

Functional a posteriori error estimates for time-periodic parabolic optimal control problems

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FUNCTIONAL A POSTERIORI ERROR ESTIMATES FOR TIME-PERIODIC PARABOLIC OPTIMAL CONTROL PROBLEMS

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ABSTRACT. This work is devoted to the functional a posteriori error analysis of multiharmonic finite element approximations to some distributed time-periodic parabolic optimal control problems. We derive easily computable, guaranteed upper bounds for both the state and co-state errors and the cost functional.

1. INTRODUCTION

We consider optimal control problems with time-periodic parabolic state equations. Problems of this type often arise in different practical applications, e.g. in electromagnetics and chemistry. The multiharmonic finite element method (MhFEM) is well adapted to this class of parabolic problems. Within the framework of this method, the state functions are expanded into Fourier series in time with coefficients depending on spatial variables. In numerical computations, these series are truncated and the Fourier coefficients are approximated by the finite element method (FEM). This scheme leads to the MhFEM also called harmonic-balanced FEM, which was successfully used for the simulation of electromagnetic devices described by nonlinear eddy current problems with harmonic excitations, see, e.g., [33, 1, 2, 3, 7] and the references therein. Later, the MhFEM has been applied to linear time-periodic parabolic boundary value and optimal control problems [13, 14, 20, 24, 32] and to linear time-periodic eddy current problems and the corresponding optimal control problems [15, 16, 17]. In the MhFEM setting, we are able to establish inf-sup and sup-sup conditions from which we deduce existence and uniqueness of the solution to parabolic time-periodic problems by applying the theorem of Babuška and Aziz. It is worth to note that, for linear time-periodic parabolic problems, the MhFEM is a natural and very efficient numerical technology because the computations of the Fourier coefficients corresponding to each single mode $k = 0, 1, \dots$ are decoupled. Hence, we can use different meshes for different modes and independently generate them by adaptive finite element approximations of the respective Fourier coefficients. At this point, we need fully computable a posteriori estimates, which provide guaranteed bounds of overall errors and reliable indicators of errors associated with the modes. Then, by prescribing certain bounds, we can finally filter out the Fourier coefficients, which are important for the numerical solution of the problem. Altogether, such an *adaptive multiharmonic finite element method (AMhFEM)* yields complete adaptivity in space and time, what is the ultimate goal in the context of a quantitative analysis of time-periodic parabolic equations. It is clear that an efficient numerical analysis of optimal control problems with state equations of the above type should be based on an AMhFEM.

This paper is aimed to make a step towards the creation of fully reliable error estimation methods for distributed time-periodic parabolic optimal control problems. We consider the multiharmonic finite element approximations of the reduced optimality system, and derive guaranteed and fully computable bounds for the discretization errors. For this purpose, we use the functional a posteriori error estimation techniques earlier introduced by S. Repin, see, e.g., the papers on parabolic problems [27, 10] as well as on optimal control problems [8, 9], the books [28, 25] and the references therein. In particular, our functional a posteriori error analysis uses the techniques close to those suggested in [27], but the analysis contains essential changes due to the MhFEM setting. In [23], the authors already derived functional a posteriori error estimates for multiharmonic finite element approximations to parabolic time-periodic boundary value problems. Similar

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results are now obtained for the multiharmonic finite element approximations to the state and co-state, which are the unique solutions of the reduced optimality system, in different settings. Let us mention that these a posteriori error estimates for the state and co-state immediately yield the corresponding a posteriori error estimates for the control. Beside these results, we provide fully computable estimates of the cost functional. In fact, we generate a new formulation of the optimal control problem, in which (unlike the original statement) the state equations are accounted in terms of penalties. It is proved that the modified cost functional attains its infimum if and only if the control function coincides with the optimal control and the state function with the respective state function. Therefore, in principle, we can use it as an object of direct minimization, whose value on each step provides a guaranteed upper bound of the cost functional.

In particular, the paper is organized as follows: In Section 2, we discuss the time-periodic parabolic optimal control problem and its reduced optimality system. We define norms and spaces in order to derive a ‘‘symmetric’’ space-time variational formulation of the optimality system. Then, the multiharmonic finite element discretization of this space-time variational problem is considered in Section 3. Section 4 is devoted to the derivation of functional a posteriori error estimates for the optimality system. Here, we derive two types of estimates including their multiharmonic setting. Finally, we present the new results on functional a posteriori estimates for the cost functional in Section 5.

2. A TIME-PERIODIC PARABOLIC OPTIMAL CONTROL PROBLEM

Let $Q_T := \Omega \times (0, T)$ denote the space-time cylinder and $\Sigma_T := \Gamma \times (0, T)$ its mantle boundary, where the spatial domain $\Omega \subset \mathbb{R}^d$, $d = \{1, 2, 3\}$, is assumed to be a bounded Lipschitz domain with boundary $\Gamma := \partial\Omega$, and $(0, T)$ is a given time interval. Moreover, let us denote the state of our optimal control problem by y and the control by u . We consider the following parabolic time-periodic optimal control problem:

$$(1) \quad \min_{y, u} \mathcal{J}(y, u) := \frac{1}{2} \int_0^T \int_{\Omega} (y(\mathbf{x}, t) - y_d(\mathbf{x}, t))^2 d\mathbf{x} dt + \frac{\lambda}{2} \int_0^T \int_{\Omega} (u(\mathbf{x}, t))^2 d\mathbf{x} dt$$

subject to the parabolic time-periodic boundary value problem

$$(2) \quad \begin{cases} \sigma(\mathbf{x}) \partial_t y(\mathbf{x}, t) - \operatorname{div}(\nu(\mathbf{x}) \nabla y(\mathbf{x}, t)) = u(\mathbf{x}, t) & (\mathbf{x}, t) \in Q_T, \\ y(\mathbf{x}, t) = 0 & (\mathbf{x}, t) \in \Sigma_T, \\ y(\mathbf{x}, 0) = y(\mathbf{x}, T) & \mathbf{x} \in \bar{\Omega}, \end{cases}$$

with uniformly bounded coefficients $\sigma(\cdot)$ and $\nu(\cdot)$ satisfying the assumptions

$$(3) \quad 0 < \underline{\sigma} \leq \sigma(\mathbf{x}) \leq \bar{\sigma}, \quad 0 < \underline{\nu} \leq \nu(\mathbf{x}) \leq \bar{\nu}, \quad \mathbf{x} \in \Omega$$

where $\underline{\sigma}$, $\bar{\sigma}$, $\underline{\nu}$ and $\bar{\nu}$ are constants. Our goal is to approach to the desired state function y_d as close as it is possible by finding a suitable control function u . The positive regularization parameter λ provides a weighting of the cost of the control in the cost functional $\mathcal{J}(\cdot, \cdot)$.

We want to formulate now the optimality system. Its solution is equivalent to the solution of the original optimal control problem (1)-(2). We denote the Lagrange multiplier by p , which is also referred as the adjoint state. The Lagrange functional for our minimization problem is given as follows:

$$(4) \quad \mathcal{L}(y, u, p) := \mathcal{J}(y, u) - \int_0^T \int_{\Omega} (\sigma \partial_t y - \operatorname{div}(\nu \nabla y) - u) p d\mathbf{x} dt,$$

and has a saddle point, see, e.g., [11, 31] and the references therein. Hence, the corresponding solutions satisfy the system of necessary conditions

$$(5) \quad \mathcal{L}_p(y, u, p) = 0, \quad \mathcal{L}_y(y, u, p) = 0, \quad \mathcal{L}_u(y, u, p) = 0.$$

Using the second condition, we can eliminate the control u from the optimality system (5), i.e.,

$$(6) \quad u = -\lambda^{-1} p \quad \text{in } Q_T.$$

From (6) it appears very natural to choose u and p from the same space. Moreover, we arrive at a reduced optimality system, written in its classical formulation as

$$(7) \quad \begin{cases} \sigma(\mathbf{x}) \partial_t y(\mathbf{x}, t) - \operatorname{div}(\nu(\mathbf{x}) \nabla y(\mathbf{x}, t)) = -\lambda^{-1} p(\mathbf{x}, t) & (\mathbf{x}, t) \in Q_T, \\ y(\mathbf{x}, t) = 0 & (\mathbf{x}, t) \in \Sigma_T, \\ y(\mathbf{x}, 0) = y(\mathbf{x}, T) & \mathbf{x} \in \bar{\Omega}, \\ -\sigma(\mathbf{x}) \partial_t p(\mathbf{x}, t) - \operatorname{div}(\nu(\mathbf{x}) \nabla p(\mathbf{x}, t)) = y(\mathbf{x}, t) - y_d(\mathbf{x}, t) & (\mathbf{x}, t) \in Q_T, \\ p(\mathbf{x}, t) = 0 & (\mathbf{x}, t) \in \Sigma_T, \\ p(\mathbf{x}, T) = p(\mathbf{x}, 0) & \mathbf{x} \in \bar{\Omega}. \end{cases}$$

In order to derive the space-time variational formulation of (7), we will introduce Sobolev spaces of functions in the space-time cylinder Q_T . At this point our notation is close to that was used in [21, 22].

Let the Sobolev spaces $H^{1,0}(Q_T) = \{v \in L^2(Q_T) : \nabla v \in [L^2(Q_T)]^d\}$ and $H^{1,1}(Q_T) = \{v \in L^2(Q_T) : \nabla v \in [L^2(Q_T)]^d, \partial_t v \in L^2(Q_T)\}$ be equipped with the norms

$$\|v\|_{1,0} := \left(\int_{Q_T} (v(\mathbf{x}, t))^2 + |\nabla v(\mathbf{x}, t)|^2 \, d\mathbf{x} \, dt \right)^{1/2},$$

$$\|v\|_{1,1} := \left(\int_{Q_T} (v(\mathbf{x}, t))^2 + |\nabla v(\mathbf{x}, t)|^2 + |\partial_t v(\mathbf{x}, t)|^2 \, d\mathbf{x} \, dt \right)^{1/2},$$

respectively, where $\nabla = \nabla_{\mathbf{x}}$ and ∂_t denote the generalized derivatives with respect to \mathbf{x} and t . The space $H^{0,1}(Q_T) = \{v \in L^2(Q_T) : \partial_t v \in L^2(Q_T)\}$ is defined analogously. Furthermore, the boundary and time-periodicity conditions are included by defining the Sobolev spaces

$$\begin{aligned} H_0^{1,0}(Q_T) &= \{v \in H^{1,0}(Q_T) : v = 0 \text{ on } \Sigma_T\}, \\ H_0^{1,1}(Q_T) &= \{v \in H^{1,1}(Q_T) : v = 0 \text{ on } \Sigma_T\}, \\ H_{per}^{0,1}(Q_T) &= \{v \in H^{0,1}(Q_T) : v(\mathbf{x}, 0) = v(\mathbf{x}, T) \text{ for a.a. } \mathbf{x} \in \Omega\}, \\ H_{per}^{1,1}(Q_T) &= \{v \in H^{1,1}(Q_T) : v(\mathbf{x}, 0) = v(\mathbf{x}, T) \text{ for a.a. } \mathbf{x} \in \Omega\}, \\ H_{0,per}^{1,1}(Q_T) &= \{v \in H_0^{1,1}(Q_T) : v(\mathbf{x}, 0) = v(\mathbf{x}, T) \text{ for a.a. } \mathbf{x} \in \Omega\}. \end{aligned}$$

All inner products and norms in L^2 related to the whole space-time domain Q_T are denoted by (\cdot, \cdot) and $\|\cdot\|$, respectively. If they are associated with the spatial domain Ω , then we write $(\cdot, \cdot)_\Omega$ and $\|\cdot\|_\Omega$. The symbols $(\cdot, \cdot)_{1,\Omega}$ and $\|\cdot\|_{1,\Omega}$ denote the standard inner products and norms of the space $H^1(\Omega)$.

The functions used in our analysis will be presented in form of Fourier series, i.e.,

$$(8) \quad v(\mathbf{x}, t) = v_0^c(\mathbf{x}) + \sum_{k=1}^{\infty} (v_k^c(\mathbf{x}) \cos(k\omega t) + v_k^s(\mathbf{x}) \sin(k\omega t))$$

with the Fourier coefficients

$$v_0^c(\mathbf{x}) = \frac{1}{T} \int_0^T v(\mathbf{x}, t) \, dt, \quad v_k^c(\mathbf{x}) = \frac{2}{T} \int_0^T v(\mathbf{x}, t) \cos(k\omega t) \, dt, \quad v_k^s(\mathbf{x}) = \frac{2}{T} \int_0^T v(\mathbf{x}, t) \sin(k\omega t) \, dt,$$

where T and $\omega = 2\pi/T$ denote the periodicity and the frequency, respectively. Note that the Fourier series approach is reasonable due to the time-periodicity condition. In what follows, we also use the spaces

$$\begin{aligned} H_{per}^{0, \frac{1}{2}}(Q_T) &= \{v \in L^2(Q_T) : \|\partial_t^{1/2} v\| < \infty\}, \\ H_{per}^{1, \frac{1}{2}}(Q_T) &= \{v \in H^{1,0}(Q_T) : \|\partial_t^{1/2} v\| < \infty\}, \\ H_{0,per}^{1, \frac{1}{2}}(Q_T) &= \{v \in H_{per}^{1, \frac{1}{2}}(Q_T) : v = 0 \text{ on } \Sigma_T\}, \end{aligned}$$

where $\|\partial_t^{1/2}v\|$ is defined in the Fourier space by the relation

$$(9) \quad \|\partial_t^{1/2}v\|^2 := |v|_{0,\frac{1}{2}}^2 := \frac{T}{2} \sum_{k=1}^{\infty} k\omega \|\mathbf{v}_k\|_{\Omega}^2,$$

where $\mathbf{v}_k = (v_k^c, v_k^s)^T$ for all $k \in \mathbb{N}$, see also [24]. These spaces will be used in order to derive a symmetrized formulation of problem (7). They are equipped with the scalar products

$$(10) \quad (\partial_t^{1/2}y, \partial_t^{1/2}v) := \frac{T}{2} \sum_{k=1}^{\infty} k\omega (\mathbf{y}_k, \mathbf{v}_k)_{\Omega}, \quad (\sigma \partial_t^{1/2}y, \partial_t^{1/2}v) := \frac{T}{2} \sum_{k=1}^{\infty} k\omega (\sigma \mathbf{y}_k, \mathbf{v}_k)_{\Omega}.$$

The seminorm and the norm of the space $H_{per}^{1,\frac{1}{2}}(Q_T)$ are defined by the relations

$$|v|_{1,\frac{1}{2}}^2 := \|\nabla v\|^2 + \|\partial_t^{1/2}v\|^2 = T \|\nabla y_0^c\|_{\Omega}^2 + \frac{T}{2} \sum_{k=1}^{\infty} (k\omega \|\mathbf{v}_k\|_{\Omega}^2 + \|\nabla \mathbf{v}_k\|_{\Omega}^2)$$

and

$$\|v\|_{1,\frac{1}{2}}^2 := \|v\|^2 + |v|_{1,\frac{1}{2}}^2 = T (\|y_0^c\|_{\Omega}^2 + \|\nabla y_0^c\|_{\Omega}^2) + \frac{T}{2} \sum_{k=1}^{\infty} ((1+k\omega) \|\mathbf{v}_k\|_{\Omega}^2 + \|\nabla \mathbf{v}_k\|_{\Omega}^2),$$

respectively. Let us define

$$(11) \quad v^{\perp}(\mathbf{x}, t) := \sum_{k=1}^{\infty} (-v_k^c(\mathbf{x}) \sin(k\omega t) + v_k^s(\mathbf{x}) \cos(k\omega t)) = \sum_{k=1}^{\infty} \underbrace{(v_k^s(\mathbf{x}), -v_k^c(\mathbf{x}))}_{=: (-\mathbf{v}_k^{\perp})^T} \cdot \begin{pmatrix} \cos(k\omega t) \\ \sin(k\omega t) \end{pmatrix},$$

where $\|\mathbf{u}_k^{\perp}\|_{\Omega} = \|\mathbf{u}_k\|_{\Omega}$ is valid. Hence, it follows that

$$\|\partial_t^{1/2}v^{\perp}\|^2 = \frac{T}{2} \sum_{k=1}^{\infty} k\omega \|\mathbf{v}_k^{\perp}\|_{\Omega}^2 = \frac{T}{2} \sum_{k=1}^{\infty} k\omega \|\mathbf{v}_k\|_{\Omega}^2 = \|\partial_t^{1/2}v\|^2 \quad \forall v \in H_{per}^{0,\frac{1}{2}}(Q_T).$$

The following lemma including its proof can be found in [23]:

Lemma 1. *The identities*

$$(12) \quad (\sigma \partial_t^{1/2}y, \partial_t^{1/2}v) = (\sigma \partial_t y, v^{\perp}) \quad \text{and} \quad (\sigma \partial_t^{1/2}y, \partial_t^{1/2}v^{\perp}) = (\sigma \partial_t y, v)$$

are valid for all $y \in H_{per}^{0,1}(Q_T)$ and $v \in H_{per}^{0,\frac{1}{2}}(Q_T)$.

