k_2\pi x_2}{b}\right) \end{pmatrix}, \end{aligned} \tag{7}$$

and the corresponding eigenvalues are given by

$$\lambda_{(k_1,k_2)}^{\varsigma} = \lambda_{(k_1,k_2)}^{\varkappa} = \lambda_{(k_1,k_2)} := \pi^2 \left(\frac{(2k_1)^2}{a^2} + \frac{k_2^2}{b^2}\right).$$

The family $\mathcal{E}$ form an orthogonal basis in $L^2(\Omega,\, \mathbb{R}^2)$.

Remark 4.1. Notice that under Lions boundary conditions we have the identities $A_L = -\nu\Pi\Delta = -\nu\Delta$, $Y_k^{\varsigma} = \nabla^\perp\psi_k^{\varsigma}$, $Y_k^{\varkappa} = \nabla^\perp\psi_k^{\varkappa}$, with $\psi_k^{\varsigma} := \sin\left(\frac{2k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right)$ and $\psi_k^{\varkappa} := \cos\left(\frac{2k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right)$; notice also that the set of stream functions $\{\psi_n^{\varsigma}, \psi_m^{\varkappa}\big|\ n \in \mathbb{N}_0^2, m \in \mathbb{N} \times \mathbb{N}_0\}, \big|\ n \in \mathbb{N}_0^2, m \in \mathbb{N} \times \mathbb{N}_0\}$ is an orthogonal and complete, in $L^2(\Omega,\, \mathbb{R})$, system of eigenfuntions of the Laplacian in $\Omega \sim (0,\, a) \times (0,\, b)$.

After some straightforwad (though a bit long) computations we can derive that, with $n \vee m := n_1 m_2 + n_2 m_1$ and $n \wedge m := n_1 m_2 - n_2 m_1$,

$$\begin{aligned} &B(Y_n^{\varsigma},\, Y_m^{\varsigma}) + B(Y_m^{\varsigma},\, Y_n^{\varsigma}) \\ = \;&+\frac{\lambda_n - \lambda_m}{\lambda_{m(++)n}} \frac{\pi^2 n\wedge m}{2ab} Y_{n(++)m}^{\varsigma} \\ &+\frac{\lambda_n - \lambda_m}{\lambda_{n(+-)m}} \frac{\pi^2 n\vee m}{2ab} \operatorname{sign}(n_2 - m_2) Y_{n(+-)m}^{\varsigma} \\ &-\frac{\lambda_n - \lambda_m}{\lambda_{n(-+)m}} \frac{\pi^2 n\vee m}{2ab} \operatorname{sign}(n_1 - m_1) Y_{n(-+)m}^{\varsigma} \\ &-\frac{\lambda_n - \lambda_m}{\lambda_{m(--)n}} \frac{\pi^2 n\wedge m}{2ab} \operatorname{sign}(n_1 - m_1) \operatorname{sign}(n_2 - m_2) Y_{n(--)m}^{\varsigma} \end{aligned} \tag{8a}$$

$$\begin{aligned} &B(Y_n^{\varkappa},\, Y_m^{\varkappa}) + B(Y_m^{\varkappa},\, Y_n^{\varkappa}) \\ = \;&-\frac{\lambda_n - \lambda_m}{\lambda_{n(++)m}} \frac{\pi^2 n\wedge m}{2ab} Y_{n(++)m}^{\varsigma} \\ &-\frac{\lambda_n - \lambda_m}{\lambda_{n(+-)m}} \frac{\pi^2 n\vee m}{2ab} \operatorname{sign}(n_2 - m_2) Y_{n(+-)m}^{\varsigma} \\ &-\frac{\lambda_n - \lambda_m}{\lambda_{n(-+)n}} \frac{\pi^2 n\vee m}{2ab} \operatorname{sign}(n_1 - m_1) Y_{n(-+)m}^{\varsigma} \\ &-\frac{\lambda_n - \lambda_m}{\lambda_{n(--)m}} \frac{\pi^2 n\wedge m}{2ab} \operatorname{sign}(n_1 - m_1) \operatorname{sign}(n_2 - m_2) Y_{n(--)m}^{\varsigma}; \end{aligned} \tag{8b}$$

