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RICAM-Report 2014-44

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GEVREY REGULARITY FOR NAVIER–STOKES EQUATIONS UNDER LIONS BOUNDARY CONDITIONS

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Abstract. The Navier–Stokes system is considered in a compact Riemannian manifold. Gevrey class regularity is proven under Lions boundary conditions in the cases of the 2D Rectangle, Cylinder, and Hemisphere. The cases of the 2D Sphere and 2D and 3D Torus are also revisited.

MSC2010: 35Q30, 76D03

Keywords: Navier–Stokes equations, Gevrey class regularity

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a connected bounded domain located locally on one side of its smooth boundary $\Gamma = \partial\Omega$. The Navier–Stokes system, in $(0, T) \times \Omega$, reads

$$\partial_t u + \langle u \cdot \nabla \rangle u - \nu \Delta u + \nabla p + h = 0, \quad \operatorname{div} u = 0, \quad \mathcal{G}u\big|_\Gamma = 0, \quad u(0, x) = u_0(x) \qquad (1)$$

where as usual $u = (u_1, \ldots, u_d)$ and p, defined for $(t, x_1, \ldots, x_d) \in I \times \Omega$, are respectively the unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity, the operators ∇ and Δ are respectively the well known gradient and Laplacian in the space variables $(x_1, \ldots, x_d)$, $\langle u \cdot \nabla \rangle v$ stands for $(u \cdot \nabla v_1, \ldots, u \cdot \nabla v_d)$, $\operatorname{div} u := \sum_{i=1}^{d} \partial_{x_i} u_i$ and h is a fixed function. Further, $\mathcal{G}$ is an appropriate linear operator imposing the boundary conditions.

The authors acknowledge support from the Austrian Science Fund (FWF): P 26034-N25.

In the case Ω is a compact Riemannian manifold, either with or without boundary, the Navier–Stokes equation reads

$$\partial_t u + \nabla_u^1 u + \nu\Delta_\Omega u + \nabla_\Omega p + h = 0, \quad \operatorname{div} u = 0, \quad \mathcal{G}u|_\Gamma = 0, \quad u(0,\,x) = u_0(x). \tag{2}$$

That is we just replace the Laplace operator by the Laplace-de Rham operator, the gradient operator by the Riemannian gradient operator, and the nonlinear term by the Levy-Civita connection. Recall that a flat (Euclidean) domain $\Omega \subset \mathbb{R}^d$ can be seen a Riemannian manifold and we have $-\Delta = \Delta_\Omega$, $\nabla = \nabla_\Omega$ and $\langle u\cdot\nabla\rangle v = \nabla_u^1 u$ (see, e.g., [Rod08, Chapter 5]). That is, (2) reads (1) in the Euclidean case. We should say that some authors consider the Navier–Stokes equation on a Riemannian manifold with a slightly different Laplacian operator and sometimes with on more term involving the (Ricci) curvature of the Riemannian manifold. In that case, we also recover (1) in the Euclidean case because the curvature vanishes. Writing the Navier–Stokes as (2), we are following [Ily91, Ily94, CRT99, FF05, Rod08, Rod07]; for other writings we refer to [Pri94, CF96].

Often system (2) can be rewritten as an evolutionary system

$$\dot{u} + B(u,\,u) + Au + h = 0, \quad u(0,\,x) = u_0(x), \tag{3}$$

where Π is a suitable projection onto a subspace H of divergence free vector fields (see, e.g., [FMRT01, Chapter II, Section 3], [Rod06, Section 4], [Rod08, Section 5.5]); formally $B(u,\,v) := \Pi\nabla_u^1 v$ and $Au = \nu\Pi\Delta_\Omega u$. Usually $\Pi\nabla = 0$, and we suppose that $h = \Pi h$ (otherwise we have just to take Πh in (3) instead).

The aim of this work is to give some sufficient conditions to guarantee that the solution of system (2) lives in a Gevrey regularity space.

For the case of periodic boundary conditions, that is, for the case $\Omega = \mathbb{T}^d$, the Gevrey regularity has been proven in the pioneering work [FT89] for a particular Gevrey class $\mathrm{D}(A^{\frac{1}{2}}\mathrm{e}^{\psi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^{\frac{1}{2}})$. These results have been extended to other Gevrey classes in [Liu92], namely $\mathrm{D}(A^s\mathrm{e}^{\psi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s > \frac{d}{4}$. The first observation is that there is a gap, for the value of s, for $d = 3$. As far as we know this gap is still open until now. Here we fill the gap, that is, for $\Omega = \mathbb{T}^3$ the Gevrey regularity holds in $\mathrm{D}(A^s\mathrm{e}^{\psi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s \geq \frac{1}{2}$. Further for $\Omega = \mathbb{T}^2$, it will follow that the Gevrey regularity holds in $\mathrm{D}(A^s\mathrm{e}^{\psi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s > 0$.

In the case of the Navier–Stokes in the 2D Sphere $\mathbb{S}^2$, from our conditions, we can recover the results obtained in [CRT99], that is to say that the Gevrey regularity holds in $\mathrm{D}(A^s\mathrm{e}^{\psi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s > \frac{1}{2}$.

In the above mentioned cases the manifolds $\mathbb{T}^d$ and $\mathbb{S}^2$ are boundaryless, which means that essentially we have no boundary conditions. Here we consider the case of manifolds with boundary and three new results are obtained under Lions boundary conditions, namely, in the cases Ω is either a 2D Rectangle $(0,\,a)\times(0,\,b)$ or a 2D Cylinder $(0,\,a)\times b\mathbb{S}^1$, or a 2D Hemisphere $\mathbb{S}^2_+$. By Lions boundary conditions, in two dimensions, we mean the vanishing both of the normal component $u\cdot\mathbf{n}$ and of the vorticity $\nabla^\perp\cdot u$ of the vector field u at the boundary; the reason of the terminology (also adopted in [Kel06, Rod06]) is the work done in [Lio69, Section 6.9]. However the terminology is not followed by all authors, for example, in [IT06, Section 3] they are just called "stress-free boundary conditions". Notice that Lions boundary conditions can be seen as a particular case of (generalized) Navier boundary conditions (cf. [Kel06, Section 1 and Corollary 4.2], cf. [Rod08, system (4.1)-(4.2) and Remark 4.4.1]). The Navier boundary conditions are also defined in three dimensions, and the particular case considered in [XX07, Equation (1.4)] would correspond to the three dimensional Lions boundary conditions. The study of Navier boundary conditions have been addressed by many authors in the last years, either because in some situations they may be more realistic than no-slip boundary conditions or because they are more appropriate in finding a solution for the Euler system as a limit of solutions for the Navier–Stoles system as ν goes to zero (cf. [XX07, WXZ12], [Kel06, Section 8]), or even the possibility to recover the solution under no-slip boundary conditions as a limit of solutions under Navier boundary conditions (cf. [JM01], and conversely (cf. [Kel06, Section 9]). We refer also to [IP06, FŇ05, CCG10, AS11] and references therein.

In both cases of the Rectangle or Cylinder, we obtain that the Gevrey regularity holds in $\mathrm{D}(A^s \mathrm{e}^{\psi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s > 0$. In the case of the Hemisphere we find that the Gevrey regularity holds in $\mathrm{D}(A^s \mathrm{e}^{\psi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s > \frac{1}{2}$.

The rest of the paper is organized as follows. In Section 2, we give the necessary conditions (as assumptions) for the existence of solutions living in a Gevrey class regularity space. In Section 3, the Gevrey class regularity is proven under the conditions on the sequence of nonrepeated eigenvalues of the Stokes operator. In Section 4, we give the corresponding conditions on the sequence of repeated eigenvalues. In Section 5, we revisit the cases where Ω is the Torus $\mathbb{T}^d$ and the Sphere $\mathbb{S}^2$ and give some new examples, namely the cases of 2D Hemisphere, 2D Rectangle, and 2D Cylinder under Lions boundary conditions. Finally, the Appendix gathers some auxiliary results used in the main text.

Notation. We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_0 \coloneqq (0, +\infty)$, and $\mathbb{N}_0 \coloneqq \mathbb{N} \setminus \{0\}$. We denote by $\Omega \subset \mathbb{R}$ as a bounded interval.

Given a Banach space X and an open subset $O \subset \mathbb{R}^n$, let us denote by $L^p(O, X)$, with either $p \in [1, +\infty)$ or $p = \infty$, the Bochner space of measurable functions $f : O \to X$, and such that $|f|_X^p$ is integrable over O, for $p \in [1, +\infty)$, and such that $\operatorname{ess\,sup}_{x \in O} |f(x)|_X < +\infty$, for $p = \infty$. In the case $X = \mathbb{R}$ we recover the usual Lebesgue spaces. By $W^{s,p}(O, \mathbb{R})$, for $s \in \mathbb{R}$, denote the usual Sobolev space of order s. In the case $p = 2$, as usual, we denote $H^s(O, \mathbb{R}) \coloneqq W^{s,2}(O, \mathbb{R})$. Recall that $H^0(O, \mathbb{R}) = L^2(O, \mathbb{R})$. For each $s > 0$, we recall also that $H^{-s}(O, \mathbb{R})$ stands for the dual space of $H_0^s(O, \mathbb{R}) =$ closure of $\{f \in C^\infty(O, \mathbb{R}) \mid \operatorname{supp} f \subset O\}$ in $H^s(O, \mathbb{R})$. Notice that $H^{-s}(O, \mathbb{R})$ is a space of distributions.

For a normed space X, we denote by $|\cdot|_X$ the corresponding norm; in the particular case $X = \mathbb{R}$ we denote $|\cdot| \coloneqq |\cdot|_{\mathbb{R}}$. By X' we denote the dual of X, and by $\langle \cdot, \cdot \rangle_{X',X}$ the duality between X' and X. The dual space is endowed with the usual dual norm: $|f|_{X'} \coloneqq \sup\{\langle f, x\rangle_{X',X} \mid x \in X \text{ and } |x|_X = 1\}$. In the case that X is a Hilbert space we denote the inner product by $(\cdot, \cdot)_X$.

Given a Riemannian manifold $\Omega = (\Omega, g)$ with Riemannian metric tensor g, we denote by $T\Omega$ the tangent bundle of Ω and by $\mathrm{d}\Omega$ the volume element of Ω. We denote by $H^s(\Omega, \mathbb{R})$ and $H^s(\Omega, T\Omega)$ respectively the Sobolev spaces of functions and vector fields defined in Ω. Recall that if $\Omega = O \subset \mathbb{R}^n$, then $H^s(O, TO) = H^s(O, \mathbb{R}^n) \sim (H^s(O, \mathbb{R}))^n$.

C, C_i, $i = 1, 2, \ldots$, stand for unessential positive constants.

2. Preliminaries

2.1. **The evolutionary Navier–Stokes system.** Given a d-dimensional compact Riemannian manifold $\Omega = (\Omega, g)$, $d \in \{2, 3\}$, we (suppose we can) write the Navier–Stokes system as an evolutionary system in a suitable closed subspace $H \subseteq \{u \in L^2(\Omega, T\Omega) \mid$

$\operatorname{div} u = 0\}$ of divergence free vector fields

$$\dot{u} + \nu A u + B u + h = 0, \quad u(0) = u_0, \tag{4}$$

where $A \coloneqq \Pi\Delta_\Omega$ is the Stokes operator and $B(u) \coloneqq B(u,\, u)$ with $B(u,\, v) \coloneqq \Pi\nabla^1_u v$ as a bilinear operator.

Here Π stands for the orthogonal projection in $L^2(\Omega,\, T\Omega)$ onto H, Δ_Ω stands for the Laplace–de Rham operator and $(u,\, v) \mapsto \nabla^1_u v$ stands for the Levi–Civita connection (cf.[Jos05, Chapter 3, Section 3.3]).

Recall that, for a domain $\Omega \in \mathbb{R}^d$, we can identify $T\Omega$ with $\mathbb{R}^d$, $\Delta_\Omega = -\Delta$ coincides with the usual Laplacian up to the minus sign, and $\nabla^1_u v = \langle u\cdot\nabla\rangle v$ (see [Rod08, Chapter 5, Sections 5.1 and 5.2], [Ily91, Section 1])

We consider H, endowed with the norm inherited from $L^2(\Omega,\, T\Omega)$, as a pivot space, that is, $H = H'$. Let $V \subseteq H$ be another Hilbert space, such that A maps V onto V'. The domain of A, in H, is denoted $\mathrm{D}(A) \coloneqq \{u \in H \mid Au \in H\}$.

The spaces H, V, and $\mathrm{D}(A)$ will depend on the boundary conditions where the fluid will be subjected to. We assume that the inclusion $V \subseteq H$ is dense, continuous, and compact. In this case, the eigenvalues of A, repeated accordingly with their multiplicity, form an increasing sequence $(\underline{\lambda}_k)_{k\in\mathbb{N}_0}$,

$$0 < \underline{\lambda}_1 \le \underline{\lambda}_2 \le \underline{\lambda}_3 \le \underline{\lambda}_4 \le \ldots,$$

with $\underline{\lambda}_k$ going to $+\infty$ with k.

Consider also the strictly increasing subsequence $(\lambda_k)_k \in \mathbb{N}_0$ of the distinct (i.e. non-repeated) eigenvalues

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots;$$

and denote by P_k the orthogonal projection in H onto the eigenspace $P_k H = \{z \in H \mid Az = \lambda_k z\}$, associated with the eigenvalue λ_k,

$$P_k \colon H \to P_k H, \quad v \mapsto P_k v; \tag{5}$$

with $v = P_k v + w$ and $(w,\, z)_H = 0$ for all $z \in P_k H$.

We define also the trilinear form

$$b(u,\, v,\, w) \coloneqq \int_\Omega g(\nabla^1_u v,\, w)\,\mathrm{d}\Omega,$$

provided the integral is finite, where $g(\cdot,\, \cdot)$ stands for the scalar product in $T\Omega$ induced by the metric tensor g.

Throughout the paper, we consider the following assumptions:

Assumption 2.1.
• $V \subset H^1(\Omega,\, T\Omega)$, and $|u|_V \coloneqq (\langle Au,\, u\rangle_{V',V})^{\frac{1}{2}}$ defines a norm equivalent to that inherited from $H^1(\Omega,\, T\Omega)$;
• $\mathrm{D}(A) \subset H^2(\Omega,\, T\Omega)$, and $|u|_{\mathrm{D}(A)} \coloneqq |Au|_H$ defines a norm equivalent to that inherited from $H^2(\Omega,\, T\Omega)$.

Assumption 2.2. The following properties hold for the trilinear form.
• $b(u,\, u,\, v) = 0$ if $u \in P_k H$ for some $k \in \mathbb{N}_0$;
• $b(u,\, v,\, w) = -b(u,\, w,\, v)$;
• $|b(u,\, v,\, w)| \le C|u|_{L^\infty(\Omega,T\Omega)}|v|_{H^1(\Omega,T\Omega)}|w|_{L^2(\Omega,T\Omega)}$;
• $|b(u,\, v,\, w)| \le C|u|_{L^2(\Omega,T\Omega)}|v|_{H^1(\Omega,T\Omega)}|w|_{L^\infty(\Omega,T\Omega)}$;
• $|b(u,\, v,\, w)| \le C|u|_{L^4(\Omega,T\Omega)}|v|_{H^1(\Omega,T\Omega)}|w|_{L^4(\Omega,T\Omega)}$.

Assumption 2.3. There are real numbers $\beta \geq 0$ and $\alpha \in (0,\,1)$ such that, for all triples $(n,\,m,\,l) \in \mathbb{N}_0^3$,

$$\begin{cases} (u,\,v,\,w) \in P_nH \times P_mH \times P_lH, \\ (B(u+v),\,w)_H \neq 0, \end{cases} \quad \text{implies} \quad \lambda_l^\alpha \leq \lambda_n^\alpha + \lambda_m^\alpha + \beta.$$

Next, for given $(n,\,m,\,l) \in \mathbb{N}_0^3$, we define the sets

$$\mathcal{F}_{n,m}^{\bullet} \coloneqq \left\{ k \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} (B(u+v),\,w)_H \neq 0, \\ \text{for some } (u,\,v,\,w) \in P_nH \times P_mH \times P_kH \text{ with } n < m \end{array} \right\};$$

$$\mathcal{F}_{n,\bullet}^{l} \coloneqq \left\{ k \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} (B(u+v),\,w)_H \neq 0, \\ \text{for some } (u,\,v,\,w) \in P_nH \times P_kH \times P_lH \text{ with } n < k \end{array} \right\}.$$

Assumption 2.4. There are $C_{\mathcal{F}} \in \mathbb{N}_0$ and $\zeta \in [0,\,+\infty)$ such that, for all $n \in \mathbb{N}_0$

$$\sup_{(m,\,l)\in\mathbb{N}_0^2} \left\{ \operatorname{card}(\mathcal{F}_{n,m}^{\bullet}),\, \operatorname{card}(\mathcal{F}_{n,\bullet}^{l}) \right\} \leq C_{\mathcal{F}} \lambda_n^\zeta,$$

where $\operatorname{card}(S)$ stands for the cardinality (i.e., the number of elements) of the set S.

Remark 2.5. Assumptions 2.1 and 2.2 are satisfied in well known settings. In contrast, assumptions 2.3 and 2.4 will be satisfied more seldom and play a key role to derive the Gevrey class regularity for the solutions of the Navier–Stokes system (4).

