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UNIFORMLY STABLE DISCONTINUOUS GALERKIN DISCRETIZATION AND ROBUST ITERATIVE SOLUTION METHODS FOR THE BRINKMAN PROBLEM

QINGGUO HONG AND JOHANNES KRAUS

ABSTRACT. We consider robust iterative methods for discontinuous Galerkin (DG) $H(\mathrm{div}, \Omega)$-conforming discretizations of the Brinkman equations. We describe a simple Uzawa iteration for the solution of this problem, which requires the solution of a nearly incompressible linear elasticity type equation with mass term on every iteration. We prove the uniform stability of the DG discretization for both problems. Then, we analyze variable V-cycle and W-cycle multigrid methods with nonnested bilinear forms. We prove that these algorithms are robust, and their convergence rates are independent of the parameters in the Brinkman problem and of the mesh size. The theoretical analysis is confirmed by numerical results.

1. INTRODUCTION

The Darcy-Stokes-Brinkman equations provide a unified mathematical model for one-phase flow of a visccous fluid in porous media (Darcy problem) coupled with free flow in a cavity (Stokes problem). Depending on the choice of parameters, these equations in the limiting cases reduce to either the Darcy, or the Stokes model. They also allow for treating the Darcy-Stokes interface problem with one set of equations by introducing a jump discontinuity of the coefficients across the interface.

The solution of the governing equations of flows in porous media might be sought at a coarse scale (e.g., if only the global pressure drop for a given flow rate is needed), a coarse scale enriched with some desirable fine-scale details, or at the fine scale (if computationally affordable and practically desirable). A subgrid (variational multiscale (VMS)) method for Brinkman's problem allowing to compute a two-scale (enriched coarse scale) solution has been introduced in [1] together with a subgrid-based two-level domain decomposition method for solving both Darcy's and Brinkman's problem in highly heterogeneous porous media at the fine scale (sometimes called iterative up-scaling). The aim of the present work, however, is to propose a uniformly stable discretization at coarse scale, and to devise and analyze optimal and robust iterative solution methods for this discretization.

Different stable discretizations of this problem have been considered in literature, which are either based on stable Stokes elements, or on $H(\mathrm{div}, \Omega)$-conforming elements, or on a stabilization of the weak formulation, see, e.g., [2].

Nonconforming elements for the Brinkman problem have been studied for example in [3], see also the references therein. The key point in this approach is to guarantee the robustness with respect to the limiting case of Darcy flow. Using the framework given in [2], necessary and sufficient conditions can be formulated to ensure the robustness of finite element methods for the Brinkman problem using (Stokes) stable finite element pairs.

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For mixed $H(\text{div}, \Omega)$-conforming schemes, e.g., Raviart-Thomas (RT) or Brezzi-Douglas-Marini (BDM) elements for the flux and discontinuous piecewise polynomials for the pressure, and using a postprocessing procedure for the pressure, optimal convergence rates were proved both in the Darcy and in the Stokes regime in [4] under proper regularity assumptions.

Quite recently, a novel mixed formulation of the Brinkman problem has been proposed in [5] in which the flow vorticity is introduced as an additional unknown. This formulation allows for a uniformly stable and conforming discretization by standard finite elements (Nedelec, Raviart-Thomas, discontinuous piecewise polynomials). The obtained stability estimates result in provably uniform scalable block-diagonal preconditioners in the constant coefficient case, see [6]. The arising subproblems can then be solved using optimal methods, e.g., auxiliary space algebraic multigrid (AMG) solvers for $H(\text{curl}; \Omega)$ and $H(\text{div}; \Omega)$ problems.

There are multigrid methods for Stokes type problems, which can roughly be classified into two categories: coupled and decoupled methods, cf. [7]. A well-known coupled approach is based on solving small saddle point systems at every grid point or on appropriate patches, cf. [8]. One classical decoupled approach is the Uzawa method [9]. There is also a well developed theory of multigrid methods for $H(\text{div}; \Omega)$ problems, see, e.g. [10], which in particular guarantees optimal solvers for the Darcy problem. There is, however, to the best of the authors' knowledge, no rigorous analysis proving the optimality of multigrid methods for uniformly stable DG discretizations of the 3D Brinkman problem up to now. For related results, for the 2D Brinkman problem discretized by H^1-conforming elements we refer to [11].

In the present paper a family of discontinuous Galerkin $H(\text{div}, \Omega)$-conforming discretizations is analyzed. A main feature of these methods is that they preserve divergence-free solutions of the Stokes problem. First introduced in [12], methods of this family were also described in [13] and, more recently, studied in the context of auxiliary space preconditioners for the Stokes problem, see [14], and optimal multigrid methods for Stokes and elasticity type equations, see [15].

The goals of this work are to prove the uniform stability of this family of DG discretizations for the Brinkman problem, and further, to devise and analyze multigrid methods that are based on nonnested bilinear forms. Using the local bounded cochain projections constructed in [16], we prove the uniform stability of a class of DG discretizations for the Brinkman problem as well as for a linear elasticity type problem with mass term, which arises in every step of a simple Uzawa iteration when used to solve the Brinkman problem. The proof of the robust (parameter-independent) convergence of the considered multigrid methods is based on a special (subspace-) decomposition of $H(\text{div}, \Omega)$. Similar decompositions have already been successfully applied in multigrid analysis in [17] and [15].

The remainder of the paper is organized as follows. In Section 2 we state the Brinkman problem in strong and weak form. The discontinuous Galerkin discretizations under consideration are discussed in Section 3 along with their approximation and stability properties and an a priori error estimate. In Section 4 we discuss multigrid methods that are based on nonnested bilinear forms. The main result is a proof of their robust and optimal convergence in the present context. We present some numerical results in Section 5 confirming our theoretical findings. Finally, we draw some concluding remarks in Section 6.

2. Problem formulation

In this section, we formulate of the Brinkman problem. Let $\Omega \subset R^d$ $(d = 2, 3)$ be a polygonal domain with boundary $\partial\Omega$, $\boldsymbol{f} \in L^2(\Omega)^d$, and $H_0^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega), u|_{\partial\Omega} = 0\}$. We need the standard Sobolev spaces $L^2(\Omega), H^1(\Omega), H^2(\Omega)$, and the corresponding norms

$$\|u\| = \Big(\int_\Omega u^2 dx\Big)^{1/2}, \|u\|_1 = \Big(\sum_{|\boldsymbol{\alpha}|\le 1}\int_\Omega |\frac{\partial^{\boldsymbol{\alpha}} u}{\partial x^{\boldsymbol{\alpha}}}|^2 dx\Big)^{1/2}, \|u\|_2 = \Big(\sum_{|\boldsymbol{\alpha}|\le 2}\int_\Omega |\frac{\partial^{\boldsymbol{\alpha}} u}{\partial x^{\boldsymbol{\alpha}}}|^2 dx\Big)^{1/2}.$$

We shall consider uniformly stable discontinuous Galerkin methods and robust iterative solution methods for the following Brinkman problem:

$$\left\{\begin{array}{c} -\epsilon^2 \operatorname{div} \varepsilon(\boldsymbol{u}) + \rho^2 \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega, \\ \operatorname{div} \boldsymbol{u} = g \text{ in } \Omega, \\ \boldsymbol{u} = 0 \text{ on } \partial\Omega. \end{array}\right.$$

Here, with the usual notation, $\boldsymbol{u}$ is the velocity field, p is the pressure, and $\varepsilon(\boldsymbol{u}) \in L^2(\Omega)^{d\times d}_{sym}$ is the symmetric (linearized) strain rate tensor defined by $\varepsilon(\boldsymbol{u}) = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T}{2}$. We also assume that the coefficients ϵ and ρ satisfy

$$0 < \epsilon^2 \leq 1 \text{ and } 0 \leq \rho^2 \leq 1. \tag{2.1}$$

Assumption (2.1) on the coefficients is reasonable since for $\epsilon^2 \geq \rho^2$, one can divide the equation by ϵ^2, and if $\epsilon^2 < \rho^2$, one can divide the equation by ρ^2. In the case $\epsilon = 1$ and $\rho = 0$ one obtains the Stokes problem. One can also consider the non-dimensional momentum equation $\frac{1}{\Delta t}\boldsymbol{u} - \frac{1}{Re} \operatorname{div} \varepsilon(\boldsymbol{u}) + \nabla p = \boldsymbol{f}$ arising from time discretization of the Navier-Stokes problem, where Re is the Reynolds number and Δt is the time step size. Let $\delta = \min\{\Delta t, Re\}$, then one can multiply the equation with δ and satisfy the assumption (2.1) for the rescaled equation $-\epsilon^2 \operatorname{div} \varepsilon(\boldsymbol{u}) + \rho^2 \boldsymbol{u} + \nabla \tilde{p} = \tilde{\boldsymbol{f}}$ with $\tilde{p} = \delta p$. Finally, let g satisfy the compatibility condition $\int_\Omega g dx = 0$.

The variational formulation of the Brinkman problem can be written as: Find $(\boldsymbol{u}, p) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$ such that

$$\left\{\begin{array}{l} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \quad \text{for all} \quad \boldsymbol{v} \in H^1_0(\Omega)^d, \\ b(\boldsymbol{u}, q) = (g, q), \quad \text{for all} \quad q \in L^2_0(\Omega), \end{array}\right. \tag{2.2}$$

The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &:= \epsilon^2 \int_\Omega \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) dx + \rho^2 \int_\Omega \boldsymbol{u}\boldsymbol{v} dx, \quad \text{for all} \quad \boldsymbol{u}, \boldsymbol{v} \in H^1_0(\Omega)^d, \\ b(\boldsymbol{u}, q) &:= \int_\Omega q \operatorname{div} \boldsymbol{u} dx, \quad \text{for all} \quad \boldsymbol{u} \in H^1_0(\Omega)^d, q \in L^2_0(\Omega). \end{aligned} \tag{2.3}$$

By Korn's inequality for functions in $H^1(\Omega)^d$, the conditions for the existence and uniqueness of the solution $(\boldsymbol{u}, p)$ of (2.2)–(2.3) are well known and understood, see, e.g. [18].