$$\begin{aligned}
&B(Y_n^\varsigma, Y_m^\varkappa) + B(Y_m^\varkappa, Y_n^\varsigma)\\
= \;& +\tfrac{\lambda_n-\lambda_m}{\lambda_{n(++)m}}\tfrac{\pi^2 n\wedge m}{2ab} Y^\varkappa_{n(++)m}\\
& +\tfrac{\lambda_n-\lambda_m}{\lambda_{n(+-)m}}\tfrac{\pi^2 n\vee m}{2ab}\operatorname{sign}(n_2-m_2) Y^\varkappa_{n(+-)m}\\
& +\tfrac{\lambda_n-\lambda_m}{\lambda_{n(-+)m}}\tfrac{\pi^2 n\vee m}{2ab} Y^\varkappa_{n(-+)m}\\
& +\tfrac{\lambda_n-\lambda_m}{\lambda_{n(--)m}}\tfrac{\pi^2 n\wedge m}{2ab}\operatorname{sign}(n_2-m_2) Y^\varkappa_{n(--)m};
\end{aligned} \tag{8c}$$

where $n(\star_1\star_2)m := (|n_1 \star_1 m_1|, |n_2 \star_2 m_2|) \in \mathbb{N}^2$, with $\{\star_1, \star_2\} \subseteq \{-, +\}$ and, for $k_1 \in \mathbb{N}$, $Y^\varsigma_{(k_1,0)} := Y^\varkappa_{(k_1,0)} := 0$. Notice that these are expressions similar to those obtained for the case of the rectangle in [Rod06, equation (6.1)], [Rod08, equation (6.4)]. Notice also that in [Rod06, section 2.3] the eigenvalues are negative because they refer to the usual Laplacean Δ in $(0, a) \times (0, b)$, instead of A_L.

From (8a) and (8b), it follows that

$$\begin{aligned}
&\mathcal{B}(Y_m^\varsigma)Y_n^\varsigma - \mathcal{B}(Y_m^\varkappa)Y_n^\varkappa\\
= \;& +2\tfrac{\lambda_n-\lambda_m}{\lambda_{m(++)n}}\tfrac{\pi^2 n\wedge m}{2ab} Y^\varsigma_{n(++)m}\\
& +2\tfrac{\lambda_n-\lambda_m}{\lambda_{n(+-)m}}\tfrac{\pi^2 n\vee m}{2ab}\operatorname{sign}(n_2-m_2) Y^\varsigma_{n(+-)m};
\end{aligned} \tag{9a}$$

$$\begin{aligned}
&\mathcal{B}(Y_m^\varsigma)Y_n^\varsigma + \mathcal{B}(Y_m^\varkappa)Y_n^\varkappa\\
= \;& -2\tfrac{\lambda_n-\lambda_m}{\lambda_{n(-+)n}}\tfrac{\pi^2 n\vee m}{2ab}\operatorname{sign}(n_1-m_1) Y^\varsigma_{n(-+)m}\\
& -2\tfrac{\lambda_n-\lambda_m}{\lambda_{n(--)m}}\tfrac{\pi^2 n\wedge m}{2ab}\operatorname{sign}(n_1-m_1)\operatorname{sign}(n_2-m_2) Y^\varsigma_{n(--)m};
\end{aligned} \tag{9b}$$

Changing the roles of n and m in (8c) we also find

$$\begin{aligned}
&B(Y_m^\varkappa)Y_n^\varsigma + B(Y_m^\varsigma)Y_n^\varkappa\\
= \;& +2\tfrac{\lambda_n-\lambda_m}{\lambda_{n(++)m}}\tfrac{\pi^2 n\wedge m}{2ab} Y^\varkappa_{n(++)m}\\
& +2\tfrac{\lambda_n-\lambda_m}{\lambda_{n(+-)m}}\tfrac{\pi^2 n\vee m}{2ab}\operatorname{sign}(n_2-m_2) Y^\varkappa_{n(+-)m};
\end{aligned} \tag{10a}$$

$$\begin{aligned}
&B(Y_m^\varkappa)Y_n^\varsigma - B(Y_m^\varsigma)Y_n^\varkappa\\
= \;& +2\tfrac{\lambda_n-\lambda_m}{\lambda_{n(-+)m}}\tfrac{\pi^2 n\vee m}{2ab} Y^\varkappa_{n(-+)m}\\
& +2\tfrac{\lambda_n-\lambda_m}{\lambda_{n(--)m}}\tfrac{\pi^2 n\wedge m}{2ab}\operatorname{sign}(n_2-m_2) Y^\varkappa_{n(--)m};
\end{aligned} \tag{10b}$$