2.2. **Some auxiliary results.** We present now some results that will be useful hereafter.

Proposition 2.6. *For given nonnegative real numbers a, b, and s, with $a + b > 0$ and $s > 0$, it holds*

$$\begin{array}{ll} 2^{s-1}(a^s + b^s) \leq (a+b)^s \leq a^s + b^s, & \text{for } 0 < s \leq 1; \\ a^s + b^s \leq (a+b)^s \leq 2^{s-1}(a^s + b^s), & \text{for } s \geq 1. \end{array}$$

The proof is given in the Appendix, Section A.1.

Remark 2.7. The constants in Proposition 2.6 are sharp, in the sense that

- for $a = b$, we have $2^{s-1}(a^s + b^s) = (a+b)^s$ for $s > 0$,
- for either $a = 0$ or $b = 0$, we have $(a+b)^s = a^s + b^s$ for $s > 0$.

Lemma 2.8. *Assumption 2.3 holds only if for all $s > 0$ there exists a nonnegative real number $C_{(s,\,\alpha,\,\beta)} > 0$ depending only on $(s,\,\alpha,\,\beta,\,\lambda_1)$ such that*

$$\begin{cases} (u,\,v,\,w) \in P_nH \times P_mH \times P_lH, \\ (B(u+v),\,w)_H \neq 0, \end{cases} \quad \textit{implies} \quad \lambda_l^s \leq C_{(s,\,\alpha,\,\beta,\,\lambda_1)}(\lambda_n^s + \lambda_m^s).$$

Proof. From Assumption 2.3, since $(\lambda_k)_{k\in\mathbb{N}_0}$ is an increasing sequence, we have that

$$\lambda_l^\alpha \leq \lambda_n^\alpha + \lambda_m^\alpha + \beta \frac{\lambda_n^\alpha + \lambda_m^\alpha}{2\lambda_1^\alpha} = \left(1 + \frac{\beta}{2\lambda_1^\alpha}\right)(\lambda_n^\alpha + \lambda_m^\alpha)\,.$$

Now for any $s > 0$, it follows that

$$\lambda_l^s \leq \left(1 + \frac{\beta}{2\lambda_1^\alpha}\right)^{\frac{s}{\alpha}} D_{\frac{s}{\alpha}}\,(\lambda_n^s + \lambda_m^s)$$

where the constant $D_{\frac{s}{\alpha}}$ depending only on $\frac{s}{\alpha}$ is given by Proposition 2.6. □

3. GEVREY CLASS REGULARITY

Here we show that, under Assumptions 2.3 and 2.4 and for suitable data $(u_0,\,h)$, the solution u of system (4) takes its values $u(t)$ in a Gevrey class regularity space. We follow the arguments in [FT89, Liu92, CRT99].

3.1. **Gevrey spaces and main theorem.** Let us set a complete orthonormal system $\{W_k \mid k \in \mathbb{N}_0\}$ of eigenfunctions of the Stokes operator A. That is,

$$AW_k = \underline{\lambda}_k W_k, \text{ for all } k \in \mathbb{N}_0.$$

We recall that any given $u \in H$ can be written in a unique way as $u = \sum_{k\in\mathbb{N}_0} u_k W_k$, with $u_k = (u, W_k)_H \in \mathbb{R}$. Now, given $s \geq 0$ we may define the power A^s of the Stokes operator as

$$A^s u := \sum_{k\in\mathbb{N}_0} \underline{\lambda}_k^s u_k W_k,$$

and we denote its domain by $\mathrm{D}\left(A^s\right) := \{u \in H \mid A^s u \in H\}$.

Analogously we may define the negative powers A^{-s} as

$$A^{-s} u := \sum_{n\in\mathbb{N}_0} \underline{\lambda}_k^{-s} u_k W_k,$$

and $\mathrm{D}\left(A^{-s}\right) := \{u \mid A^{-s}u \in H\}$, more precisely $\mathrm{D}\left(A^{-s}\right)$ is the closure of H in the norm $|u|_{\mathrm{D}(A^{-s})} := \left(\sum_{k\in\mathbb{N}_0} \underline{\lambda}_k^{-2s} u_k^2\right)^{\frac{1}{2}}$.

We recall that for $s = \frac{1}{2}$ we have $\mathrm{D}(A^{\frac{1}{2}}) = V$. For a more complete discussion on the fractional powers of a compact operator we refer to [Tem97, Chapter II, Section 2.1].

Given two more nonnegative real numbers σ and α, we define the Gevrey operator

$$A^s e^{\sigma A^\alpha} u := \sum_{k\in\mathbb{N}_0} e^{\sigma \underline{\lambda}_k^\alpha} \underline{\lambda}_k^s u_k W_k,$$

which domain is the Gevrey space $\mathrm{D}\left(A^s e^{\sigma A^\alpha}\right) := \left\{u \in H \mid A^s e^{\sigma A^\alpha} u \in H\right\}$.

Notice that, for given $s \geq 0$, $\sigma \geq 0$, and $\alpha \geq 0$ the functions in $\{W_k \mid k \in \mathbb{N}_0\}$ are also eigenfunctions for A^s and for $A^s e^{\sigma A^\alpha}$. Indeed for any $k \in \mathbb{N}_0$ it follows that

$$A^s W_k = \underline{\lambda}_k^s W_k \quad \text{and} \quad A^s e^{\sigma A^\alpha} W_k = e^{\sigma \underline{\lambda}_k^\alpha} \underline{\lambda}_k^s W_k.$$

Furthermore the operators A^s and $A^s e^{\sigma A^\alpha}$ are selfadjoint; indeed

$$\begin{array}{rcccl} (A^s u, v)_H &=& \sum\limits_{k\in\mathbb{N}_0} \underline{\lambda}_k^s u_k v_k &=& (u, A^s v)_H, \\ (A^s e^{\sigma A^\alpha} u, v)_H &=& \sum\limits_{k\in\mathbb{N}_0} e^{\sigma \underline{\lambda}_k^\alpha} \underline{\lambda}_k^s u_k v_k &=& (u, A^s e^{\sigma A^\alpha} v)_H. \end{array}$$

Theorem 3.1. *Suppose that the Assumptions 2.1, 2.2, 2.3 and 2.4 hold, and let the strictly increasing sequence of (nonrepeated) eigenvalues $(\lambda_k)_{k\in\mathbb{N}_0}$ of the Stokes operator A satisfy, for some positive real numbers ρ and ξ, the relation*

$$\lambda_k > \rho k^\xi, \quad \textit{for all } k \in \mathbb{N}_0. \tag{6}$$

Further, let us be given $\alpha \in (0, 1)$ as in Assumption 2.3, $C_\mathcal{F}$ and $\zeta \geq 0$ as in Assumption 2.4, $\sigma > 0$, $s > \frac{d+2(\xi^{-1}+2\zeta-1)}{4}$, $h \in L^\infty(\mathbb{R}_0, \mathrm{D}(A^{s-\frac{1}{2}} e^{\sigma A^\alpha}))$, and $u_0 \in \mathrm{D}(A^s)$. Then, there are $T^* > 0$ and a unique solution

$$u \in L^\infty\left((0, T^*), D\left(A^s e^{\sigma A^\alpha}\right)\right) \cap L^2\left((0, T^*), D\left(A^{s+\frac{1}{2}} e^{\sigma A^\alpha}\right)\right), \tag{7}$$

for the Navier Stokes system (4).

Further, T^* depends on the data $\left(|h|_{L^\infty\left(\mathbb{R}_0, D\left(A^{s-\frac{1}{2}} e^{\sigma A^\alpha}\right)\right)}, |A^s u_0|_H\right)$ and also on the constants ν, λ_1, d, s, σ, α, β, $C_\mathcal{F}$, ζ, ρ, and ξ.

The proof is given below, in Section 3.3.

3.2. **Some preliminary results.** We derive some preliminary results that we will need in the proof of Theorem 3.1. Let u solve system (4) and let $\sigma > 0$, $\alpha \in (0, 1)$ and ζ be real numbers as in Theorem 3.1, and set $\varphi(t) := \min(\sigma, t)$. Following the Remark in [FT89, Section 2.3(iii)], we can see that the function $u^*(t) = e^{\varphi(t)A^\alpha}u(t)$ satisfies $\partial_t u^* = \frac{d\varphi}{dt}A^\alpha e^{\varphi A^\alpha}u + e^{\varphi A^\alpha}\partial_t u$, and denoting $h^*(t) := e^{\varphi(t)A^\alpha}h(t)$, it follows that u^* solves

$$\partial_t u^* + \nu A u^* + e^{\varphi A^\alpha}B(u) + h^* - \frac{d\varphi}{dt}A^\alpha u^* = 0, \tag{8a}$$
$$u^*(0) = u_0. \tag{8b}$$

Now, let $s \geq 0$ be another nonnegative number and multiply (8a) by $A^{2s}u^*$, formally we obtain

$$\begin{aligned}&\left(\partial_t u^*, A^{2s}u^*\right)_H + \nu\left(Au^*, A^{2s}u^*\right)_H\\ &= -\left(e^{\varphi A^\alpha}B(u), A^{2s}u^*\right)_H - \left(h^*, A^{2s}u^*\right)_H + \frac{d\varphi}{dt}\left(A^\alpha u^*, A^{2s}u^*\right)_H.\end{aligned}$$

From the fact that $\left(e^{\varphi A^\alpha}B(u), A^{2s}u^*\right)_H = \left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H$ and $\left|\frac{d\varphi}{dt}\right| \leq 1$ for all $t \geq 0$, it follows

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left|A^s u^*\right|_H^2 + \nu\left|A^{s+\frac{1}{2}}u^*\right|_H^2\\ &\leq \left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right| + \left|A^{s-\frac{1}{2}}h^*\right|_H\left|A^{s+\frac{1}{2}}u^*\right|_H + \left|A^{s+\alpha-\frac{1}{2}}u^*\right|_H\left|A^{s+\frac{1}{2}}u^*\right|_H.\end{aligned} \tag{9}$$

Now, we find an appropriate bound for the term $\left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right|$. Recall the strictly increasing sequence $(\lambda_k)_{k\in\mathbb{N}_0}$ of all the distinct eigenvalues of the Stokes operator A and the orthogonal projections $P_k : H \to P_k H$ onto the λ_k-eigenspace; see (5) above. We observe that for any $u \in H$, we may write

$$u = \sum_{k\in\mathbb{N}_0} P_k u. \tag{10}$$

Remark 3.2. Given nonnegative real numbers s, α, and σ, $u \in \mathrm{D}\left(A^s e^{\sigma A^\alpha}\right)$, and $l \in \mathbb{N}_0$, we have $P_l(A^s e^{\sigma A^\alpha}u) = \lambda_l^s e^{\sigma\lambda_l^\alpha}P_l u$, and $|u|^2_{\mathrm{D}\left(A^s e^{\sigma A^\alpha}\right)} = \sum\limits_{k\in\mathbb{N}_0} e^{2\sigma\lambda_k^\alpha}\lambda_k^{2s}\left|P_k u\right|^2$.

From (10) and Assumption 2.2, we may write

$$\begin{aligned}&\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H = \sum_{(m,n,l)\in\mathbb{N}_0^3} b\left(P_m u, P_n u, P_l(A^{2s}e^{\varphi A^\alpha}u^*)\right)\\ &= \frac{1}{2}\sum_{(m,n,l)\in\mathbb{N}_0^3}\left(B(P_m u + P_n u),\, \lambda_l^{2s}e^{2\varphi\lambda_l^\alpha}P_l u\right)_H = \sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,\,m}}}\left(B(P_m u + P_n u),\, \lambda_l^{2s}e^{2\varphi\lambda_l^\alpha}P_l u\right)_H\\ &= -\sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,\,m}}} b\left(P_n u, \lambda_l^{2s}e^{2\varphi\lambda_l^\alpha}P_l u, P_m u\right) - \sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,\,m}}} b\left(P_n u, \lambda_l^{2s}e^{2\varphi\lambda_l^\alpha}P_l u, P_m u\right).\end{aligned}$$

Hence by Assumptions 2.1, 2.2 and 2.3, we can derive that

$$\left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right| \le 2C \sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,m}}} |P_n u|_{L^\infty(\Omega,T\Omega)}\, \lambda_l^{2s} e^{2\varphi\lambda_l^\alpha} |P_l u|_{H^1(\Omega,T\Omega)} |P_m u|_{L^2(\Omega,T\Omega)}$$

$$\le 2C \sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,m}}} |P_n u|_{L^\infty(\Omega,T\Omega)}\, \lambda_l^{2s} \mathrm{e}^{\varphi(\lambda_l^\alpha+\lambda_n^\varsigma+\lambda_m^\alpha+\beta)} \left|A^{\frac12}P_l u\right|_{L^2(\Omega,T\Omega)} |P_m u|_{L^2(\Omega,T\Omega)}$$

$$\le 2C \sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,m}}} \mathrm{e}^{\varphi\beta} |P_n u^*|_{L^\infty(\Omega,T\Omega)}\, \lambda_l^{2s+\frac12} |P_l u^*|_H |P_m u^*|_H\,.$$

Finally, from a suitable Agmon inequality (cf. [Tem95], Section 2.3), it follows that $|P_n u^*|_{L^\infty(\Omega,T\Omega)} \le C_1 |P_n u^*|_{L^2(\Omega,T\Omega)}^{\frac{4-d}{4}} |P_n u^*|_{H^2(\Omega,T\Omega)}^{\frac{d}{4}}$ and

$$\left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right| \le C_2\mathrm{e}^{\sigma\beta} \sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,m}}} \lambda_n^{\frac d4}\lambda_l^{2s+\frac12} |P_n u^*|_H |P_m u^*|_H |P_l u^*|_H. \tag{11}$$

Remark 3.3. Notice that the Agmon inequalities we find in [Tem95, Section 2.3] concern the case Ω is a subset of $\mathbb{R}^d$. However they hold also for a boundaryless manifold $\mathcal{C}$, because we can cover $\mathcal{C}$ by a finite number of charts and use a partition of unity argument. Recall that the Sobolev spaces on a manifold may be defined by means of an atlas of $\mathcal{C}$ (cf. [Tay97, Chapter 4, Section 3]). They hold also for smooth manifolds Ω with smooth boundary $\partial\Omega$ (cf. the discussion after Equation (4.11) in [Tay97, Chapter 4, Section 4]).

Lemma 3.4. *Suppose that the Assumptions 2.1, 2.2, 2.3 and 2.4 hold, and let the strictly increasing sequence of (nonrepeated) eigenvalues $(\lambda_k)_{k\in\mathbb{N}_0}$ of the Stokes operator A satisfy* (6)*. Then, for any given $s > \frac{d+2\left(\xi^{-1}+2\zeta-1\right)}{4}$, there exists $C_B \in \mathbb{R}_0$ such that*

$$\left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right| \le C_B |A^s u^*|^2 \left|A^{s+\frac12}u^*\right|, \qquad \textit{if } 4s \ge d + 2\left(\xi^{-1}+2\zeta+1\right);$$

$$\left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right| \le C_B |A^s u^*|^{\frac{6-\left(d-4s+2\xi^{-1}+22\zeta\right)}{4}} \left|A^{s+\frac12}u^*\right|^{\frac{6+d-4s+2\xi^{-1}+22\zeta}{4}},$$
$$\textit{if } 4s < d + 2\left(\xi^{-1}+2\zeta+1\right).$$

Further, C_B depends on d, s, σ, α, β, $C_{\mathcal{F}}$, ζ, ρ, and ξ.