Furthermore, the orthogonality relations

$$(13) \quad \begin{cases} (\sigma \partial_t y, y) = 0 & \text{and} & (\sigma y^{\perp}, y) = 0 & \forall y \in H_{per}^{0,1}(Q_T), \\ (\sigma \partial_t^{1/2}y, \partial_t^{1/2}y^{\perp}) = 0 & \text{and} & (\nu \nabla y, \nabla y^{\perp}) = 0 & \forall y \in H_{per}^{1,\frac{1}{2}}(Q_T), \end{cases}$$

hold, see [24, 32]. The identity

$$(14) \quad \int_0^T \xi \partial_t^{1/2}v^{\perp} dt = - \int_0^T \partial_t^{1/2}\xi^{\perp} v dt \quad \forall \xi, v \in H_{per}^{0,\frac{1}{2}}(Q_T)$$

is defined in the Fourier space yielding the definition

$$(15) \quad (\xi, \partial_t^{1/2}v) := \frac{T}{2} \sum_{k=1}^{\infty} (k\omega)^{1/2} (\xi_k, \mathbf{v}_k)_{\Omega}.$$

We note that for functions presented in terms of Fourier series the standard Friedrichs inequality holds in the form

$$(16) \quad \begin{aligned} \|\nabla u\|^2 &= \int_{Q_T} |\nabla u|^2 d\mathbf{x} dt = T \|\nabla u_0^c\|_{\Omega}^2 + \frac{T}{2} \sum_{k=1}^{\infty} \|\nabla \mathbf{u}_k\|_{\Omega}^2 \\ &\geq \frac{1}{C_F^2} \left(T \|u_0^c\|_{\Omega}^2 + \frac{T}{2} \sum_{k=1}^{\infty} \|\mathbf{u}_k\|_{\Omega}^2 \right) = \frac{1}{C_F^2} \|u\|^2. \end{aligned}$$

In order to derive the space-time variational formulation of the parabolic time-periodic optimal control problem (1)-(2), we multiply the parabolic partial differential equations (7) by test functions $v, q \in H_{0,per}^{1,\frac{1}{2}}(Q_T)$ and integrate over the space-time cylinder Q_T . After integration by parts with respect to the spatial and time variables, the following ‘‘symmetric’’ space-time variational formulation of the reduced optimality system (7) is obtained: Given the desired state $y_d \in L^2(Q_T)$, find y and p from $H_{0,per}^{1,\frac{1}{2}}(Q_T)$ such that

$$(17) \quad \begin{cases} \int_{Q_T} (y v - \nu(\mathbf{x}) \nabla p \cdot \nabla v + \sigma(\mathbf{x}) \partial_t^{1/2} p \partial_t^{1/2} v^\perp) d\mathbf{x} dt = \int_{Q_T} y_d v d\mathbf{x} dt, \\ \int_{Q_T} (\nu(\mathbf{x}) \nabla y \cdot \nabla q + \sigma(\mathbf{x}) \partial_t^{1/2} y \partial_t^{1/2} q^\perp + \lambda^{-1} p q) d\mathbf{x} dt = 0, \end{cases}$$

for all test functions $v, q \in H_{0,per}^{1,\frac{1}{2}}(Q_T)$. Now, let

$$(18) \quad \begin{aligned} \mathcal{B}((y, p), (v, q)) = & \int_{Q_T} (y v - \nu(\mathbf{x}) \nabla p \cdot \nabla v + \sigma(\mathbf{x}) \partial_t^{1/2} p \partial_t^{1/2} v^\perp \\ & + \nu(\mathbf{x}) \nabla y \cdot \nabla q + \sigma(\mathbf{x}) \partial_t^{1/2} y \partial_t^{1/2} q^\perp + \lambda^{-1} p q) d\mathbf{x} dt \end{aligned}$$

denote the space-time bilinear form of the variational problem (17).

3. MULTIHARMONIC FINITE ELEMENT APPROXIMATION

In order to solve the optimal control problem (1)-(2) approximately, we discretize the optimality system (17) by the MhFEM, see [24]. Inserting the Fourier series ansatz (8) into the space-time variational formulation (17) and exploiting the orthogonality of the functions $\cos(k\omega t)$ and $\sin(k\omega t)$, we arrive at the following system, which has to be solved for every mode $k \in \mathbb{N}$: Find $\mathbf{y}_k, \mathbf{p}_k \in \mathbb{V} := V \times V = (H_0^1(\Omega))^2$ such that

$$(19) \quad \begin{cases} \int_{\Omega} (\mathbf{y}_k \cdot \mathbf{v}_k - \nu(\mathbf{x}) \nabla \mathbf{p}_k \cdot \nabla \mathbf{v}_k + k\omega \sigma(\mathbf{x}) \mathbf{p}_k \cdot \mathbf{v}_k^\perp) d\mathbf{x} = \int_{\Omega} \mathbf{y}_{dk} \cdot \mathbf{v}_k d\mathbf{x}, \\ \int_{\Omega} (\nu(\mathbf{x}) \nabla \mathbf{y}_k \cdot \nabla \mathbf{q}_k + k\omega \sigma(\mathbf{x}) \mathbf{y}_k \cdot \mathbf{q}_k^\perp + \lambda^{-1} \mathbf{p}_k \cdot \mathbf{q}_k) d\mathbf{x} = 0, \end{cases}$$

for all test functions $\mathbf{v}_k, \mathbf{q}_k \in \mathbb{V}$. In the case of $k = 0$, we obtain the following optimality system: Find $y_0^c, p_0^c \in V$ such that

$$(20) \quad \begin{cases} \int_{\Omega} (y_0^c \cdot v_0^c - \nu(\mathbf{x}) \nabla p_0^c \cdot \nabla v_0^c) d\mathbf{x} = \int_{\Omega} y_{d0}^c \cdot v_0^c d\mathbf{x}, \\ \int_{\Omega} (\nu(\mathbf{x}) \nabla y_0^c \cdot \nabla q_0^c + \lambda^{-1} p_0^c \cdot q_0^c) d\mathbf{x} = 0, \end{cases}$$

for all test functions $v_0^c, q_0^c \in V$. The variational problems (19) and (20) have unique solutions due to the Babuška-Aziz theorem, see [24, 32]. In order to solve the problems numerically, the Fourier series are truncated at a finite index N and the unknown Fourier coefficients $\mathbf{y}_k = (y_k^c, y_k^s)^T$, $\mathbf{p}_k = (p_k^c, p_k^s)^T \in \mathbb{V}$ are approximated by finite element functions

$$\mathbf{y}_{kh} = (y_{kh}^c, y_{kh}^s)^T, \mathbf{p}_{kh} = (p_{kh}^c, p_{kh}^s)^T \in \mathbb{V}_h = V_h \times V_h \subset \mathbb{V},$$

where $V_h = \text{span}\{\varphi_1, \dots, \varphi_n\}$ and $\{\varphi_i(\mathbf{x}) : i = 1, 2, \dots, n_h\}$ is the standard nodal basis. We denote by h the usual discretization parameter such that $n = n_h = \dim V_h = O(h^{-d})$ and use continuous, piecewise linear finite elements on the finite elements on a regular triangulation \mathcal{T}_h to construct V_h and its basis, see, e.g., [4, 6, 12, 30]. This leads to the following infinite dimensional problem for every single mode $k = 1, 2, \dots, N$:

$$(21) \quad \begin{pmatrix} M_h & 0 & -K_{h,\nu} & k\omega M_{h,\sigma} \\ 0 & M_h & -k\omega M_{h,\sigma} & -K_{h,\nu} \\ -K_{h,\nu} & -k\omega M_{h,\sigma} & -\lambda^{-1} M_h & 0 \\ k\omega M_{h,\sigma} & -K_{h,\nu} & 0 & -\lambda^{-1} M_h \end{pmatrix} \begin{pmatrix} y_k^c \\ y_k^s \\ p_k^c \\ p_k^s \end{pmatrix} = \begin{pmatrix} y_{dk}^c \\ y_{dk}^s \\ 0 \\ 0 \end{pmatrix},$$

which has to be solved with respect to the nodal parameter vectors

$$\underline{y}_k^c = (y_{k,i}^c)_{i=1,\dots,n}, \underline{y}_k^s = (y_{k,i}^s)_{i=1,\dots,n}, \underline{p}_k^c = (p_{k,i}^c)_{i=1,\dots,n}, \underline{p}_k^s = (p_{k,i}^s)_{i=1,\dots,n} \in \mathbb{R}^n$$

of the finite element approximations $y_{kh}^c(\mathbf{x}) = \sum_{i=1}^n y_{k,i}^c \varphi_i(\mathbf{x})$, $y_{kh}^s(\mathbf{x}) = \sum_{i=1}^n y_{k,i}^s \varphi_i(\mathbf{x})$, $p_{kh}^c(\mathbf{x}) = \sum_{i=1}^n p_{k,i}^c \varphi_i(\mathbf{x})$ and $p_{kh}^s(\mathbf{x}) = \sum_{i=1}^n p_{k,i}^s \varphi_i(\mathbf{x})$ to the unknown Fourier coefficients $y_k^c(\mathbf{x})$, $y_k^s(\mathbf{x})$, $p_k^c(\mathbf{x})$ and $p_k^s(\mathbf{x})$. The matrices M_h , $M_{h,\sigma}$ and $K_{h,\nu}$ correspond to the mass matrix, the weighted mass matrix and the stiffness matrix, respectively. Their entries are computed by the formulas

$$M_h^{ij} = \int_{\Omega} \varphi_i \varphi_j \, d\mathbf{x}, \quad M_{h,\sigma}^{ij} = \int_{\Omega} \sigma \varphi_i \varphi_j \, d\mathbf{x}, \quad K_{h,\nu}^{ij} = \int_{\Omega} \nu \nabla \varphi_i \cdot \nabla \varphi_j \, d\mathbf{x}, \quad i, j = 1, \dots, n.$$

The right hand side is computed by

$$\underline{y}_{dk}^c = \left[\int_{\Omega} y_{dk}^c \varphi_j \, d\mathbf{x} \right]_{j=1,\dots,n} \quad \text{and} \quad \underline{y}_{dk}^s = \left[\int_{\Omega} y_{dk}^s \varphi_j \, d\mathbf{x} \right]_{j=1,\dots,n}.$$

In the case $k = 0$, the following linear system arising from the variational problem (20) is obtained:

$$(22) \quad \begin{pmatrix} M_h & -K_{h,\nu} \\ -K_{h,\nu} & -\lambda^{-1} M_h \end{pmatrix} \begin{pmatrix} \underline{y}_0^c \\ \underline{p}_0^c \end{pmatrix} = \begin{pmatrix} \underline{y}_{d0}^c \\ 0 \end{pmatrix}.$$

Fast and robust solvers for the linear systems (19) and (20) can be found in [14, 18, 24, 32], which we use in order to obtain the multiharmonic finite element approximations

$$(23) \quad \begin{cases} y_{Nh}(\mathbf{x}, t) = y_{0h}^c(\mathbf{x}) + \sum_{k=1}^N (y_{kh}^c(\mathbf{x}) \cos(k\omega t) + y_{kh}^s(\mathbf{x}) \sin(k\omega t)), \\ p_{Nh}(\mathbf{x}, t) = p_{0h}^c(\mathbf{x}) + \sum_{k=1}^N (p_{kh}^c(\mathbf{x}) \cos(k\omega t) + p_{kh}^s(\mathbf{x}) \sin(k\omega t)). \end{cases}$$

In the next section, we will present an a posteriori error analysis for the error between the unknown solution (y, p) and its multiharmonic finite element approximation (y_{Nh}, p_{Nh}) .

4. FUNCTIONAL A POSTERIORI ERROR ESTIMATES FOR THE OPTIMALITY SYSTEM

First, we present inf-sup and sup-sup conditions for the space-time bilinear form (18).

Lemma 2. *The space-time bilinear form $\mathcal{B}(\cdot, \cdot)$ defined by (18) meets the following inf-sup and sup-sup conditions:*

$$(24) \quad \mu_1 \|(y, p)\|_{1, \frac{1}{2}} \leq \sup_{0 \neq (v, q) \in (H_{0,per}^{1, \frac{1}{2}}(Q_T))^2} \frac{\mathcal{B}((y, p), (v, q))}{\|(v, q)\|_{1, \frac{1}{2}}} \leq \mu_2 \|(y, p)\|_{1, \frac{1}{2}} \quad \forall y, p \in H_{0,per}^{1, \frac{1}{2}}(Q_T)$$

with positive constants $\mu_1 = \frac{\min\{\frac{1}{\sqrt{\lambda}}, \underline{\nu}, \underline{\sigma}\} \min\{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\}}{\sqrt{1+2 \max\{\lambda, \frac{1}{\lambda}\}}}$ and $\mu_2 = \max\{1, \frac{1}{\lambda}, \bar{\nu}, \bar{\sigma}\}$.