Theorem 4.2. *Under Navier boundary conditions, the set*

$$\mathcal{C} := \left\{ Y_n^\varsigma, Y_m^\varkappa \;\middle|\; \begin{array}{l} 1 \le n_1 \le 4,\ 1 \le n_2 \le 4,\ n \ne (4, 4) \\ 0 \le m_1 \le 4,\ 1 \le m_2 \le 4,\ m \ne (4, 4) \end{array} \right\}$$

is V-saturating, and $B(W, W) = 0$ for all $W \in \mathcal{C}$.

Proof. From (8a) it follows that $2B(Y_n^\varsigma, Y_n^\varsigma) = B(Y_n^\varsigma, Y_n^\varsigma) + B(Y_n^\varsigma, Y_n^\varsigma) = 0$ and similarly from (8b) it follows that $B(Y_n^\varkappa, Y_n^\varkappa) = 0$.

Consider the sequence of subspaces $(\mathcal{G}^j)_{j\in\mathbb{N}}$, defined recursively as in Definition 3.1. Then to prove that $\mathcal{C}$ is V-saturating it suffices to prove that for all $j \in \mathbb{N}_0$ the set

$$\mathcal{C}^j := \left\{ Y_n^\varsigma, Y_m^\varkappa \;\middle|\; \begin{array}{l} 1 \le n_1 \le 4+j,\ 1 \le n_2 \le 4+j \\ 0 \le m_1 \le 4+j,\ 1 \le m_2 \le 4+j \\ n \ne (4+j, 4+j) \ne m \end{array} \right\}$$

is contained in $\mathcal{G}^j$. We prove this by Induction: by definition we have $\mathcal{C}^0 = \mathcal{C} \subset \mathcal{G}^0$; next supposing that for a given $j \in \mathbb{N}_0$ we have $\mathcal{C}^j \subset \mathcal{G}^j$, it remains to prove that $\mathcal{C}^{j+1} \subset \mathcal{G}^{j+1}$.

To simplify the exposition, let us define the sets

$$\mathcal{S}^j := \left\{ k \in \mathbb{N}\times\mathbb{N}_0 \;\middle|\; \begin{array}{l} 0 \le k_1 \le 4+j,\ 1 \le k_2 \le 4+j \\ k \ne (4+j, 4+j) \end{array} \right\},$$

$$\mathcal{S}^j_0 := \{k \in \mathcal{S}^j \mid k_1 \ne 0\};$$

we can write $\mathcal{C}^j = \left\{Y^{\varsigma}_{n},\, Y^{\varkappa}_{m} \,\middle|\, (n,\,m) \in \mathcal{S}^j_0 \times \mathcal{S}^j\right\}$ and, by the inductive hypothesis, we conclude that $\mathcal{C}^{j+1} \subset \mathcal{G}^{j+1}$ if $\mathcal{C}^{j+1} \subset \mathcal{G}^j + \operatorname{span}\mathcal{B}(\mathcal{C}^0)\mathcal{C}^j$. We select the following family $\mathcal{F}^j_0 \subset \mathcal{S}^0_0 \times \mathcal{S}^j_0$,

$$\begin{aligned}\mathcal{F}^j_0 \coloneqq \{&((1,\,2),\,(j+3,\,j+2))\}\\ &\cup \{((1,\,1),\,(k-1,\,j+4)) \mid k = 2,\,3,\,\ldots,j+4\}\\ &\cup \{((2,\,1),\,(3,\,j+4))\}\\ &\cup \{((1,\,1),\,(j+4,\,k-1)) \mid k = 2,\,3,\,\ldots,j+4\}\\ &\cup \{((2,\,1),\,(j+3,\,2))\}\,.\end{aligned}$$