Proof. From (11), Assumption 2.3 and Lemma 2.8, it follows that

$$\left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right| \le K \sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,m}}} \lambda_n^{\frac d4}\lambda_l^{s+\frac12}\lambda_m^s |P_n u^*|_H |P_m u^*|_H |P_l u^*|_H,$$

with $K = K(s,\, \sigma,\, \alpha,\, \beta,\, \lambda_1)$. Now we notice that for any triple $(m,n,l) \in \mathbb{N}_0^3$ with $n<m$ we have that

$$l \in \mathcal{F}^\bullet_{n,m} \Leftrightarrow (B(P_nH + P_mH),\, P_lH)_H \ne \emptyset \Leftrightarrow m \in \mathcal{F}^l_{n,\bullet};$$

thus, by the Cauchy inequality, we obtain that

$$\begin{aligned}&\left|\left(B(u),A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right|^2\\ &\le K\Bigg(\sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,m}}}\lambda_n^{\frac{d}{4}}\left|P_nu^*\right|_H\lambda_m^{2s}\left|P_mu^*\right|_H^2\Bigg)^{\frac12}\Bigg(\sum_{\substack{l\in\mathbb{N}_0\\ n<m\\ m\in\mathcal{F}^l_{n,\bullet}}}\lambda_n^{\frac{d}{4}}\left|P_nu^*\right|_H\lambda_l^{2s+1}\left|P_lu^*\right|_H^2\Bigg)^{\frac12}.\end{aligned}$$

From Assumption 2.4 we obtain

$$\begin{aligned}&\left|\left(B(u),A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right|^2\\ &\le KC_{\mathcal{F}}\Bigg(\sum_{n\in\mathbb{N}_0}\lambda_n^{\frac{d}{4}+\zeta}\left|P_nu^*\right|_H\Bigg)\Bigg(\sum_{m\in\mathbb{N}_0}\lambda_m^{2s}\left|P_mu^*\right|_H^2\Bigg)^{\frac12}\Bigg(\sum_{l\in\mathbb{N}_0}\lambda_l^{2s+1}\left|P_lu^*\right|_H^2\Bigg)^{\frac12}.\end{aligned}$$

Now, again thanks to the Cauchy inequality, for $\gamma\in\mathbb{R}$ we find

$$\begin{aligned}&\left|\left(B(u),A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right|\\ &\le KC_{\mathcal{F}}\Bigg(\sum_{n\in\mathbb{N}_0}\lambda_n^{\frac{d}{2}+2\zeta-2s-\gamma}\Bigg)^{\frac12}\left|A^{s+\frac{\gamma}{2}}u^*\right|_H\left|A^su^*\right|_H\left|A^{s+\frac12}u^*\right|_H.\end{aligned}\tag{12}$$

Since $s>\frac{d+2\xi^{-1}+4\zeta-2}{4}$, we obtain that $\frac{d}{2}-2s+2\zeta<1-\xi^{-1}$. Thus, we may set $\gamma\in\left(\frac{d}{2}-2s+\xi^{-1}+2\zeta,1\right)$; which implies that $\frac{d}{2}-2s+2\zeta-\gamma<-\xi^{-1}$ and that $\delta:=\left(\frac{d}{2}-2s+2\zeta-\gamma\right)\xi<-1$. From (6), it follows that

$$\sum_{n\in\mathbb{N}_0}\lambda_n^{\frac{d}{2}-2s+2\zeta-\gamma}\le\rho^{\frac{d}{2}-2s+2\zeta-\gamma}\sum_{n\in\mathbb{N}_0}n^\delta=:C_{d,s,\rho,\xi,\zeta,\gamma}<+\infty.\tag{13}$$

and, choosing in particular $\gamma=\bar\gamma:=\frac{d-4s+2\left(\xi^{-1}+2\zeta+1\right)}{4}$, from (12) and (13), it follows that

$$\left|\left(B(u),A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right|\le KC_{d,s,\rho,\xi,\zeta}\left|A^{s+\frac{\bar\gamma}{2}}u^*\right|_H\left|A^su^*\right|_H\left|A^{s+\frac12}u^*\right|_H.$$

If $\bar\gamma\le0$, that is if $4s\ge d+2\left(\xi^{-1}+2\zeta+1\right)$, then

$$\left|\left(B(u^*),A^{2s}u\right)_H\right|\le KC_{d,s,\rho,\xi}\left|A^su^*\right|_H^2\left|A^{s+\frac12}u^*\right|_H.$$

If $\bar\gamma\in(0,1)$, that is if $d+2\left(\xi^{-1}+2\zeta-1\right)<4s<d+2\left(\xi^{-1}+2\zeta+1\right)$, then by an interpolation argument (cf. [LM72], Chapter 1), we can obtain that

$$\begin{aligned}\left|\left(B(u),A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right|&\le KC_{d,s,\rho,\xi,\zeta}C_1\left|A^su^*\right|_H^{1+(1-\bar\gamma)}\left|A^{s+\frac12}u^*\right|_H^{1+\bar\gamma}\\ &=KC_{d,s,\rho,\xi,\zeta}C_1\left|A^su^*\right|_H^{\frac{-d+4s-2\xi^{-1}-4\zeta+6}{4}}\left|A^{s+\frac12}u^*\right|_H^{\frac{d-4s+2\xi^{-1}+4\zeta+6}{4}},\end{aligned}$$

which completes the proof of the lemma. □

3.3. **Proof of Theorem 3.1.** We look for u in the form $u=e^{-\varphi(t)A^\alpha}u^*$ where u^* solves (8). We will use Lemma 3.4, which suggests us to consider two cases.

3.3.1. *The case* $4s < d + 2\left(\xi^{-1} + 2\zeta + 1\right)$. *Existence.* We start by observing that

$$\left|A^{s+\alpha-\frac{1}{2}}u^*\right|_H \le \lambda_1^{\alpha-\frac{1}{2}}\left|A^s u^*\right|_H, \quad \text{if } \alpha \le \frac{1}{2},$$

and, by an interpolation argument

$$\left|A^{s+\alpha-\frac{1}{2}}u^*\right|_H \le \left|A^s u^*\right|_H^{2(1-\alpha)}\left|A^{s+\frac{1}{2}}u^*\right|_H^{2\alpha-1}, \quad \text{if } \frac{1}{2} < \alpha < 1.$$

Next, since $4s > d + 2\left(\xi^{-1} + 2\zeta - 1\right)$, we have $\frac{6+d-4s+2\xi^{-1}+4\zeta}{4} < 2$. Thus, we can set $p = \frac{8}{6+d-4s+2\xi^{-1}+4\zeta} > 1$, and q such that $\frac{1}{p} + \frac{1}{q} = 1$, that is, $\frac{1}{q} = \frac{2-\left(d-4s+2\xi^{-1}+4\zeta\right)}{8}$.

From (9), Lemma 3.4, and suitable Young inequalities, we derive that

$$\begin{aligned}\frac{d}{dt}\left|A^s u^*\right|_H^2 + \frac{4\nu}{3}\left|A^{s+\frac{1}{2}}u^*\right|_H^2 &\le 2^q\left(\frac{3}{\nu}\right)^{\frac{q}{p}} C_B^q \left|A^s u^*\right|_H^{\left(\frac{-d+4s-2\xi^{-1}-4\zeta+6}{4}\right)q} + \frac{3}{\nu}\left|A^{s-\frac{1}{2}}h^*\right|_H^2\\ &\quad + \left|A^{s+\alpha-\frac{1}{2}}u^*\right|_H\left|A^{s+\frac{1}{2}}u^*\right|_H.\end{aligned} \tag{14}$$

Notice that in the case $\alpha \in (0, \frac{1}{2}]$ we have

$$\left|A^{s+\alpha-\frac{1}{2}}u^*\right|_H\left|A^{s+\frac{1}{2}}u^*\right|_H \le C_\nu\left|A^s u^*\right|_H^2 + \frac{\nu}{3}\left|A^{s+\frac{1}{2}}u^*\right|_H^2,$$

and in the case $\alpha \in (\frac{1}{2}, 1)$ we have

$$\begin{aligned}\left|A^{s+\alpha-\frac{1}{2}}u^*\right|_H\left|A^{s+\frac{1}{2}}u^*\right|_H &\le C_{\nu,\alpha}\left|A^s u^*\right|_H^{2(1-\alpha)\frac{2}{3-2\alpha}} + \frac{\nu}{3}\left|A^{s+\frac{1}{2}}u^*\right|_H^2\\ &\le C_{\nu,\alpha}\left(\left|A^s u^*\right|_H + 1\right)^2 + \frac{\nu}{3}\left|A^{s+\frac{1}{2}}u^*\right|_H^2\\ &\le 2C_{\nu,\alpha}\left(\left|A^s u^*\right|_H^2 + 1\right) + \frac{\nu}{3}\left|A^{s+\frac{1}{2}}u^*\right|_H^2,\end{aligned}$$

because $0 < \frac{4(1-\alpha)}{3-2\alpha} < 2$.

Next we observe that $\left(\frac{-d+4s-2\xi^{-1}-4\zeta+6}{4}\right)q = 2 + q > 3$, and from Proposition 2.6 it follows $\left|A^s u^*\right|_H^{2+q} + 1 \le \left(\left|A^s u^*\right|_H^2 + 1\right)^{\frac{2+q}{2}}$. Therefore, from (14), we can obtain

$$\begin{aligned}&\frac{d}{dt}\left|A^s u^*\right|_H^2 + \nu\left|A^{s+\frac{1}{2}}u^*\right|_H^2\\ &\le K_1\left|A^s u^*\right|_H^{2+q} + \frac{3}{\nu}\left|A^{s-\frac{1}{2}}h^*\right|_H^2 + K_2(\left|A^s u^*\right|_H^2 + 1)\\ &\le (K_1 + K_2)\left(\left|A^s u^*\right|_H^2 + 1\right)^{\frac{2+q}{2}} + \frac{3}{\nu}\left|A^{s-\frac{1}{2}}h^*\right|_H^2,\end{aligned}$$

with $K_1 + K_2$ depending on ν, λ_1, d, s, σ, α, β, ρ, ξ, ζ, and $C_{\mathcal{F}}$.

Now, setting $K_3 := K_1 + K_2 + \frac{3}{\nu}\left|A^{s-\frac{1}{2}}h^*\right|_{L^\infty((0,+\infty),\,H)}^2$, we arrive to

$$\frac{d}{dt}\left|A^s u^*\right|_H^2 + \nu\left|A^{s+\frac{1}{2}}u^*\right|_H^2 \le K_3\left(\left|A^s u^*\right|_H^2 + 1\right)^{\frac{2+q}{2}} \tag{15}$$

and, in particular, to

$$\frac{d}{dt}y \le K_3 y^{\frac{2+q}{2}}, \quad \text{with } y(t) := \left|A^s u^*(t)\right|_H^2 + 1,$$

that is, $\frac{d}{dt}y^\gamma \geq \gamma K_3$ with $\gamma := 1 - (\frac{2+q}{2}) = -\frac{q}{2} < 0$. Integrating over the interval $(0,\, t)$, it follows that $y^\gamma(t) \geq y^\gamma(0) - (\frac{q}{2})K_3 t$. If we set T^* such that $(\frac{q}{2})K_{B,\nu}T^* \leq (\frac{1}{2})y^\gamma(0)$, that is if $T^* \leq \frac{y^\gamma(0)}{qK_{B,\nu}}$, then $y^{-\gamma}(t) \leq 2y^{-\gamma}(0)$, for all $t \in [0, T^*]$. Thus, we obtain

$$|A^s u^*(t)|_H^2 + 1 \leq 4^{\frac{1}{q}} \left(|A^s u(0)|_H^2 + 1\right) \quad \text{for all } t \in [0, T^*]\,,$$

from which, together with $u(0) = u_0 \in \mathrm{D}(A^s)$ and (15), we can conclude that

$$u^* \in L^\infty\left((0, T^*)\,, D(A^s)\right) \cap L^2\left((0, T^*)\,, D(A^{s+\frac{1}{2}})\right) \tag{16}$$

which implies (7).

3.3.2. *The case* $4s \geq d + 2\left(\xi^{-1} + 2\zeta + 1\right)$. *Existence.* Using the corresponding inequality from 3.4, it is straightforward to check that all the arguments from the first case, $4s < d + 2\left(\xi^{-1} + 2\zeta + 1\right)$, can be repeated by taking $p = q = 2$. We will arrive again to the conclusions (16), and (7).

3.3.3. *Uniqueness.* It remains to check the uniqueness of u. Let v be another solution for (4), and set $\eta = v - u$. We start by noticing that, from (7), with nonnegative $(s,\, \sigma,\, \alpha) \in [0,\, +\infty)^3$, we have in particular that u is a weak solution:

$$u \in L^\infty\left((0, T^*), H\right) \cap L^2\left((0, T^*), D\left(A^{\frac{1}{2}}\right)\right).$$

In the case $d = 2$, it is well known that the uniqueness of u will follow from the estimate

$$\begin{aligned}|(B(v) - B(u),\, \eta)_H| &= |b(\eta,\, u,\, \eta)| \leq |\eta|_{L^4(\Omega, T\Omega)}|\eta|_{L^4(\Omega, T\Omega)}|u|_{H^1(\Omega, T\Omega)}\\ &\leq C|\eta|^2_{H^{\frac{1}{2}}(\Omega, T\Omega)}|u|_{H^1(\Omega, T\Omega)} \leq C_1|\eta|_H|A^{\frac{1}{2}}\eta|_H|A^{\frac{1}{2}}u|_H\end{aligned}$$

(see, e.g., [Tem01, Chapter 3, Section 3.3, Theorem 3.2]).

In the case $d = 3$. Since $s > \frac{d-2}{4} = \frac{1}{4}$, again from (7), we also have that

$$u \in L^\infty\left((0, T^*), D\left(A^{s_1}\right)\right) \subseteq L^\infty\left((0, T^*), H^{2s_1}(\Omega,\, \mathbb{R}^3)\right) \subset L^{r_1}\left((0, T^*), L^{r_2}(\Omega,\, \mathbb{R}^3)\right)$$

with $s_1 < s$ and $s_1 \in (\frac{1}{4},\, \frac{1}{2}]$, $r_1 > 1$ and $r_2 = \frac{2d}{d-4s_1} > 3$, by the Sobolev embedding Theorem (cf. [DD12, Section 4.4, Corollary 4.53]). Now, the uniqueness of u follows from the fact that for r_1 big enough we have that $\frac{2}{r_1} + \frac{d}{r_2} \leq 1$, and from [Lio69, Chapter 1, Section 6.8, Theorem 6.9]). □

Remark 3.5. For simplicity we have restricted ourselves to the above formal computations, but those computations will hold for the Galerkin approximations based on the eigenfunctions of A, which means that they can be made rigorous. See, for example, [Lio69, Chapter 1, Section 6.4] and [Tem01, Chapter 3, Section 3]

4. Considering repeated eigenvalues

In some cases it will be more convenient to work with the sequence $(\underline{\lambda}_k)_{k\in\mathbb{N}_0}$ of repeated eigenvalues. In that case we have to adjust our assumptions to obtain the corresponding version of the Theorem 3.1. Consider the system of eigenfunctions $\{W_k \mid k \in \mathbb{N}_0\}$.

Assumption 4.1. There are real numbers $\alpha > 0$ and $\beta \geq 0$, such that for all triples $(n,\, m,\, l) \in \mathbb{N}_0^3$

$$(B(W_n + W_m),\, W_l)_H \neq 0, \quad \text{implies} \quad \underline{\lambda}_l^\alpha \leq \underline{\lambda}_n^\alpha + \underline{\lambda}_m^\alpha + \beta.$$

For given $(n, m, l) \in \mathbb{N}_0^3$, we define the sets

$$\underline{\mathcal{F}}_{n,m}^{\bullet} := \{k \in \mathbb{N}_0 \mid (B(W_n + W_m), W_k)_H \neq 0, \text{ with } n < m\};$$
$$\underline{\mathcal{F}}_{n,\bullet}^{l} := \{k \in \mathbb{N}_0 \mid (B(W_n + W_k), W_l)_H \neq 0, \text{ with } n < k\}.$$

Assumption 4.2. There are $C_{\mathcal{F}} \in \mathbb{N}_0$ and $\zeta \in [0, +\infty)$, such that for all $n \in \mathbb{N}_0$ we have

$$\sup_{(m,l)\in\mathbb{N}_0^2} \left\{\mathrm{card}(\underline{\mathcal{F}}_{n,m}^{\bullet}), \mathrm{card}(\underline{\mathcal{F}}_{n,\bullet}^{l})\right\} \leq C_{\mathcal{F}} \underline{\lambda}_n^{\zeta}.$$

Theorem 4.3. *Suppose that the Assumptions 2.1, 2.2, 4.1 and 4.2 hold, and let the increasing sequence of (repeated) eigenvalues $(\underline{\lambda}_k)_{k\in\mathbb{N}_0}$ of the Stokes operator A satisfy, for some positive real numbers ρ, ξ,*

$$\underline{\lambda}_k > \rho k^{\xi}, \quad \textit{for all } k \in \mathbb{N}_0.$$

Further, let us be given $\alpha \in (0, 1)$ as in Assumption 4.1, $C_{\mathcal{F}}$ and $\zeta \geq 0$ as in Assumption 4.2, $s > \frac{d+2(\xi^{-1}+2\zeta-1)}{4}$, $\sigma > 0$, $h \in L^\infty(\mathbb{R}_0, \mathrm{D}(A^{s-\frac{1}{2}}e^{\sigma A^\alpha}))$, and $u_0 \in \mathrm{D}(A^s)$.
Then, there are $T^* > 0$ and a unique solution

$$u \in L^\infty\left((0,T^*), D\left(A^s e^{\sigma A^\alpha}\right)\right) \cap L^2\left((0,T^*), D\left(A^{s+\frac{1}{2}} e^{\sigma A^\alpha}\right)\right)$$

for the Navier Stokes system (4).
Further, T^* depends on the data $\left(|h|_{L^\infty(\mathbb{R}_0, D\left(A^{s-\frac{1}{2}}e^{\sigma A^\alpha}\right)}, |A^s u_0|_H\right)$ and also on the constants ν, d, s, σ, α, ρ, ξ, ζ, and $C_{\mathcal{F}}$.

The proof can be done following line by line that of Theorem 3.1.

Remark 4.4. If we can find a bound $|P_n u|_{L^\infty(\Omega, T\Omega)} \leq C\lambda_n^\theta |P_n u|_H$ with $\theta < \frac{d}{4}$ and C independent of n, then we can take θ in the place of $\frac{d}{4}$ in (11). As a corollary, we can replace d by 4θ in Theorem 3.1, provided s satisfies $s \geq 0$ in the case $d = 2$ and $s > \frac{1}{4}$ in the case $d = 3$, in order to guarantee the uniqueness of the solution. The analogous conclusion holds for Theorem 4.3, if we can find a bound $|W_n|_{L^\infty(\Omega, T\Omega)} \leq C\lambda_n^\theta$.