For convenience, in this paper, we assume that the domain Ω is such that the following regularity estimate holds for $\boldsymbol{f} \in L^2(\Omega)^d$ and $g = 0$ (see e.g. [18, 19]) :

$$\epsilon^2 \|\boldsymbol{u}\|_2 + \epsilon\rho \|\boldsymbol{u}\|_1 + \|p\|_1 \lesssim \|\boldsymbol{f}\|. \tag{2.4}$$

Lemma 2.1. *Let Ω be an open bounded domain with $C^{1,1}$-boundary or a convex polyhedron in $R^d (d = 2, 3)$. Then the solution of the problem* (2.2)–(2.3) *has the regularity property* (2.4).

In equation (2.4) and throughout the presentation that follows, the hidden constants in $\lesssim$, $\gtrsim$ and $\eqsim$ are independent of ϵ, ρ, λ and the mesh size h.

3. Discontinuous Galerkin discretization

In this section, we first give some preliminaries and notation for a DG formulations. Next, we derive the DG discretization of the Brinkman problem and describe the Uzawa method for solving the saddle piont system (2.2). Finally, we analyze the stability and approximation properties of this discretization.

3.1. **Preliminaries and notation.** We denote by T_h a shape-regular triangulation of mesh-size h of the domain Ω into triangles $\{K\}$. We further denote by E_h^I the set of all interior edges (or faces) of T_h and by E_h^B the set of all boundary edges (or faces); we set $E_h = E_h^I \cup E_h^B$.

For $s \geq 1$, we define

$$H^s(T_h) = \{\phi \in L^2(\Omega), \text{ such that } \phi|_K \in H^s(K) \text{ for all } K \in T_h\}.$$

Let us recall the definitions of the spaces to be used herein:

$$H(\text{div};\Omega) := \{\boldsymbol{v} \in L^2(\Omega) : \text{div}\, \boldsymbol{v} \in L^2(\Omega)\},$$

with the norm

$$\|\boldsymbol{v}\|^2_{H(\text{div};\Omega)} := \|\boldsymbol{v}\|^2 + \|\text{div}\,\boldsymbol{v}\|^2.$$

As commonly used with DG method, we define some trace operators. Let $e = \partial K_1 \cap \partial K_2$ be the common boundary (interface) of two subdomains K_1 and K_2 in T_h, and $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be unit normal vectors to e pointing to the exterior of K_1 and K_2, respectively. For any edge (or face) $e \in E_h^I$ and a scalar $q \in H^1(T_h)$, vector $\boldsymbol{v} \in H^1(T_h)^d$ and tensor $\boldsymbol{\tau} \in H^1(T_h)^{d\times d}$, we define the averages

$$\{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}|_{\partial K_1\cap e} \cdot \boldsymbol{n}_1 - \boldsymbol{v}|_{\partial K_2\cap e} \cdot \boldsymbol{n}_2), \quad \{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}|_{\partial K_1\cap e}\boldsymbol{n}_1 - \boldsymbol{\tau}|_{\partial K_2\cap e}\boldsymbol{n}_2),$$
$$\langle\boldsymbol{\tau}\rangle = \frac{1}{2}(\boldsymbol{\tau}|_{\partial K_1\cap e} + \boldsymbol{\tau}|_{\partial K_2\cap e}),$$

and jumps

$$[q] = q|_{\partial K_1\cap e} - q|_{\partial K_2\cap e}, \quad [\boldsymbol{v}] = \boldsymbol{v}|_{\partial K_1\cap e} - \boldsymbol{v}|_{\partial K_2\cap e}, \quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}|_{\partial K_1\cap e} \odot \boldsymbol{n}_1 + \boldsymbol{v}|_{\partial K_2\cap e} \odot \boldsymbol{n}_2,$$

where $\boldsymbol{v} \odot \boldsymbol{n} = \frac{1}{2}(\boldsymbol{v}\boldsymbol{n}^T + \boldsymbol{n}\boldsymbol{v}^T)$ is the symmetric part of the tensor product of $\boldsymbol{v}$ and $\boldsymbol{n}$.

When $e \in E_h^B$ then the above quantities are defined as

$$\{\boldsymbol{v}\} = \boldsymbol{v}|_e \cdot \boldsymbol{n},\ \{\boldsymbol{\tau}\} = \boldsymbol{\tau}|_e\boldsymbol{n},\ \langle\boldsymbol{\tau}\rangle = \boldsymbol{\tau}|_e,\ [q] = q|_e,\ [\boldsymbol{v}] = \boldsymbol{v}|_e,\ [\![\boldsymbol{v}]\!] = \boldsymbol{v}|_e \odot \boldsymbol{n}.$$

Since $\boldsymbol{n}_1 = -\boldsymbol{n}_2$, $\{\varepsilon(\boldsymbol{u})\} = \langle\varepsilon(\boldsymbol{u})\rangle\boldsymbol{n}_1$ and $[\![\boldsymbol{v}]\!] = [\boldsymbol{v}] \odot \boldsymbol{n}_1$, it follows that

$$\begin{aligned} \langle\varepsilon(\boldsymbol{u})\rangle : [\![\boldsymbol{v}]\!] &= \text{trace}([\![\boldsymbol{v}]\!]^T\langle\varepsilon(\boldsymbol{u})\rangle) = \text{trace}([\boldsymbol{v}]\{\varepsilon(\boldsymbol{u})\}^T) \\ &= \{\varepsilon(\boldsymbol{u})\} \cdot [\boldsymbol{v}], \quad \text{for all} \quad \boldsymbol{u},\boldsymbol{v} \in H^1(T_h)^d. \end{aligned} \tag{3.1}$$

If we denote by $\boldsymbol{n}_K$ the outward unit normal to ∂K, it is easy to check that

$$\sum_{K\in T_h}\int_{\partial K} \boldsymbol{v}\cdot\boldsymbol{n}_K q ds = \sum_{e\in E_h}\int_e \{\boldsymbol{v}\}[q]ds, \quad \text{for all} \quad \boldsymbol{v} \in H(\text{div};\Omega), \quad \text{for all} \quad q \in H^1(T_h). \tag{3.2}$$

Also for $\boldsymbol{\tau} \in H^1(\Omega)^{d\times d}$ and for all $\boldsymbol{v} \in H^1(T_h)^d$, we have

$$\sum_{K\in T_h}\int_{\partial K}(\boldsymbol{\tau}\boldsymbol{n}_K)\cdot\boldsymbol{v}ds = \sum_{e\in E_h}\int_e\{\boldsymbol{\tau}\}\cdot[\boldsymbol{v}]ds. \tag{3.3}$$

The finite element spaces are denoted by

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in H(\text{div};\Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}(K),\ K \in T_h;\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$

$$S_h = \{q \in L^2(\Omega) : q|_K \in Q(K),\ K \in T_h;\ \int_\Omega q dx = 0\}.$$

For the DG method, we denote the RT pair $RT_l(K)/P_l(K)$ or the BDM pair $BDM_l(K)/P_{l-1}(K)$ or the BDFM pair $BDFM_l(K)/P_{l-1}(K)$ as $\boldsymbol{V}(K)/Q(K)$. Note that $\text{div}\,\boldsymbol{V}(K) = Q(K)$, which implies that the divergence-free velocity fields are preserved (see [14]).

We recall the basic approximation properties of these spaces: for all $K \in T_h$ and for all $\boldsymbol{v} \in H^s(K)^d$, there exists $\boldsymbol{v}_I \in \boldsymbol{V}(K)$ such that

$$\|\boldsymbol{v}-\boldsymbol{v}_I\|_{0,K} + h_K|\boldsymbol{v}-\boldsymbol{v}_I|_{1,K} + h_K^2|\boldsymbol{v}-\boldsymbol{v}_I|_{2,K} \lesssim h_K^s|\boldsymbol{v}|_{s,K},\ 2 \leq s \leq l+1. \tag{3.4}$$

3.2. **DG formulations.** We note that according to the definition of $\boldsymbol{V}_h$, the normal component of any $\boldsymbol{v} \in \boldsymbol{V}_h$ is continuous on the internal edges and vanishes on the boundary edges. Therfore, by splitting a vector $\boldsymbol{v} \in \boldsymbol{V}_h$ into its normal and tangential components $\boldsymbol{v}_n$ and $\boldsymbol{v}_t$

$$\boldsymbol{v}_n := (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}, \quad \boldsymbol{v}_t := \boldsymbol{v} - \boldsymbol{v}_n, \tag{3.5}$$

we find that for all $e \in E_h$,

$$\int_e [\boldsymbol{v}_n] \cdot \boldsymbol{w} ds = 0, \quad \text{for all} \quad \boldsymbol{w} \in H^1(T_h)^d, \boldsymbol{v} \in \boldsymbol{V}_h, \tag{3.6}$$

and thus

$$\int_e [\boldsymbol{v}] \cdot \boldsymbol{w} ds = \int_e [\boldsymbol{v}_t] \cdot \boldsymbol{w} ds = 0, \quad \text{for all} \quad \boldsymbol{w} \in H^1(T_h)^d, \boldsymbol{v} \in V_h. \tag{3.7}$$

A direct computation, similar to the one given in (3.1), shows that

$$\begin{aligned} [\boldsymbol{u}_t] \odot \boldsymbol{n} : [\boldsymbol{v}_t] \odot \boldsymbol{n} &= (([\boldsymbol{u}_t] \odot \boldsymbol{n})\boldsymbol{n}) \cdot [\boldsymbol{v}_t] = \frac{1}{2}\left(([\boldsymbol{u}_t]\boldsymbol{n}^T + \boldsymbol{n}[\boldsymbol{u}_t]^T) \cdot \boldsymbol{n}\right) \cdot [\boldsymbol{v}_t] \\ &= \frac{1}{2}\left([\boldsymbol{u}_t] + ([\boldsymbol{u}_t] \cdot \boldsymbol{n})\boldsymbol{n}\right) \cdot [\boldsymbol{v}_t] = \frac{1}{2}[\boldsymbol{u}_t] \cdot [\boldsymbol{v}_t], \end{aligned}$$

implying that

$$[\![\boldsymbol{u}_t]\!] : [\![\boldsymbol{v}_t]\!] = \frac{1}{2}[\boldsymbol{u}_t] \cdot [\boldsymbol{v}_t]. \tag{3.8}$$