Proof. Using the triangle and Cauchy-Schwarz inequalities, we obtain the estimate

$$\begin{aligned} |\mathcal{B}((y, p), (v, q))| &= \left| \int_0^T \int_{\Omega} \left(y v - \nu(\mathbf{x}) \nabla p \cdot \nabla v + \sigma(\mathbf{x}) \partial_t^{1/2} p \partial_t^{1/2} v^\perp \right. \right. \\ &\quad \left. \left. + \nu(\mathbf{x}) \nabla y \cdot \nabla q + \sigma(\mathbf{x}) \partial_t^{1/2} y \partial_t^{1/2} q^\perp + \lambda^{-1} p q \right) d\mathbf{x} dt \right| \\ &\leq \|y\| \|v\| + \bar{\nu} \|\nabla p\| \|\nabla v\| + \bar{\sigma} \|\partial_t^{1/2} p\| \|\partial_t^{1/2} v\| \\ &\quad + \bar{\nu} \|\nabla y\| \|\nabla q\| + \bar{\sigma} \|\partial_t^{1/2} y\| \|\partial_t^{1/2} q\| + \lambda^{-1} \|p\| \|q\| \\ &\leq \mu_2 \|(y, p)\|_{1, \frac{1}{2}} \|(v, q)\|_{1, \frac{1}{2}} \end{aligned}$$

with $\mu_2 = \max\{1, \frac{1}{\lambda}, \bar{\nu}, \bar{\sigma}\}$. Hence, the right hand-side inequality in (24) is proved. In order to prove the left hand-side inequality, we choose the test function

$$(v, q) = \left(y - \frac{1}{\sqrt{\lambda}} p - \frac{1}{\sqrt{\lambda}} p^\perp, p + \sqrt{\lambda} y - \sqrt{\lambda} y^\perp \right).$$

Using the σ - and ν -weighted orthogonality relations (13), we obtain the following relations:

$$\begin{aligned}\mathcal{B}((y, p), (y, p)) &= \|y\|^2 + \lambda^{-1} \|p\|^2, \\ \mathcal{B}((y, p), (-\frac{1}{\sqrt{\lambda}}p, \sqrt{\lambda}y)) &= \frac{1}{\sqrt{\lambda}}(\nu \nabla p, \nabla p) + \sqrt{\lambda}(\nu \nabla y, \nabla y), \\ \mathcal{B}((y, p), (-\frac{1}{\sqrt{\lambda}}p^\perp, -\sqrt{\lambda}y^\perp)) &= \frac{1}{\sqrt{\lambda}}(\sigma \partial_t^{1/2} p, \partial_t^{1/2} p) + \sqrt{\lambda}(\sigma \partial_t^{1/2} y, \partial_t^{1/2} y),\end{aligned}$$

which lead to the following estimate from below:

$$\mathcal{B}((y, p), (y - \frac{1}{\sqrt{\lambda}}p - \frac{1}{\sqrt{\lambda}}p^\perp, p + \sqrt{\lambda}y - \sqrt{\lambda}y^\perp)) \geq \min\{\frac{1}{\sqrt{\lambda}}, \underline{\nu}, \underline{\sigma}\} \min\{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\} \|(y, p)\|_{1, \frac{1}{2}}^2.$$

Due to

$$\|(v, q)\|_{1, \frac{1}{2}}^2 \leq \left(1 + 2 \max\{\lambda, \frac{1}{\lambda}\}\right) \|(y, p)\|_{1, \frac{1}{2}}^2,$$

we arrive at the following estimate of the supremum from below:

$$\sup_{0 \neq (v, q) \in (H_{0, per}^{1, \frac{1}{2}}(Q_T))^2} \frac{\mathcal{B}((y, p), (v, q))}{\|(v, q)\|_{1, \frac{1}{2}}} \geq \frac{\min\{\frac{1}{\sqrt{\lambda}}, \underline{\nu}, \underline{\sigma}\} \min\{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\} \|(y, p)\|_{1, \frac{1}{2}}^2}{\sqrt{1 + 2 \max\{\lambda, \frac{1}{\lambda}\}} \|(y, p)\|_{1, \frac{1}{2}}} = \mu_1 \|(y, p)\|_{1, \frac{1}{2}}$$

with the constant $\mu_1 = \frac{\min\{\frac{1}{\sqrt{\lambda}}, \underline{\nu}, \underline{\sigma}\} \min\{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\}}{\sqrt{1 + 2 \max\{\lambda, \frac{1}{\lambda}\}}}$. \square

Note that the norm $|\cdot|_{1, \frac{1}{2}}$ is equivalent to the norm $\|\cdot\|_{1, \frac{1}{2}}$ due to the Friedrichs inequality.

Lemma 3. *The space-time bilinear form $\mathcal{B}(\cdot, \cdot)$ defined by (18) satisfies the following inf-sup and sup-sup conditions:*

$$(25) \quad \tilde{\mu}_1 |(y, p)|_{1, \frac{1}{2}} \leq \sup_{0 \neq (v, q) \in (H_{0, per}^{1, \frac{1}{2}}(Q_T))^2} \frac{\mathcal{B}((y, p), (v, q))}{|(v, q)|_{1, \frac{1}{2}}} \leq \tilde{\mu}_2 |(y, p)|_{1, \frac{1}{2}} \quad \forall y, p \in H_{0, per}^{1, \frac{1}{2}}(Q_T)$$

with positive constants $\tilde{\mu}_1 = \frac{\min\{\underline{\nu}, \underline{\sigma}\} \min\{\lambda, \frac{1}{\lambda}\}}{\sqrt{2}}$ and $\tilde{\mu}_2 = \max\{1, \frac{1}{\lambda}, \bar{\nu}, \bar{\sigma}\} \max\{1, C_F^2 + 1\}$, where C_F is the Friedrichs constant.

Proof. The right hand-side inequality in (25) is proven by using again the triangle and Cauchy-Schwarz inequalities and additionally Friedrichs inequality (16):

$$\begin{aligned}|\mathcal{B}((y, p), (v, q))| &\leq \max\{1, \frac{1}{\lambda}, \bar{\nu}, \bar{\sigma}\} \left((C_F^2 + 1) \|\nabla y\|^2 + \|\partial_t^{1/2} y\|^2 + (C_F^2 + 1) \|\nabla p\|^2 + \|\partial_t^{1/2} p\|^2 \right)^{1/2} \\ &\quad \times \left((C_F^2 + 1) \|\nabla v\|^2 + \|\partial_t^{1/2} v\|^2 + (C_F^2 + 1) \|\nabla q\|^2 + \|\partial_t^{1/2} q\|^2 \right)^{1/2} \\ &\leq \tilde{\mu}_2 |(y, p)|_{1, \frac{1}{2}} |(v, q)|_{1, \frac{1}{2}}\end{aligned}$$

with the constant $\tilde{\mu}_2 = \max\{1, \frac{1}{\lambda}, \bar{\nu}, \bar{\sigma}\} \max\{1, C_F^2 + 1\}$. The left hand-side inequality in (25) is proven by selecting the test function

$$(v, q) = \left(-\frac{1}{\sqrt{\lambda}}p - \frac{1}{\sqrt{\lambda}}p^\perp, \sqrt{\lambda}y - \sqrt{\lambda}y^\perp\right),$$

and using the σ - and ν -weighted orthogonality relations (13). We obtain the following estimate from below:

$$\mathcal{B}((y, p), (-\frac{1}{\sqrt{\lambda}}p - \frac{1}{\sqrt{\lambda}}p^\perp, \sqrt{\lambda}y - \sqrt{\lambda}y^\perp)) \geq \min\{\underline{\nu}, \underline{\sigma}\} \min\{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\} |(y, p)|_{1, \frac{1}{2}}^2.$$

Together with

$$|(v, q)|_{1, \frac{1}{2}}^2 \leq 2 \max\{\lambda, \frac{1}{\lambda}\} |(y, p)|_{1, \frac{1}{2}}^2,$$

this justifies the left-hand side inequality in (25), since

$$\sup_{0 \neq (v,q) \in (H_{0,per}^{1,\frac{1}{2}}(Q_T))^2} \frac{\mathcal{B}((y,p), (v,q))}{\|(v,q)\|_{1,\frac{1}{2}}} \geq \frac{\min\{\underline{\nu}, \underline{\sigma}\} \min\{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\} |(y,p)|_{1,\frac{1}{2}}^2}{\sqrt{2} \max\{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\} |(y,p)|_{1,\frac{1}{2}}} = \tilde{\mu}_1 |(y,p)|_{1,\frac{1}{2}}$$

with the constant $\tilde{\mu}_1 = \frac{\min\{\underline{\nu}, \underline{\sigma}\} \min\{\lambda, \frac{1}{\lambda}\}}{\sqrt{2}}$. \square

Error majorant of the first type. Let (η, ζ) be an approximation of (y, p) , which is a bit more regular with respect to the time variable than the exact solution (y, p) . We set $\eta, \zeta \in H_{0,per}^{1,1}(Q_T)$ that is of course true for the multiharmonic finite element approximations y_{N_h} and p_{N_h} defined in (23). Our goal is to deduce a computable upper bound of the error $e := (y, p) - (\eta, \zeta) = (y - \eta, p - \zeta)$ in $H_{0,per}^{1,\frac{1}{2}}(Q_T) \times H_{0,per}^{1,\frac{1}{2}}(Q_T)$. From (17), it follows that

$$(26) \quad \left\{ \begin{array}{l} \int_{Q_T} \left((y - \eta)v - \nu(\mathbf{x})\nabla(p - \zeta) \cdot \nabla v + \sigma(\mathbf{x})\partial_t^{1/2}(p - \zeta) \partial_t^{1/2}v^\perp \right. \\ \quad \left. + \nu(\mathbf{x})\nabla(y - \eta) \cdot \nabla q + \sigma(\mathbf{x})\partial_t^{1/2}(y - \eta) \partial_t^{1/2}q^\perp + \lambda^{-1}(p - \zeta)q \right) d\mathbf{x} dt \\ = \int_{Q_T} \left(y_d v - \eta v + \nu(\mathbf{x})\nabla\zeta \cdot \nabla v - \sigma(\mathbf{x})\partial_t^{1/2}\zeta \partial_t^{1/2}v^\perp \right. \\ \quad \left. - \nu(\mathbf{x})\nabla\eta \cdot \nabla q - \sigma(\mathbf{x})\partial_t^{1/2}\eta \partial_t^{1/2}q^\perp - \lambda^{-1}\zeta q \right) d\mathbf{x} dt \end{array} \right.$$

is valid for all $v, q \in H_{0,per}^{1,\frac{1}{2}}(Q_T)$. The linear functional

$$\mathcal{F}_{(\eta,\zeta)}(v, q) = \int_{Q_T} \left(y_d v - \eta v + \nu(\mathbf{x})\nabla\zeta \cdot \nabla v - \sigma(\mathbf{x})\partial_t^{1/2}\zeta \partial_t^{1/2}v^\perp \right. \\ \left. - \nu(\mathbf{x})\nabla\eta \cdot \nabla q - \sigma(\mathbf{x})\partial_t^{1/2}\eta \partial_t^{1/2}q^\perp - \lambda^{-1}\zeta q \right) d\mathbf{x} dt$$

is defined on $(v, q) \in H_{0,per}^{1,\frac{1}{2}}(Q_T) \times H_{0,per}^{1,\frac{1}{2}}(Q_T)$. Getting an upper bound of the error is reduced to the estimation of

$$(27) \quad \sup_{0 \neq (v,q) \in (H_{0,per}^{1,\frac{1}{2}}(Q_T))^2} \frac{\mathcal{F}_{(\eta,\zeta)}(v, q)}{\|(v, q)\|_{1,\frac{1}{2}}} \quad \text{or} \quad \sup_{0 \neq (v,q) \in (H_{0,per}^{1,\frac{1}{2}}(Q_T))^2} \frac{\mathcal{F}_{(\eta,\zeta)}(v, q)}{|(v, q)|_{1,\frac{1}{2}}}.$$

Therefor, we reconstruct the functional $\mathcal{F}_{(\eta,\zeta)}$ using the identity

$$(28) \quad (\sigma\partial_t^{1/2}\eta, \partial_t^{1/2}v^\perp) = (\sigma\partial_t\eta, v) \quad \forall \eta \in H_{0,per}^{1,1}(Q_T) \quad \forall v \in H_{0,per}^{1,\frac{1}{2}}(Q_T)$$

as stated in (12). The identities

$$\int_{\Omega} \operatorname{div} \boldsymbol{\tau} v d\mathbf{x} = - \int_{\Omega} \boldsymbol{\tau} \cdot \nabla v d\mathbf{x} \quad \text{and} \quad \int_{\Omega} \operatorname{div} \boldsymbol{\rho} q d\mathbf{x} = - \int_{\Omega} \boldsymbol{\rho} \cdot \nabla q d\mathbf{x}$$

are valid for any $v, q \in H_0^1(\Omega)$ and any

$$\boldsymbol{\tau}, \boldsymbol{\rho} \in H(\operatorname{div}, Q_T) := \{\boldsymbol{\tau} \in [L^2(Q_T)]^d : \operatorname{div}_{\mathbf{x}} \boldsymbol{\tau}(\cdot, t) \in L^2(\Omega) \text{ for a.e. } t \in (0, T)\}.$$

For ease of notation, the index \mathbf{x} in $\operatorname{div}_{\mathbf{x}}$ will be henceforth omitted, i.e., $\operatorname{div} = \operatorname{div}_{\mathbf{x}}$ denotes the generalized spatial divergence. Using the Cauchy-Schwarz inequality leads to

$$\begin{aligned} \mathcal{F}_{(\eta,\zeta)}(v, q) &= \int_{Q_T} \left(y_d v - \eta v + \nu(\mathbf{x})\nabla\zeta \cdot \nabla v - \sigma(\mathbf{x})\partial_t\zeta v + \operatorname{div} \boldsymbol{\tau} v + \boldsymbol{\tau} \cdot \nabla v \right. \\ &\quad \left. - \nu(\mathbf{x})\nabla\eta \cdot \nabla q - \sigma(\mathbf{x})\partial_t\eta q - \lambda^{-1}\zeta q + \operatorname{div} \boldsymbol{\rho} q + \boldsymbol{\rho} \cdot \nabla q \right) d\mathbf{x} dt \\ &\leq \|\mathcal{R}_1(\eta, \zeta, \boldsymbol{\tau})\| \|v\| + \|\mathcal{R}_2(\zeta, \boldsymbol{\tau})\| \|\nabla v\| + \|\mathcal{R}_3(\eta, \zeta, \boldsymbol{\rho})\| \|q\| + \|\mathcal{R}_4(\eta, \boldsymbol{\rho})\| \|\nabla q\|, \end{aligned}$$

where

$$(29) \quad \begin{cases} \mathcal{R}_1(\eta, \zeta, \boldsymbol{\tau}) = \sigma\partial_t\zeta + \eta - \operatorname{div} \boldsymbol{\tau} - y_d, & \mathcal{R}_2(\zeta, \boldsymbol{\tau}) = \boldsymbol{\tau} + \nu\nabla\zeta, \\ \mathcal{R}_3(\eta, \zeta, \boldsymbol{\rho}) = \sigma\partial_t\eta + \lambda^{-1}\zeta - \operatorname{div} \boldsymbol{\rho}, & \mathcal{R}_4(\eta, \boldsymbol{\rho}) = \boldsymbol{\rho} - \nu\nabla\eta. \end{cases}$$