Remark 4.5. In some situations like in the case of general Navier boundary conditions it may be useful to split the Stokes operator $\Pi\Delta$ as $\Pi\Delta = A + C$ (cf. [Rod08, Chapter 4, Section 4.2]), or it may be interesting to consider an additional linear external forcing (like a Coriolis forcing as in [CRT99]). In these cases we will have the system

$$\dot{u} + B(u, u) + Au + Cu + h = 0, \quad u(0, x) = u_0(x),$$

instead of (3). Notice that Theorems 3.1 and 4.3 will hold in these cases provided we have the estimate $(Cu, A^{2s}u)_{V',V} \leq C_1|A^s u|_H |A^{s+\frac{1}{2}}u|_H$. A better estimate holds in the case of the two-dimensional Navier–Stokes equation under the action of a Coriolis force $\tilde{C}u$: from [CRT99, Lemma 1] we know that for $s > \frac{1}{2}$ it holds $(Cu, A^{2s}u)_{V',V} = (A^{s+\frac{1}{2}}Cu, A^{s-\frac{1}{2}}u)_H \leq C_1|A^s|_H |A^{s-\frac{1}{2}}|_H$, for $Cu := \Pi\tilde{C}u$.

5. Examples

We start by revisiting the cases where Ω is the Torus $\mathbb{T}^d$ and the Sphere $\mathbb{S}^2$. Then we give some new examples in two dimensions, namely the cases of Hemisphere, Rectangle and Cylinder under Lions boundary conditions.

5.1. **Torus.** We consider the torus $\mathbb{T}^d = \Pi_{i=1}^d \mathbb{S}^1 \sim (0, 2\pi]^d$, $d \in \{2, 3\}$. This case corresponds to the case where we take periodic boundary conditions in $\mathbb{R}^d$ with period 2π in each direction x_i, $i \in \{1, \ldots, d\}$. We also assume that the average $\int_{\mathbb{T}^2} u(t) \,\mathrm{d}\mathbb{T}^d$ vanishes for (a.e.) $t \geq 0$ (cf. [FMRT01, Chapter II, eq. (2.5)], [AS05, Section 2.1]). In this case the Navier–Stokes system can be rewritten as an evolutionary equation in the space of divergence free and zero averaged vector fields $H = \{u \in L^2(\mathbb{T}^d, T\mathbb{T}^d) \sim L^2(\mathbb{T}^d, \mathbb{R}^d) \mid \operatorname{div} u = 0 \text{ and } \int_{\mathbb{T}^2} u \,\mathrm{d}\mathbb{T}^d = 0\}$, with the spaces V and $\mathrm{D}(A)$, defined in Section 2.1 given by $V = H \cap H^1(\mathbb{T}^d, T\mathbb{T}^d)$ and $\mathrm{D}(A) = H \cap H^2(\mathbb{T}^d, T\mathbb{T}^d)$.

We will show that in this case we can take $\alpha = \frac{1}{2}$, $\xi = \frac{1}{2}$, and $\zeta = 0$ in Theorem 4.3, and $\theta = 0$ in Remark 4.4. That is, we can take $s > 0$, in Theorem 4.3.

To simplify the writing we will denote the usual Euclidean scalar product $(u, v)_{\mathbb{R}^d}$ in $\mathbb{R}^d$ by $u \cdot v := \sum_{i=1}^d u_i v_i$. It is well known that a vector field can be written as

$$u = \sum_{k \in \mathbb{Z}^d \setminus \{0_d\}} u_k \mathrm{e}^{\mathbf{i} k \cdot x},$$

where 0_d stands for the zero element $(0, \ldots, 0) \in \mathbb{R}^d$, $\mathbf{i} \sim 0 + 1\mathbf{i}$ is the imaginary complex unit, and the coefficients satisfy $k \cdot u_k = 0$ and $u_{-k} = \overline{u_k}$, where the overline stands for the complex conjugate. The condition $k \cdot u_k = 0$ comes from the divergence free condition, and $u_{-k} = \overline{u_k}$ comes from the fact that u is a function with (real) values in $\mathbb{R}^3$. Thus

$$u = \sum_{k \in \mathbb{Z}^d;\, k > 0_d} \mathrm{Re}(u_k) \cos(k \cdot x) - \mathrm{Im}(u_k) \sin(k \cdot x),$$

where $k > 0_d$ is understood in the lexicographical order, that is either $k_1 > 0$, or $k_1 = 0$ and $k_2 > 0$, or $(k_1, k_{d-1}) = (0, 0)$ and $k_d > 0$, and that a basis of vector fields in H is given by

$$\mathcal{W} = \{w_k^j \cos(k \cdot x),\, w_k^j \sin(k \cdot x) \mid k \in \mathbb{Z}^d,\, k > 0_d \text{ and } j \in \{1, d-1\}\}$$

where for each $k \in \mathbb{Z}^d$, $k > 0_d$, $\{w_k^1, w_k^{d-1}\}$ is a basis for the orthogonal space $\{k\}^\perp$ of $\{k\}$, in $\mathbb{R}^d$. That is, $\operatorname{span}\{w_k^1, w_k^{d-1}\} = \{k\}^\perp$ (cf. [Rod08, Chapter 6, Section 1] for the case $d = 2$). Moreover we may choose the vectors w_k^j so that the basis above is orthonormal, that is, we can write

$$u = \sum_{\substack{k \in \mathbb{Z}^d;\, k > 0_d \\ j \in \{1, d-1\}}} u_{k,j}^c w_k^j \cos(k \cdot x) + u_{k,j}^s w_k^j \sin(k \cdot x).$$

Since the cardinality of $\{k \in \mathbb{Z}^d \mid k > 0_d\} \times \{1, d-1\}$ is equal to that of $\mathbb{N}_0$ we could write the previous sum as $u = \sum_{k \in \mathbb{N}_0} u_k W_k$, as in the preceding text (cf. Section 3). However we can check the Assumptions 2.1, 2.2, 4.1, and 4.2 without performing that writing explicitly.

Checking Assumptions 2.1 and 2.2. Assumptions 2.1 is well known to hold under periodic boundary conditions. The same holds for all the points in Assumption 2.2 (cf. [Tem95, Section 2.3]); we only check the first one, or taking into account Remark 4.4, we check that $B\left(w_n^j \cos(n \cdot x),\, w_n^j \cos(n \cdot x)\right) = B\left(w_n^j \sin(n \cdot x),\, w_n^j \sin(n \cdot x)\right) = 0$. Indeed from $w_n^j \cdot n = 0$, it follows

$$\begin{aligned}
B(w_n^j \cos(n \cdot x),\, w_n^j \cos(n \cdot x)) &= -\Pi(w_n^j \cdot n) \cos(n \cdot x) \sin(n \cdot x) w_n^j = 0, \\
B(w_n^j \sin(n \cdot x),\, w_m^j \sin(n \cdot x)) &= \Pi(w_n^j \cdot n) \sin(n \cdot x) \cos(n \cdot x) w_n^j = 0,
\end{aligned}$$

where Π stands for the orthogonal projection in $L^2(\mathbb{T}^d, T\mathbb{T}^d)$ onto H.

Checking Assumptions 4.1 and 4.2. We proceed as follows: first we obtain

$$\begin{aligned}
B(w_n^j\cos(n\cdot x),\, w_m^i\cos(m\cdot x)) &= -\Pi(w_n^j\cdot m)\cos(n\cdot x)\sin(m\cdot x)w_m^i,\\
B(w_n^j\cos(n\cdot x),\, w_m^i\sin(m\cdot x)) &= \Pi(w_n^j\cdot m)\cos(n\cdot x)\cos(m\cdot x)w_m^i,\\
B(w_n^j\sin(n\cdot x),\, w_m^i\cos(m\cdot x)) &= -\Pi(w_n^j\cdot m)\sin(n\cdot x)\sin(m\cdot x)w_m^i,\\
B(w_n^j\sin(n\cdot x),\, w_m^i\sin(m\cdot x)) &= \Pi(w_n^j\cdot m)\sin(n\cdot x)\cos(m\cdot x)w_m^i,
\end{aligned}$$

from which we can find that

$$\begin{aligned}
&B(w_n^j\cos(n\cdot x)+w_m^i\cos(m\cdot x))\\
&= B(w_n^j\cos(n\cdot x),\, w_m^i\cos(m\cdot x)) + B(w_m^i\cos(m\cdot x),\, w_n^j\cos(n\cdot x))\\
&\quad + B(w_n^j\cos(n\cdot x),\, w_n^j\cos(n\cdot x)) + B(w_m^i\cos(m\cdot x),\, w_m^i\cos(m\cdot x))\\
&= -\Pi w_m^i(w_n^j\cdot m)\cos(n\cdot x)\sin(m\cdot x) - \Pi w_n^j(w_m^j\cdot n)\cos(m\cdot x)\sin(n\cdot x)\\
&= \frac{1}{2}\Pi(-w_m^i(w_n^j\cdot m) - w_n^j(w_m^j\cdot n))\sin((m+n)\cdot x)\\
&\quad + \frac{1}{2}\Pi(-w_m^i(w_n^j\cdot m) + w_n^j(w_m^j\cdot n))\sin((m-n)\cdot x),
\end{aligned}$$

then, it is straightforward to check that $B(w_n^j\cos(n\cdot x)+w_m^i\cos(m\cdot x))$ is orthogonal in $L^2(\mathbb{T}^d,\, T\mathbb{T}^d)$ to all the elements in $\mathcal{W}$ except those in

$$\{w_{m+n}^j\sin\left((m+n)\cdot x\right),\, w_{[m-n]}^j\sin\left(([m-n])\cdot x\right) \mid j\in\{1,\, d-1\}\},$$

where we denote

$$[m-n] = \begin{cases} m-n & \text{if } m-n>0_d\\ n-m & \text{if } n-m>0_d \text{ or } n-m=0_d \end{cases}.$$

In other words, we can conclude that $(B(w_n^j\cos(n\cdot x)+w_m^i\cos(m\cdot x)),\, v)_H \neq 0$ only if

$$v\in \operatorname{span}\{w_{m+n}^j\sin\left((m+n)\cdot x\right),\, w_{[m-n]}^j\sin\left(([m-n])\cdot x\right) \mid j\in\{1,\, d-1\}\}.$$

Analogously, we can conclude that $(B(w_n^j\sin(n\cdot x)+w_m^i\sin(m\cdot x)),\, v)_H \neq 0$ only if

$$v\in \operatorname{span}\{w_{m+n}^j\sin\left((m+n)\cdot x\right),\, w_{[m-n]}^j\sin\left(([m-n])\cdot x\right) \mid j\in\{1,\, d-1\}\}.$$

Besides that $(B(w_n^j\sin(n\cdot x)+w_m^i\cos(m\cdot x)),\, v)_H \neq 0$ only if

$$v\in \operatorname{span}\{w_{m+n}^j\cos\left((m+n)\cdot x\right),\, w_{[m-n]}^j\cos\left(([m-n])\cdot x\right) \mid j\in\{1,\, d-1\}\};$$

and that $(B(w_n^j\cos(n\cdot x)+w_m^i\sin(m\cdot x)),\, v)_H \neq 0$ only if

$$v\in \operatorname{span}\{w_{m+n}^j\cos\left((m+n)\cdot x\right),\, w_{[m-n]}^j\cos\left(([m-n])\cdot x\right) \mid j\in\{1,\, d-1\}\}.$$

Therefore we can conclude that $\operatorname{card}(\underline{\mathcal{F}}_{n,m}^{\bullet}) \leq 4$ and that necessarily $\operatorname{card}(\underline{\mathcal{F}}_{n,\bullet}^{l}) \leq 4$. That is, we can take $C_{\mathcal{F}}=4$ and $\zeta=0$ in Assumption 4.2.

Assumption 4.1 follows from the fact that the eigenvalue associated to $w_n^j\sin(n\cdot x)$ and $w_n^j\cos(n\cdot x)$, is given by $|n|_{\mathbb{R}^d}^2 = n\cdot n$, and by the triangle inequality, $|n\pm m|_{\mathbb{R}^d} \leq |n|_{\mathbb{R}^d}+|m|_{\mathbb{R}^d}$, which implies that Assumption 4.1 holds with $\alpha=\frac{1}{2}$ and $\beta=0$.

Looking for the value θ in Remark 4.4. From $|w_n^j\sin(n\cdot x)|_{L^\infty(\mathbb{T}^d,T\mathbb{T}^d)} \leq |w_n^j|$ and $|w_n^j\sin(n\cdot x)|_{L^2(\mathbb{T}^d,T\mathbb{T}^d)} = 1$, we obtain $|w_n^j|^2 = |\sin(n\cdot x)|_{L^2(\mathbb{T}^d,T\mathbb{T}^d)}^{-2} = \pi^{-d}$ and $|w_n^j\sin(n\cdot x)|_{L^\infty(\mathbb{T}^d,T\mathbb{T}^d)} \leq \pi^{-\frac{d}{2}}$, and similarly $|w_n^j\cos(n\cdot x)|_{L^\infty(\mathbb{T}^d,T\mathbb{T}^d)} \leq \pi^{-\frac{d}{2}}$. Hence, we can take $\theta=0$ in Remark 4.4.

Asymptotic behavior of the (repeated) eigenvalues. From [FMRT01, Chapter II, Section 6] we know that the asymptotic behavior of the (repeated) eigenvalues of the Stokes operator in $\mathbb{T}^d$ satisfy $\lambda_k \sim \lambda_1 k^{\frac{2}{d}}$ and more precisely

$$\lim_{k\to+\infty} \frac{\lambda_k}{\lambda_1 k^{\frac{2}{d}}} =: q > 0;$$

then in particular there is $k_0 \in \mathbb{N}_0$ such that $\frac{\lambda_k}{\lambda_1 k^{\frac{2}{d}}} \ge \frac{q}{2}$ for all $k > k_0$, which implies that for all $k \in \mathbb{N}_0$ we have $\lambda_k > \rho k^{\frac{2}{d}}$ if $\rho < \lambda_1 \min\{\frac{q}{2},\, q_0\}$, with $q_0 := \min_{k\le k_0} \frac{\lambda_k}{\lambda_1 k^{\frac{2}{d}}}$. That is, we can take $\xi = \frac{2}{d}$ in Theorem 4.3.

Conclusion. Taking into account Remark 4.4, we conclude that Theorem 4.3 holds with $\alpha = \frac{1}{2}$ and $s > \frac{d-2}{4}$. This improves the results in [FT89, Liu92], from whose we already knew that s could be taken in $[\frac{1}{2},\, +\infty)$ for $d = 2$, and in $\{\frac{1}{2}\} \cup [\frac{3}{4},\, +\infty)$ for $d = 3$.

5.2. **Sphere.** Let $\Omega = \mathbb{S}^2 := \{(x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ be the two-dimensional sphere with the Riemannian metric induced from the usual Euclidean metric in $\mathbb{R}^3$.

In this case we can write the Navier–Stokes system as an evolutionary equation in the space $H := \{u \in L^2(\Omega,\, T\Omega) \mid \nabla\cdot u = 0\} \cap \{\nabla^\perp \psi \mid \psi \in H^1(\mathbb{S}^2,\, \mathbb{R})\}$, with $V := H \cap H^1(\Omega,\, T\Omega)$ and $\mathrm{D}(A) := H \cap H^2(\Omega,\, T\Omega)$ (cf. [Rod08, Section 5.6], [CRT99, Section 2]).

Remark 5.1. Notice that in [CRT99, Section 2] and [Rod08, Section 5.6] the definitions and notations of the curl of a function f are different; in the former reference it is denoted $\operatorname{Curl} f$ and in the latter $\nabla^\perp f$; we shall show that $\operatorname{Curl} f = -\nabla^\perp f$ in the Appendix, Section A.3.

In this case we will use Theorem 3.1 and Remark 4.4, and show that there we can take $\theta = \frac{1}{4}$, $\xi = 2$, $\zeta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $s > \frac{1}{2}$. In particular we recover the result in [CRT99].

The complete system of eigenfunctions and respective eigenvalues for $A = -\nu\Pi\Delta$, in H, is presented in [CRT99, Section 2], and it is given by

$$\{Z_n^m(\vartheta,\phi) = \lambda_n^{-\frac{1}{2}} \nabla^\perp Y_n^m(\vartheta,\phi) \mid n \in \mathbb{N}_0, m \in \mathbb{Z}, \text{ and } |m| \le n\} \tag{17}$$

where $\vartheta \in (0,\pi)$, $\phi \in (0, 2\pi)$, and for each $Y_n^m(\vartheta,\phi) := C_n^m e^{im\phi} P_n^m(\cos\vartheta)$ is a normalized eigenfunctions of the Laplacian in $L^2(\mathbb{S}^2,\, \mathbb{R})$ associated with the eigenvalue $\lambda_n = n(n+1)$, with $C_n^m := \left(\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n-|m|)!}\right)^{\frac{1}{2}}$ and P_n^m is the Ferrers' associated Legendre function of the first kind

$$P_n^m(x) := \frac{(1-x^2)^{\frac{m}{2}}}{2^n n!}\frac{\mathrm{d}^{n+m}(x^2-1)^n}{\mathrm{d}x^{n+m}}, \quad P_n^{-m}(x) := P_n^m(x) \tag{18}$$

for $m \in \{k \in \mathbb{N} : k \le n\}$, defined for $|x| \le 1$. For further details on these functions we refer to [WW69, Chapter XV, Section 15.5].