From (9a) and (9b) it follows that the vectors $\mathcal{B}(Y^{\varsigma}_{m})Y^{\varsigma}_{n} \pm \mathcal{B}(Y^{\varkappa}_{m})Y^{\varkappa}_{n}$ are in $\operatorname{span}\mathcal{B}(\mathcal{C}^0)\mathcal{C}^j$ if $(m,\,n) \in \mathcal{F}^j_0$. Hence

$$(1-\Pi^j)\big(\mathcal{B}(Y^{\varsigma}_{m})Y^{\varsigma}_{n} \pm \mathcal{B}(Y^{\varkappa}_{m})Y^{\varkappa}_{n}\big) \in \mathcal{G}^{j+1},$$

where Π^j stands for the orthogonal projection in H onto $\operatorname{span}\mathcal{C}^j$ (notice that, from the inductive hypothesis, $\operatorname{span}\mathcal{C}^j \subseteq \mathcal{G}^j$). Denoting

$$\begin{aligned}C^{++}_{m,n} &\coloneqq \frac{\lambda_n - \lambda_m}{\lambda_{m(++)n}}\frac{\pi^2 n \wedge m}{ab}, &\qquad C^{+-}_{m,n} &\coloneqq \frac{\lambda_n - \lambda_m}{\lambda_{m(+-)n}}\frac{\pi^2 n \vee m}{ab},\\ C^{-+}_{m,n} &\coloneqq \frac{\lambda_n - \lambda_m}{\lambda_{m(-+)n}}\frac{\pi^2 n \vee m}{ab}, &\qquad C^{--}_{m,n} &\coloneqq \frac{\lambda_n - \lambda_m}{\lambda_{m(--)n}}\frac{\pi^2 n \wedge m}{ab},\end{aligned}$$

we can conclude, from (9a) and (9b), that the vectors

$$C^{++}_{(1,2),\,(j+3,\,j+2)}Y^{\varsigma}_{(j+4,\,j+4)}; \tag{11a}$$
$$C^{++}_{(1,1),\,(k-1,\,j+4)}Y^{\varsigma}_{(k,\,j+5)}, \quad k = 2,\,3,\,\ldots,j+4; \tag{11b}$$
$$C^{-+}_{(2,1),\,(3,\,j+4)}Y^{\varsigma}_{(1,\,j+5)}; \tag{11c}$$
$$C^{++}_{(1,1),\,(j+4,\,1)}Y^{\varsigma}_{(j+5,\,2)}; \tag{11d}$$
$$C^{++}_{(1,1),\,(j+4,\,k-1)}Y^{\varsigma}_{(j+5,\,k)} + C^{+-}_{(1,1),\,(j+4,\,k-1)}Y^{\varsigma}_{(j+5,\,k-2)}, \quad k = 3,\,4,\,\ldots,j+4; \tag{11e}$$
$$C^{++}_{(2,1),\,(j+3,\,2)}Y^{\varsigma}_{(j+5,\,3)} + C^{+-}_{(2,1),\,(j+3,\,2)}Y^{\varsigma}_{(j+5,\,1)}. \tag{11f}$$

are in $\mathcal{B}(\mathcal{C}^0)\mathcal{C}^j \subseteq \mathcal{G}^{j+1}$. Notice that for the subscript indices $m,\,n$ of the coefficients $C^{++}_{m,\,n}$ in (11), we have that $m < n$ in lexycographical order and that $n_2 \geq m_2$. From [Rod06, Corollary 7.1], it follows that all the coefficients $C^{++}_{m,\,n}$ in (11) are nonzero, with the exception of $C^{++}_{(2,\,1),\,(j+3,\,2)}$ which vanishes if, and only if, $j = 1$. Therefore, from (11a) it follows that $Y^{\varsigma}_{(j+4,\,j+4)} \in \mathcal{G}^{j+1}$ and from (11b) and (11c), it follows that $Y^{\varsigma}_{(k,\,j+5)} \in \mathcal{G}^{j+1}$, for all $1 \leq k \leq j+4$. From (11d) and all the vectors in (11e) corresponding to even k, it follows that $Y^{\varsigma}_{(j+5,\,r)} \in \mathcal{G}^{j+1}$, for all even r, $1 \leq r \leq j+4$. From (11f) and (11e), with $k = 3$, we have that the vectors