Then, applying (16) yields

$$\mathcal{F}_{(\eta, \zeta)}(v, q) \leq \left(C_F \|\mathcal{R}_1(\eta, \zeta, \boldsymbol{\tau})\| + \|\mathcal{R}_2(\zeta, \boldsymbol{\tau})\| \right) \|\nabla v\| + \left(C_F \|\mathcal{R}_3(\eta, \zeta, \boldsymbol{\rho})\| + \|\mathcal{R}_4(\eta, \boldsymbol{\rho})\| \right) \|\nabla q\|.$$

From

$$\sup_{0 \neq (v, q) \in (H_{0, per}^{1, \frac{1}{2}}(Q_T))^2} \frac{\mathcal{F}_{(\eta, \zeta)}(v, q)}{|(v, q)|_{1, \frac{1}{2}}} \leq C_F \|\mathcal{R}_1(\eta, \zeta, \boldsymbol{\tau})\| + \|\mathcal{R}_2(\zeta, \boldsymbol{\tau})\| + C_F \|\mathcal{R}_3(\eta, \zeta, \boldsymbol{\rho})\| + \|\mathcal{R}_4(\eta, \boldsymbol{\rho})\|$$

and the inf-sup condition in (25), we obtain

$$|e|_{1, \frac{1}{2}} \leq \frac{1}{\tilde{\mu}_1} \sup_{0 \neq (v, q) \in (H_{0, per}^{1, \frac{1}{2}}(Q_T))^2} \frac{\mathcal{B}(e, (v, q))}{|(v, q)|_{1, \frac{1}{2}}} = \frac{1}{\tilde{\mu}_1} \sup_{0 \neq (v, q) \in (H_{0, per}^{1, \frac{1}{2}}(Q_T))^2} \frac{\mathcal{F}_{(\eta, \zeta)}(v, q)}{|(v, q)|_{1, \frac{1}{2}}},$$

which leads to the following result:

Theorem 1. *Let $\eta, \zeta \in H_{0, per}^{1, 1}(Q_T)$ and the bilinear form $\mathcal{B}(\cdot, \cdot)$ defined by (18) meet the inf-sup condition (25). Then,*

$$(30) \quad \begin{cases} |e|_{1, \frac{1}{2}} \leq \frac{1}{\tilde{\mu}_1} (C_F \|\mathcal{R}_1(\eta, \zeta, \boldsymbol{\tau})\| + \|\mathcal{R}_2(\zeta, \boldsymbol{\tau})\| \\ \quad + C_F \|\mathcal{R}_3(\eta, \zeta, \boldsymbol{\rho})\| + \|\mathcal{R}_4(\eta, \boldsymbol{\rho})\|) =: \mathcal{M}_{|\cdot|}^{\oplus}(\eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho}), \end{cases}$$

where $e = (y - \eta, p - \zeta) \in (H_{0, per}^{1, \frac{1}{2}}(Q_T))^2$, $\tilde{\mu}_1 = \frac{\min\{\nu, \sigma\} \min\{\lambda, \frac{1}{\lambda}\}}{\sqrt{2}}$ and $\boldsymbol{\tau}, \boldsymbol{\rho} \in H(\text{div}, Q_T)$.

Remark 1. *For computational reasons, it is common to reformulate majorants in such a way that they are given by quadratic functionals, see, e.g., [8]. This is done by introducing some parameters $\alpha, \beta, \gamma > 0$ and applying Young's inequality. For the error majorant $\mathcal{M}_{|\cdot|}^{\oplus}(\eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho})$, this leads to*

$$\begin{aligned} \mathcal{M}_{|\cdot|}^{\oplus}(\eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho})^2 &\leq \mathcal{M}_{|\cdot|}^{\oplus}(\alpha, \beta, \gamma; \eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho})^2 \\ &= \frac{1}{\mu_1^2} \left(C_F^2 (1 + \alpha)(1 + \beta) \|\mathcal{R}_1(\eta, \zeta, \boldsymbol{\tau})\|^2 + \frac{(1 + \alpha)(1 + \beta)}{\beta} \|\mathcal{R}_2(\zeta, \boldsymbol{\tau})\|^2 \right. \\ &\quad \left. + C_F^2 \frac{(1 + \alpha)(1 + \gamma)}{\alpha} \|\mathcal{R}_3(\eta, \zeta, \boldsymbol{\rho})\|^2 + \frac{(1 + \alpha)(1 + \gamma)}{\alpha \gamma} \|\mathcal{R}_4(\eta, \boldsymbol{\rho})\|^2 \right). \end{aligned}$$

A similar estimate as (30) for $|\cdot|_{1, \frac{1}{2}}$ can be proven for $\|\cdot\|_{1, \frac{1}{2}}$ by using the inf-sup condition in (24).

Theorem 2. *Let $\eta, \zeta \in H_{0, per}^{1, 1}(Q_T)$ and the bilinear form $\mathcal{B}(\cdot, \cdot)$ defined by (18) satisfy (24). Then,*

$$(31) \quad \begin{cases} \|e\|_{1, \frac{1}{2}} \leq \frac{1}{\mu_1} \left(\|\mathcal{R}_1(\eta, \zeta, \boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\zeta, \boldsymbol{\tau})\|^2 \right. \\ \quad \left. + \|\mathcal{R}_3(\eta, \zeta, \boldsymbol{\rho})\|^2 + \|\mathcal{R}_4(\eta, \boldsymbol{\rho})\|^2 \right)^{1/2} =: \mathcal{M}_{\|\cdot\|}^{\oplus}(\eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho}), \end{cases}$$

where $e = (y - \eta, p - \zeta) \in (H_{0, per}^{1, \frac{1}{2}}(Q_T))^2$, $\mu_1 = \frac{\min\{\frac{1}{\sqrt{\lambda}}, \nu, \sigma\} \min\{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\}}{\sqrt{1 + 2 \max\{\lambda, \frac{1}{\lambda}\}}}$ and $\boldsymbol{\tau}, \boldsymbol{\rho} \in H(\text{div}, Q_T)$.

The multiharmonic approximation. The desired state $y_d \in L^2(Q_T)$ can be expanded into a Fourier series. Moreover, we choose our approximations η and ζ of the exact state y and adjoint state p , respectively, as well as the vector-valued functions $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ to be truncated Fourier series, e.g.,

$$\begin{aligned} \eta(\mathbf{x}, t) &= \eta_0^c(\mathbf{x}) + \sum_{k=1}^N (\eta_k^c(\mathbf{x}) \cos(k\omega t) + \eta_k^s(\mathbf{x}) \sin(k\omega t)), \\ \boldsymbol{\tau}(\mathbf{x}, t) &= \boldsymbol{\tau}_0^c(\mathbf{x}) + \sum_{k=1}^N (\boldsymbol{\tau}_k^c(\mathbf{x}) \cos(k\omega t) + \boldsymbol{\tau}_k^s(\mathbf{x}) \sin(k\omega t)), \end{aligned}$$

where all Fourier coefficients are at least from the space $L^2(\Omega)$. Hence, we get

$$\begin{aligned}\partial_t \eta(\mathbf{x}, t) &= \sum_{k=1}^N (k\omega \eta_k^s(\mathbf{x}) \cos(k\omega t) - k\omega \eta_k^c(\mathbf{x}) \sin(k\omega t)), \\ \nabla \eta(\mathbf{x}, t) &= \nabla \eta_0^c(\mathbf{x}) + \sum_{k=1}^N (\nabla \eta_k^c(\mathbf{x}) \cos(k\omega t) + \nabla \eta_k^s(\mathbf{x}) \sin(k\omega t)), \\ \operatorname{div} \boldsymbol{\tau}(\mathbf{x}, t) &= \operatorname{div} \boldsymbol{\tau}_0^c(\mathbf{x}) + \sum_{k=1}^N (\operatorname{div} \boldsymbol{\tau}_k^c(\mathbf{x}) \cos(k\omega t) + \operatorname{div} \boldsymbol{\tau}_k^s(\mathbf{x}) \sin(k\omega t)),\end{aligned}$$

and the $L^2(Q_T)$ -norms of the functions \mathcal{R}_1 , \mathcal{R}_2 , \mathcal{R}_3 and \mathcal{R}_4 defined in (29) can be easily computed. Thus, we arrive at

$$\begin{aligned}\|\mathcal{R}_1(\eta, \zeta, \boldsymbol{\tau})\|^2 &= T \|\eta_0^c - \operatorname{div} \boldsymbol{\tau}_0^c - y_{d0}^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N (\|k\omega \sigma \zeta_k^s + \eta_k^c - \operatorname{div} \boldsymbol{\tau}_k^c - y_{dk}^c\|_\Omega^2 \\ &\quad + \| -k\omega \sigma \zeta_k^c + \eta_k^s - \operatorname{div} \boldsymbol{\tau}_k^s - y_{dk}^s\|_\Omega^2) + \frac{T}{2} \sum_{k=N+1}^\infty (\|y_{dk}^c\|_\Omega^2 + \|y_{dk}^s\|_\Omega^2) \\ &= T \|\eta_0^c - \operatorname{div} \boldsymbol{\tau}_0^c - y_{d0}^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N \| -k\omega \sigma \zeta_k^\perp + \boldsymbol{\eta}_k - \mathbf{div} \boldsymbol{\tau}_k - \mathbf{y}_{dk}\|_\Omega^2 \\ &\quad + \frac{T}{2} \sum_{k=N+1}^\infty \|\mathbf{y}_{dk}\|_\Omega^2,\end{aligned}$$

where $\mathbf{div} \boldsymbol{\tau}_k = (\operatorname{div} \boldsymbol{\tau}_k^c, \operatorname{div} \boldsymbol{\tau}_k^s)^T$, and

$$\begin{aligned}\|\mathcal{R}_2(\zeta, \boldsymbol{\tau})\|^2 &= T \|\boldsymbol{\tau}_0^c + \nu \nabla \zeta_0^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N \|\boldsymbol{\tau}_k + \nu \nabla \zeta_k\|_\Omega^2, \\ \|\mathcal{R}_3(\eta, \zeta, \boldsymbol{\rho})\|^2 &= T \|\lambda^{-1} \zeta_0^c - \operatorname{div} \boldsymbol{\rho}_0^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N \| -k\omega \sigma \boldsymbol{\eta}_k^\perp + \lambda^{-1} \zeta_k - \operatorname{div} \boldsymbol{\rho}_k\|_\Omega^2, \\ \|\mathcal{R}_4(\eta, \boldsymbol{\rho})\|^2 &= T \|\boldsymbol{\rho}_0^c - \nu \nabla \eta_0^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N \|\boldsymbol{\rho}_k - \nu \nabla \boldsymbol{\eta}_k\|_\Omega^2.\end{aligned}$$

Remark 2. *The remainder term*

$$(32) \quad \mathcal{E}_N := \frac{T}{2} \sum_{k=N+1}^\infty \|\mathbf{y}_{dk}\|_\Omega^2 = \frac{T}{2} \sum_{k=N+1}^\infty (\|y_{dk}^c\|_\Omega^2 + \|y_{dk}^s\|_\Omega^2)$$

is always computable due to knowledge on the given data y_d .

All the L^2 -norms of \mathcal{R}_1 , \mathcal{R}_2 , \mathcal{R}_3 and \mathcal{R}_4 corresponding to every single mode $k = 0, \dots, N$ are decoupled. We define the functions

$$\begin{aligned}\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \zeta_k, \boldsymbol{\tau}_k) &= -k\omega \sigma \zeta_k^\perp + \boldsymbol{\eta}_k - \mathbf{div} \boldsymbol{\tau}_k - \mathbf{y}_{dk} = (\mathcal{R}_{1k}^c(\eta_k^c, \zeta_k^c, \boldsymbol{\tau}_k^c), \mathcal{R}_{1k}^s(\eta_k^s, \zeta_k^s, \boldsymbol{\tau}_k^s))^T \\ &= (k\omega \sigma \zeta_k^s + \eta_k^c - \operatorname{div} \boldsymbol{\tau}_k^c - y_{dk}^c, -k\omega \sigma \zeta_k^c + \eta_k^s - \operatorname{div} \boldsymbol{\tau}_k^s - y_{dk}^s)^T, \\ \mathcal{R}_{2k}(\zeta_k, \boldsymbol{\tau}_k) &= \boldsymbol{\tau}_k + \nu \nabla \zeta_k = (\mathcal{R}_{2k}^c(\zeta_k^c, \boldsymbol{\tau}_k^c), \mathcal{R}_{2k}^s(\zeta_k^s, \boldsymbol{\tau}_k^s))^T = (\boldsymbol{\tau}_k^c + \nu \nabla \zeta_k^c, \boldsymbol{\tau}_k^s + \nu \nabla \zeta_k^s)^T, \\ \mathcal{R}_{3k}(\boldsymbol{\eta}_k, \zeta_k, \boldsymbol{\rho}_k) &= -k\omega \sigma \boldsymbol{\eta}_k^\perp + \lambda^{-1} \zeta_k - \operatorname{div} \boldsymbol{\rho}_k = (\mathcal{R}_{3k}^c(\eta_k^c, \zeta_k^c, \boldsymbol{\rho}_k^c), \mathcal{R}_{3k}^s(\eta_k^s, \zeta_k^s, \boldsymbol{\rho}_k^s))^T \\ &= (k\omega \sigma \boldsymbol{\eta}_k^s + \lambda^{-1} \zeta_k^c - \operatorname{div} \boldsymbol{\rho}_k^c, -k\omega \sigma \boldsymbol{\eta}_k^c + \lambda^{-1} \zeta_k^s - \operatorname{div} \boldsymbol{\rho}_k^s)^T, \\ \mathcal{R}_{4k}(\boldsymbol{\eta}_k, \boldsymbol{\rho}_k) &= \boldsymbol{\rho}_k - \nu \nabla \boldsymbol{\eta}_k = (\mathcal{R}_{4k}^c(\eta_k^c, \boldsymbol{\rho}_k^c), \mathcal{R}_{4k}^s(\eta_k^s, \boldsymbol{\rho}_k^s))^T = (\boldsymbol{\rho}_k^c - \nu \nabla \boldsymbol{\eta}_k^c, \boldsymbol{\rho}_k^s - \nu \nabla \boldsymbol{\eta}_k^s)^T,\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_{10}^c(\eta_0^c, \boldsymbol{\tau}_0^c) &= \eta_0^c - \operatorname{div} \boldsymbol{\tau}_0^c - y_{d0}^c, & \mathcal{R}_{20}^c(\zeta_0^c, \boldsymbol{\tau}_0^c) &= \boldsymbol{\tau}_0^c + \nu \nabla \zeta_0^c, \\ \mathcal{R}_{30}^c(\zeta_0^c, \boldsymbol{\rho}_0^c) &= \lambda^{-1} \zeta_0^c - \operatorname{div} \boldsymbol{\rho}_0^c, & \mathcal{R}_{40}^c(\eta_0^c, \boldsymbol{\rho}_0^c) &= \boldsymbol{\rho}_0^c - \nu \nabla \eta_0^c.\end{aligned}$$