Therefore, the discretization of the Brinkman problem (2.2) is given by: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times S_h$ such that

$$\begin{cases} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b_h(\boldsymbol{u}_h, q_h) = (g, q_h), \quad \text{for all} \quad q_h \in S_h, \end{cases} \tag{3.9}$$

where

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) &= \epsilon^2\Big(\sum_{K \in T_h} \int_K \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) dx - \sum_{e \in E_h} \int_e \{\varepsilon(\boldsymbol{u})\} \cdot [\boldsymbol{v}_t] ds \\ &\quad - \sum_{e \in E_h} \int_e \{\varepsilon(\boldsymbol{v})\} \cdot [\boldsymbol{u}_t] ds + \sum_{e \in E_h} \int_e \eta h_e^{-1} [\boldsymbol{u}_t] \cdot [\boldsymbol{v}_t] ds\Big) + \rho^2 \int_\Omega \boldsymbol{u}\boldsymbol{v} dx, \end{aligned} \tag{3.10}$$

$$b_h(\boldsymbol{u}, q) = \int_\Omega \nabla \cdot \boldsymbol{u} q dx, \tag{3.11}$$

and η is a properly chosen penalty parameter independent of the mesh size h such that $a_h(\cdot, \cdot)$ is positive definite.

Remark 3.1. *Noting the identities* (3.1) *and* (3.8)*, we can rewrite* $a_h(\cdot, \cdot)$ *as*

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) &= \epsilon^2\Big(\sum_{K \in T_h} \int_K \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) dx - \sum_{e \in E_h} \int_e \langle\varepsilon(\boldsymbol{u})\rangle : [\![\boldsymbol{v_t}]\!] ds \\ &\quad - \sum_{e \in E_h} \int_e \langle\varepsilon(\boldsymbol{v})\rangle : [\![\boldsymbol{u_t}]\!] ds + \sum_{e \in E_h} \int_e 2\eta h_e^{-1} [\![\boldsymbol{u}_t]\!] : [\![\boldsymbol{v}_t]\!] ds\Big) + \rho^2 \int_\Omega \boldsymbol{u}\boldsymbol{v} dx, \end{aligned} \tag{3.12}$$

which generalizes the bilinear form in [12] *since the normal component of* $\boldsymbol{u} \in \boldsymbol{V}_h$ *is continuous. When compared to the bilinear form in* [14]*, we can see that the only difference is that in* (3.12) *the jumps of* $\boldsymbol{u}_t$ *on the boundary edges are included for the Dirichlet boundary, which is due to the fact that the condition* $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ *is included in the definition of the space* $\boldsymbol{V}_h$*.*

Noting that $\operatorname{div} \boldsymbol{V}_h = S_h$ we can rewrite the above system in the following equivalent form: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times S_h$ such that

$$\left\{ \begin{array}{l} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \lambda b_h(\boldsymbol{u}_h, \operatorname{div} \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) + \lambda(g, \operatorname{div} \boldsymbol{v}_h), \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b_h(\boldsymbol{u}_h, q_h) = (g, q_h), \quad \text{for all} \quad q_h \in S_h. \end{array} \right. \tag{3.13}$$

By the definition of $b_h(\cdot,\cdot)$, namely (3.11), the Uzawa method for (3.13) reads: Given $(\boldsymbol{u}_h^l, p^l)$, the new iterate $(\boldsymbol{u}_h^{l+1}, p^{l+1})$ is obtained by solving the following system:

$$\left\{ \begin{array}{l} a_h(\boldsymbol{u}_h^{l+1}, \boldsymbol{v}_h) + \lambda(\operatorname{div} \boldsymbol{u}_h^{l+1}, \operatorname{div} \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) - b_h(\boldsymbol{v}_h, p_h^l) + \lambda(g, \operatorname{div} \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ p_h^{l+1} = p_h^l - \lambda \operatorname{div} \boldsymbol{u}_h^{l+1} - \lambda g, \end{array} \right. \tag{3.14}$$

where $\lambda > 1$ is a damping parameter.
Convergence of this method has been discussed in several works, see, e.g., [9, 20, 21, 22] indicating that for large λ, the iterates converge rapidly to the solution of problem (3.9). As a consequence, the major computational cost lies in solving the discrete problem arising from the following linear elasticity type equation with mass term: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{F}, \boldsymbol{v}_h), \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.15}$$

where $A_h(\cdot,\cdot)$ is defined by

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \lambda(\operatorname{div} \boldsymbol{u}_h, \operatorname{div} \boldsymbol{v}_h), \tag{3.16}$$

and $a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$ is given in (3.10).

Remark 3.2. *In fact,* (3.15) *is the DG discretization of the following problem: Find $\boldsymbol{u} \in H_0^1(\Omega)$ such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) + \lambda(\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{F}, \boldsymbol{v}), \textit{for all } \boldsymbol{v} \in H_0^1(\Omega). \tag{3.17}$$

Noting that $\operatorname{div} \boldsymbol{V}_h = S_h$, it is immediately seen that problem (3.15)–(3.16) has the following equivalent formulation: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times S_h$ such that

$$\left\{ \begin{array}{l} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = (\boldsymbol{F}, \boldsymbol{v}_h), \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b_h(\boldsymbol{u}_h, q_h) - (\lambda^{-1} p_h, q_h) = 0, \quad \text{for all} \quad q_h \in S_h, \end{array} \right. \tag{3.18}$$

where $a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$ and $b_h(\boldsymbol{u}_h, q_h)$ are defined by (3.10) and (3.11), respectively.

It is clear that (3.15) is the DG discretization of the linear elasticity equation with a mass term which reads: Find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$\left\{ \begin{array}{l} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{F}, \boldsymbol{v}), \quad \text{for all} \quad \boldsymbol{v} \in H_0^1(\Omega)^d, \\ b(\boldsymbol{u}, q) - (\lambda^{-1} p, q) = 0, \quad \text{for all} \quad q \in L^2(\Omega), \end{array} \right. \tag{3.19}$$

where $a(\boldsymbol{u}, \boldsymbol{v})$ and $b(\boldsymbol{u}, q)$ are defined in (2.3).

3.3. Approximation and stability properties.

In this subsection, we analyze the approximation and stability properties of the discrete problems (3.9) and (3.15)–(3.16).

For any $\boldsymbol{u} \in H^1(T_h)^d$, we define the mesh dependent norms:

$$\begin{aligned} \|\boldsymbol{u}\|_h^2 &= \sum_{K \in T_h} \|\varepsilon(\boldsymbol{u})\|_{0,K}^2 + \sum_{e \in E_h} h_e^{-1} \|[\boldsymbol{u}_t]\|_{0,e}^2, \\ \|\boldsymbol{u}\|_{1,h}^2 &= \sum_{K \in T_h} \|\nabla \boldsymbol{u}\|_{0,K}^2 + \sum_{e \in E_h} h_e^{-1} \|[\boldsymbol{u}_t]\|_{0,e}^2. \end{aligned}$$

Next, for $\boldsymbol{u} \in H^2(T_h)^d$, we define the "DG"-norm, the "$\|\cdot\|_{\epsilon,\rho}$"-norm and the $\|\cdot\|_{\epsilon,h}$"-norm as follows

$$\|\boldsymbol{u}\|_{DG}^2 = \|\boldsymbol{u}\|_{1,h}^2 + \sum_{K \in T_h} h_K^2 |\boldsymbol{u}|_{2,K}^2, \tag{3.20}$$

$$\|\boldsymbol{u}\|_{\epsilon,\rho}^2 = \epsilon^2\|\boldsymbol{u}\|_{DG}^2 + \rho^2\|\boldsymbol{u}\|^2, \tag{3.21}$$

$$\|\boldsymbol{u}\|_{\epsilon,h}^2 = \|\boldsymbol{u}\|_{\epsilon,\rho}^2 + \|\operatorname{div}\boldsymbol{u}\|^2. \tag{3.22}$$

From the discrete version of the Korn's inequality (see [23, Equation (1.12)]) we have the following norm equivalence result.

Lemma 3.1. *The norms $\|\cdot\|_{DG}$, $\|\cdot\|_h$, and $\|\cdot\|_{1,h}$ are equivalent in $\boldsymbol{V}_h$, namely*

$$\|\boldsymbol{u}\|_{DG} \eqsim \|\boldsymbol{u}\|_h \eqsim \|\boldsymbol{u}\|_{1,h}, \quad \textit{for all } \boldsymbol{u} \in \boldsymbol{V}_h. \tag{3.23}$$

Proof. By the inverse inequality, we clearly have that $\|\boldsymbol{u}\|_{DG} \eqsim \|\boldsymbol{u}\|_{1,h}$. We show now that $\|\boldsymbol{u}\|_h \eqsim \|\boldsymbol{u}\|_{1,h}$. From the definitions we immediately get $\|\boldsymbol{u}\|_h \leq \|\boldsymbol{u}\|_{1,h}$.