$$\begin{aligned}C^{++}_{(1,1),\,(j+4,\,2)}Y^{\varsigma}_{(j+5,\,3)} &+ C^{+-}_{(1,1),\,(j+4,\,2)}Y^{\varsigma}_{(j+5,\,1)},\\ C^{++}_{(2,1),\,(j+3,\,2)}Y^{\varsigma}_{(j+5,\,3)} &+ C^{+-}_{(2,1),\,(j+3,\,2)}Y^{\varsigma}_{(j+5,\,1)},\end{aligned}$$

are in $\mathcal{G}^{j+1}$. These vectors are linearly independent because

$$\begin{aligned}\frac{C^{++}_{(2,\,1),\,(j+3,\,2)}}{C^{++}_{(1,\,1),\,(j+4,\,2)}} &= \frac{\left((2j^2+12j+2)\,b^2+3a^2\right)(j-1)}{\left((2j^2+16j+28)\,b^2+3a^2\right)(j+2)}\\ &\neq \frac{\left((2j^2+12j+2)\,b^2+3a^2\right)(j+7)}{\left((2j^2+16j+28)\,b^2+3a^2\right)(j+6)}\\ &= \frac{C^{+-}_{(2,\,1),\,(j+3,\,2)}}{C^{+-}_{(1,\,1),\,(j+4,\,2)}}.\end{aligned}$$

This implies that both $Y^{\varsigma}_{(j+5,\,1)}$ and $Y^{\varsigma}_{(j+5,\,3)}$ are in $\mathcal{G}^{j+1}$. Then, from the vectors in (11e) corresponding to odd $k \geq 5$, we can conclude that $Y^{\varsigma}_{(j+5,\,r)} \in \mathcal{G}^{j+1}$, for all odd r, $1 \leq r \leq j+4$.

Therefore, we have that $Y^{\varsigma}_{k} \in \mathcal{G}^{j+1}$ for all $k \in \mathcal{S}^{j+1}_0$.

Since for all the subscript indices m, n of the coefficients $C^{++}_{m,\,n}$ in (11), we have that $m_1 < n_1$, then for those indices the expressions in (9) and (10) are identical; hence we can repeat the above argument to prove that $Y^{\varkappa}_{k} \in \mathcal{G}^{j+1}$ for all $k \in \mathcal{S}^{j+1}_0$.

Finally, from (10b) we have that

$$C^{-+}_{(1,\,1),\,(1,\,j+4)} Y^{\varkappa}_{(0,\,j+5)} \in \mathcal{G}^{j+1}$$

and, since $C^{-+}_{(1,\,1),\,(1,\,j+4)} \neq 0$, we can conclude that $Y^{\varkappa}_{(0,\,j+5)} \in \mathcal{G}^{j+1}$. Thus, we indeed have the inclusion $\mathcal{C}^j = \left\{ Y^{\varsigma}_{n},\, Y^{\varkappa}_{m} \,\middle|\, (n,\, m) \in \mathcal{S}^j_0 \times \mathcal{S}^j \right\} \subset \mathcal{G}^{j+1}$. □

From Corollary 3.5 and Theorem 4.2, it follows:

Corollary 4.3. *Given $T > 0$, the Navier–Stokes system* (1) *in a cylinder, under Navier boundary conditions, is $(\mathcal{G}^0,\, H)$-approximately controllable at time T.*

5. Approximate controllability for 1D Burgers

Let us consider the Burgers system in a nonempty open interval $\Omega = (0,\, L)$, $L > 0$, and let $\emptyset \neq \mathcal{O} = (l_1,\, l_2) \subseteq \Omega$. In this case we have $V = H^1_0(\Omega,\, \mathbb{R})$ and $H = L^2(\Omega,\, \mathbb{R})$.