For any $(u, v, w) \in P_n H \times P_m H \times P_l H$, there are scalar functions (ψ_u, ψ_v, ψ_w) called stream functions associated with (u, v, w) respectively such that

$$u = -\nabla^\perp \psi_u, \quad v = -\nabla^\perp \psi_v \quad w = -\nabla^\perp \psi_w$$

where

$$\psi_u = \sum_{|i|\le n} \psi_u^i Y_n^i, \quad \psi_v = \sum_{|j|\le m} \psi_v^j Y_m^j, \quad \psi_w = \sum_{|k|\le l} \psi_w^k Y_l^k,$$

and $\psi_u^i = \overline{\psi_u^{-i}}$, $\psi_v^j = \overline{\psi_v^{-j}}$, and $\psi_w^k = \overline{\psi_w^{-k}}$.

Checking Assumptions 2.1 and 2.2. Assumption 2.1 and the estimates in Assumption 2.2 follow straightforward. Now we show that $Bu = 0$ if u is an eigenfunction. From the discussion after Corollary 5.6.3 in [Rod08, Chapter 5, Section 5.6] we have that $\nabla^{\perp}\cdot(Bu) = g(\nabla\Delta^{-1}\nabla^{\perp}\cdot u,\, \nabla^{\perp}\nabla^{\perp}\cdot u)$, where $g(\cdot,\,\cdot)$ is the scalar product in $T\mathbb{S}^2$ inherited from the Euclidean scalar product in $\mathbb{R}^3$ and $\Delta^{-1}f$ denotes the solution g of the Poisson system $\Delta g = f,\ g|_{\partial\Omega} = 0$. If u is an eigenfunction from (17) with associated eigenvalue λ_u and associated stream function ψ_u, then $\nabla^{\perp}\cdot u = \Delta\psi_u$, and we find $\nabla^{\perp}\cdot B(u) = g(\nabla\psi_u,\, \nabla^{\perp}\Delta\psi_u) = \lambda_u g(\nabla\psi_u,\, \nabla^{\perp}\psi_u) = 0$, this implies that $ABu = \Delta Bu = \nabla^{\perp}\nabla^{\perp}\cdot(Bu) = 0$, and necessarily $Bu \in H$ is orthogonal to all eigenfunctions in (17), $(Bu, Z_n^m)_H = \lambda_n^{-1}(Bu, AZ_n^m)_H = \lambda_n^{-1}(ABu, Z_n^m)_H = 0$, that is, $B(u) = 0$.

Finally, for the skew-symmetry property $b(u,\, v,\, w) = -b(u,\, w,\, v)$ we refer to [Arn66, Section 8, Equation (59)] [Rod08, Chapter 5, Corollary 5.5.2].

Checking the Assumptions 2.3 and 2.4. Following [CRT99, Section 3, Lemma 2] (cf. [Rod08, Chapter 5, corollary 5.6.3] [Arn66, Section 9, Equation (90)]), for eigenfunctions $u \in P_nH$, $v \in P_mH$, and $w \in P_lH$ we obtain

$$\begin{aligned}
\left|(B(u+v), w)_H\right| &= \left|\left(\Pi\left(\Delta\psi_v\nabla\psi_u\right), \nabla^{\perp}\psi_w\right)_H + \left(\Pi\left(\Delta\psi_u\nabla\psi_v\right), \nabla^{\perp}\psi_w\right)_H\right| \\
&= \left|\left(\Pi\left(\sum_{|j|\le m}\psi_v^j\Delta Y_m^j\sum_{|i|\le n}\psi_u^i\nabla Y_n^i\right), \sum_{|k|\le l}\psi_w^l\nabla^{\perp}Y_l^k\right)_H\right. \\
&\qquad \left. + \left(\Pi\left(\sum_{|j|\le m}\psi_v^j\Delta Y_m^j\sum_{|i|\le n}\psi_u^i\nabla Y_n^i\right), \sum_{|k|\le l}\psi_w^l\nabla^{\perp}Y_l^k\right)_H\right| \\
&= \left|\sum_{|i|\le n}\sum_{|j|\le m}\sum_{|k|\le l}\psi_u^i\psi_v^j\psi_w^l\left(\Delta Y_m^j\nabla Y_n^i, \nabla^{\perp}Y_l^k\right)_H\right. \\
&\qquad \left. + \sum_{|i|\le n}\sum_{|j|\le m}\sum_{|k|\le l}\psi_u^i\psi_v^j\psi_w^l\left(\Delta Y_m^j\nabla Y_n^i, \nabla^{\perp}Y_l^k\right)_H\right|
\end{aligned}$$

An explicit expression for the scalar product $\left(\Delta Y_m^j\nabla Y_n^i, \nabla^{\perp}Y_l^k\right)_H$ is given in [FF05, Theorem 5.3], that expression involves the so-called Wigner-3j symbols. For this symbols we refer also to [Edm96, Section 3.7] and [RY04, Section 2]. From that expession in [FF05, Theorem 5.3], recalling that the Wigner-3j symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ vanishes unless all the conditions

i. $m_1 + m_2 + m_3 = 0$,
ii. $j_1 + j_2 + j_3$ is an integer (if $m_1 = m_2 = m_3 = 0$, $j_1 + j_2 + j_3$ is an even integer),
iii. $|m_k| \le j_k$, and
iv. $|j_1 - j_2| \le j_3 \le j_1 + j_2$

are satisfied, we can conclude that $(u, v, w) \in P_nH \times P_mH \times P_lH$ and $(B(u+v), w)_H \ne 0$ only if $m - n \le l < m + n$ and $m + n + l$ is odd (cf. [FF05, Corollary 5.4]).

Therefore, we obtain that necessarily $\mathrm{card}(\mathcal{F}_{n,m}^{\bullet}) \le 2n$ and $\mathrm{card}(\mathcal{F}_{n,\bullet}^{l}) \le 2n$, that is, Assumption 2.4 holds for $C_{\mathcal{F}} = 2$ and $\zeta = \frac{1}{2}$.

Since, for $(u, v, w) \in P_nH \times P_mH \times P_lH$ and $(B(u+v), w)_H \ne 0$ we have $l \in [m-n, m+n]$, then $\lambda_l < \lambda_{n+m}$, and from Lemma A.1 and (A.2) in the Appendix (setting $p(x) = x(x+1)$), we have that $\lambda_{m+n}^{\frac{1}{2}} \le \lambda_m^{\frac{1}{2}} + \lambda_n^{\frac{1}{2}} + 2$, and it follows that Assumption 2.3 holds with $\alpha = \frac{1}{2}$.

The parameters θ and ξ. From [CRT99, Section 3, Lemma 2], we can take $\theta = \frac{1}{4}$ in 4.4, and from $\lambda_k = k(k+1) > k^2$ it follows that (6) holds with $\xi = 2$.

Conclusion. Taking into account Remark 4.4, we conclude that Theorem 3.1 holds with $\alpha = \frac{1}{2}$ and $s > \frac{1}{2}$. This agrees with the results in [CRT99].

5.3. **Hemisphere.** Let Ω be the Hemisphere $\mathbb{S}^2_+ = \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_3 > 0\}$. On the boundary $\partial\mathbb{S}^2_+ = \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_3 = 0\}$ we impose the Lions boundary conditions, that is, we consider the evolutionary Navier–Stokes equation in $H := \{u \in L^2(\Omega, T\Omega) \mid \nabla \cdot u = 0 \text{ and } g(u, \mathbf{n}) = 0\} \cap \{\nabla^\perp \psi \mid \psi \in H^1(\mathbb{S}^2_+, \mathbb{R})\}$, with $V := H \cap H^1(\Omega, T\Omega)$ and $\mathrm{D}(A) := V \cap \{u \in H^2(\Omega, T\Omega) \mid \nabla^\perp \cdot u = 0 \text{ on } \partial\mathbb{S}^2_+\}$ (cf. [Rod08, Sections 5.5 and 6.4]).

In this case we will use Theorem 3.1 and Remark 4.4, and show that as in the case of the Sphere, in Section 5.2, there we can take $\theta = \frac{1}{4}$, $\xi = 2$, $\zeta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $s > \frac{1}{2}$.

In spherical coordinates $\mathbb{S}^2 \sim (\vartheta, \phi) \in [0, \pi] \times [0, 2\pi)$ the Hemisphere corresponds to $\mathbb{S}^2_+ \sim (\vartheta, \phi) \in [0, \frac{\pi}{2}) \times [0, 2\pi)$. It turns out that from the system (17) we can construct a complete system in H formed by eigenfunctions of A, it is

$$\left\{ Z_n^m(\vartheta, \phi)|_{\vartheta\in[0,\frac{\pi}{2})} = \lambda_n^{-\frac{1}{2}} \nabla^\perp Y_n^m(\vartheta, \phi)|_{\vartheta\in[0,\frac{\pi}{2})} \;\middle|\; \begin{array}{l} n \in \mathbb{N}_0,\ m \in \mathbb{Z}, \\ |m| \le n,\ |m| + n \text{ is odd} \end{array} \right\} \tag{19}$$

(cf. [Rod08, Proposition 6.4.2]). Let us show that the system is complete. For $Z_n^m(\vartheta, \phi)$ in (17) we have that $\nabla^\perp \cdot Z_n^m(\vartheta, \phi) = \lambda_n^{-\frac{1}{2}} \Delta Y_n^m(\vartheta, \phi) = \lambda_n^{\frac{1}{2}} Y_n^m(\vartheta, \phi)$, and if $|m| + n$ is odd we have that $Y_n^m(\frac{\pi}{2}, \phi) = 0$, that is, $Z_n^m(\vartheta, \phi)|_{\vartheta\in[0,\frac{\pi}{2})}$ is in $\mathrm{D}(A)$. Further we have that for $\vartheta_1 \in [0, \frac{\pi}{2}]$,

$$\begin{array}{rcll} Z_n^m(\frac{\pi}{2} - \vartheta_1, \phi) & = & Z_n^m(\frac{\pi}{2} + \vartheta_1, \phi), & \text{if } |m| + n \text{ is odd;} \\ Z_n^m(\frac{\pi}{2} - \vartheta_1, \phi) & = & -Z_n^m(\frac{\pi}{2} + \vartheta_1, \phi), & \text{if } |m| + n \text{ is even.} \end{array}$$

Notice that from (18), we can see that $P_n^m(-x) = -P_n^m(x)$ if $|m|+n$ is odd, and $P_n^m(-x) = P_n^m(x)$ if $|m| + n$ is even.

To show that (19) is complete in H, it is sufficient to show that the stream functions $\{Y_n^m(\vartheta, \phi)|_{\vartheta\in[0,\frac{\pi}{2})} \mid n \in \mathbb{N}_0,\ m \in \mathbb{Z},\ |m| \le n, \text{ and } |m|+n \text{ is odd}\}$ form a complete system in $L^2(\mathbb{S}^2_+, \mathbb{R})$. Let $g(\vartheta, \phi)$ be a function defined on the Hemisphere $[0, \frac{\pi}{2}) \times [0, 2\pi)$; we extend it to a function $\tilde{g}$ defined on the Sphere as follows

$$\tilde{g}(\vartheta, \phi) = \begin{cases} g(\vartheta, \phi) & \text{if } \vartheta \in [0, \frac{\pi}{2}), \\ -g(\pi - \vartheta, \phi) & \text{if } \vartheta \in (\frac{\pi}{2}, \pi]. \end{cases}$$

We know that we can write $\tilde{g} = \sum_{(n,m)\in\mathcal{S}} (\tilde{g}, Y_n^m)_{L^2(\mathbb{S}^2,\mathbb{R})} Y_n^m$ where $\mathcal{S} := \{(n, m) \in \mathbb{Z}^2 \mid n \in \mathbb{N} \text{ and } |m| \le n\}$.

By using spherical coordinates, we find for even $|m| + n$

$$\begin{aligned} &\int_{\frac{\pi}{2}}^{\pi} \tilde{g}(\vartheta, \phi) Y_n^m(\vartheta, \phi) \sin(\vartheta) \mathrm{d}\vartheta = \int_0^{\frac{\pi}{2}} \tilde{g}(\tfrac{\pi}{2} + \vartheta_1, \phi) Y_n^m(\tfrac{\pi}{2} + \vartheta_1, \phi) \sin(\tfrac{\pi}{2} + \vartheta_1) \mathrm{d}\vartheta_1 \\ &= \int_0^{\frac{\pi}{2}} -g(\tfrac{\pi}{2} - \vartheta_1, \phi) Y_n^m(\tfrac{\pi}{2} - \vartheta_1, \phi) \sin(\tfrac{\pi}{2} - \vartheta_1) \mathrm{d}\vartheta_1 \\ &= -\int_{\frac{\pi}{2}}^{0} g(\vartheta_2, \phi) Y_n^m(\vartheta_2, \phi) \sin(\vartheta_2)(-\mathrm{d}\vartheta_2) = -\int_0^{\frac{\pi}{2}} g(\vartheta_2, \phi) Y_n^m(\vartheta_2, \phi) \sin(\vartheta_2) \mathrm{d}\vartheta_2, \end{aligned}$$

which implies $\int_0^\pi g(\vartheta, \phi) Y_n^m(\vartheta, \phi) \sin(\vartheta) \mathrm{d}\vartheta = 0$. Hence, for even $|m|+n$, $(\tilde{g}, Y_n^m)_{L^2(\mathbb{S}^2,\mathbb{R})} = \int_0^{2\pi} d\phi \int_0^\pi \tilde{g}(\vartheta, \phi) Y_n^m(\vartheta, \phi) \sin(\vartheta) \mathrm{d}\vartheta = 0$, that is, $\tilde{g} = \sum_{(n,m)\in\mathcal{S}_+} (\tilde{g}, Y_n^m)_{L^2(\mathbb{S}^2,\mathbb{R})} Y_n^m$, with $\mathcal{S}_+ := \{(n, m) \in \mathcal{S} \mid |m| + n \text{ is odd}\}$, and

$$g = \tilde{g}|_{\vartheta\in[0,\frac{\pi}{2})} = \sum_{(n,m)\in\mathcal{S}_+} (\tilde{g}, Y_n^m)_{L^2(\mathbb{S}^2,\mathbb{R})} Y_n^m|_{\vartheta\in[0,\frac{\pi}{2})},$$

which shows that the set $\{Y_n^m(\vartheta,\phi)|_{\vartheta\in[0,\,\frac{\pi}{2})} \mid (n,\,m)\in\mathcal{S}\}$ is complete in $L^2(\mathbb{S}^2_+,\,\mathbb{R})$.

Now, procceding as above for the extension $\tilde{g}$ and for odd $|m|+n$ we have that $(\tilde{g},Y_n^m)_{L^2(\mathbb{S}^2,\mathbb{R})} = 2(g,Y_n^m|_{\vartheta\in[0,\,\frac{\pi}{2})})_{L^2(\mathbb{S}^2_+,\mathbb{R})}$, and also $Y_n^m = \tilde{h}$ with $h = Y_n^m|_{\vartheta\in[0,\,\frac{\pi}{2})}$. In particular we conclude that the family $\{Y_n^m(\vartheta,\phi)|_{\vartheta\in[0,\,\frac{\pi}{2})} \mid (n,\,m)\in\mathcal{S}\}$ is orthogonal in $L^2(\mathbb{S}^2_+,\,\mathbb{R})$ and then it forms a basis in $L^2(\mathbb{S}^2_+,\,\mathbb{R})$.

As a consequence we can conclude that the family (19) form a complete system in H. Notice that for $n=0$, Y_0^0 is a constant function, and the vector field $\nabla^\perp Y_0^0\in L^2(\mathbb{S}^2,\,T\mathbb{S}^2)$ vanishes. From the fact that $(Y_n^m,Y_n^m)_{L^2(\mathbb{S}^2,\mathbb{R})} = 2(Y_n^m|_{\vartheta\in[0,\,\frac{\pi}{2})}\,,Y_n^m|_{\vartheta\in[0,\,\frac{\pi}{2})})_{L^2(\mathbb{S}^2_+,\mathbb{R})}$, we can normalize that system as

$$\left\{ \sqrt{2}Z_n^m(\vartheta,\phi)|_{\vartheta\in[0,\,\frac{\pi}{2})} = \lambda_n^{-\frac{1}{2}}\nabla^\perp Y_n^m(\vartheta,\phi)|_{\vartheta\in[0,\,\frac{\pi}{2})} \;\middle|\; \begin{array}{l} n\in\mathbb{N}_0,\ m\in\mathbb{Z}, \\ |m|\le n,\ |m|+n \text{ is odd} \end{array} \right\} \tag{20}$$

Conclusion. We can follow the arguments in the case of the Sphere, in Section 5.2, to conclude that Theorem 3.1 holds with $\alpha=\frac{1}{2}$ and $s>\frac{1}{2}$.

5.4. **Rectangle.** Let Ω be the two-dimensional Rectangle $\Omega=(0,a)\times(0,b)\subset\mathbb{R}^2$. On the boundary we impose the Lions boundary conditions, that is, we consider the evolutionary Navier–Stokes equation in $H\coloneqq\{u\in L^2(\Omega,\,\mathbb{R}^2)\mid \nabla\cdot u=0 \text{ and } u\cdot\mathbf{n}=0 \text{ on } \partial\Omega\}$, with $V\coloneqq H\cap H^1(\Omega,\,\mathbb{R}^2)$ and $\mathrm{D}(A)\coloneqq V\cap\{u\in H^2(\Omega,\,\mathbb{R}^2)\mid \nabla^\perp\cdot u=0 \text{ on } \partial\Omega\}$ (cf. [Rod06] and [Rod08, Section 6.3]).