To prove the inequality in the other direction, we use [23, Equation (1.22)], namely,

$$\begin{aligned}\|\boldsymbol{u}\|^2 + \sum_{K\in T_h}\|\nabla\boldsymbol{u}\|_{0,K}^2 \quad &\lesssim \quad \Big(\sum_{K\in T_h}\|\varepsilon(\boldsymbol{u})\|_{0,K}^2 + \sup_{\substack{\boldsymbol{m}\in RM(\Omega)\\ \|\boldsymbol{m}\|_{L^2(\partial\Omega)=1}}}(\int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{m}ds)^2 \\ &\quad + \sum_{e\in E_h^I} h_e^{-1}\|\pi_e[\boldsymbol{u}]_e\|_{0,e}^2\Big).\end{aligned} \tag{3.24}$$

Here RM(Ω) denotes the space of rigid body motions,

$$\mathrm{RM}(\Omega) = \left\{\boldsymbol{a} + \boldsymbol{A}\boldsymbol{x} \;\middle|\; \boldsymbol{a}\in R^d,\ \boldsymbol{A}\in R^{d\times d},\ \boldsymbol{A} = -\boldsymbol{A}^T\right\},$$

and the operator π_e is the $L^2(e)$-orthogonal projection operator onto $(P_1(e))^d$, the space of vector-valued linear polynomials on e.

For the second term on the right hand side of (3.24) we have

$$\sup_{\substack{\boldsymbol{m}\in RM(\Omega)\\ \|\boldsymbol{m}\|_{L^2(\partial\Omega)=1}}}(\int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{m}ds)^2 \leq \int_{\partial\Omega}\boldsymbol{u}^2 ds = \sum_{e\in E_h^B}\|\boldsymbol{u}\|_{0,e}^2 \leq \sum_{e\in E_h^B} h_e^{-1}\|[\boldsymbol{u}]_e\|_{0,e}^2.$$

Since π_e is an orthogonal projection, for the third term on the right hand side of (3.24) we obtain

$$\sum_{e\in E_h^I} h_e^{-1}\|\pi_e[\boldsymbol{u}]_e\|_{0,e}^2 \leq \sum_{e\in E_h^I} h_e^{-1}\|[\boldsymbol{u}]_e\|_{0,e}^2.$$

Finally, combining the two inequalities above completes the proof. □

Moreover, both bilinear forms, $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$, introduced above are continuous and we have

$$\begin{aligned}|a_h(\boldsymbol{u},\boldsymbol{v})| \quad &\lesssim \quad \|\boldsymbol{u}\|_{\epsilon,h}\|\boldsymbol{v}\|_{\epsilon,h}\ , \quad \text{for all} \quad \boldsymbol{u},\ \boldsymbol{v}\in H^2(T_h)^d,\\ |b_h(\boldsymbol{u},q)| \quad &\leq \quad \|\boldsymbol{u}\|_{\epsilon,h}\|q\|\ , \quad \text{for all } \boldsymbol{u}\in\boldsymbol{V}_h, q\in L_0^2(\Omega).\end{aligned}$$

For our choice of the finite element spaces $\boldsymbol{V}_h$ and S_h, noting that $\|\operatorname{div}\boldsymbol{u}\| \lesssim \|\boldsymbol{u}\|_{1,h}$ and the proof of Lemma 3.1, which implies $\|\boldsymbol{u}\| \lesssim \|\boldsymbol{u}\|_{1,h}$, we have the following inf-sup condition for $b_h(\cdot,\cdot)$ (see, e.g., [14, 24]).

Lemma 3.2. *There exist positive constants β_0, β_1 independent of ϵ, ρ and the mesh size h, such that*

$$\inf_{q_h\in S_h}\sup_{\boldsymbol{u}_h\in\boldsymbol{V}_h}\frac{(\operatorname{div}\boldsymbol{u}_h, q_h)}{\|\boldsymbol{u}_h\|_{1,h}\|q_h\|} \geq \beta_0 \ \textit{and}\ \inf_{q_h\in S_h}\sup_{\boldsymbol{u}_h\in\boldsymbol{V}_h}\frac{(\operatorname{div}\boldsymbol{u}_h, q_h)}{\|\boldsymbol{u}_h\|_{\epsilon,h}\|q_h\|} \geq \beta_1. \tag{3.25}$$

Now for any given $g\in L_0^2(\Omega)$, we define

$$\boldsymbol{Z}_h(g) = \{\boldsymbol{u}_h\in\boldsymbol{V}_h:\ b_h(\boldsymbol{u}_h, q_h) = (g, q_h), \forall q_h\in S_h\}. \tag{3.26}$$

Noting that $\operatorname{div}\boldsymbol{V}_h = S_h$, it follows that $a_h(\cdot,\cdot)$ is coercive on $\boldsymbol{Z}_h(0)$, namely, we have the following Lemma whose proof follows the lines of similar arguments in [14, 25].

Lemma 3.3. *For sufficiently large η, independent of the mesh size h, we have*

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \gtrsim \|\boldsymbol{u}_h\|_{\epsilon,\rho}^2, \quad \text{for all } \boldsymbol{u}_h \in \boldsymbol{V}_h, \tag{3.27}$$

and hence

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \gtrsim \|\boldsymbol{u}_h\|_{\epsilon,h}^2, \quad \text{for all } \boldsymbol{u}_h \in \boldsymbol{Z}_h(0). \tag{3.28}$$

By the equivalence of the norms shown in (3.23) and also by the standard theory for solvability of mixed problems [26], we obtain the following theorem that is similar to a result in [27].

Theorem 3.1. *The discrete problem* (3.9) *has a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times S_h$ that satisfies*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\epsilon,\rho} \lesssim \inf_{v_h \in \boldsymbol{Z}_h(g)} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\epsilon,\rho}, \quad \|p - p_h\| \lesssim \inf_{q_h \in S_h} \|p - q_h\| + \inf_{v_h \in \boldsymbol{Z}_h(g)} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\epsilon,\rho} \tag{3.29}$$

with $(\boldsymbol{u}, p)$ being the solution of (2.2).

The bilinear forms $a_h(\cdot,\cdot)$ and $A_h(\cdot,\cdot)$ are coercive and also define norms on $\boldsymbol{V}_h$, i.e.,

$$\|\boldsymbol{u}\|_{a_h}^2 = a_h(\boldsymbol{u}, \boldsymbol{u}), \quad \|\boldsymbol{u}\|_{A_h}^2 = A_h(\boldsymbol{u}, \boldsymbol{u}).$$

We now introduce the canonical interpolation operator $\Pi_h^{\text{div}} : H^1(\Omega)^d \mapsto \boldsymbol{V}_h$ and denote the L^2-projection on S_h by Q_h. The following Lemma summarizes some of the properties of Π_h^{div} and Q_h needed later.

Lemma 3.4. *For all $\boldsymbol{w} \in H^1(K)^d$ we have*

$$\begin{aligned}
&\operatorname{div} \Pi_h^{\text{div}} = Q_h \operatorname{div}\,; \quad |\Pi_h^{\text{div}} \boldsymbol{w}|_{1,K} \lesssim |\boldsymbol{w}|_{1,K};\\
&\|\boldsymbol{w} - \Pi_h^{\text{div}} \boldsymbol{w}\|_{0,\partial K}^2 \lesssim h_K |\boldsymbol{w}|_{1,K}^2; \quad \|\operatorname{div}(\boldsymbol{w} - \Pi_h^{\text{div}} \boldsymbol{w})\|_{-1} \lesssim h_K \|\operatorname{div} \boldsymbol{w}\|,
\end{aligned}$$

where $\|r\|_{-1} = \sup_{\chi \in H^1} \frac{(\chi, r)}{\|\chi\|_1}$.

Proof. The proof of the commutativity of Π_h^{div} and div and the first two inequalities are well known and we refer the reader to [28] for the details.

The last inequality follows from the approximation properties of the L^2-orthogonal projection, that is,

$$\begin{aligned}
\|\operatorname{div} \boldsymbol{w} - \operatorname{div} \Pi_h^{\text{div}} \boldsymbol{w}\|_{-1} &= \sup_{\chi \in H^1} \frac{((I - Q_h) \operatorname{div} \boldsymbol{w}, \chi)}{\|\chi\|_1} = \sup_{\chi \in H^1} \frac{(\operatorname{div} \boldsymbol{w}, (I - Q_h)\chi)}{\|\chi\|_1}\\
&\lesssim \sup_{\chi \in H^1} \frac{\|\operatorname{div} \boldsymbol{w}\| \|(I - Q_h)\chi)\|}{\|\chi\|_1} \lesssim h_K \|\operatorname{div} \boldsymbol{w}\|.
\end{aligned}$$

□

Remark 3.3. *For all $\boldsymbol{w} \in H_0(\text{div}; \Omega) \cap H^1(T_h)^d$, by noting that $H_0(\text{div}; \Omega) \cap H^1(T_h)^d$ is a subset of $L^3(\Omega) \cap H(\text{div}; \Omega)$, we conclude that Π_h^{div} is well-defined for $\boldsymbol{w}$, see, e.g.,* [28] *and further by Lemma 3.4, we have the estimate*

$$\|\Pi_h^{\text{div}} \boldsymbol{w}\|_{1,h} \lesssim \|\boldsymbol{w}\|_{1,h}. \tag{3.30}$$

The above canonical interpolation is not bounded in $H(\text{div})$ norm. For that reason, we will need to use the following local bounded cochain projection operators.

For any $K \in T_h$, we denote by $\Delta(K)$ the set of all subsimplexes of K, and by $\Delta_m(K)$ all subsimplexes of dimension m. We further denote by $\Delta(T_h)$ the set of all subsimplexes of all dimensions of the triangulation T_h, and correspondingly by $\Delta_m(T_h)$ the set of all subsimplices of dimension m. Now for each $f \in \Delta_m(T_h)$, we let Ω_f be the associated macroelement consisting of the union of the elements of T_h containing f, i.e.,

$$\Omega_f = \cup\{K | K \in T_h, f \in \Delta(K)\}. \tag{3.31}$$

In addition to macroelements Ω_f it will also be convenient to introduce the notion of an extended macroelement Ω_f^e defined for $f \in \Delta(T_h)$ by

$$\Omega_f^e = \bigcup_{i \in \Delta_0(f)} \Omega_i. \tag{3.32}$$

We let $T_{f,h}$ denote the restriction of T_h to Ω_f. Define $D_{m,K} \subset \Omega$ by

$$D_{m,K} = \cup \{ D_{m-1,K'} \mid K' \in T_{f,h}, f \in \Delta_m(K) \}, \; D_{0,K} = \Omega_K^e. \tag{3.33}$$

and finally let

$$D_K = D_{d,K} \; (d = 2, 3). \tag{3.34}$$

Lemma 3.5. *There exist projection operators* $\pi_h^{div} : H_0(\mathrm{div}; \Omega) \to \boldsymbol{V}_h$ *and* $\pi_h : L^2(\Omega) \to S_h$ *satisfying* $\mathrm{div}\, \pi_h^{div} = \pi_h \,\mathrm{div}$.