Corollary 1. *The error majorants $\mathcal{M}_{|\cdot|}^\oplus(\eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho})$ and $\mathcal{M}_{\|\cdot\|}^\oplus(\eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho})$ defined in (30) and (31), respectively, can be represented as*

$$\begin{aligned}\mathcal{M}_{|\cdot|}^\oplus(\eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho}) &= \frac{1}{\tilde{\mu}_1} \left(C_F (T \|\mathcal{R}_{10}^c(\eta_0^c, \boldsymbol{\tau}_0^c)\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N \|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\zeta}_k, \boldsymbol{\tau}_k)\|_\Omega^2 + \mathcal{E}_N)^{1/2} \right. \\ &\quad + (T \|\mathcal{R}_{20}^c(\zeta_0^c, \boldsymbol{\tau}_0^c)\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N \|\mathcal{R}_{2k}(\boldsymbol{\zeta}_k, \boldsymbol{\tau}_k)\|_\Omega^2)^{1/2} \\ &\quad + C_F (T \|\mathcal{R}_{30}^c(\zeta_0^c, \boldsymbol{\rho}_0^c)\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N \|\mathcal{R}_{3k}(\boldsymbol{\eta}_k, \boldsymbol{\zeta}_k, \boldsymbol{\rho}_k)\|_\Omega^2)^{1/2} \\ &\quad \left. + (T \|\mathcal{R}_{40}^c(\eta_0^c, \boldsymbol{\rho}_0^c)\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N \|\mathcal{R}_{4k}(\boldsymbol{\eta}_k, \boldsymbol{\rho}_k)\|_\Omega^2)^{1/2} \right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_{\|\cdot\|}^\oplus(\eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho}) &= \frac{1}{\mu_1} \left(T (\|\mathcal{R}_{10}^c(\eta_0^c, \boldsymbol{\tau}_0^c)\|_\Omega^2 + \|\mathcal{R}_{20}^c(\zeta_0^c, \boldsymbol{\tau}_0^c)\|_\Omega^2 + \|\mathcal{R}_{30}^c(\zeta_0^c, \boldsymbol{\rho}_0^c)\|_\Omega^2 \right. \\ &\quad + \|\mathcal{R}_{40}^c(\eta_0^c, \boldsymbol{\rho}_0^c)\|_\Omega^2) + \frac{T}{2} \sum_{k=1}^N (\|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\zeta}_k, \boldsymbol{\tau}_k)\|_\Omega^2 + \|\mathcal{R}_{2k}(\boldsymbol{\zeta}_k, \boldsymbol{\tau}_k)\|_\Omega^2 \\ &\quad \left. + \|\mathcal{R}_{3k}(\boldsymbol{\eta}_k, \boldsymbol{\zeta}_k, \boldsymbol{\rho}_k)\|_\Omega^2 + \|\mathcal{R}_{4k}(\boldsymbol{\eta}_k, \boldsymbol{\rho}_k)\|_\Omega^2) + \mathcal{E}_N \right)^{1/2},\end{aligned}$$

where $\tilde{\mu}_1 = \frac{\min\{\nu, \sigma\} \min\{\lambda, \frac{1}{\lambda}\}}{\sqrt{2}}$ and $\mu_1 = \frac{\min\{\frac{1}{\sqrt{\lambda}}, \nu, \sigma\} \min\{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\}}{\sqrt{1+2 \max\{\lambda, \frac{1}{\lambda}\}}}$.

Remark 3. *Let y_d has a multiharmonic representation, i.e.,*

$$(33) \quad y_d(\mathbf{x}, t) = y_{d0}^c(\mathbf{x}) + \sum_{k=1}^{N_d} (f_k^c(\mathbf{x}) \cos(k\omega t) + y_{dk}^s(\mathbf{x}) \sin(k\omega t)),$$

where $N_d \in \mathbb{N}$ is defined by y_d . If $N \geq N_d$, then (η, ζ) is the exact solution of problem (17) and $(\boldsymbol{\tau}, \boldsymbol{\rho})$ is the exact flux if and only if the error majorants vanish, i.e.,

$$\mathcal{R}_{j0}^c = 0 \quad \text{and} \quad \mathcal{R}_{jk} = 0 \quad \forall k = 1, \dots, N_d, \quad \forall j \in \{1, 2, 3, 4\}.$$

Indeed, let the error majorants vanish. Then, $\eta_0^c - \operatorname{div} \boldsymbol{\tau}_0^c = y_{d0}^c$, $\boldsymbol{\tau}_0^c = -\nu \nabla \zeta_0^c$, $\lambda^{-1} \zeta_0^c - \operatorname{div} \boldsymbol{\rho}_0^c = 0$, $\boldsymbol{\rho}_0^c = \nu \nabla \eta_0^c$ and

$$\begin{aligned}k\omega \sigma \zeta_k^s + \eta_k^c - \operatorname{div} \boldsymbol{\tau}_k^c &= y_{dk}^c, & -k\omega \sigma \zeta_k^c + \eta_k^s - \operatorname{div} \boldsymbol{\tau}_k^s &= y_{dk}^s, \\ k\omega \sigma \eta_k^s + \lambda^{-1} \zeta_k^c - \operatorname{div} \boldsymbol{\rho}_k^c &= 0, & -k\omega \sigma \eta_k^c + \lambda^{-1} \zeta_k^s - \operatorname{div} \boldsymbol{\rho}_k^s &= 0, \\ \boldsymbol{\tau}_k^c &= -\nu \nabla \zeta_k^c, & \boldsymbol{\tau}_k^s &= -\nu \nabla \zeta_k^s, & \boldsymbol{\rho}_k^c &= \nu \nabla \eta_k^c, & \boldsymbol{\rho}_k^s &= \nu \nabla \eta_k^s,\end{aligned}$$

for all $k = 1, \dots, N_d$, so that collecting the N_d+1 harmonics leads to multiharmonic representations for η , ζ , $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ of the form (33) satisfying the equations $\sigma \partial_t \eta - \operatorname{div} \boldsymbol{\rho} + \lambda^{-1} \zeta = 0$, $\boldsymbol{\rho} = \nu \nabla \eta$, $\sigma \partial_t \zeta - \operatorname{div} \boldsymbol{\tau} + \eta = y_d$, $\boldsymbol{\tau} = -\nu \nabla \zeta$. Since η and ζ also meet the boundary conditions, we conclude that $\eta = y$ and $\zeta = p$.

Error majorant of the second type. In this section, we deduce another upper bound of the error $e := (y - \eta, p - \zeta)$, which is valid for approximations that are less regular with respect to the time, i.e., $\eta, \zeta \in H_{0,per}^{1, \frac{1}{2}}(Q_T)$. In fact, our multiharmonic finite element approximations y_{Nh} and p_{Nh} as η and ζ , respectively, are, of course, more regular in time, but the abstract functional

a posteriori error estimates, which are obtained in this section, can be used in a more general setting. Let us again consider the functional

$$\mathcal{F}_{(\eta,\zeta)}(v,q) = \int_{Q_T} \left(y_d v - \eta v + \nu(\mathbf{x}) \nabla \zeta \cdot \nabla v - \sigma(\mathbf{x}) \partial_t^{1/2} \zeta \partial_t^{1/2} v^\perp - \nu(\mathbf{x}) \nabla \eta \cdot \nabla q - \sigma(\mathbf{x}) \partial_t^{1/2} \eta \partial_t^{1/2} q^\perp - \lambda^{-1} \zeta q \right) d\mathbf{x} dt$$

defined for all $v, q \in H_{0,per}^{1,\frac{1}{2}}(Q_T)$. In addition to the vector-valued functions $\boldsymbol{\tau}, \boldsymbol{\rho} \in H(\operatorname{div}, Q_T)$, we introduce the functions $\xi, \chi \in H_{per}^{0,\frac{1}{2}}(Q_T)$, which fulfill the identity (14). We rearrange the functional $\mathcal{F}_{(\eta,\zeta)}(v,q)$ in the following way:

$$\begin{aligned} \mathcal{F}_{(\eta,\zeta)}(v,q) &= \int_{Q_T} \left((y_d - \eta + \operatorname{div} \boldsymbol{\tau} + \sigma \partial_t^{1/2} \xi^\perp) v + (\sigma(\xi - \partial_t^{1/2} \zeta)) \partial_t^{1/2} v^\perp \right. \\ &\quad \left. + (\boldsymbol{\tau} + \nu \nabla \zeta) \cdot \nabla v + (\boldsymbol{\rho} - \nu \nabla \eta) \cdot \nabla q \right. \\ &\quad \left. + (\sigma(\chi - \partial_t^{1/2} \eta)) \partial_t^{1/2} q^\perp + (\operatorname{div} \boldsymbol{\rho} + \sigma \partial_t^{1/2} \chi^\perp - \lambda^{-1} \zeta) q \right) d\mathbf{x} dt \end{aligned}$$

for all $v, q \in H_{0,per}^{1,\frac{1}{2}}(Q_T)$. Let

$$\begin{aligned} \mathcal{R}_1(\boldsymbol{\tau}, \xi, \eta) &= y_d - \eta + \operatorname{div} \boldsymbol{\tau} + \sigma \partial_t^{1/2} \xi^\perp, & \mathcal{R}_4(\boldsymbol{\rho}, \chi, \zeta) &= \operatorname{div} \boldsymbol{\rho} + \sigma \partial_t^{1/2} \chi^\perp - \lambda^{-1} \zeta, \\ \mathcal{R}_2(\boldsymbol{\tau}, \zeta) &= \boldsymbol{\tau} + \nu \nabla \zeta, & \mathcal{R}_5(\boldsymbol{\rho}, \eta) &= \boldsymbol{\rho} - \nu \nabla \eta, \\ \mathcal{R}_3(\xi, \zeta) &= \sigma(\xi - \partial_t^{1/2} \zeta), & \mathcal{R}_6(\chi, \eta) &= \sigma(\chi - \partial_t^{1/2} \eta). \end{aligned}$$

Using the Cauchy-Schwarz inequality, the functional $\mathcal{F}_{(\eta,\zeta)}(v,q)$ can be estimated from above as follows

$$\begin{aligned} \mathcal{F}_{(\eta,\zeta)}(v,q) &\leq \|\mathcal{R}_1(\boldsymbol{\tau}, \xi, \eta)\| \|v\| + \|\mathcal{R}_2(\boldsymbol{\tau}, \zeta)\| \|\nabla v\| + \|\mathcal{R}_3(\xi, \zeta)\| \|\partial_t^{1/2} v\| \\ &\quad + \|\mathcal{R}_4(\boldsymbol{\rho}, \chi, \zeta)\| \|q\| + \|\mathcal{R}_5(\boldsymbol{\rho}, \eta)\| \|\nabla q\| + \|\mathcal{R}_6(\chi, \eta)\| \|\partial_t^{1/2} q\| \\ &\leq \left(\|\mathcal{R}_1(\boldsymbol{\tau}, \xi, \eta)\|^2 + \|\mathcal{R}_2(\boldsymbol{\tau}, \zeta)\|^2 + \|\mathcal{R}_3(\xi, \zeta)\|^2 \right. \\ &\quad \left. + \|\mathcal{R}_4(\boldsymbol{\rho}, \chi, \zeta)\|^2 + \|\mathcal{R}_5(\boldsymbol{\rho}, \eta)\|^2 + \|\mathcal{R}_6(\chi, \eta)\|^2 \right)^{1/2} \|(v,q)\|_{1,\frac{1}{2}}. \end{aligned}$$

Altogether, we obtain the upper bound

$$(34) \quad \left\{ \begin{array}{l} \sup_{0 \neq (v,q) \in (H_{0,per}^{1,\frac{1}{2}}(Q_T))^2} \frac{\mathcal{F}_{(\eta,\zeta)}(v,q)}{\|(v,q)\|_{1,\frac{1}{2}}} \leq \left(\|\mathcal{R}_1(\boldsymbol{\tau}, \xi, \eta)\|^2 + \|\mathcal{R}_2(\boldsymbol{\tau}, \zeta)\|^2 + \|\mathcal{R}_3(\xi, \zeta)\|^2 \right. \\ \left. + \|\mathcal{R}_4(\boldsymbol{\rho}, \chi, \zeta)\|^2 + \|\mathcal{R}_5(\boldsymbol{\rho}, \eta)\|^2 + \|\mathcal{R}_6(\chi, \eta)\|^2 \right)^{1/2}. \end{array} \right.$$

From (24) follows that

$$\|e\|_{1,\frac{1}{2}} \leq \frac{1}{\mu_1} \sup_{0 \neq (v,q) \in (H_{0,per}^{1,\frac{1}{2}}(Q_T))^2} \frac{\mathcal{B}(e, (v,q))}{\|(v,q)\|_{1,\frac{1}{2}}} = \frac{1}{\mu_1} \sup_{0 \neq (v,q) \in (H_{0,per}^{1,\frac{1}{2}}(Q_T))^2} \frac{\mathcal{F}_{(\eta,\zeta)}(v,q)}{\|(v,q)\|_{1,\frac{1}{2}}},$$

which leads together with (34) to the following result:

Theorem 3. *Let $\eta, \zeta \in H_{0,per}^{1,\frac{1}{2}}(Q_T)$ and the bilinear form $\mathcal{B}(\cdot, \cdot)$ satisfy (24). Then,*

$$(35) \quad \left\{ \begin{array}{l} \|e\|_{1,\frac{1}{2}} \leq \frac{1}{\mu_1} \left(\|\mathcal{R}_1(\boldsymbol{\tau}, \xi, \eta)\|^2 + \|\mathcal{R}_2(\boldsymbol{\tau}, \zeta)\|^2 + \|\mathcal{R}_3(\xi, \zeta)\|^2 \right. \\ \left. + \|\mathcal{R}_4(\boldsymbol{\rho}, \chi, \zeta)\|^2 + \|\mathcal{R}_5(\boldsymbol{\rho}, \eta)\|^2 + \|\mathcal{R}_6(\chi, \eta)\|^2 \right)^{1/2} \\ =: \mathcal{M}_{\|\cdot\|}^\oplus(\eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho}, \xi, \chi), \end{array} \right.$$

where $\boldsymbol{\tau}, \boldsymbol{\rho} \in H(\operatorname{div}, Q_T)$, $\xi, \chi \in H_{per}^{0,\frac{1}{2}}(Q_T)$ and $\mu_1 = \frac{\min\{\frac{1}{\sqrt{\lambda}}, \nu, \sigma\} \min\{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\}}{\sqrt{1+2 \max\{\lambda, \frac{1}{\lambda}\}}}$.

Remark 4. If $\mathcal{R}_1(\boldsymbol{\tau}, \xi, \eta) = 0$, $\mathcal{R}_2(\boldsymbol{\tau}, \zeta) = 0$, $\mathcal{R}_3(\xi, \zeta) = 0$, $\mathcal{R}_4(\boldsymbol{\rho}, \chi, \zeta) = 0$, $\mathcal{R}_5(\boldsymbol{\rho}, \eta) = 0$ and $\mathcal{R}_6(\chi, \eta) = 0$, then

$$\begin{aligned} -\sigma \partial_t^{1/2} \xi^\perp - \operatorname{div} \boldsymbol{\tau} + \eta &= y_d, & -\sigma \partial_t^{1/2} \chi^\perp - \operatorname{div} \boldsymbol{\rho} + \lambda^{-1} \zeta &= 0, \\ \boldsymbol{\tau} &= -\nu \nabla \zeta, & \boldsymbol{\rho} &= \nu \nabla \eta, \\ \xi &= \partial_t^{1/2} \zeta, & \chi &= \partial_t^{1/2} \eta. \end{aligned}$$

Since η and ζ satisfy the Dirichlet condition on Σ_T , (η, ζ) is the solution. In other words, the majorant $\mathcal{M}_{\|\cdot\|}^\oplus(\eta, \zeta, \boldsymbol{\tau}, \boldsymbol{\rho}, \xi, \chi)$ vanishes if and only if (η, ζ) is the exact solution, $(\boldsymbol{\tau}, \boldsymbol{\rho})$ the exact flux and (ξ, χ) the exact half time derivative. Moreover, if $\eta, \zeta \in H_{0,per}^{1,1/2}(Q_T)$, we derive the optimality system

$$\begin{aligned} \sigma \partial_t \zeta + \operatorname{div}(\nu \nabla \zeta) + \eta &= y_d & \text{in } Q_T, \\ \sigma \partial_t \eta - \operatorname{div}(\nu \nabla \eta) + \lambda^{-1} \zeta &= 0 & \text{in } Q_T, \end{aligned}$$

in the weak sense.