We will show that in this case we can take $\alpha=\frac{1}{2}$, $\xi=\frac{1}{2}$, and $\zeta=0$ in Theorem 4.3, and $\theta=0$ in Remark 4.4. That is, we can take $s>0$, in Theorem 4.3.

The complete system of eigenfunctions $\{Y_{(k_1,\,k_2)}\mid(k_1,\,k_2)\in\mathbb{N}_0^2\}$ and respective eigenvalues $\{\lambda_{(k_1,\,k_2)}\mid(k_1,\,k_2)\in\mathbb{N}_0^2\}$ of A, can be found in [Rod06, Sections 2.2 and 2.3], they are given by

$$Y_{(k_1,k_2)}\coloneqq\begin{pmatrix} -\frac{k_2\pi}{b}\sin\left(\frac{k_1\pi x_1}{a}\right)\cos\left(\frac{k_2\pi x_2}{b}\right) \\ \frac{k_1\pi}{a}\cos\left(\frac{k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right)\end{pmatrix},\quad \lambda_{(k_1,\,k_2)}\coloneqq\pi^2\left(\frac{k_1^2}{a^2}+\frac{k_2^2}{b^2}\right). \tag{21}$$

Though, the above systems are indexed over $\mathbb{N}_0^2$, like in Section 5.1, we can check the Assumptions 2.1, 2.2, 4.1, and 4.2 without rewriting the families as indexed over $\mathbb{N}_0$.

We may normalize the family (21), obtaining the system $\{W_{(k_1,k_2)}\mid(k_1,\,k_2)\in\mathbb{N}_0^2\}$, with

$$W_{(k_1,k_2)}\coloneqq 2(ab\lambda_{(k_1,\,k_2)})^{-\frac{1}{2}}\begin{pmatrix} -\frac{k_2\pi}{b}\sin\left(\frac{k_1\pi x_1}{a}\right)\cos\left(\frac{k_2\pi x_2}{b}\right) \\ \frac{k_1\pi}{a}\cos\left(\frac{k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right)\end{pmatrix}. \tag{22}$$

Checking Assumptions 2.1 and 2.2. We need only to check that $Bu=0$ if u is an eigenfuntion; this follows from [Rod06, equation (6.1)]. For the other points we refer to [Rod06] and [Tem95, Section 2.3].

Checking Assumptions 4.1 and 4.2. From [Rod06, equation (6.1)] (cf. [Rod08, equation (6.4)]), we can derive that $(B(Y_{(n_1,\,n_2)}+Y_{(m_1,\,m_2)}),\,Y_{(l_1,\,l_2)})_H = (B(Y_{(n_1,\,n_2)},\,Y_{(m_1,\,m_2)})+B(Y_{(m_1,\,m_2)},\,Y_{(n_1,\,n_2)}),\ Y_{(l_1,\,l_2)})_H\neq 0$ only if

$$l_1=|n_1\pm m_1| \text{ and } l_2=|n_2\pm m_2|,$$

which implies that $\mathrm{card}(\underline{\mathcal{F}}^{\bullet}_{n,m})\le 4$ and $\mathrm{card}(\underline{\mathcal{F}}^{l}_{n,\bullet})\le 4$. That is, Assumption 4.2 holds with $\zeta=0$. We also see that necessarily $\lambda_{(l_1,\,l_2)}\le\lambda_{(n_1+m_1,\,n_2+m_2)}$; noticing that $(k_1,\,k_2)\mapsto\lambda_{(k_1,\,k_2)}$ is a scalar product, or using Lemma A.1, we conclude that $\lambda^{\frac{1}{2}}_{(l_1,\,l_2)}\le\lambda^{\frac{1}{2}}_{(n_1,\,n_2)}+\lambda^{\frac{1}{2}}_{(m_1,\,m_2)}$, that is, Assumption 4.1 holds with $\alpha=\frac{1}{2}$.

Looking for the value θ in Remark 4.4. We have that

$$|W_{(k_1,\,k_2)}|^2_{L^\infty(\Omega,\,\mathbb{R}^2)} = \max_{(x_1,\,x_2)\in\Omega} |W_{(k_1,\,k_2)}(x_1,\,x_2)|^2_{\mathbb{R}^2} \le 4(ab\lambda_{(k_1,\,k_2)})^{-1}(\tfrac{k_2^2\pi^2}{b^2} + \tfrac{k_1^2\pi^2}{a^2}) = 4(ab)^{-1},$$

that is, we can take $\theta = 0$.

Asymptotic behavior of the (repeated) eigenvalues. We recall that for an open domain $\Omega \subset \mathbb{R}^2$, under Lions boundary conditions, the eigenvalues of the Stokes operator $A : \mathrm{D}(A) \to H$ are those of the Dirichlet Laplacian $\Delta H^2(\Omega,\,\mathbb{R}) \cap H^1_0(\Omega,\,\mathbb{R}) \to L^2(\Omega,\,\mathbb{R})$, that is, $Au = \lambda u$ if, and only if, $\Delta\nabla^\perp u = \lambda\nabla^\perp u$. Thus, from [LY83, Corollary 1] we have that we can take $\rho < \frac{2\pi}{ab}$ and $\xi = 1$ in Theorem 4.3.

For the sake of completeness we would like also to refer to the results in [Ily09], and references therein, for the case of no-slip boundary conditions.

Conclusion. Taking into account Remark 4.4, we conclude that Theorem 4.3 holds with $\alpha = \frac{1}{2}$ and $s > 0$.

5.5. **Cylinder.** Let Ω be a two-dimensional Cylinder $\Omega = \left(\frac{a}{2\pi}\mathbb{S}^1\right) \times (0,\,b) \sim (0,\,a)\times(0,\,b)$. On the boundary $(0,\,a) \times \{0,\,b\}$ we impose the Lions boundary conditions, that is, we consider the evolutionary Navier–Stokes equation in $H := \{u \in L^2(\Omega,\,\mathbb{R}^2) \mid \nabla\cdot u = 0 \text{ and } u\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$, with $V := H \cap H^1(\Omega,\,\mathbb{R}^2)$ and $\mathrm{D}(A) := V \cap \{u \in H^2(\Omega,\,\mathbb{R}^2) \mid \nabla^\perp\cdot u = 0 \text{ on } \partial\Omega\}$. We can see the domain Ω as an infinite channel $\mathbb{R}\times(0,\,b)$ where we take a-periodic boundary conditions on the infinite direction $x_1 \in \mathbb{R}$ and Lions boundary conditions on the boundary $\mathbb{R}\times\{0,\,b\}$.

We will show that in this case we can take $\alpha = \frac{1}{2}$, $\xi = \frac{1}{2}$, and $\zeta = 0$ in Theorem 4.3, and $\theta = 0$ in Remark 4.4. That is, we can take $s > 0$, in Theorem 4.3.

A complete system of orthogonal eigenfunctions of A $\{Y^\varsigma_n, Y^\varkappa_m \mid n \in \mathbb{N}_0^2, m \in \mathbb{N}\times\mathbb{N}_0\}$, and corresponding eigenvalues $\{\lambda^\varsigma_n, \lambda^\varkappa_m \mid n \in \mathbb{N}_0^2, m \in \mathbb{N}\times\mathbb{N}_0\}$, are given by

$$\begin{aligned} Y^\varsigma_k = Y^\varsigma_{(k_1,k_2)} &= \begin{pmatrix} -\frac{k_2\pi}{b}\sin\left(\frac{2k_1\pi x_1}{a}\right)\cos\left(\frac{k_2\pi x_2}{b}\right) \\ \frac{2k_1\pi}{a}\cos\left(\frac{2k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right) \end{pmatrix}, \\ Y^\varkappa_k = Y^\varkappa_{(k_1,k_2)} &= \begin{pmatrix} -\frac{k_2\pi}{b}\cos\left(\frac{2k_1\pi x_1}{a}\right)\cos\left(\frac{k_2\pi x_2}{b}\right) \\ -\frac{2k_1\pi}{a}\sin\left(\frac{2k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right) \end{pmatrix}, \end{aligned} \tag{23}$$

and $\lambda^\varsigma_{(k_1,k_2)} = \lambda^\varkappa_{(k_1,k_2)} = \lambda_{(k_1,k_2)} := \pi^2\left(\frac{(2k_1)^2}{a^2} + \frac{k_2^2}{b^2}\right)$.

Remark 5.2. Notice that $Y^\varsigma_n = \nabla^\perp\psi^\varsigma_n$, $Y^\varkappa_m = \nabla^\perp\psi^\varkappa_n$, with $\psi^\varsigma_n := \sin\left(\frac{2k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right)$ and $\psi^\varkappa_n := \cos\left(\frac{2k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right)$; notice also that the set of stream functions $\{Y^\varsigma_n, Y^\varkappa_m \mid n \in \mathbb{N}_0^2, m \in \mathbb{N}\times\mathbb{N}_0\}, \mid n \in \mathbb{N}_0^2, m \in \mathbb{N}\times\mathbb{N}_0\}$ is an orthogonal and complete, in $L^2(\Omega,\,\mathbb{R}^2)$, system of eigenfunctions of the Laplacian in $\Omega \sim (0,\,a)\times(0,\,b)$.

We may normalize the family, obtaining the normalized system $\{W^\varsigma_n, W^\varkappa_m \mid n \in \mathbb{N}_0^2, m \in \mathbb{N}\times\mathbb{N}_0\}$, with

$$W^\varsigma_k := 2(ab\lambda_k)^{-\frac{1}{2}}Y^\varsigma_k, \quad W^\varkappa_k := 2(ab\lambda_k)^{-\frac{1}{2}}Y^\varkappa_k. \tag{24}$$

Now we check our assumptions, proceeding as in the case of the Rectangle, in Section 5.4.

Checking Assumptions 2.1 and 2.2. The assumptions follow by reasoning as in the case of the Sphere in Section 5.2, where now $g(\cdot,\,\cdot) = (\cdot,\,\cdot)_{\mathbb{R}^2}$ is the usual Euclidean scalar product in $\mathbb{R}^2$.

Checking Assumptions 4.1 and 4.2. From the discussion following Corollary 5.6.3 in [Rod08] we can conclude that $\nabla^\perp\cdot(B(u,\,v)+B(u,\,v)) = (\nabla\Delta^{-1}\nabla^\perp\cdot v,\,\nabla^\perp\nabla^\perp\cdot u)_{\mathbb{R}^2} +$

$(\nabla\Delta^{-1}\nabla^\perp\cdot u,\, \nabla^\perp\nabla^\perp\cdot v)_{\mathbb{R}^2}$. If u and v are eigenfunctions from (23) with associated eigenvalues λ_u and λ_v, and associated eigenfunctions ψ_u and ψ_v, we obtain

$$\begin{aligned}\nabla^\perp\cdot(B(u,\,v)+B(u,\,v)) &= \lambda_u(\nabla\psi_v,\,\nabla^\perp\psi_u)_{\mathbb{R}^2}+\lambda_v(\nabla\psi_u,\,\nabla^\perp\psi_v)_{\mathbb{R}^2}\\ &= (\lambda_u-\lambda_v)(\nabla^\perp\psi_u,\,\nabla\psi_v)_{\mathbb{R}^2}.\end{aligned}\tag{25}$$

From straightforward computations, we find the following expressions

$$\begin{aligned}&(\nabla^\perp\psi^\varsigma_n,\,\nabla\psi^\varsigma_m)_{\mathbb{R}^2}\\ =\;&\begin{pmatrix}-\frac{n_2\pi}{b}\sin\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\\ \frac{2n_1\pi}{a}\cos\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\end{pmatrix}\cdot\begin{pmatrix}\frac{2m_1\pi}{a}\cos\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\ \frac{m_2\pi}{b}\sin\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right)\end{pmatrix}\\ =\;&-\frac{2\pi^2n_2m_1}{ab}\sin\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\cos\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\ &+\frac{2\pi^2n_1m_2}{ab}\cos\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\sin\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right);\end{aligned}$$

$$\begin{aligned}&(\nabla^\perp\psi^\varsigma_n,\,\nabla\psi^\varkappa_m)_{\mathbb{R}^2}\\ =\;&\begin{pmatrix}-\frac{n_2\pi}{b}\sin\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\\ \frac{2n_1\pi}{a}\cos\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\end{pmatrix}\cdot\begin{pmatrix}-\frac{2m_1\pi}{a}\sin\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\ \frac{m_2\pi}{b}\cos\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right)\end{pmatrix}\\ =\;&\frac{2\pi^2n_2m_1}{ab}\sin\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\sin\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\ &+\frac{2\pi^2n_1m_2}{ab}\cos\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\cos\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right);\end{aligned}$$

$$\begin{aligned}&(\nabla^\perp\psi^\varkappa_n,\,\nabla\psi^\varkappa_m)_{\mathbb{R}^2}\\ =\;&\begin{pmatrix}-\frac{n_2\pi}{b}\cos\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\\ -\frac{2n_1\pi}{a}\sin\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\end{pmatrix}\cdot\begin{pmatrix}-\frac{2m_1\pi}{a}\sin\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\ \frac{m_2\pi}{b}\cos\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right)\end{pmatrix}\\ =\;&\frac{2\pi^2n_2m_1}{ab}\cos\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\sin\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\ &-\frac{2\pi^2n_1m_2}{ab}\sin\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\cos\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right).\end{aligned}$$

Thus, if we denote $\varsigma_c^{z,\,l} := \sin\left(\frac{l\pi z}{c}\right)$ and $\varkappa_c^{z,\,l} := \cos\left(\frac{l\pi z}{c}\right)$; $m\wedge n = m_1n_2 - m_2n_1$ and $m\vee n = m_1n_2 + m_2n_1$, we obtain

$$\begin{aligned}&(\nabla^\perp\psi^\varsigma_n,\,\nabla\psi^\varsigma_m)_{\mathbb{R}^2}\\ =\;&-\frac{\pi^2n_2m_1}{2ab}\left(\varsigma_a^{x_1,\,2(n_1+m_1)}+\varsigma_a^{x_1,\,2(n_1-m_1)}\right)\left(\varsigma_b^{x_2,\,n_2+m_2}-\varsigma_b^{x_2,\,n_2-m_2}\right)\\ &+\frac{\pi^2n_1m_2}{2ab}\left(\varsigma_a^{x_1,\,2(n_1+m_1)}-\varsigma_a^{x_1,\,2(n_1-m_1)}\right)\left(\varsigma_b^{x_2,\,m_2+n_2}+\varsigma_b^{x_2,\,n_2-m_2}\right)\\ =\;&-\frac{\pi^2m\wedge n}{2ab}\varsigma_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2+m_2}+\frac{\pi^2m\vee n}{2ab}\varsigma_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2-m_2}\\ &-\frac{\pi^2m\vee n}{2ab}\varsigma_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2+m_2}+\frac{\pi^2m\wedge n}{2ab}\varsigma_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2-m_2}.\end{aligned}\tag{26}$$

$$\begin{aligned}&(\nabla\psi^\varsigma_n,\,\nabla^\perp\psi^\varkappa_m)_{\mathbb{R}^2}\\ =\;&+\frac{\pi^2n_2m_1}{2ab}\left(\varkappa_a^{x_1,\,2(n_1-m_1)}-\varkappa_a^{x_1,\,2(n_1+m_1)}\right)\left(\varsigma_b^{x_2,\,n_2+m_2}-\varsigma_b^{x_2,\,n_2-,m_2}\right)\\ &+\frac{\pi^2n_1m_2}{2ab}\left(\varkappa_a^{x_1,\,2(n_1+m_1)}+\varkappa_a^{x_1,\,2(n_1-m_1)}\right)\left(\varsigma_b^{x_2,\,n_2+m_2}+\varsigma_b^{x_2,\,n_2-m_2}\right)\\ =\;&-\frac{\pi^2m\wedge n}{2ab}\varkappa_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2+m_2}+\frac{\pi^2m\vee n}{2ab}\varkappa_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2-m_2}\\ &+\frac{\pi^2m\vee n}{2ab}\varkappa_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2+m_2}-\frac{\pi^2m\wedge n}{2ab}\varkappa_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2-m_2};\end{aligned}\tag{27}$$

$$\begin{aligned}&(\nabla\psi^\varkappa_n,\,\nabla^\perp\psi^\varkappa_m)_{\mathbb{R}^2}\\ =\;&\frac{\pi^2n_2m_1}{2ab}\left(\varsigma_a^{x_1,\,2(n_1+m_1)}-\varsigma_a^{x_1,\,2(n_1-m_1)}\right)\left(\varsigma_b^{x_2,\,n_2+m_2}-\varsigma_b^{x_2,\,n_2-m_2}\right)\\ &-\frac{\pi^2n_1m_2}{2ab}\left(\varsigma_a^{x_1,\,2(n_1+m_1)}+\varsigma_a^{x_1,\,2(n_1-m_1)}\right)\left(\varsigma_b^{x_2,\,m_2+n_2}+\varsigma_b^{x_2,\,n_2-m_2}\right)\\ =\;&\frac{\pi^2m\wedge n}{2ab}\varsigma_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2+m_2}-\frac{\pi^2m\vee n}{2ab}\varsigma_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2-m_2}\\ &-\frac{\pi^2m\vee n}{2ab}\varsigma_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2+m_2}+\frac{\pi^2m\wedge n}{2ab}\varsigma_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2-m_2}.\end{aligned}\tag{28}$$