The construction of π_h^{div} and π_h in differential form can be found in [16]. Now we show the local boundedness of these operators. Let D_K be defined in (3.34). Then the following estimates hold, see [16].

Lemma 3.6. *([16]) For any* $\boldsymbol{u} \in H_0(\mathrm{div}, \Omega)$ *and* $p \in L_0^2(\Omega)$*, we have*

$$\begin{aligned} &\|\pi_h^{div} \boldsymbol{u}\|_{0,K} \lesssim \|\boldsymbol{u}\|_{0,D_K} + h_K \|\mathrm{div}\, \boldsymbol{u}\|_{0,D_K}, \; \forall K \in T_h, \\ &\|\pi_h p\|_{0,K} \lesssim \|p\|_{0,D_K}, \; \forall K \in T_h. \end{aligned} \tag{3.35}$$

Using Lemma 3.6 and Lemma 3.4, we can prove the following boundedness of π_h^{div} in DG norm.

Lemma 3.7. *Let* $\boldsymbol{u} \in H_0(\mathrm{div}, \Omega) \cap H^1(T_h)^d$ *and denote* $|\boldsymbol{u}|_{1,D_K} = \sum_{K \in D_K} |\boldsymbol{u}|_{1,K}$*. Then we have*

$$\begin{aligned} &|\pi_h^{div} \boldsymbol{u}|_{1,K} \lesssim |\boldsymbol{u}|_{1,D_K}, \; \forall K \in T_h, \\ &\|\boldsymbol{u} - \pi_h^{\mathrm{div}} \boldsymbol{u}\|_{0,\partial K} \lesssim h_K^{\frac{1}{2}} |\boldsymbol{u}|_{1,D_K}, \; \forall K \in T_h. \end{aligned} \tag{3.36}$$

Furthermore,

$$\|\pi_h^{div} \boldsymbol{u}\|_{1,h} \lesssim \|\boldsymbol{u}\|_{1,h}. \tag{3.37}$$

Proof. We prove the first inequality first. Noting that π_h^{div} is a projection, by the triangle inequality, it follows that

$$|\pi_h^{div} \boldsymbol{u}|_{1,K} \le |\pi_h^{div}(\boldsymbol{u} - \Pi_h^{div} \boldsymbol{u})|_{1,K} + |\Pi_h^{div} \boldsymbol{u}|_{1,K}. \tag{3.38}$$

Now by a standard inverse inequality and the H^1-stability of Π_h^{div} given in Lemma 3.4, we have

$$|\pi_h^{div} \boldsymbol{u}|_{1,K} \le h_K^{-1} \|\pi_h^{div}(\boldsymbol{u} - \Pi_h^{div} \boldsymbol{u})\|_{0,K} + |\boldsymbol{u}|_{1,K}. \tag{3.39}$$

Next we use Lemma 3.6 and the error estimate (3.4) for Π_h^{div} in order to obtain

$$\begin{aligned} |\pi_h^{div} \boldsymbol{u}|_{1,K} &\lesssim h_K^{-1} (\|\boldsymbol{u} - \Pi_h^{div} \boldsymbol{u}\|_{0,D_K} + h_K \|\mathrm{div}\, \boldsymbol{u} - Q_h \,\mathrm{div}\, \boldsymbol{u}\|_{0,D_K}) + |\boldsymbol{u}|_{1,K} \\ &\lesssim |\boldsymbol{u}|_{1,D_K} + \|\mathrm{div}\, \boldsymbol{u}\|_{0,D_K} + |\boldsymbol{u}|_{1,K} \\ &\le |\boldsymbol{u}|_{1,D_K} + |\boldsymbol{u}|_{1,D_K} + |\boldsymbol{u}|_{1,D_K} \lesssim |\boldsymbol{u}|_{1,D_K}. \end{aligned} \tag{3.40}$$

Now we prove the second inequality. By the trace theorem, inequality (3.40), and using the fact that π_h^{div} is a projection and also Lemma 3.4 and Lemma 3.6, we get

$$\begin{aligned}\|\boldsymbol{u}-\pi_h^{\rm div}\boldsymbol{u}\|_{0,\partial K}^2 &\lesssim h_K|\boldsymbol{u}-\pi_h^{\rm div}\boldsymbol{u}|_{1,K}^2+h_K^{-1}\|\boldsymbol{u}-\pi_h^{\rm div}\boldsymbol{u}\|_{0,K}^2\\ &\lesssim h_K|\boldsymbol{u}|_{1,K}^2+h_K|\boldsymbol{u}|_{1,D_K}^2+h_K^{-1}\|(I-\pi_h^{\rm div})(\boldsymbol{u}-\Pi_h^{div}\boldsymbol{u})\|_{0,K}\\ &\lesssim h_K|\boldsymbol{u}|_{1,D_K}^2+h_K^{-1}\|\boldsymbol{u}-\Pi_h^{div}\boldsymbol{u}\|_{0,K}^2+h_K^{-1}\|\pi_h^{div}(\boldsymbol{u}-\Pi_h^{div}\boldsymbol{u})\|_{0,K}^2\\ &\lesssim h_K|\boldsymbol{u}|_{1,D_K}^2+h_K|\boldsymbol{u}|_{1,K}^2\\ &+h_K^{-1}(\|\boldsymbol{u}-\Pi_h^{div}\boldsymbol{u}\|_{0,D_K}^2+h_K^2\|\operatorname{div}\boldsymbol{u}-Q_h\operatorname{div}\boldsymbol{u}\|_{0,D_K}^2)\\ &\lesssim h_K|\boldsymbol{u}|_{1,D_K}^2.\end{aligned}\tag{3.41}$$

Finally we prove the third inequality. From the inequality (3.40), it is obvious that

$$\sum_{K\in T_h}|\pi_h^{div}\boldsymbol{u}|_{1,K}^2\lesssim\sum_{K\in T_h}|\boldsymbol{u}|_{1,K}^2.\tag{3.42}$$

Now set $e=\partial K_1\cap\partial K_2$. Then from (3.41), we have

$$\begin{aligned}\|[\pi_h^{\rm div}\boldsymbol{u}]\|_{0,e}^2 &\lesssim \|[\pi_h^{\rm div}\boldsymbol{u}-\boldsymbol{u}]\|_{0,e}^2+\|[\boldsymbol{u}]\|_{0,e}^2\\ &\lesssim \|\pi_h^{\rm div}\boldsymbol{u}-\boldsymbol{u}\|_{0,\partial K_1}^2+\|\pi_h^{\rm div}\boldsymbol{u}-\boldsymbol{u}\|_{0,\partial K_2}^2+\|[\boldsymbol{u}]\|_{0,e}^2\\ &\lesssim h_{K_1}|\boldsymbol{u}|_{1,D_{K_1}}^2+h_{K_2}|\boldsymbol{u}|_{1,D_{K_2}}^2+\|[\boldsymbol{u}]\|_{0,e}^2.\end{aligned}\tag{3.43}$$

In view of the definition of the norm $\|\cdot\|_{1,h}$, combining the two inequalities (3.42) and (3.43) completes the proof of (3.37). □

The following result shows that the approximation proposed by the DG discretization (3.15) is uniform.

Theorem 3.2. *Let $\boldsymbol{u}$ be the solution of* (3.19) *and $\boldsymbol{u}_h$ be the solution of* (3.15). *Then we have the following estimate*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\epsilon,\rho}^2+\lambda\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{u}_h)\|^2\lesssim\inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\Big(\|\boldsymbol{u}-\boldsymbol{v}\|_{\epsilon,\rho}^2+\lambda\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2\Big).$$

Proof. If $(\boldsymbol{u},p)$ is the solution of the continuous problem (3.19) and $(\boldsymbol{u}_h,p_h)$ is the solution of the discrete problem (3.18) we have that $p=\lambda\operatorname{div}\boldsymbol{u}$, and, since $\operatorname{div}\boldsymbol{V}_h=S_h$ we also have that $p_h=\lambda\operatorname{div}\boldsymbol{u}_h$. The left hand side of the first equation in (3.18) then is given by the bilinear form (3.16), and, since this discrete problem is consistent, we have

$$A_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})=0,\ \text{for all }\boldsymbol{v}\in\boldsymbol{V}_h.$$