Similar results to the ones obtained in Theorem 3 can be obtained using the following weighted $H^{1, \frac{1}{2}}$ -norms characterized as $\|\cdot\|_V$:

$$\begin{aligned} \|y\|_V &:= \left(\|y\|^2 + \sqrt{\lambda}(\nu \nabla y, \nabla y) + \sqrt{\lambda}(\sigma \partial_t^{1/2} y, \partial_t^{1/2} y) \right)^{1/2}, \\ \|(y, p)\|_V &:= \left(\|y\|_V^2 + \lambda^{-1} \|p\|_V^2 \right)^{1/2}. \end{aligned}$$

By using these V -norms, we can get rid of all parameters from the inf-sup and sup-sup constants. The proof of the following lemma can be found in [32].

Lemma 4. The space-time bilinear form $\mathcal{B}(\cdot, \cdot)$ defined by (18) meets the following inf-sup and sup-sup conditions:

$$(36) \quad \hat{\mu}_1 \|(y, p)\|_V \leq \sup_{0 \neq (v, q) \in (H_{0,per}^{1, \frac{1}{2}}(Q_T))^2} \frac{\mathcal{B}((y, p), (v, q))}{\|(v, q)\|_V} \leq \hat{\mu}_2 \|(y, p)\|_V$$

for all $y, p \in H_{0,per}^{1, \frac{1}{2}}(Q_T)$ with $\hat{\mu}_1 = 1/\sqrt{3}$ and $\hat{\mu}_2 = 1$.

In order to find an upper bound of

$$(37) \quad \sup_{0 \neq (v, q) \in (H_{0,per}^{1, \frac{1}{2}}(Q_T))^2} \frac{\mathcal{F}_{(\eta, \zeta)}(v, q)}{\|(v, q)\|_V},$$

we introduce two vector-valued functions $\tilde{\boldsymbol{\tau}}$ and $\tilde{\boldsymbol{\rho}}$ satisfying the identity

$$\int_{\Omega} \operatorname{div}(\nu \tilde{\boldsymbol{\tau}}) v \, d\mathbf{x} = - \int_{\Omega} (\nu \tilde{\boldsymbol{\tau}}) \cdot \nabla v \, d\mathbf{x} \quad \forall v \in H_0^1(\Omega).$$

Hence, $(\nu \tilde{\boldsymbol{\tau}}), (\nu \tilde{\boldsymbol{\rho}}) \in H(\operatorname{div}, Q_T)$ and the functional $\mathcal{F}_{(\eta, \zeta)}(v, q)$ can be rearranged as follows

$$\begin{aligned} \mathcal{F}_{(\eta, \zeta)}(v, q) &= \int_{Q_T} \left((y_d - \eta + \operatorname{div}(\nu \tilde{\boldsymbol{\tau}}) + \sigma \partial_t^{1/2} \xi^\perp) v + \sigma (\xi - \partial_t^{1/2} \zeta) \partial_t^{1/2} v^\perp + \nu (\tilde{\boldsymbol{\tau}} + \nabla \zeta) \cdot \nabla v \right. \\ &\quad \left. + \nu (\tilde{\boldsymbol{\rho}} - \nabla \eta) \cdot \nabla q + \sigma (\chi - \partial_t^{1/2} \eta) \partial_t^{1/2} q^\perp + (\operatorname{div}(\nu \tilde{\boldsymbol{\rho}}) + \sigma \partial_t^{1/2} \chi^\perp - \lambda^{-1} \zeta) q \right) d\mathbf{x} dt \end{aligned}$$

for all $v, q \in H_{0,per}^{1, \frac{1}{2}}(Q_T)$. Using the Cauchy-Schwarz inequality as well as a proper weighting with λ , we can estimate the functional $\mathcal{F}_{(\eta, \zeta)}(v, q)$ from above as follows

$$\begin{aligned} \mathcal{F}_{(\eta, \zeta)}(v, q) &\leq \|y_d - \eta + \operatorname{div}(\nu \tilde{\boldsymbol{\tau}}) + \sigma \partial_t^{1/2} \xi^\perp\| \|v\| + (\sigma (\xi - \partial_t^{1/2} \zeta), \xi - \partial_t^{1/2} \zeta)^{1/2} (\sigma \partial_t^{1/2} v, \partial_t^{1/2} v)^{1/2} \\ &\quad + (\nu (\tilde{\boldsymbol{\tau}} + \nabla \zeta), \tilde{\boldsymbol{\tau}} + \nabla \zeta)^{1/2} (\nu \nabla v, \nabla v)^{1/2} + (\nu (\tilde{\boldsymbol{\rho}} - \nabla \eta), \tilde{\boldsymbol{\rho}} - \nabla \eta)^{1/2} (\nu \nabla q, \nabla q)^{1/2} \\ &\quad + (\sigma (\chi - \partial_t^{1/2} \eta), \chi - \partial_t^{1/2} \eta)^{1/2} (\sigma \partial_t^{1/2} q, \partial_t^{1/2} q)^{1/2} + \|\operatorname{div}(\nu \tilde{\boldsymbol{\rho}}) + \sigma \partial_t^{1/2} \chi^\perp - \lambda^{-1} \zeta\| \|q\| \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_{(\eta, \zeta)}(v, q) &\leq \left(\|\mathcal{R}_1(\tilde{\tau}, \xi, \eta)\|^2 + \frac{1}{\sqrt{\lambda}}(\nu \mathcal{R}_2(\tilde{\tau}, \zeta), \mathcal{R}_2(\tilde{\tau}, \zeta)) + \frac{1}{\sqrt{\lambda}}(\sigma \mathcal{R}_3(\xi, \zeta), \mathcal{R}_3(\xi, \zeta)) \right. \\ &\quad \left. + \lambda \|\mathcal{R}_4(\tilde{\rho}, \chi, \zeta)\|^2 + \sqrt{\lambda}(\nu \mathcal{R}_5(\tilde{\rho}, \eta), \mathcal{R}_5(\tilde{\rho}, \eta)) + \sqrt{\lambda}(\sigma \mathcal{R}_6(\chi, \eta), \mathcal{R}_6(\chi, \eta)) \right)^{1/2} \|(v, q)\|_V, \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_1(\tilde{\tau}, \xi, \eta) &= y_d - \eta + \operatorname{div}(\nu \tilde{\tau}) + \sigma \partial_t^{1/2} \xi^\perp, & \mathcal{R}_4(\tilde{\rho}, \chi, \zeta) &= \operatorname{div}(\nu \tilde{\rho}) + \sigma \partial_t^{1/2} \chi^\perp - \lambda^{-1} \zeta, \\ \mathcal{R}_2(\tilde{\tau}, \zeta) &= \tilde{\tau} + \nabla \zeta, & \mathcal{R}_5(\tilde{\rho}, \eta) &= \tilde{\rho} - \nabla \eta, & \mathcal{R}_3(\xi, \zeta) &= \xi - \partial_t^{1/2} \zeta, & \mathcal{R}_6(\chi, \eta) &= \chi - \partial_t^{1/2} \eta. \end{aligned}$$

From (36) follows that

$$\|e\|_V \leq \frac{1}{\hat{\mu}_1} \sup_{0 \neq (v, q) \in (H_{0, \text{per}}^{1, \frac{1}{2}}(Q_T))^2} \frac{\mathcal{B}(e, (v, q))}{\|(v, q)\|_V} = \frac{1}{\hat{\mu}_1} \sup_{0 \neq (v, q) \in (H_{0, \text{per}}^{1, \frac{1}{2}}(Q_T))^2} \frac{\mathcal{F}_{(\eta, \zeta)}(v, q)}{\|(v, q)\|_V}$$

with $\hat{\mu}_1 = 1/\sqrt{3}$, which leads to the following a posteriori error result for the V -norm:

Theorem 4. *Let $\eta, \zeta \in H_{0, \text{per}}^{1, \frac{1}{2}}(Q_T)$ and the bilinear form $\mathcal{B}(\cdot, \cdot)$ defined by (18) satisfy (36). Then,*

$$(38) \quad \left\{ \begin{aligned} \|e\|_V &\leq \frac{1}{\hat{\mu}_1} \left(\|\mathcal{R}_1(\tilde{\tau}, \xi, \eta)\|^2 + \frac{1}{\sqrt{\lambda}}(\nu \mathcal{R}_2(\tilde{\tau}, \zeta), \mathcal{R}_2(\tilde{\tau}, \zeta)) + \frac{1}{\sqrt{\lambda}}(\sigma \mathcal{R}_3(\xi, \zeta), \mathcal{R}_3(\xi, \zeta)) \right. \\ &\quad \left. + \lambda \|\mathcal{R}_4(\tilde{\rho}, \chi, \zeta)\|^2 + \sqrt{\lambda}(\nu \mathcal{R}_5(\tilde{\rho}, \eta), \mathcal{R}_5(\tilde{\rho}, \eta)) + \sqrt{\lambda}(\sigma \mathcal{R}_6(\chi, \eta), \mathcal{R}_6(\chi, \eta)) \right)^{1/2} \\ &=: \mathcal{M}_{\|\cdot\|_V}^\oplus(\eta, \zeta, \tilde{\tau}, \tilde{\rho}, \xi, \chi), \end{aligned} \right.$$

where $(\nu \tilde{\tau}), (\nu \tilde{\rho}) \in H(\operatorname{div}, Q_T)$, $\xi, \chi \in H_{\text{per}}^{0, \frac{1}{2}}(Q_T)$ and $\hat{\mu}_1 = 1/\sqrt{3}$.

Now we briefly discuss a posteriori estimates for Fourier modes in the context of V -norms. They are derived by the same arguments as before. Therefore, we present only the results and comments related to specific features of this case. We consider the problems (19), i.e.,

$$(39) \quad \left\{ \begin{aligned} &\int_{\Omega} ((\mathbf{y}_k - \boldsymbol{\eta}_k) \cdot \mathbf{v}_k - \nu(\mathbf{x}) \nabla(\mathbf{p}_k - \boldsymbol{\zeta}_k) \cdot \nabla \mathbf{v}_k + k\omega \sigma(\mathbf{x})(\mathbf{p}_k - \boldsymbol{\zeta}_k) \cdot \mathbf{v}_k^\perp \\ &\quad + \nu(\mathbf{x}) \nabla(\mathbf{y}_k - \boldsymbol{\eta}_k) \cdot \nabla \mathbf{q}_k + k\omega \sigma(\mathbf{x})(\mathbf{y}_k - \boldsymbol{\eta}_k) \cdot \mathbf{q}_k^\perp + \lambda^{-1}(\mathbf{p}_k - \boldsymbol{\zeta}_k) \cdot \mathbf{q}_k) d\mathbf{x} \\ &= \int_{\Omega} (\mathbf{y}_{d_k} \cdot \mathbf{v}_k - \boldsymbol{\eta}_k \cdot \mathbf{v}_k + \nu(\mathbf{x}) \nabla \boldsymbol{\zeta}_k \cdot \nabla \mathbf{v}_k - k\omega \sigma(\mathbf{x}) \boldsymbol{\zeta}_k \cdot \mathbf{v}_k^\perp \\ &\quad - \nu(\mathbf{x}) \nabla \boldsymbol{\eta}_k \cdot \nabla \mathbf{q}_k - k\omega \sigma(\mathbf{x}) \boldsymbol{\eta}_k \cdot \mathbf{q}_k^\perp - \lambda^{-1} \boldsymbol{\zeta}_k \cdot \mathbf{q}_k) d\mathbf{x}, \end{aligned} \right.$$

for every single mode $k = 1, \dots, N$, and, in the case $k = 0$, we obtain the variational problem (20), i.e.,

$$(40) \quad \left\{ \begin{aligned} &\int_{\Omega} ((y_0^c - \eta_0^c) v_0^c - \nu(\mathbf{x}) \nabla(p_0^c - \zeta_0^c) \cdot \nabla v_0^c + \nu(\mathbf{x}) \nabla(y_0^c - \eta_0^c) \cdot \nabla q_0^c + \lambda^{-1}(p_0^c - \zeta_0^c) q_0^c) d\mathbf{x} \\ &= \int_{\Omega} (y_{d_0}^c v_0^c - \eta_0^c v_0^c + \nu(\mathbf{x}) \nabla \zeta_0^c \cdot \nabla v_0^c - \nu(\mathbf{x}) \nabla \eta_0^c \cdot \nabla q_0^c - \lambda^{-1} \zeta_0^c q_0^c) d\mathbf{x}. \end{aligned} \right.$$

We define the left hand sides of (39) and (40) by

$$\mathcal{B}_k((\mathbf{y}_k - \boldsymbol{\eta}_k, \mathbf{p}_k - \boldsymbol{\zeta}_k), (\mathbf{v}_k, \mathbf{q}_k)) \quad \text{and} \quad \mathcal{B}_0((y_0^c - \eta_0^c, p_0^c - \zeta_0^c), (v_0^c, q_0^c)),$$

respectively. Let us start with the case $k = 1, \dots, N$ and introduce the V -norm corresponding to the Fourier modes as follows

$$\|\mathbf{y}_k\|_V^2 := \sqrt{\lambda}(\nu \nabla \mathbf{y}_k, \nabla \mathbf{y}_k)_\Omega + k\omega \sqrt{\lambda}(\sigma \mathbf{y}_k, \mathbf{y}_k)_\Omega + \|\mathbf{y}_k\|_\Omega^2$$

and

$$\|(\mathbf{y}_k, \mathbf{p}_k)\|_V^2 := \|\mathbf{y}_k\|_V^2 + \lambda^{-1} \|\mathbf{p}_k\|_V^2,$$

where the following inf-sup condition holds:

$$(41) \quad \sup_{0 \neq (\mathbf{v}_k, \mathbf{q}_k) \in (H_0^1(\Omega))^4} \frac{\mathcal{B}_k(\mathbf{e}_k, (\mathbf{v}_k, \mathbf{q}_k))}{\|(\mathbf{v}_k, \mathbf{q}_k)\|_V} \geq \hat{c} \|\mathbf{e}_k\|_V$$

with the parameter-independent constant $\hat{c} = 1/\sqrt{3}$ and $\mathbf{e}_k := (\mathbf{y}_k - \boldsymbol{\eta}_k, \mathbf{p}_k - \boldsymbol{\zeta}_k)^T$. In addition to $\tilde{\boldsymbol{\tau}}_k = (\tilde{\boldsymbol{\tau}}_k^c, \tilde{\boldsymbol{\tau}}_k^s)^T$ and $\tilde{\boldsymbol{\rho}}_k = (\tilde{\boldsymbol{\rho}}_k^c, \tilde{\boldsymbol{\rho}}_k^s)^T$ with $(\nu \tilde{\boldsymbol{\tau}}_k^c), (\nu \tilde{\boldsymbol{\tau}}_k^s), (\nu \tilde{\boldsymbol{\rho}}_k^c), (\nu \tilde{\boldsymbol{\rho}}_k^s) \in H(\operatorname{div}, \Omega)$, we introduce the functions $\boldsymbol{\xi}_k = (\boldsymbol{\xi}_k^c, \boldsymbol{\xi}_k^s)^T$, $\boldsymbol{\chi}_k = (\boldsymbol{\chi}_k^c, \boldsymbol{\chi}_k^s)^T \in (L^2(\Omega))^2$, which fulfill both the simple orthogonality relation

$$\int_{\Omega} k\omega \sigma(\mathbf{x}) \boldsymbol{\xi}_k \cdot \mathbf{v}^\perp d\mathbf{x} = - \int_{\Omega} k\omega \sigma(\mathbf{x}) \boldsymbol{\xi}_k^\perp \cdot \mathbf{v} d\mathbf{x} \quad \forall \mathbf{v} \in (L^2(\Omega))^2.$$