Hence from $u = \nabla^\perp \psi_u = \nabla^\perp \Delta^{-1} \nabla^\perp \cdot u$, (25), (26), (27) and (28), we obtain

$$\begin{aligned}
&B(Y^\varsigma_n, Y^\varsigma_m) + B(Y^\varsigma_m, Y^\varsigma_n)\\
= \;& \tfrac{\lambda_n - \lambda_m}{\lambda_{m(++)n}} \tfrac{\pi^2 n\wedge m}{2ab} Y^\varsigma_{n(++)m} + \tfrac{\lambda_n - \lambda_m}{\lambda_{n(+-)m}} \tfrac{\pi^2 n\vee m}{2ab} \mathrm{sign}(n_2 - m_2) Y^\varsigma_{n(+-)m}\\
& - \tfrac{\lambda_n - \lambda_m}{\lambda_{n(-+)m}} \tfrac{\pi^2 n\vee m}{2ab} \mathrm{sign}(n_1 - m_1) Y^\varsigma_{n(-+)m} - \tfrac{\lambda_n - \lambda_m}{\lambda_{m(--)n}} \tfrac{\pi^2 n\wedge m}{2ab} \mathrm{sign}(n_1 - m_1)\, \mathrm{sign}(n_2 - m_2) Y^\varsigma_{n(--)m};\\
\\
&B(Y^\varsigma_n, Y^\varkappa_m) + B(Y^\varkappa_m, Y^\varsigma_n)\\
= \;& \tfrac{\lambda_n - \lambda_m}{\lambda_{n(++)m}} \tfrac{\pi^2 n\wedge m}{2ab} Y^\varkappa_{n(++)m} + \tfrac{\lambda_n - \lambda_m}{\lambda_{n(+-)m}} \tfrac{\pi^2 n\vee m}{2ab} \mathrm{sign}(n_2 - m_2) Y^\varkappa_{n(+-)m}\\
& + \tfrac{\lambda_n - \lambda_m}{\lambda_{n(-+)n}} \tfrac{\pi^2 n\vee m}{2ab} Y^\varkappa_{n(-+)m} + \tfrac{\lambda_n - \lambda_m}{\lambda_{n(--)m}} \tfrac{\pi^2 n\wedge m}{2ab} \mathrm{sign}(n_2 - m_2) Y^\varkappa_{n(+-)m};\\
\\
&B(Y^\varkappa_n, Y^\varkappa_m) + B(Y^\varkappa_m, Y^\varkappa_n)\\
= \;& -\tfrac{\lambda_n - \lambda_m}{\lambda_{n(++)m}} \tfrac{\pi^2 n\wedge m}{2ab} Y^\varsigma_{n(++)m} - \tfrac{\lambda_n - \lambda_m}{\lambda_{n(+-)m}} \tfrac{\pi^2 n\vee m}{2ab} \mathrm{sign}(n_2 - m_2) Y^\varsigma_{n(+-)m}\\
& - \tfrac{\lambda_n - \lambda_m}{\lambda_{n(-+)n}} \tfrac{\pi^2 n\vee m}{2ab} \mathrm{sign}(n_1 - m_1) Y^\varsigma_{n(-+)m} - \tfrac{\lambda_n - \lambda_m}{\lambda_{n(--)m}} \tfrac{\pi^2 n\wedge m}{2ab} \mathrm{sign}(n_1 - m_1)\, \mathrm{sign}(n_2 - m_2) Y^\varsigma_{n(--)m};
\end{aligned} \tag{29}$$

where $n(\star_1\star_2)m := (|n_1 \star_1 m_1|, |n_2 \star_2 m_2|) \in \mathbb{N}^2$, with $\{\star_1, \star_2\} \in \{-, +\}^2$ and for $k_1 \in \mathbb{N}$, $Y^\varsigma_{(k_1, 0)} := Y^\varkappa_{(k_1, 0)} := 0$. Notice that these are expressions similar to that obtained for the case of the Rectangle in [Rod06, equation (6.1)], [Rod08, equation (6.4)]. Notice also that in [Rod06, Section 2.3] the eigenvalues are negative and here they are positive, this is because in [Rod06] it is considered the usual Laplacian Δ in $(0, a) \times (0, b)$ and here (cf. the discussion following Equation (2)) we consider the Laplace–de Rham operator $\Delta_\Omega = -\Delta = A$.

From (29), and (24), we conclude that $\mathrm{card}(\underline{\mathcal{F}}^\bullet_{n, m}) \le 4$ and $\mathrm{card}(\underline{\mathcal{F}}^l_{n, \bullet}) \le 8$. That is, Assumption 4.2 holds with $\zeta = 0$. We also see that necessarily $\lambda_{(l_1, l_2)} \le \lambda_{(n_1+m_1, n_2+m_2)}$; and from Lemma A.1 we conclude that $\lambda^{\frac12}_{(l_1, l_2)} \le \lambda^{\frac12}_{(n_1, n_2)} + \lambda^{\frac12}_{(m_1, m_2)}$, that is, Assumption 4.1 holds with $\alpha = \frac12$.

Looking for the value θ in Remark 4.4. We can take $\theta = 0$, because, proceeding as in the case of the Rectangle, in Section 5.4, we obtain

$$|W^\varsigma_k|^2_{L^\infty(\Omega, \mathbb{R}^2)} \le 4(ab)^{-1}, \quad |W^\varkappa_k|^2_{L^\infty(\Omega, \mathbb{R}^2)} \le 4(ab)^{-1}.$$

Asymptotic behavior of the (repeated) eigenvalues. Notice that the family $\{\lambda_k \mid k \in N_0^2\} = \{\lambda^\varsigma_k \mid k \in \mathbb{N}_0^2\} = \{\lambda^\varkappa_k \mid k \in \mathbb{N}_0^2\}$ is a subset of $\{\lambda^R_k \mid k \in \mathbb{N}_0^2\}$ where λ^R_k are the eigenvalues of the Dirichlet Laplacian on the Rectangle $R := (0, a) \times (0, b)$, in Section 5.4. Hence, ordering the families as $\{\lambda_k \mid k \in \mathbb{N}_0^2\} = \{\tilde\lambda_n \mid n \in \mathbb{N}_0\}$ and $\{\lambda^R_k \mid k \in \mathbb{N}_0^2\} = \{\tilde\lambda^R_n \mid n \in \mathbb{N}_0\}$ such that $\tilde\lambda_n \le \tilde\lambda_{n+1}$ and $\tilde\lambda^R_n \le \tilde\lambda^R_{n+1}$, we can conclude that $(\tilde\lambda_n)_{n\in\mathbb{N}_0}$ is a subsequence of $(\tilde\lambda^R_n)_{n\in\mathbb{N}_0}$. Now we already know that

$$\tilde\lambda^R_n \ge \frac{2\pi}{ab} n,$$

which implies $\tilde\lambda_n \ge \tilde\lambda^R_n \ge \frac{2\pi}{ab} n$ for all $n \in \mathbb{N}_0$. The family $\{\lambda_k \mid k \in \mathbb{N}_0^2\}$ is repeated twice $\lambda_k = \lambda^\varsigma_k = \lambda^\varkappa_k$ for $k \in \mathbb{N}_0^2$. Then for the ordered families, we can write

$$\tilde\lambda^\varsigma_n \ge \tilde\lambda^R_n \ge \frac{2\pi}{ab} n^2 \text{ and } \tilde\lambda^\varkappa_n \ge \tilde\lambda^R_n \ge \frac{2\pi}{ab} n^2 \text{ for all } n \in \mathbb{N}_0.$$

Finally the family of eigenvalues $\{\tilde\lambda^{\varkappa, 0}_n := \lambda^\varkappa_{(0, n)} \mid n \in N_0\}$ satisfies

$$\tilde\lambda^{\varkappa, 0}_n = \frac{\pi^2}{b^2} n^2 \text{ for all } n \in \mathbb{N}_0.$$

In particular ordering the set $\{\tilde\lambda^\varsigma_n, \tilde\lambda^\varkappa_n, \tilde\lambda^{\varkappa, 0}_n \mid n \in \mathbb{N}_0\}$, in a nondecreasing way, we obtain the sequence of repeated eigenvalues $(\underline{\lambda}_n)_{n\in\mathbb{N}_0}$ in the case of the Cylinder. Moreover setting $\varrho = \min\{\frac{2\pi}{ab}, \frac{\pi^2}{b^2}\}$ we find that there are at most $3n$ elements in the set $\{\underline{\lambda}_n \mid n \in \mathbb{N}_0\}$ that are not bigger than $\varrho(n+1)$; which implies that $\underline{\lambda}_{3n+1} \geq \varrho(n+1)$. Hence, since $\underline{\lambda}_{3(n+1)} \geq \underline{\lambda}_{3n+2} \geq \underline{\lambda}_{3n+1}$ we can conclude that for $m \geq 4$, $\underline{\lambda}_m \geq \varrho\lfloor\frac{m+2}{3}\rfloor$ where for a positive real number r, $\lfloor r\rfloor$ stands for the biggest integer below r, that is $r \in \mathbb{N}$ and $r = \lfloor r\rfloor + r_1$ with $r_1 \in [0, 1)$. In particular, from $\lfloor\frac{m+2}{3}\rfloor \geq \frac{m-1}{3} = \frac{m-1}{3m}m$, we find $\underline{\lambda}_m \geq \frac{\varrho}{4}m$ for $m \geq 4$. So for $\varrho_0 := \min_{k\in\{1,2,3\}}\{\underline{\lambda}_k\}$ and $\rho_1 := \min\{\frac{\varrho}{4}, \varrho_0\}$, we have that $\underline{\lambda}_m \geq \rho_1 m$ for all $m \in \mathbb{N}_0$. Thus we can take $\rho < \rho_1$ and $\xi = 1$ in Theorem 4.3.

Conclusion. Taking into account Remark 4.4, we conclude that Theorem 4.3 holds with $\alpha = \frac{1}{2}$ and $s > 0$

— APPENDIX —

A.1. **Proof of Proposition 2.6.** The inequalities in Proposition 2.6 are clear for $s = 0$ and $s = 1$. Now let $s > 0$, $s \neq 1$, and consider the quotient $f(x, y) := \frac{(x+y)^s}{x^s+y^s}$ for $x \geq 0$, $y \geq 0$, and $(x, y) \neq (0, 0)$. The gradient of f is given by

$$\begin{aligned}\nabla f &= \begin{bmatrix} s(x+y)^{s-1}(x^s+y^s)^{-1} - s(x+y)^s x^{s-1}(x^s+y^s)^{-2} \\ s(x+y)^{s-1}(x^s+y^s)^{-1} - s(x+y)^s y^{s-1}(x^s+y^s)^{-2} \end{bmatrix} \\ &= s(x+y)^{s-1}(x^s+y^s)^{-2}\begin{bmatrix} (x^s+y^s) - (x+y)x^{s-1} \\ (x^s+y^s) - (x+y)y^{s-1} \end{bmatrix} \\ &= s(x+y)^{s-1}(x^s+y^s)^{-2}(y^{s-1} - x^{s-1})\begin{bmatrix} y & -x \end{bmatrix}^\perp.\end{aligned}$$

Notice that $\nabla f|_{(x,y)}$ is orthogonal to $\begin{bmatrix} x & y \end{bmatrix}^\perp \sim (x, y)$. Which means that the trajectories associated with the vector field ∇f are pieces of spheres. Moreover, $\nabla f|_{(x,y)}$ vanishes only at the straight lines $x = y$, $x = 0$, and $y = 0$, and we observe that f is constant in those lines. Now, for $s > 1$ we have that $(y^{s-1} - x^{s-1}) > 0$ if, and only if, $y > x$, and it is straightforward to conclude that, in the sphere containing a point (a, b), with $a \geq 0$, $b \geq 0$, and $(a, b) \neq (0, 0)$, the function f attains its minimum either at the line $x = 0$ or at the line $y = 0$; and attains its maximum at the line $x = y$. Hence we can conclude that $1 \leq f(a, b) \leq 2^{s-1}$.

Analogously, for $s < 1$ we have that $(y^{s-1} - x^{s-1}) > 0$ if, and only if, $y < x$, which gives us $2^{s-1} \leq f(a, b) \leq 1$. □

A.2. **A remark on the square root of a quadratic polynomial.** Let $p(x)$ be a polynomial, of degree two, in the variable $x \in \mathbb{R}^n$.

Lemma A.1. *If the Hessian matrix $\mathcal{H}$ of p is positive definite, then there is a constant $K \in \mathbb{R}$ such that for any $x, y \in \mathbb{R}^n$ we have*

$$|p(x+y)|_{\mathbb{R}}^{\frac{1}{2}} \leq |p(x)|_{\mathbb{R}}^{\frac{1}{2}} + |p(y)|_{\mathbb{R}}^{\frac{1}{2}} + K. \tag{A.1}$$

Proof. We can see that the derivative $\mathrm{d}_{x_0}p : \mathbb{R}^n \to \mathbb{R}$ can be rewritten as $x_0^\top\mathcal{H} + G$, for a suitable row matrix G, thus there is (a unique) $\bar{x} \in \mathbb{R}^n$ such that $\bar{x}^\top\mathcal{H} + G = 0$. Now we define the function $q(w) := p(w + \bar{x}) - p(\bar{x})$. From the Taylor formula we find that

$$q(w) = \tfrac{1}{2}w^\top\mathcal{H}w,$$

and we see that $q(w) = Q(w, w)$, where $Q(w, z) := \frac{1}{2}w^\top\mathcal{H}z$ is a scalar product in $\mathbb{R}^n$.

Now, from the identity $p(x+y) = q(x+y-\bar{x}) + p(\bar{x})$, the inequality $(a+b)^{\frac{1}{2}} \leq a^{\frac{1}{2}} + b^{\frac{1}{2}}$ (for $a, b \geq 0$, cf. Proposition 2.6), and the triangle inequality $q(w+z)^{\frac{1}{2}} \leq q(w)^{\frac{1}{2}} + q(z)^{\frac{1}{2}}$, it follows that

$$\begin{aligned}|p(x+y)|_{\mathbb{R}}^{\frac{1}{2}} &\le q(x+y-\bar{x})^{\frac{1}{2}} + |p(\bar{x})|_{\mathbb{R}}^{\frac{1}{2}}\\ &\le q(x-\bar{x})^{\frac{1}{2}} + q(y-\bar{x})^{\frac{1}{2}} + q(\bar{x})^{\frac{1}{2}} + |p(\bar{x})|_{\mathbb{R}}^{\frac{1}{2}}\\ &\le |p(x)|_{\mathbb{R}}^{\frac{1}{2}} + |p(y)|_{\mathbb{R}}^{\frac{1}{2}} + |p(0)|_{\mathbb{R}}^{\frac{1}{2}} + 4|p(\bar{x})|_{\mathbb{R}}^{\frac{1}{2}}.\end{aligned}$$

Therefore, we may take

$$K = |p(0)|_{\mathbb{R}}^{\frac{1}{2}} + 4|p(\bar{x})|_{\mathbb{R}}^{\frac{1}{2}} \tag{A.2}$$

in (A.1). □

A.3. **On the curl operator in the Sphere.** Here we show that, in the case of the Sphere $\mathbb{S}^2$, the definitions of the curl operators in [CRT99] and in [Rod08] are equivalent up to a minus sign (cf. Remark 5.1). Familiarity with basic tools from differential geometry is assumed; we refer to [Car67, dC94, Jos05, Tra84] (we follow the notations from [Rod08, Chapter 5, Section 5.7]). Since the curl is a local operator it is enough to check those definitions on local charts. We consider the chart

$$\begin{gathered}\Phi : \mathbf{C} \to \mathbf{B}\\ \Phi(w^1,\, w^2,\, w^3) \mapsto (x^1,\, x^2,\, x^3) \coloneqq (1+w^3)(w^1,\, w^2,\, \Phi^0(w^1,\, w^2))\end{gathered}$$

with $\Phi^0(w^1,\, w^2) \coloneqq (1-(w^1)^2-(w^2)^2)^{\frac{1}{2}}$, mapping the set $\mathbf{C} \coloneqq \{(w^1,\, w^2,\, w^3) \in \mathbb{R}^3 \mid (w^1)^2+(w^2)^2 < \frac{1}{2}$ and $w^3 \le \frac{1}{2}\}$ onto $\mathbf{B} \coloneqq \Phi(\mathbf{C}) \subset \{(x^1,\, x^2,\, x^3) \in \mathbb{R}^3 \mid \frac{1}{2} < (x^1)^2 + (x^2)^2 + (x^3)^2 < \frac{3}{2}\}$. Notice that we can cover the entire Sphere $\mathbb{S}^2$ with similar charts.