Consider now the interpolation $\pi_h^{\rm div}\boldsymbol{u}\in\boldsymbol{V}_h$ of $\boldsymbol{u}$ and set $q=\lambda\operatorname{div}\pi_h^{\rm div}\boldsymbol{u}$. Recall that $p=\lambda\operatorname{div}\boldsymbol{u}$, and $p_h=\lambda\operatorname{div}\boldsymbol{u}_h$ and hence (by Lemma 3.5) $q=\lambda\pi_h\operatorname{div}\boldsymbol{u}=\pi_h p$. Then with $\boldsymbol{e}_h=(\boldsymbol{u}_h-\pi_h^{\rm div}\boldsymbol{u})$ from the coercivity of $a_h(\cdot,\cdot)$ we get

$$\begin{aligned}&\|\boldsymbol{e}_h\|_{\epsilon,\rho}^2+\lambda^{-1}\|p_h-q\|^2=\epsilon^2\|\boldsymbol{e}_h\|_{DG}^2+\rho^2\|\boldsymbol{e}_h\|^2+\lambda\|\operatorname{div}\boldsymbol{e}_h\|^2\\ &\lesssim A_h(\boldsymbol{e}_h,\boldsymbol{e}_h)=A_h(\boldsymbol{u}-\pi_h^{\rm div}\boldsymbol{u},\boldsymbol{e}_h)=a_h(\boldsymbol{u}-\pi_h^{div}\boldsymbol{u},\boldsymbol{e}_h)+\lambda(\operatorname{div}(\boldsymbol{u}-\pi_h^{div}\boldsymbol{u}),\operatorname{div}\boldsymbol{e}_h)\\ &\lesssim\|\boldsymbol{u}-\pi_h^{\rm div}\boldsymbol{u}\|_{\epsilon,\rho}\|\boldsymbol{e}_h\|_{\epsilon,\rho}+\lambda^{-1}\|p-q\|\|p_h-q\|\\ &\lesssim(\|\boldsymbol{u}-\pi_h^{\rm div}\boldsymbol{u}\|_{\epsilon,\rho}^2+\lambda^{-1}\|p-q\|^2)^{1/2}(\|\boldsymbol{e}_h\|_{\epsilon,\rho}^2+\lambda^{-1}\|p_h-q\|^2)^{1/2},\end{aligned}\tag{3.44}$$

which means that

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\epsilon,\rho}^2+\lambda^{-1}\|p-p_h\|^2\lesssim\|\boldsymbol{u}-\pi_h^{\rm div}\boldsymbol{u}\|_{\epsilon,\rho}^2+\lambda^{-1}\|p-q\|^2.\tag{3.45}$$

Hence for any $\boldsymbol{v} \in \boldsymbol{V}_h$, we have

$$\begin{aligned} &\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\epsilon,\rho}^2 + \lambda\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{u}_h)\|^2 \\ &\lesssim \|\boldsymbol{u}-\boldsymbol{v}-\pi_h^{\text{div}}(\boldsymbol{u}-\boldsymbol{v})\|_{\epsilon,\rho}^2 + \lambda\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{v}) - \pi_h \operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2 \\ &\lesssim \|\boldsymbol{u}-\boldsymbol{v}\|_{\epsilon,\rho}^2 + \lambda\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2 + \|\pi_h^{\text{div}}(\boldsymbol{u}-\boldsymbol{v})\|_{\epsilon,\rho}^2 + \lambda\|\pi_h \operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2. \end{aligned} \tag{3.46}$$

By the definition of the norm $\|\cdot\|_{\epsilon,\rho}$ and Lemma 3.7, we get

$$\begin{aligned} &\|\pi_h^{\text{div}}(\boldsymbol{u}-\boldsymbol{v})\|_{\epsilon,\rho}^2 + \lambda\|\pi_h \operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2 \\ &\lesssim \epsilon^2\|\pi_h^{\text{div}}(\boldsymbol{u}-\boldsymbol{v})\|_{1,h}^2 + \rho^2\|\pi_h^{\text{div}}(\boldsymbol{u}-\boldsymbol{v})\|^2 + \lambda\|\pi_h \operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2 \\ &\lesssim \epsilon^2\|\boldsymbol{u}-\boldsymbol{v}\|_{1,h}^2 + \rho^2\|\pi_h^{\text{div}}(\boldsymbol{u}-\boldsymbol{v})\|^2 + \lambda\|\pi_h \operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2. \end{aligned} \tag{3.47}$$

Now, since $h \lesssim 1, \rho \le 1, \lambda \gtrsim 1$ we see that

$$\begin{aligned} &\|\pi_h^{\text{div}}(\boldsymbol{u}-\boldsymbol{v})\|_{\epsilon,\rho}^2 + \lambda\|\pi_h \operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2 \\ &\lesssim \epsilon^2\|\boldsymbol{u}-\boldsymbol{v}\|_{1,h}^2 + \rho^2(\|\boldsymbol{u}-\boldsymbol{v}\|^2 + h^2\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2) + \lambda\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2 \\ &\lesssim \epsilon^2\|\boldsymbol{u}-\boldsymbol{v}\|_{1,h}^2 + \rho^2\|\boldsymbol{u}-\boldsymbol{v}\|^2 + \rho^2 h^2\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2 + \lambda\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2 \\ &\lesssim \epsilon^2\|\boldsymbol{u}-\boldsymbol{v}\|_{1,h}^2 + \rho^2\|\boldsymbol{u}-\boldsymbol{v}\|^2 + \lambda\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2 \\ &\lesssim \|\boldsymbol{u}-\boldsymbol{v}\|_{\epsilon,\rho}^2 + \lambda\|\operatorname{div}(\boldsymbol{u}-\boldsymbol{v})\|^2, \end{aligned} \tag{3.48}$$

where we have used Lemma 3.6.

Taking the infimum over $\boldsymbol{v}$ then yields the desired result. □

Remark 3.4. *Let us set*

$$a_h^s(\boldsymbol{u},\boldsymbol{v}) = \sum_{K\in T_h}\int_K \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})dx - \sum_{e\in E_h}\int_e \{\varepsilon(\boldsymbol{u})\}\cdot[\boldsymbol{v}_t]ds - \sum_{e\in E_h}\int_e \{\varepsilon(\boldsymbol{v})\}\cdot[\boldsymbol{u}_t]ds + \sum_{e\in E_h}\int_e \eta h_e^{-1}[\boldsymbol{u}_t]\cdot[\boldsymbol{v}_t]ds,$$

and

$$B_\lambda((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)) = a_h^s(\boldsymbol{u}_h,\boldsymbol{v}_h) - (\operatorname{div}\boldsymbol{u}_h,q_h) - (\operatorname{div}\boldsymbol{v}_h,p_h) - \lambda^{-1}(p_h,q_h).$$

Then for any given $(\boldsymbol{u}_h,p_h)$, choosing $(\boldsymbol{v}_h,q_h) = (\boldsymbol{u}_h,-p_h)$, by the coercivity of $a_h(\cdot,\cdot)$ with $\epsilon = 1, \rho = 0$, it is straightforward to show that the inf-sup condition for $B_\lambda(\cdot,\cdot)$ holds, namely, for any $(\boldsymbol{u}_h,p_h) \in \boldsymbol{V}_h \times S_h$ we have

$$\sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h\times S_h} \frac{B_\lambda((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h))}{\|\boldsymbol{v}_h\|_{1,h} + \lambda^{-1/2}\|q_h\|} \gtrsim \|\boldsymbol{u}_h\|_{1,h} + \lambda^{-1/2}\|p_h\|. \tag{3.49}$$

For the Stokes equation, we have from [26, Theorem 8.2.1] *and* [29, 30] *that*

$$\sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h\times S_h} \frac{B_\infty((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h))}{\|\boldsymbol{v}_h\|_{1,h} + \|q_h\|} \gtrsim \|\boldsymbol{u}_h\|_{1,h} + \|p_h\|. \tag{3.50}$$

3.4. **An *a priori* estimate for the discrete problem.** The next lemma is an *a priori* estimate of the L^2-norm of the solution of a discrete problem which is later used to prove the so called "smoothing property" – an essential part in the multigrid convergence analysis. We state and prove this estimate here (before the multigrid analysis), since it could be of independent interest.

We consider the finite element spaces introduced earlier: $\boldsymbol{V}_h \subset H(\operatorname{div};\Omega)$ and $S_h \subset L_0^2(\Omega)$. Let $\boldsymbol{w}_1 \in \boldsymbol{V}_h$ and $\boldsymbol{w}_2 \in \boldsymbol{V}_h$ be given and let $\widetilde{\boldsymbol{u}} \in \boldsymbol{V}_h, \widetilde{p} \in S_h$ solve the discrete problem

$$\begin{aligned} a_h^s(\widetilde{\boldsymbol{u}},\boldsymbol{v}) - (\operatorname{div}\boldsymbol{v},\widetilde{p}) &= a_h^s(\boldsymbol{w}_1,\boldsymbol{v}), \qquad &&\text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}_h, \\ (\operatorname{div}\widetilde{\boldsymbol{u}},q) &= (\operatorname{div}\boldsymbol{w}_2,q), &&\text{for all} \quad q \in S_h. \end{aligned} \tag{3.51}$$

We note that the inf-sup condition (3.50) implies that

$$\begin{aligned}\|\widetilde{\boldsymbol{u}}\|_{1,h}+\|\widetilde{p}\| &\lesssim \sup_{(\boldsymbol{v},q)\in \boldsymbol{V}_h\times S_h}\frac{a_h^s(\widetilde{\boldsymbol{u}},\boldsymbol{v})-(\operatorname{div}\boldsymbol{v},\widetilde{p})-(\operatorname{div}\widetilde{\boldsymbol{u}},q)}{\|\boldsymbol{v}\|_{1,h}+\|q\|}\\ &= \sup_{(\boldsymbol{v},q)\in \boldsymbol{V}_h\times S_h}\frac{a_h^s(\boldsymbol{w}_1,\boldsymbol{v})-(\operatorname{div}\boldsymbol{w}_2,q)}{\|\boldsymbol{v}\|_{1,h}+\|q\|}\lesssim \|\boldsymbol{w}_1\|_{1,h}+\|\operatorname{div}\boldsymbol{w}_2\|. \end{aligned}\tag{3.52}$$

Lemma 3.8. *For the solution of* (3.51) *we have the estimate:*

$$\|\widetilde{\boldsymbol{u}}\|\lesssim \|\boldsymbol{w}_1\|+\|\operatorname{div}\boldsymbol{w}_2\|_{-1}.\tag{3.53}$$

Proof. We consider the following dual problem: Find $\boldsymbol{\phi}\in (H_0^1(\Omega))^d$ and $\theta\in L_0^2(\Omega)$ such that

$$\begin{aligned} a^s(\boldsymbol{v},\boldsymbol{\phi})-(\operatorname{div}\boldsymbol{v},\theta)&=(\widetilde{\boldsymbol{u}},\boldsymbol{v}), &&\text{for all}\quad \boldsymbol{v}\in (H_0^1(\Omega))^d,\\ (\operatorname{div}\boldsymbol{\phi},q)&=0, &&\text{for all}\quad q\in L_0^2(\Omega),\end{aligned}\tag{3.54}$$

where $a^s(\boldsymbol{v},\boldsymbol{\phi})=\int_\Omega \varepsilon(\boldsymbol{v}):\varepsilon(\boldsymbol{\phi})dx$.