Let

$$\begin{aligned} \mathcal{R}_{1k} &= \mathcal{R}_{1k}(\boldsymbol{\eta}_k, \tilde{\boldsymbol{\tau}}_k, \boldsymbol{\xi}_k) = \mathbf{y}_{d_k} - \boldsymbol{\eta}_k + \operatorname{div}(\nu \tilde{\boldsymbol{\tau}}_k) + k\omega \sigma \boldsymbol{\xi}_k^\perp, \\ \mathcal{R}_{4k} &= \mathcal{R}_{4k}(\boldsymbol{\zeta}_k, \tilde{\boldsymbol{\rho}}_k, \boldsymbol{\chi}_k) = \operatorname{div}(\nu \tilde{\boldsymbol{\rho}}_k) + k\omega \sigma \boldsymbol{\chi}_k^\perp - \lambda^{-1} \boldsymbol{\zeta}_k, \\ \mathcal{R}_{2k} &= \mathcal{R}_{2k}(\boldsymbol{\zeta}_k, \tilde{\boldsymbol{\tau}}_k) = \tilde{\boldsymbol{\tau}}_k + \nabla \boldsymbol{\zeta}_k, & \mathcal{R}_{5k} &= \mathcal{R}_{5k}(\boldsymbol{\eta}_k, \tilde{\boldsymbol{\rho}}_k) = \tilde{\boldsymbol{\rho}}_k - \nabla \boldsymbol{\eta}_k, \\ \mathcal{R}_{3k} &= \mathcal{R}_{3k}(\boldsymbol{\zeta}_k, \boldsymbol{\xi}_k) = \boldsymbol{\xi}_k - \boldsymbol{\zeta}_k, & \mathcal{R}_{6k} &= \mathcal{R}_{6k}(\boldsymbol{\eta}_k, \boldsymbol{\chi}_k) = \boldsymbol{\chi}_k - \boldsymbol{\eta}_k. \end{aligned}$$

By arguments similar to those used for proving the theorems before, we deduce the following upper bounds for every single mode $k = 1, \dots, N$ with the V -norm:

Theorem 5. *Let $\boldsymbol{\eta}_k, \boldsymbol{\zeta}_k \in (H_0^1(\Omega))^2$ and the bilinear form $\mathcal{B}_k(\cdot, \cdot)$ in (39) satisfy (41). Then,*

$$(42) \quad \left\{ \begin{aligned} \|\mathbf{e}_k\|_V &\leq \frac{1}{\hat{c}} \left(\|\mathcal{R}_{1k}\|_{\Omega}^2 + \frac{1}{\sqrt{\lambda}} (\nu \mathcal{R}_{2k}, \mathcal{R}_{2k})_{\Omega} + \frac{1}{\sqrt{\lambda}} k\omega (\sigma \mathcal{R}_{3k}, \mathcal{R}_{3k})_{\Omega} \right. \\ &\quad \left. + \lambda \|\mathcal{R}_{4k}\|_{\Omega}^2 + \sqrt{\lambda} (\nu \mathcal{R}_{5k}, \mathcal{R}_{5k})_{\Omega} + \sqrt{\lambda} k\omega (\sigma \mathcal{R}_{6k}, \mathcal{R}_{6k})_{\Omega} \right)^{1/2} \\ &=: \mathcal{M}_{\|\cdot\|_V}^{\oplus k}(\boldsymbol{\eta}_k, \boldsymbol{\zeta}_k, \tilde{\boldsymbol{\tau}}_k, \tilde{\boldsymbol{\rho}}_k, \boldsymbol{\xi}_k, \boldsymbol{\chi}_k), \end{aligned} \right.$$

where $(\nu \tilde{\boldsymbol{\tau}}_k), (\nu \tilde{\boldsymbol{\rho}}_k) \in H(\operatorname{div}, \Omega)^2$, $\boldsymbol{\xi}_k, \boldsymbol{\chi}_k \in (L^2(\Omega))^2$ and $\hat{c} = 1/\sqrt{3}$.

In the case $k = 0$, we define the V -norm as follows:

$$\|(y, p)\|_V^2 = \|y\|_{\Omega}^2 + \sqrt{\lambda} (\nu \nabla y, \nabla y)_{\Omega} + \lambda^{-1} (\|p\|_{\Omega}^2 + \sqrt{\lambda} (\nu \nabla p, \nabla p)_{\Omega}).$$

So, the following inf-sup condition holds:

$$(43) \quad \sup_{0 \neq (v_0^c, q_0^c) \in (H_0^1(\Omega))^2} \frac{\mathcal{B}_0(e_0^c, (v_0^c, q_0^c))}{\|(v_0^c, q_0^c)\|_V} \geq \hat{c} \|e_0^c\|_V$$

with the inf-sup constant $\hat{c} = 1/\sqrt{2}$ and $e_0^c := (y_0^c - \eta_0^c, p_0^c - \zeta_0^c)^T$. Again, we introduce the vector-valued functions $\tilde{\boldsymbol{\tau}}_0^c$ and $\tilde{\boldsymbol{\rho}}_0^c$ with $(\nu \tilde{\boldsymbol{\tau}}_0^c), (\nu \tilde{\boldsymbol{\rho}}_0^c) \in H(\operatorname{div}, \Omega)$, and define

$$\begin{aligned} \mathcal{R}_{10}^c &= \mathcal{R}_{10}^c(\eta_0^c, \boldsymbol{\tau}_0^c) = y_{d_0}^c - \eta_0^c + \operatorname{div}(\nu \tilde{\boldsymbol{\tau}}_0^c), & \mathcal{R}_{20}^c &= \mathcal{R}_{20}^c(\zeta_0^c, \boldsymbol{\tau}_0^c) = \tilde{\boldsymbol{\tau}}_0^c + \nabla \zeta_0^c, \\ \mathcal{R}_{30}^c &= \mathcal{R}_{30}^c(\eta_0^c, \boldsymbol{\rho}_0^c) = \tilde{\boldsymbol{\rho}}_0^c - \nabla \eta_0^c, & \mathcal{R}_{40}^c &= \mathcal{R}_{40}^c(\zeta_0^c, \boldsymbol{\rho}_0^c) = \operatorname{div}(\nu \tilde{\boldsymbol{\rho}}_0^c) - \lambda^{-1} \zeta_0^c. \end{aligned}$$

Finally, we arrive at the following upper bounds in the case $k = 0$ for the V -norm:

Theorem 6. *Let $\eta_0^c, \zeta_0^c \in H_0^1(\Omega)$ and the bilinear form $\mathcal{B}_0(\cdot, \cdot)$ in (40) satisfy (43). Then,*

$$(44) \quad \left\{ \begin{aligned} \|e_0^c\|_V &\leq \frac{1}{\hat{c}} \left(\|\mathcal{R}_{10}^c(\eta_0^c, \boldsymbol{\tau}_0^c)\|_{\Omega}^2 + \frac{1}{\sqrt{\lambda}} (\nu \mathcal{R}_{20}^c(\zeta_0^c, \boldsymbol{\tau}_0^c), \mathcal{R}_{20}^c(\zeta_0^c, \boldsymbol{\tau}_0^c))_{\Omega} \right. \\ &\quad \left. + \sqrt{\lambda} (\nu \mathcal{R}_{30}^c(\eta_0^c, \tilde{\boldsymbol{\rho}}_0^c), \mathcal{R}_{30}^c(\eta_0^c, \tilde{\boldsymbol{\rho}}_0^c))_{\Omega} + \lambda \|\mathcal{R}_{40}^c(\zeta_0^c, \tilde{\boldsymbol{\rho}}_0^c)\|_{\Omega}^2 \right)^{1/2} \\ &=: \mathcal{M}_{\|\cdot\|_V}^{\oplus 0}(\eta_0^c, \zeta_0^c, \tilde{\boldsymbol{\tau}}_0^c, \tilde{\boldsymbol{\rho}}_0^c), \end{aligned} \right.$$

where $(\nu \tilde{\boldsymbol{\tau}}_0^c), (\nu \tilde{\boldsymbol{\rho}}_0^c) \in H(\operatorname{div}, \Omega)$ and $\hat{c} = 1/\sqrt{2}$.

Finally, we present the error majorant $\mathcal{M}_{\|\cdot\|_V}^{\oplus}(\boldsymbol{\eta}, \boldsymbol{\zeta}, \tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\rho}}, \boldsymbol{\xi}, \boldsymbol{\chi})$ of Theorem 4 in the Fourier space as follows:

Corollary 2. *The error majorant $\mathcal{M}_{\|\cdot\|_V}^{\oplus}(\eta, \zeta, \tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\rho}}, \xi, \chi)$ is given by*

$$\begin{aligned} \mathcal{M}_{\|\cdot\|_V}^{\oplus}(\eta, \zeta, \tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\rho}}, \xi, \chi) &= \frac{1}{\hat{\mu}_1} \left(\|\mathcal{R}_1(\tilde{\boldsymbol{\tau}}, \xi, \eta)\|^2 + \frac{1}{\sqrt{\lambda}} (\nu \mathcal{R}_2(\tilde{\boldsymbol{\tau}}, \zeta), \mathcal{R}_2(\tilde{\boldsymbol{\tau}}, \zeta)) \right. \\ &\quad + \frac{1}{\sqrt{\lambda}} (\sigma \mathcal{R}_3(\xi, \zeta), \mathcal{R}_3(\xi, \zeta)) + \lambda \|\mathcal{R}_4(\tilde{\boldsymbol{\rho}}, \chi, \zeta)\|^2 \\ &\quad \left. + \sqrt{\lambda} (\nu \mathcal{R}_5(\tilde{\boldsymbol{\rho}}, \eta), \mathcal{R}_5(\tilde{\boldsymbol{\rho}}, \eta)) + \sqrt{\lambda} (\sigma \mathcal{R}_6(\chi, \eta), \mathcal{R}_6(\chi, \eta)) \right)^{1/2} \\ &= \frac{1}{\hat{\mu}_1} \left(T (\|\mathcal{R}_{1_0^c}\|_{\Omega}^2 + \frac{1}{\sqrt{\lambda}} (\nu \mathcal{R}_{2_0^c}, \mathcal{R}_{2_0^c})_{\Omega} + \sqrt{\lambda} (\nu \mathcal{R}_{3_0^c}, \mathcal{R}_{3_0^c})_{\Omega} + \lambda \|\mathcal{R}_{4_0^c}\|_{\Omega}^2) \right. \\ &\quad + \frac{T}{2} \sum_{k=1}^N \left(\|\mathcal{R}_{1_k}\|_{\Omega}^2 + \frac{1}{\sqrt{\lambda}} (\nu \mathcal{R}_{2_k}, \mathcal{R}_{2_k})_{\Omega} + \frac{1}{\sqrt{\lambda}} k \omega (\sigma \mathcal{R}_{3_k}, \mathcal{R}_{3_k})_{\Omega} \right. \\ &\quad \left. + \lambda \|\mathcal{R}_{4_k}\|_{\Omega}^2 + \sqrt{\lambda} (\nu \mathcal{R}_{5_k}, \mathcal{R}_{5_k})_{\Omega} + \sqrt{\lambda} k \omega (\sigma \mathcal{R}_{6_k}, \mathcal{R}_{6_k})_{\Omega} \right) + \frac{T}{2} \sum_{k=N+1}^{\infty} \|\mathbf{y}_{d_k}\|_{\Omega}^2 \Big)^{1/2}, \end{aligned}$$

where $\hat{\mu}_1 = 1/\sqrt{3}$.