Let $\frac{\partial}{\partial w^i}$ be the vector field induced in $\mathbf{B}$ by the new coordinate function w^i; we find

$$\begin{aligned}\tfrac{\partial}{\partial w^i}\big|_{(w^1,\, w^2,\, w^3)} &= (1+w^3)\left(\tfrac{\partial}{\partial x^i} + \tfrac{\partial\Phi^0_p}{\partial w^i}\big|_{(w^1,\, w^2,\, w^3)}\tfrac{\partial}{\partial x^3}\right) \text{ for } i=1,\,2\\ \tfrac{\partial}{\partial w^3}\big|_{(w^1,\, w^2,\, w^3)} &= w^1\tfrac{\partial}{\partial x^1} + w^2\tfrac{\partial}{\partial x^2} + \Phi^0(w^1,\, w^2)\tfrac{\partial}{\partial x^3} = \mathbf{n}_{(w^1,\, w^2,\, \Phi^0_p(w^1,\, w^2))}\end{aligned} \tag{A.3}$$

where $\mathbf{n}_q$ stands for the outward normal vector at the point $q \in \mathbb{S}^2$ (cf. [Rod14, Appendix], recalling that the outward normal vector at a point $q = (q_1,\, q_2,\, q_3) \in \mathbb{S}^2$ is given by $\mathbf{n}_q = q_1\frac{\partial}{\partial x^1} + q_2\frac{\partial}{\partial x^2} + q_3\frac{\partial}{\partial x^3} \sim q$). Notice that $(w^1,\, w^2,\, \Phi^0(w^1,\, w^2) \in \mathbb{S}^2$.

Reasoning, for example, as in [Rod14, Appendix], we can see the Euclidean set $\mathbf{B}$ as the Riemannian manifold $(\mathbf{C},\, g)$ with the metric tensor

$$\begin{aligned}g = {}&\frac{1-(w^2)^2}{\Phi^0(w^1,\, w^2)^2}\mathrm{d}w^1\otimes\mathrm{d}w^1 + \frac{w^1w^2}{\Phi^0(w^1,\, w^2)^2}(\mathrm{d}w^1\otimes\mathrm{d}w^2 + \mathrm{d}w^2\otimes\mathrm{d}w^1)\\ &+ \frac{1-(w^1)^2}{\Phi^0(w^1,\, w^2)^2}\mathrm{d}w^2\otimes\mathrm{d}w^2 + \mathrm{d}w^3\otimes\mathrm{d}w^3\end{aligned}$$

and the Euclidean volume element in $\mathbf{B}$ may then be written as $\mathrm{d}\mathbf{C} = \sqrt{\bar{g}}\,\mathrm{d}w^1\wedge\mathrm{d}w^2\wedge\mathrm{d}w^3$, with $\bar{g} \coloneqq \frac{1}{\Phi^0(w^1,w^2)^2}$.

Moreover the mapping $\Phi^0 : \mathbf{C}_0 \to \mathbf{B}_0$ maps the disc $\mathbf{C}_0 \coloneqq \{(w^1,\, w^2) \in \mathbb{R}^2 \mid (w^1)^2 + (w^2)^2 < \frac{1}{2}\}$ onto $\mathbf{B}_0 \coloneqq \mathbf{B}\cap\mathbb{S}^2$. Hence we can see the subset $\mathbf{B}_0$ with the metric inherited from $\mathbb{R}^3$ as the Riemannian manifold $(\mathbf{C}_0,\, g_0)$ with the metric tensor

$$\begin{aligned}g_0 = {}&\frac{1-(w^2)^2}{\Phi^0(w^1,\, w^2)^2}\mathrm{d}w^1\otimes\mathrm{d}w^1 + \frac{w^1w^2}{\Phi^0(w^1,\, w^2)^2}(\mathrm{d}w^1\otimes\mathrm{d}w^2 + \mathrm{d}w^2\otimes\mathrm{d}w^1)\\ &+ \frac{1-(w^1)^2}{\Phi^0(w^1,\, w^2)^2}\mathrm{d}w^2\otimes\mathrm{d}w^2,\end{aligned}$$

and volume (i.e., area) element $\mathrm{d}\mathbf{C}_0 = \sqrt{\bar{g}_0}\,\mathrm{d}w^1\wedge\mathrm{d}w^2$ with $\bar{g}_0 \coloneqq \frac{1}{\Phi^0(w^1,\, w^2)^2} = \bar{g}$.

On the curl of a function. In [Rod08, Section 5.7] the curl vector field $\operatorname{curl} f$ of a function f on the Sphere is the vector field denoted $\nabla^\perp f$ and defined as $\nabla^\perp f = (*\mathrm{d}f)^\flat$, where in coordinates $(w^1,\, w^2)$, we denote by $[g^{ij}]$ the inverse matrix $[g_{ij}]^{-1}$ and $(a_i \mathrm{d}w^i)^\flat := g^{ij} a_j \frac{\partial}{\partial w^i}$. We obtain

$$\nabla^\perp f = \frac{1}{\sqrt{g}} \left(\frac{\partial f}{\partial w^1} \frac{\partial}{\partial w^2} - \frac{\partial f}{\partial w^2} \frac{\partial}{\partial w^1} \right), \tag{A.4}$$

while in [CRT99, Definition 1] it is denoted $\operatorname{Curl} f$ and can be obtained as follows: first we extend f to $\mathbf{B}$; then we consider the extension $\tilde f \tilde{\mathbf{n}}$, where $\tilde{\mathbf{n}} := x_1 \frac{\partial}{\partial x^1} + x_2 \frac{\partial}{\partial x^2} + x_3 \frac{\partial}{\partial x^3}$ is an extension to $\mathbf{B}$ of the outward normal vector $\mathbf{n}$ to $\mathbb{S}^2 \supset \mathbf{B}_0$; finally we set $\operatorname{Curl} f := (\operatorname{curl} \tilde f \tilde{\mathbf{n}})|_{\mathbf{B}_0}$, where $\operatorname{curl} = \nabla\times$ is the standard curl vector in $\mathbb{R}^3$ (see also [Ily91, Definition 1.1], [Ily94, Definition 2.1]); Notice that we can write $\tilde f \tilde{\mathbf{n}}|_{(x^1,\, x^2,\, x^3)} = \frac{\tilde f}{|\tilde{\mathbf{n}}|_{\mathbb{R}^3}}|_{(x^1,\, x^2,\, x^3)}\, \mathbf{n}|_{(x^1,\, x^2,\, \Phi^0(x^1,\, x^2))}$. We find

$$\begin{aligned}
\operatorname{curl} \tilde f \tilde{\mathbf{n}} &= \operatorname{curl} \left(\tilde f x_1 \frac{\partial}{\partial x^1} + \tilde f x_2 \frac{\partial}{\partial x^2} + \tilde f x_3 \frac{\partial}{\partial x^3} \right) \\
&= \left(x_3 \frac{\partial \tilde f}{\partial x^2} - x_2 \frac{\partial \tilde f}{\partial x^3} \right) \frac{\partial}{\partial x^1} + \left(x_1 \frac{\partial \tilde f}{\partial x^3} - x_3 \frac{\partial \tilde f}{\partial x^1} \right) \frac{\partial}{\partial x^2} + \left(x_2 \frac{\partial \tilde f}{\partial x^1} - x_1 \frac{\partial \tilde f}{\partial x^2} \right) \frac{\partial}{\partial x^3}.
\end{aligned}$$

and, from $x_3|_{\mathbf{B}_0} = \Phi^0(x^1,\, x^2)$ we have

$$\begin{aligned}
&\operatorname{Curl} f \\
&= \left(\Phi^0 \frac{\partial \tilde f}{\partial x^2} - x_2 \frac{\partial \tilde f}{\partial x^3} \right) \frac{\partial}{\partial x^1} + \left(x_1 \frac{\partial \tilde f}{\partial x^3} - \Phi^0 \frac{\partial \tilde f}{\partial x^1} \right) \frac{\partial}{\partial x^2} + \left(x_2 \frac{\partial \tilde f}{\partial x^1} - x_1 \frac{\partial \tilde f}{\partial x^2} \right) \frac{\partial}{\partial x^3}.
\end{aligned} \tag{A.5}$$

On the other hand, from (A.4), (A.3), the identity $\sqrt{g} = \frac{1}{\Phi^0}$, and from the fact that the vector fields $\frac{\partial}{\partial w^1}$ and $\frac{\partial}{\partial w^2}$ are tangent to $\mathbf{B}_0$, we obtain

$$\begin{aligned}
\nabla^\perp f &= \frac{1}{\sqrt{g}} \left(\frac{\partial \tilde f}{\partial w^1} \frac{\partial}{\partial w^2} - \frac{\partial \tilde f}{\partial w^2} \frac{\partial}{\partial w^1} \right)\Bigg|_{\{w^3=0\}} \\
&= \left(x_2 \frac{\partial \tilde f}{\partial x^3} - \Phi^0 \frac{\partial \tilde f}{\partial x^2} \right) \frac{\partial}{\partial x^1} + \left(\Phi^0 \frac{\partial \tilde f}{\partial x^1} - x_1 \frac{\partial \tilde f}{\partial x^3} \right) \frac{\partial}{\partial x^2} + \left(x_1 \frac{\partial \tilde f}{\partial x^2} - x_2 \frac{\partial \tilde f}{\partial x^1} \right) \frac{\partial}{\partial x^3}.
\end{aligned} \tag{A.6}$$

That is, from (A.5), we have $\nabla^\perp f = -\operatorname{Curl} f$.

On the curl of a vector field. In [Rod14, Appendix], the curl of a vector field $u \in T\mathbf{C}_0$, in the manifold $(\mathbf{C}_0,\, g_0)$, is the function defined and denoted as $\nabla^\perp \cdot u := *\mathrm{d}u^\sharp$, with $\sharp = \flat^{-1}$, that is, in local coordinates $(V^i \frac{\partial}{\partial w_i})^\sharp = g_{ij} V^j \mathrm{d}w^i$. In [CRT99, Definition 1], [Ily94, Section 2], the curl of $u \in T\mathbf{C}_0$ is the function defined and denoted as $\operatorname{Curl}_{\mathbf{n}} u := ((\operatorname{curl} \tilde u)|_{\mathbf{B}_0},\, \mathbf{n})_{\mathbb{R}^3}$, where $\tilde u$ is an extension from $\mathbf{B}_0$ to $\mathbf{B}$ of u. Now we can show that $\nabla^\perp \cdot u = \operatorname{Curl}_{\mathbf{n}} u$ up to an additive constant; we proceed as follows: first we notice that the Laplacian in the two-dimensional manifold $(\mathbf{C}_0,\, g_0)$ is defined by $\Delta u = (-\mathrm{d} * \mathrm{d} * u^\sharp - *\mathrm{d} * \mathrm{d}u^\sharp)^\flat$ in [Rod14, Appendix], and given by $(\mathrm{d} * \mathrm{d} * u^\sharp + *\mathrm{d} * \mathrm{d}u^\sharp)^\flat = -\Delta u = (\mathrm{d} * \mathrm{d} * u^\sharp)^\flat - \operatorname{Curl} \operatorname{Curl}_{\mathbf{n}} u$ in[Ily94, Section 2]. Necessarily, we have that $*\mathrm{d} * \mathrm{d}u^\sharp = -(\operatorname{Curl} \operatorname{Curl}_{\mathbf{n}} u)^\sharp$ for all $u \in T\mathbf{C}_0$. Since we already know that $\nabla^\perp f = -\operatorname{Curl} f$, it follows that $*\mathrm{d}(*\mathrm{d}u^\sharp - \operatorname{Curl}_{\mathbf{n}} u) = 0$, which implies that $\nabla^\perp \cdot u = *\mathrm{d}u^\sharp = \operatorname{Curl}_{\mathbf{n}} u$ up to an additive constant.

Further, for a nonharmonic divergence free vector field we have that $*\mathrm{d}u^\sharp = \operatorname{Curl}_{\mathbf{n}} u$. Indeed, if u is divergence free, that is if $- * \mathrm{d} * u^\sharp = 0$, then $\Delta u^\sharp = (\Delta u)^\sharp = - * \mathrm{d} * \mathrm{d}u^\sharp = (\operatorname{Curl} \operatorname{Curl}_{\mathbf{n}} u)^\sharp = - * \mathrm{d} \operatorname{Curl}_{\mathbf{n}} u$. Now, for given constants c_1 and c_2, we have

$\Delta(c_2 - c_1)u^\sharp = -*\mathrm{d}(c_2 * \mathrm{d}u^\sharp - c_1 \operatorname{Curl}_{\mathbf{n}} u)$; and if we write $*\mathrm{d}u^\sharp = z_1 + \frac{\int_{\mathbb{S}^2} *\mathrm{d}u^\sharp \mathrm{d}\mathbb{S}^2}{\int_{\mathbb{S}^2} 1 \mathrm{d}\mathbb{S}^2 u}$ and $\operatorname{Curl}_{\mathbf{n}} u = z_2 + \frac{\int_{\mathbb{S}^2} \operatorname{Curl}_{\mathbf{n}} u \mathrm{d}\mathbb{S}^2}{\int_{\mathbb{S}^2} 1 \mathrm{d}\mathbb{S}^2 u}$ we have that z_j is zero averaged, $\int_{\mathbb{S}^2} z_j \mathrm{d}\mathbb{S}^2 = 0$, for $j \in \{1, 2\}$. Choosing $c_1 = \frac{\int_{\mathbb{S}^2} *\mathrm{d}u^\sharp \mathrm{d}\mathbb{S}^2}{\int_{\mathbb{S}^2} 1 \mathrm{d}\mathbb{S}^2 u}$ and $c_2 = \frac{\int_{\mathbb{S}^2} \operatorname{Curl}_{\mathbf{n}} u \mathrm{d}\mathbb{S}^2}{\int_{\mathbb{S}^2} 1 \mathrm{d}\mathbb{S}^2 u}$, we obtain $(c_2 - c_1)\Delta u^\sharp = -*\mathrm{d}(c_2 z_1 - c_1 z_2)$ and $c_2 z_1 - c_1 z_2 = c_2 * \mathrm{d}u^\sharp - c_1 \operatorname{Curl}_{\mathbf{n}} u$. Since $c_2 z_1 - c_1 z_2$ is constant and zero averaged, necessarily $c_2 z_1 - c_1 z_2 = 0$, which implies $c_1 = c_2$, because u is nonharmonic.

If $c_1 = c_2 \neq 0$, we have that $\nabla^\perp \cdot u = *\mathrm{d}u^\sharp = z_1 + c_1 = z_2 + c_2 = \operatorname{Curl}_{\mathbf{n}} u$; if $c_1 = c_2 = 0$ we have that $*\mathrm{d}u^\sharp - \operatorname{Curl}_{\mathbf{n}} u = z_1 - z_2$ is constant and zero averaged, so again $\nabla^\perp \cdot u = *\mathrm{d}u^\sharp = \operatorname{Curl}_{\mathbf{n}} u$.

Furthermore, notice that by our choice of the space H in Section 5, harmonic vector fields are orthogonal to the space H. Indeed under Lions boundary conditions we have $A = \Delta$ and the eigenfunctions $\{W_k \mid k \in \mathbb{N}_0\}$ of the Stokes operator A with positive eigenvalues form a basis in H, and any divergence free harmonic vector field $\mathcal{W}$ satisfies $(\mathcal{W}, W_k)_H = (\mathcal{W}, \lambda_k^{-1}\Delta W_k)_H = \lambda_k^{-1}(\mathcal{W}, \Delta W_k)_H = \lambda_k^{-1}(\Delta\mathcal{W}, W_k)_H = 0$. Therefore we can conclude that

$$\nabla^\perp \cdot u = \operatorname{Curl}_{\mathbf{n}} u \in H^{-1}(\mathbb{S}^2, \mathbb{R}) \coloneqq H^1(\mathbb{S}^2, \mathbb{R})' \quad \text{for all } u \in H$$

where $(\nabla^\perp \cdot u, v)_{H^{-1}(\mathbb{S}^2,\mathbb{R}), H^1(\mathbb{S}^2,\mathbb{R})} \coloneqq -(u, \nabla^\perp v)_H$. Notice that for smoother data $u \in H^1(\mathbb{S}^2, T\mathbb{S}^2)$ and $v \in H^1(\mathbb{S}^2, \mathbb{R})$, we can write

$$\begin{aligned}
&(\nabla^\perp \cdot u, v)_{H^{-1}(\mathbb{S}^2,\mathbb{R}), H^1(\mathbb{S}^2,\mathbb{R})} = (\nabla^\perp \cdot u, v)_{L^2(\mathbb{S}^2,\mathbb{R})} = \int_{\mathbb{S}^2} (*\mathrm{d}u^\sharp) v \,\mathrm{d}\mathbb{S}^2 = \int_{\mathbb{S}^2} *(v\mathrm{d}u^\sharp)\,\mathrm{d}\mathbb{S}^2 \\
&= \int_{\mathbb{S}^2} \mathrm{d}(vu^\sharp) - \int_{\mathbb{S}^2} \mathrm{d}v \wedge u^\sharp = 0 - \int_{\mathbb{S}^2} \iota_u(*\mathrm{d}v)\,\mathrm{d}\mathbb{S}^2 = -\int_{\mathbb{S}^2} g(u, (*\mathrm{d}v)^\flat)\,\mathrm{d}\mathbb{S}^2 \\
&= -(u, \nabla^\perp v)_{L^2(\mathbb{S}^2, T\mathbb{S}^2)}.
\end{aligned}$$

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