Let Π_h^{div} be the interpolation operator introduced earlier in Section 3.3. Recall that $\operatorname{div}\boldsymbol{\phi}=0$ and hence $(\operatorname{div}\Pi_h^{\text{div}}\boldsymbol{\phi},\widetilde{p})=0$. From equations (3.51) we then have

$$\begin{aligned} 0 &= a_h^s(\boldsymbol{w}_1,\Pi_h^{\text{div}}\boldsymbol{\phi})-a_h^s(\widetilde{\boldsymbol{u}},\Pi_h^{\text{div}}\boldsymbol{\phi})+(\operatorname{div}\Pi_h^{\text{div}}\boldsymbol{\phi},\widetilde{p})\\ &= a_h^s(\boldsymbol{w}_1,\boldsymbol{\phi})-a_h^s(\boldsymbol{w}_1,\boldsymbol{\phi}-\Pi_h^{\text{div}}\boldsymbol{\phi})-a_h^s(\widetilde{\boldsymbol{u}},\Pi_h^{\text{div}}\boldsymbol{\phi}).\end{aligned}\tag{3.55}$$

Observing that $a^s(\boldsymbol{\phi},\boldsymbol{v})=a_h^s(\boldsymbol{\phi},\boldsymbol{v})$ for all $\boldsymbol{v}\in\boldsymbol{V}_h$, from (3.54) and (3.55) we obtain

$$\|\widetilde{\boldsymbol{u}}\|^2=a_h^s(\boldsymbol{\phi},\widetilde{\boldsymbol{u}})-(\operatorname{div}\widetilde{\boldsymbol{u}},\theta)+a_h^s(\boldsymbol{w}_1,\boldsymbol{\phi})-a_h^s(\boldsymbol{w}_1,\boldsymbol{\phi}-\Pi_h^{\text{div}}\boldsymbol{\phi})-a_h^s(\widetilde{\boldsymbol{u}},\Pi_h^{\text{div}}\boldsymbol{\phi}).\tag{3.56}$$

Combining the first and the last term, using the triangle inequality and the continuity of $a_h^s(\cdot,\cdot)$ then shows that

$$\begin{aligned}\|\widetilde{\boldsymbol{u}}\|^2 &\le |(\operatorname{div}\widetilde{\boldsymbol{u}},\theta)|+|a_h^s(\boldsymbol{w}_1,\boldsymbol{\phi})|+|a_h^s(\boldsymbol{w}_1,\boldsymbol{\phi}-\Pi_h^{\text{div}}\boldsymbol{\phi})|+|a_h^s(\widetilde{\boldsymbol{u}},\boldsymbol{\phi}-\Pi_h^{\text{div}}\boldsymbol{\phi})|\\ &\lesssim |(\operatorname{div}\widetilde{\boldsymbol{u}},\theta)|+|a_h^s(\boldsymbol{w}_1,\boldsymbol{\phi})|+(\|\boldsymbol{w}_1\|_{1,h}+\|\widetilde{\boldsymbol{u}}\|_{1,h})\|\boldsymbol{\phi}-\Pi_h^{\text{div}}\boldsymbol{\phi}\|_{1,h}.\end{aligned}$$

As we have that $\operatorname{div}\widetilde{\boldsymbol{u}}=\operatorname{div}\boldsymbol{w}_2$ for the first term on the right side we get

$$|(\operatorname{div}\widetilde{\boldsymbol{u}},\theta)|=|(\operatorname{div}\boldsymbol{w}_2,\theta)|\le\|\theta\|_1\sup_{\chi\in H^1}\frac{(\operatorname{div}\boldsymbol{w}_2,\chi)}{\|\chi\|_1}=\|\operatorname{div}\boldsymbol{w}_2\|_{-1}\|\theta\|_1.$$

For the second term, by the regularity estimate (2.4) with $\epsilon=1,\rho=0$, we have that $\boldsymbol{\phi}\in(H^2(\Omega))^d$, and, thus, $\boldsymbol{\phi}$ is continuous and $[\boldsymbol{\phi}]=0$. Now, integrating by parts and combining the interface terms from neighboring elements shows that

$$\begin{aligned} a_h^s(\boldsymbol{\phi},\boldsymbol{w}_1) &= \sum_{K\in T_h}\int_K\varepsilon(\boldsymbol{\phi}):\varepsilon(\boldsymbol{w}_1)dx-\sum_{e\in E_h}\int_e\{\varepsilon(\boldsymbol{\phi})\}\cdot[(\boldsymbol{w}_1)_t]ds\\ &\quad -\sum_{e\in E_h}\int_e\{\varepsilon(\boldsymbol{w}_1)\}\cdot[\boldsymbol{\phi}_t]ds+\sum_{e\in E_h}\int_e\eta h_e^{-1}[\boldsymbol{\phi}_t]\cdot[(\boldsymbol{w}_1)_t]ds,\\ &= \sum_{K\in T_h}\int_K\varepsilon(\boldsymbol{\phi}):\varepsilon(\boldsymbol{w}_1)dx-\sum_{e\in E_h}\int_e\{\varepsilon(\boldsymbol{\phi})\}\cdot[(\boldsymbol{w}_1)_t]ds\\ &= -\sum_{T\in\mathcal{T}_h}\int_T\operatorname{div}\varepsilon(\boldsymbol{\phi})\cdot\boldsymbol{w}_1\le\|\boldsymbol{\phi}\|_2\|\boldsymbol{w}_1\|.\end{aligned}$$

Finally, the desired result follows from the interpolation estimates in Lemma 4.3, the regularity estimate $\|\boldsymbol{\phi}\|_2+\|\theta\|_1\lesssim\|\widetilde{\boldsymbol{u}}\|$, inequality (3.52), and the inverse inequalities $\|\boldsymbol{w}_1\|_{1,h}\lesssim h^{-1}\|\boldsymbol{w}_1\|$ and $\|\operatorname{div}\boldsymbol{w}_2\|\lesssim h^{-1}\|\operatorname{div}\boldsymbol{w}_2\|_{-1}$. □

4. Multigrid method

In this section, we design a multigrid algorithm to solve the discrete system (3.15)–(3.16) arising from the DG discretization of the equation $-\epsilon^2 \operatorname{div} \varepsilon(\boldsymbol{u}) + \rho^2 \boldsymbol{u} + \lambda \nabla\nabla \cdot \boldsymbol{u} = \boldsymbol{F}$. We will show that the algorithm is robust with respect to the parameters ϵ, ρ, λ when $h_k \leq \epsilon^2$ or $\lambda \gtrsim \min\{\epsilon^{-1}, h_k^{-1}\}$. Hence, by combining it with the Uzawa method and choosing λ to be large enough, we can also solve the discrete system (3.9) arising from the DG discretization of the Brinkman problem very efficiently.

4.1. **Preliminaries.** Let $\{T_k\}_{k=0}^J$ be a family of partitions of Ω and denote the finest partition by $T_h = T_J$. The edges (faces) of T_k are denoted by E_k. We assume that all the partitions $T_k, k = 0, 1, \ldots, J$, are quasi-uniform with characteristic mesh size h_k and $h_k = \gamma h_{k-1}$, $\gamma \in (0, 1)$, where $h_0 = \mathcal{O}(1)$. Note that the penalty term in the bilinear form $a_h(\cdot,\cdot)$ depends on the mesh size of the partition.

Thus, for every partition T_k we need to specify the space $\boldsymbol{V}_h$ at level k. A natural choice is the space $\boldsymbol{M}_k$ defined as follows:

$$\boldsymbol{M}_k = \{\boldsymbol{v} \in H(\mathrm{div};\Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}(K),\ K \in T_k;\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}.$$

Moreover, we denote the pressure space S_h at level k by

$$S_k = \left\{ q \in L^2(\Omega) : q|_K \in Q(K), \quad K \in T_k;\ \int_\Omega q dx = 0 \right\}.$$

Thus, corresponding to the set of refined triangulations $\{T_k\}_{k=0}^J$, we also have a sequence of nested $H(\mathrm{div},\Omega)$-conforming finite element spaces

$$\boldsymbol{M}_0 \subseteq \boldsymbol{M}_1 \subseteq \boldsymbol{M}_2 \subseteq \cdots \subseteq \boldsymbol{M}_J \subseteq H(\mathrm{div},\Omega).$$

With every space we associate a bilinear form $a_k(\cdot,\cdot)$ which corresponds to $a_h(\cdot,\cdot)$ at level k, i.e.,

$$\begin{aligned} a_k(\boldsymbol{u},\boldsymbol{v}) = \epsilon^2 \Big(\sum_{K\in T_k} \int_K \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) dx - \sum_{e\in E_k} \int_e \{\varepsilon(\boldsymbol{u})\}\cdot[\boldsymbol{v}_{\boldsymbol{t}}] ds \\ - \sum_{e\in E_k} \int_e \{\varepsilon(\boldsymbol{v})\}\cdot[\boldsymbol{u}_{\boldsymbol{t}}] ds + \sum_{e\in E_k} \int_e \eta h_k^{-1} [\boldsymbol{u}_t]\cdot[\boldsymbol{v}_t] ds \Big) + \rho^2(\boldsymbol{u},\boldsymbol{v}). \end{aligned}$$

Adding the divergence term then gives the bilinear form corresponding to $A_h(\cdot,\cdot)$ at level k i.e.,

$$A_k(\boldsymbol{u},\boldsymbol{v}) = a_k(\boldsymbol{u},\boldsymbol{v}) + \lambda(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v}), \quad \text{for all} \quad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{M}_k.$$