5. FUNCTIONAL A POSTERIORI ESTIMATES FOR COST FUNCTIONALS OF PARABOLIC TIME-PERIODIC OPTIMAL CONTROL PROBLEMS

This section is aimed at deriving a posteriori estimates for the cost functional. Similar results for elliptic optimal control problems can be found, e.g., in [8, 28]. In the following, we will present majorants for the cost functional $\mathcal{J}(y, u)$, which are sharp for the exact solution of the optimal control problem. The cost functional $\mathcal{J}(y, u)$ defined in (1) is given in the Fourier space by

$$\mathcal{J}(y(u), u) = T \mathcal{J}_0(y_0^c(u_0^c), u_0^c) + \frac{T}{2} \sum_{k=1}^{\infty} \mathcal{J}_k(\mathbf{y}_k(\mathbf{u}_k), \mathbf{u}_k),$$

where

$$\mathcal{J}_0(y_0^c(u_0^c), u_0^c) = \frac{1}{2} \int_{\Omega} (y_0^c(\mathbf{x}) - y_{d_0}^c(\mathbf{x}))^2 d\mathbf{x} + \frac{\lambda}{2} \int_{\Omega} (u_0^c(\mathbf{x}))^2 d\mathbf{x} = \frac{1}{2} \|y_0^c - y_{d_0}^c\|_{\Omega}^2 + \frac{\lambda}{2} \|u_0^c\|_{\Omega}^2$$

and

$$\mathcal{J}_k(\mathbf{y}_k(\mathbf{u}_k), \mathbf{u}_k) = \frac{1}{2} \int_{\Omega} (\mathbf{y}_k(\mathbf{x}) - \mathbf{y}_{d_k}(\mathbf{x}))^2 d\mathbf{x} + \frac{\lambda}{2} \int_{\Omega} (\mathbf{u}_k(\mathbf{x}))^2 d\mathbf{x} = \frac{1}{2} \|\mathbf{y}_k - \mathbf{y}_{d_k}\|_{\Omega}^2 + \frac{\lambda}{2} \|\mathbf{u}_k\|_{\Omega}^2.$$

We want to determine majorants for the cost functional $\mathcal{J}(y, u)$ by using some of the results presented in [23], which are obtained for the time-periodic boundary value problem (2). In [23], the following functional a posteriori error estimate for problem (2) is shown:

$$(45) \quad |y - \eta|_{1, \frac{1}{2}} \leq \frac{1}{\mu_{1,a}} (C_F \|\mathcal{R}_1(\eta, \boldsymbol{\tau})\| + \|\mathcal{R}_2(\eta, \boldsymbol{\tau})\|)$$

for arbitrary functions $\eta \in H_{0,per}^{1,1}(Q_T)$ and $\boldsymbol{\tau} \in H(\text{div}, Q_T)$, where $\mathcal{R}_1(\eta, \boldsymbol{\tau}) := \sigma \partial_t \eta - \text{div } \boldsymbol{\tau} - u$ and $\mathcal{R}_2(\eta, \boldsymbol{\tau}) := \boldsymbol{\tau} - \nu \nabla \eta$ for a given $u \in L^2(Q_T)$. For this result, the following space-time variational problem of problem (2) has to be derived: Find $y \in H_{0,per}^{1, \frac{1}{2}}(Q_T)$ such that

$$a(y, v) = \int_{Q_T} u(\mathbf{x}, t) v(\mathbf{x}, t) d\mathbf{x} dt \quad \forall v \in H_{0,per}^{1, \frac{1}{2}}(Q_T),$$

where

$$a(y, v) := \int_{Q_T} \left(\sigma(\mathbf{x}) \partial_t^{1/2} y(\mathbf{x}, t) \partial_t^{1/2} v^{\perp}(\mathbf{x}, t) + \nu(\mathbf{x}) \nabla y(\mathbf{x}, t) \cdot \nabla v(\mathbf{x}, t) \right) d\mathbf{x} dt$$

denotes the space-time bilinear form of problem (2). This bilinear-form meets the following inf-sup condition:

$$\sup_{0 \neq v \in H_{0,per}^{1,\frac{1}{2}}(Q_T)} \frac{a(y, v)}{|v|_{1,\frac{1}{2}}} \geq \mu_{1,a} |y|_{1,\frac{1}{2}} \quad \forall y \in H_{0,per}^{1,\frac{1}{2}}(Q_T),$$

where $\mu_{1,a} = \min\{\underline{\nu}, \underline{\sigma}\}$ is a positive constant. The proof of the inf-sup condition can also be found in [23]. Now, adding and subtracting η as well as applying the triangle and Friedrichs inequalities to the cost functional $\mathcal{J}(y, u)$ yields the estimate

$$\mathcal{J}(y(u), u) \leq \frac{1}{2} (\|\eta - y_d\| + C_F \|\nabla y - \nabla \eta\|)^2 + \frac{\lambda}{2} \|u\|^2.$$

Since

$$\|\nabla y - \nabla \eta\|^2 \leq |y - \eta|_{1,\frac{1}{2}}^2 = \|\nabla y - \nabla \eta\|^2 + \|\partial_t^{1/2} y - \partial_t^{1/2} \eta\|^2,$$

we obtain the estimate

$$\mathcal{J}(y(u), u) \leq \frac{1}{2} \left(\|\eta - y_d\| + C_F |y - \eta|_{1,\frac{1}{2}} \right)^2 + \frac{\lambda}{2} \|u\|^2.$$

Together with (45) this leads to the estimate

$$\mathcal{J}(y(u), u) \leq \frac{1}{2} \left(\|\eta - y_d\| + \frac{C_F}{\mu_{1,a}} \|\mathcal{R}_2(\eta, \boldsymbol{\tau})\| + \frac{C_F^2}{\mu_{1,a}} \|\mathcal{R}_1(\eta, \boldsymbol{\tau}, u)\| \right)^2 + \frac{\lambda}{2} \|u\|^2,$$

where $\mu_{1,a} = \min\{\underline{\nu}, \underline{\sigma}\}$, $\boldsymbol{\tau} \in H(\operatorname{div}, Q_T)$, $\mathcal{R}_1(\eta, \boldsymbol{\tau}, u) = \sigma \partial_t \eta - \operatorname{div} \boldsymbol{\tau} - u$ and $\mathcal{R}_2(\eta, \boldsymbol{\tau}) = \boldsymbol{\tau} - \nu \nabla \eta$. By introducing parameters $\alpha, \beta > 0$ and applying Young's inequality, we can reformulate the estimate such that the right-hand side is given by a quadratic functional as follows:

$$\mathcal{J}(y(u), u) \leq \mathcal{J}^\oplus(\alpha, \beta; \eta, \boldsymbol{\tau}, u) \quad \forall u \in L^2(Q_T)$$

with the majorant

$$(46) \quad \begin{cases} \mathcal{J}^\oplus(\alpha, \beta; \eta, \boldsymbol{\tau}, u) := \frac{1+\alpha}{2} \|\eta - y_d\|^2 + \frac{(1+\alpha)(1+\beta)C_F^2}{2\alpha\mu_{1,a}^2} \|\mathcal{R}_2(\eta, \boldsymbol{\tau})\|^2 \\ \quad + \frac{(1+\alpha)(1+\beta)C_F^4}{2\alpha\beta\mu_{1,a}^2} \|\mathcal{R}_1(\eta, \boldsymbol{\tau}, u)\|^2 + \frac{\lambda}{2} \|u\|^2. \end{cases}$$

The majorant (46) provides a sharp upper bound of the cost functional, if it is minimized over η , $\boldsymbol{\tau}$, u and $\alpha, \beta > 0$, cf. [8]. This important result is summarized in the following theorem:

Theorem 7. *The exact lower bound of the majorant \mathcal{J}^\oplus defined in (46) coincides with the optimal value of the cost functional of problem (1)-(2), i.e.,*

$$(47) \quad \inf_{\substack{\eta \in H_{0,per}^{1,1}(Q_T), \boldsymbol{\tau} \in H(\operatorname{div}, Q_T) \\ u \in L^2(Q_T), \alpha, \beta > 0}} \mathcal{J}^\oplus(\alpha, \beta; \eta, \boldsymbol{\tau}, u) = \mathcal{J}(y(\bar{u}), \bar{u}).$$

Proof. The infimum of \mathcal{J}^\oplus is attained for the optimal control \bar{u} , its corresponding state $\bar{y} = y(\bar{u})$ and its exact flux $(\nu \nabla \bar{y})$, for which \mathcal{R}_1 and \mathcal{R}_2 vanish, and for α going to zero. \square

Corollary 3. *From Theorem 7, we obtain the following estimate:*

$$(48) \quad \mathcal{J}(\bar{y}, \bar{u}) \leq \mathcal{J}^\oplus(\alpha, \beta; \eta, \boldsymbol{\tau}, u) \quad \forall \eta \in H_{0,per}^{1,1}(Q_T), \boldsymbol{\tau} \in H(\operatorname{div}, Q_T), u \in L^2(Q_T), \alpha, \beta > 0.$$

Now, it is easy to derive an a posteriori estimate. Let η be the multiharmonic finite element approximation y_{N_h} to the state y . Since the control u can be chosen arbitrarily in (46), we choose a multiharmonic finite element approximation u_{N_h} for it as well. More precisely, we can compute the multiharmonic finite element approximation u_{N_h} for the control from the multiharmonic finite element approximation p_{N_h} of the adjoint state, since $u_{N_h} = -\lambda^{-1} p_{N_h}$, by solving the optimality system, from which we obtain y_{N_h} as well. Hence, we arrive at the estimate

$$(49) \quad \mathcal{J}(\bar{y}, \bar{u}) \leq \mathcal{J}^\oplus(\alpha, \beta; y_{N_h}, \boldsymbol{\tau}, u_{N_h}) \quad \forall \boldsymbol{\tau} \in H(\operatorname{div}, Q_T), \alpha, \beta > 0.$$

Next, we have to reconstruct the flux $\boldsymbol{\tau}$, which can be done by different techniques, see, e.g., [28, 25] and the references therein. For that, we first choose the vector-valued function $\boldsymbol{\tau}$ to be some multiharmonic finite element function $\boldsymbol{\tau}_{Nh}$ as well. Then the majorant $\mathcal{J}^\oplus(\alpha, \beta; y_{Nh}, \boldsymbol{\tau}_{Nh}, u_{Nh})$ is given by

$$\begin{aligned} \mathcal{J}^\oplus(\alpha, \beta; y_{Nh}, \boldsymbol{\tau}_{Nh}, u_{Nh}) &= \frac{1+\alpha}{2} \|y_{Nh} - y_d\|^2 + \frac{(1+\alpha)(1+\beta)C_F^2}{2\alpha\mu_{1,a}^2} \|\mathcal{R}_2(y_{Nh}, \boldsymbol{\tau}_{Nh})\|^2 \\ &\quad + \frac{(1+\alpha)(1+\beta)C_F^4}{2\alpha\beta\mu_{1,a}^2} \|\mathcal{R}_1(y_{Nh}, \boldsymbol{\tau}_{Nh}, u_{Nh})\|^2 + \frac{\lambda}{2} \|u_{Nh}\|^2, \end{aligned}$$

which can be written in the Fourier space as

$$\begin{aligned} \mathcal{J}^\oplus(\alpha, \beta; y_{Nh}, \boldsymbol{\tau}_{Nh}, u_{Nh}) &= \frac{1+\alpha}{2} \left(T \|y_{0h}^c - y_{d0}^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N (\|y_{kh}^c - y_{dk}^c\|_\Omega^2 + \|y_{kh}^s - y_{dk}^s\|_\Omega^2) + \mathcal{E}_N \right) \\ &\quad + \frac{(1+\alpha)(1+\beta)C_F^2}{2\alpha\mu_{1,a}^2} \left(T \|\mathcal{R}_{20}^c(y_{0h}^c, \boldsymbol{\tau}_{0h}^c)\|_\Omega^2 \right. \\ &\quad \quad \left. + \frac{T}{2} \sum_{k=1}^N (\|\mathcal{R}_{2k}^c(y_{kh}^c, \boldsymbol{\tau}_{kh}^c)\|_\Omega^2 + \|\mathcal{R}_{2k}^s(y_{kh}^s, \boldsymbol{\tau}_{kh}^s)\|_\Omega^2) \right) \\ &\quad + \frac{(1+\alpha)(1+\beta)C_F^4}{2\alpha\beta\mu_{1,a}^2} \left(T \|\mathcal{R}_{10}^c(\boldsymbol{\tau}_{0h}^c, u_{0h}^c)\|_\Omega^2 \right. \\ &\quad \quad \left. + \frac{T}{2} \sum_{k=1}^N (\|\mathcal{R}_{1k}^c(y_{kh}^s, \boldsymbol{\tau}_{kh}^c, u_{kh}^c)\|_\Omega^2 + \|\mathcal{R}_{1k}^s(y_{kh}^c, \boldsymbol{\tau}_{kh}^s, u_{kh}^s)\|_\Omega^2) \right) \\ &\quad + \frac{\lambda}{2} \left(T \|u_{0h}^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N (\|u_{kh}^c\|_\Omega^2 + \|u_{kh}^s\|_\Omega^2) \right) \end{aligned}$$

with $\mathcal{R}_{10}^c(\boldsymbol{\tau}_{0h}^c, u_{0h}^c) = \operatorname{div} \boldsymbol{\tau}_{0h}^c + u_{0h}^c$ and $\mathcal{R}_{20}^c(y_{0h}^c, \boldsymbol{\tau}_{0h}^c) = \boldsymbol{\tau}_{0h}^c - \nu \nabla y_{0h}^c$ as well as

$$\begin{aligned} \mathcal{R}_{1k}(\mathbf{y}_{kh}, \boldsymbol{\tau}_{kh}, \mathbf{u}_{kh}) &= k\omega \boldsymbol{\sigma} \mathbf{y}_{kh}^\perp + \operatorname{div} \boldsymbol{\tau}_{kh} + \mathbf{u}_{kh} \\ &= (-k\omega \boldsymbol{\sigma} \mathbf{y}_{kh}^s + \operatorname{div} \boldsymbol{\tau}_{kh}^c + u_{kh}^c, k\omega \boldsymbol{\sigma} \mathbf{y}_{kh}^c + \operatorname{div} \boldsymbol{\tau}_{kh}^s + u_{kh}^s)^T \\ &= (\mathcal{R}_{1k}^c(y_{kh}^s, \boldsymbol{\tau}_{kh}^c, u_{kh}^c), \mathcal{R}_{1k}^s(y_{kh}^c, \boldsymbol{\tau}_{kh}^s, u_{kh}^s))^T \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_{2k}(\mathbf{y}_{kh}, \boldsymbol{\tau}_{kh}) &= \boldsymbol{\tau}_{kh} - \nu \nabla \mathbf{y}_{kh} \\ &= (\boldsymbol{\tau}_{kh}^c - \nu \nabla y_{kh}^c, \boldsymbol{\tau}_{kh}^s - \nu \nabla y_{kh}^s)^T \\ &= (\mathcal{R}_{2k}^c(y_{kh}^c, \boldsymbol{\tau}_{kh}^c), \mathcal{R}_{2k}^s(y_{kh}^s, \boldsymbol{\tau}_{kh}^s))^T. \end{aligned}$$

Note that the computations are as straightforward as using the formulation with the optimality system. The remainder term (32) also stays the same.

Remark 5. *Since all the terms corresponding to every single mode k in the majorant \mathcal{J}^\oplus are decoupled, we arrive at some majorants \mathcal{J}_k^\oplus , for which we can, of course, introduce positive parameters α_k and β_k for every single mode k as well.*

Next, we have to reconstruct the fluxes $\boldsymbol{\tau}_{0h}^c$ and $\boldsymbol{\tau}_{kh}$ for all $k = 1, \dots, N$, which we denote by

$$\boldsymbol{\tau}_{kh} = R_h^{\text{flux}}(\nu \nabla \mathbf{y}_{kh}).$$

This can be done by various techniques as already mentioned. In [23], we have used Raviart-Thomas elements of the lowest order, see, e.g., [26, 5, 29], in order to regularize the fluxes by a post-processing operator, which maps the L^2 -functions into $H(\operatorname{div}, \Omega)$. Collecting all the fluxes corresponding to the single modes k together yields the reconstructed flux

$$\boldsymbol{\tau}_{Nh} = R_h^{\text{flux}}(\nu \nabla y_{Nh}).$$

After performing a simple minimization of the majorant $\mathcal{J}^\oplus(\alpha, \beta; y_{Nh}, \tau_{Nh}, u_{Nh})$ with respect to α and β , we finally arrive at the a posteriori estimate

$$(50) \quad \mathcal{J}(\bar{y}, \bar{u}) \leq \mathcal{J}^\oplus(\bar{\alpha}, \bar{\beta}; y_{Nh}, \tau_{Nh}, u_{Nh}),$$

where $\bar{\alpha}$ and $\bar{\beta}$ denote the optimized positive parameters. This majorant provides a guaranteed upper bound for the cost functional. Alternatively, but more costly, we can mode-wise minimize the majorant leading to $H(\text{div})$ -problems, see, e.g., [28, 19].

Remark 6. *In this work, we do not consider any inequality constraints imposed neither on the control nor on the state, but inequality constraints imposed on the Fourier coefficients of the control or the state can easily be included into the multiharmonic finite element approach, see [13], and, hence, may be considered in the a posteriori error analysis of parabolic time-periodic optimal control problems as well.*

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