Let us denote by $\mathcal{I}_{\mathcal{O}}\colon H^1(\Omega,\, \mathbb{R}) \to L^2(\Omega,\, \mathbb{R})$, $f \mapsto \mathcal{I}_{\mathcal{O}} f$ be the indicator operator defined by

$$\mathcal{I}_{\mathcal{O}} f(x) \coloneqq \begin{cases} f(x), & \text{if } x \in \mathcal{O};\\ 0, & \text{if } x \in \Omega \setminus \mathcal{O}.\end{cases}$$

For simplicity let us denote $\sigma_i(x) \coloneqq \sin(i\pi \frac{x-l_1}{l_2-l_1})$, $x \in \Omega$.

Theorem 5.1. *The set $\mathcal{C} \coloneqq \{\mathcal{I}_{\mathcal{O}}\sigma_1\}$ is $V_{\mathcal{O}}$-saturating.*

Proof. We prove by induction that for $j \geq 1$, $\mathcal{I}_{\mathcal{O}}\sigma_j \in \mathcal{G}^{j-1}$. For $j = 1$ we find $\mathcal{I}_{\mathcal{O}}\sigma_1 \in \mathcal{C} \subset \mathcal{G}^0$. On the other hand if $\mathcal{I}_{\mathcal{O}}\sigma_n \in \mathcal{G}^{n-1}$ for all $n \leq j$, then we can find $\mathcal{B}(\mathcal{I}_{\mathcal{O}}\sigma_1)\mathcal{I}_{\mathcal{O}}\sigma_j = \partial_x(\mathcal{I}_{\mathcal{O}}\sigma_1\mathcal{I}_{\mathcal{O}}\sigma_j) = \mathcal{I}_{\mathcal{O}}\partial_x(\sigma_1\sigma_j) = \frac{\pi}{2(l_2-l_1)}\mathcal{I}_{\mathcal{O}}\left((j+1)\sigma_{j+1} - (j-1)\sigma_{j-1}\right)$, that is $\mathcal{I}_{\mathcal{O}}\sigma_{j+1} \in \mathcal{G}^{j-1+1} = \mathcal{G}^{(j+1)-1}$. □

Corollary 5.2. *If system* (2) *is approximately controllable at time T, by means of controls $\zeta \in L^2((0,\, T),\, L^2(\mathcal{O},\, \mathbb{R}))$, then it is $(\mathcal{G}^1,\, H)$-approximately controllable at time T.*

Proof. Given a control $\zeta \in L^2((0,\, T),\, L^2(\mathcal{O},\, \mathbb{R}))$, then $P^j\zeta \in L^2((0,\, T),\, \mathcal{G}^j)$ will be close to ζ in $L^2((0,\, T),\, V')$, for big enough j. So we can conclude that the assumption of Theorem 3.6 is satisfied. □

Remark 5.3. In the case $\mathcal{O} = \Omega$ we have $V_\Omega = V$; thus it follows that system (2) is $(\mathcal{G}^1,\, H)$-approximately controllable at time T, from Theorem 3.4. Notice that $\mathcal{G}^1 = \operatorname{span}\{\mathcal{I}_{\mathcal{O}}\sigma_1,\, \mathcal{I}_{\mathcal{O}}\sigma_2\}$.

Remark 5.4. The more interesting case is, of course, the case $\mathcal{O} \subset \Omega$. Though, in this case, we do not know whether the assumption in Corollary 5.2 holds for system (2), there are some related results on this direction we would like to mention, for instance, that it can be proven that the system is locally exactly controllable to trajectories (following, e.g., the idea in [FCGIP04]) by controls $\zeta \in L^2((0, T), L^2(\mathcal{O}, \mathbb{R}))$. The local nature of this result is due to the fact that the result is derived from a linearization based method, while the method we follow here is purely based in the properties of the nonlinear term. So, one question could be whether we can combine the ideas of the two methods to derive the desired $(\mathcal{G}^1, H)$-approximate controllability result.

6. Concluding Remarks

We have proven that the Navier–Stokes system in a cylinder under Navier boundary conditions is approximately controllable with a few controls, and that the approximate controllability of the Burgers system with two controls supported in a small subset will follow if the same property holds by means of general square integrable controls supported in that subset.

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