Our goal is to analyze the variable V-cycle and W-cycle multigrid algorithms for the solution of the following problem: Given $\boldsymbol{F} \in \boldsymbol{M}_J$, find $\boldsymbol{v} \in \boldsymbol{M}_J$ satisfying

$$A_J(\boldsymbol{v},\boldsymbol{\phi}) = (\boldsymbol{F},\boldsymbol{\phi}), \quad \text{for all} \quad \boldsymbol{\phi} \in \boldsymbol{M}_J. \tag{4.1}$$

To define the algorithm, let us first clarify the notation. For $k = 0, \cdots, J$, define the operator $\mathbb{A}_k : \boldsymbol{M}_k \to \boldsymbol{M}_k$ by

$$(\mathbb{A}_k \boldsymbol{w},\boldsymbol{\phi}) = A_k(\boldsymbol{w},\boldsymbol{\phi}), \quad \text{for all} \quad \boldsymbol{\phi} \in \boldsymbol{M}_k.$$

The norms on $\boldsymbol{M}_k$ induced by $A_k(\cdot,\cdot)$ and $a_k(\cdot,\cdot)$ are denoted by $\|\cdot\|_{A_k}^2$, and $\|\cdot\|_{a_k}^2$ respectively, i.e.,

$$\|\boldsymbol{u}\|_{A_k}^2 = A_k(\boldsymbol{u},\boldsymbol{u}), \quad \|\boldsymbol{u}\|_{a_k}^2 = a_k(\boldsymbol{u},\boldsymbol{u}), \quad \text{for all} \quad \boldsymbol{u} \in \boldsymbol{M}_k.$$

We also need the L^2-orthogonal projections on $\boldsymbol{M}_k$, and S_k, which will be denoted by $\boldsymbol{Q}_k : \boldsymbol{L}^2(\Omega) \mapsto \boldsymbol{M}_k$ and $Q_k : \boldsymbol{L}^2(\Omega) \mapsto S_k$ and the canonical interpolation $\Pi_k : [H_0^1(\Omega)]^2 \mapsto \boldsymbol{M}_k$. According to the notation of the previous section, Π_k and Q_k are just shorthands for $\Pi_{h_k}^{\mathrm{div}}$ and Q_{h_k}, i.e.

$\Pi_k = \Pi^{\text{div}}_{h_k}, Q_k = Q_{h_k}$, and we recall that $Q_k \operatorname{div} = \operatorname{div} \Pi_k$. Further, we introduce the operator $P_{k-1}: \boldsymbol{M}_k \to \boldsymbol{M}_{k-1}$ defined by

$$A_{k-1}(P_{k-1}\boldsymbol{w}, \boldsymbol{\phi}) = A_k(\boldsymbol{w}, \boldsymbol{\phi}), \quad \text{for all} \quad \boldsymbol{\phi} \in \boldsymbol{M}_{k-1}. \tag{4.2}$$

Noting that the bilinear forms A_k are nonnested, P_{k-1} is not a projection. Finally, we denote the norm $\|\cdot\|_{1,h}$ at level k as $\|\cdot\|_{1,k}$.

To define the smoothing process, we require linear operators $R_k : \boldsymbol{M}_k \to \boldsymbol{M}_k$ for $k = 1, \cdots, J$. These operators may be symmetric or nonsymmetric with respect to the inner product $(\cdot,\cdot)$. If R_k is nonsymmetric, then we define R_k^t to be its adjoint and set

$$R_k^{(l)} = \begin{cases} R_k \text{ if } l \text{ is odd,} \\ R_k^t \text{ if } l \text{ is even.} \end{cases}$$

4.2. **Multigrid algorithm.** The multigrid operator $\mathbb{B}_k : \boldsymbol{M}_k \to \boldsymbol{M}_k$ is defined by induction and is given as follows, see, e.g., [31].

Multigrid algorithm Set $\mathbb{B}_0 = \mathbb{A}_0^{-1}$. Assume that $\mathbb{B}_{k-1}$ has already been defined, let $\mathbb{B}_k \boldsymbol{g}$ for $\boldsymbol{g} \in \boldsymbol{M}_k$ be defined as follows:

(1) Set $\boldsymbol{x}^0 = 0$ and $\boldsymbol{q}^0 = 0$.
(2) Define $\boldsymbol{x}^l$ for $l = 1, \cdots, m(k)$ by

$$\boldsymbol{x}^l = \boldsymbol{x}^{l-1} + R_k^{(l+m(k))}(\boldsymbol{g} - \mathbb{A}_k \boldsymbol{x}^{l-1}). \tag{4.3}$$

(3) Set $\boldsymbol{y}^{m(k)} = \boldsymbol{x}^{m(k)} + \boldsymbol{q}^p$, where $\boldsymbol{q}^i$ for $i = 1, \cdots, p$ is defined by

$$\boldsymbol{q}^i = \boldsymbol{q}^{i-1} + \mathbb{B}_{k-1}[\boldsymbol{Q}_{k-1}(\boldsymbol{g} - \mathbb{A}_k \boldsymbol{x}^{m(k)}) - \mathbb{A}_{k-1}\boldsymbol{q}^{i-1}]. \tag{4.4}$$

(4) Define $\boldsymbol{y}^l$ for $l = m(k)+1, \cdots, 2m(k)$ by

$$\boldsymbol{y}^l = \boldsymbol{y}^{l-1} + R_k^{(l+m(k))}(\boldsymbol{g} - \mathbb{A}_k \boldsymbol{y}^{l-1}).$$

(5) Set $\mathbb{B}_k \boldsymbol{g} = \boldsymbol{y}^{2m(k)}$.

In this algorithm, $m(k)$ is a positive integer which may vary from level to level and determines the number of smoothing iterations at that level and p is another positive integer. We shall study the cases $p = 1$ and $p = 2$, which correspond to the symmetric V- and W-cycles of the multigrid algorithm, respectively.

4.3. **Multigrid convergence.** Set $K_k = I - R_k \mathbb{A}_k$, then $K_k^* = I - R_k^t \mathbb{A}_k$ is the adjoint of K_k with respect to $A_k(\cdot,\cdot)$. Further, set

$$\widetilde{K}_k^{(m)} = \begin{cases} (K_k^* K_k)^{m/2} \text{ if } l \text{ is odd,} \\ (K_k^* K_k)^{(m-1)/2} K_k^* \text{ if } l \text{ is even,} \end{cases}$$

and denote by $(\widetilde{K}_k^{(m)})^*$ the adjoint of $\widetilde{K}_k^{(m)}$ with respect to $A_k(\cdot,\cdot)$.

For convergence estimates, we shall make the following a priori assumptions:

- (A0) The spectrum of $K_k^* K_k$ is in the interval $[0, 1)$.

In order to analyze the approximation property and the smoothing property of the multigrid algorithm, for any $\boldsymbol{u} \in \boldsymbol{M}_k$, we need to define a norm at level k which is similar to the one in [17] and given by

(4.5)

$$\|\boldsymbol{u}\|_{k,0}^2 := \begin{cases} \epsilon^2 \|\boldsymbol{u}\|^2 + \rho^2 h_k^2 \|\boldsymbol{u}\|^2 + \lambda h_k^2 \|\operatorname{div} \boldsymbol{u}\|^2 + \lambda^2 \max\{h_k^2, \epsilon^2\} \|Q_{k-1} \operatorname{div} \boldsymbol{u}\|^2 \quad \text{if } h_k \geq \epsilon^2, \\ \epsilon^2 \|\boldsymbol{u}\|^2 + \rho^2 h_k^2 \|\boldsymbol{u}\|^2 + \lambda h_k^2 \|\operatorname{div} \boldsymbol{u}\|^2 + \lambda^2 \epsilon^{-2} h_k^2 \|Q_{k-1} \operatorname{div} \boldsymbol{u}\|^2 \quad \text{if } h_k \leq \epsilon^2. \end{cases}$$

The second assumption is an approximation property in $\|\cdot\|_{k,0}$-norm (also known as approximation and regularity assumption, cf [31]), i.e.,

- (A1) $\|(I-P_{k-1})\boldsymbol{u}\|_{k,0} \lesssim h_k\|\boldsymbol{u}\|_{A_k}$, for all $\boldsymbol{u}\in \boldsymbol{M}_k$.

The third assumption is a requirement on the smoother (smoothing property) and reads,

- (A2) $\|(\widetilde{K}_k^{(m)})^*\boldsymbol{u}\|_{A_k} \lesssim m^{-1/4}h_k^{-1}\|\boldsymbol{u}\|_{k,0}$, for all $\boldsymbol{u}\in \boldsymbol{M}_k$.

The next Lemma is an analogue of a result given in Bramble, Pasciak, Xu [31, Lemma 4.1]. We include its short proof for the sake of self-containedness.

Lemma 4.1. *Assume that (A0), (A1) and (A2) hold and let $\widetilde{\boldsymbol{u}} = \widetilde{K}_k^{(m)}\boldsymbol{u}$. Then we have the estimate*

$$-A_k((I-P_{k-1})\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{u}}) \lesssim m^{-1/4}\|\boldsymbol{u}\|_{A_k}^2, \quad \textit{for all} \quad \boldsymbol{u}\in\boldsymbol{M}_k.$$

Proof. By the Cauchy-Schwarz inequality and assumption (A2), we have

$$\begin{aligned} -A_k((I-P_{k-1})\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{u}}) &= -A_k((I-P_{k-1})\widetilde{K}_k^{(m)}\boldsymbol{u},\widetilde{K}_k^{(m)}\boldsymbol{u}) \\ &= -A_k((\widetilde{K}_k^{(m)})^*(I-P_{k-1})\widetilde{K}_k^{(m)}\boldsymbol{u},\boldsymbol{u}) \\ &\le \|(\widetilde{K}_k^{(m)})^*(I-P_{k-1})\widetilde{\boldsymbol{u}}\|_{A_k}\|\boldsymbol{u}\|_{A_k} \\ &\lesssim m^{-1/4}h_k^{-1}\|(I-P_{k-1})\widetilde{\boldsymbol{u}}\|_{k,0}\|\boldsymbol{u}\|_{A_k}. \end{aligned}$$

Next, by assumptions (A1) and (A0) (applied in that order) we have

$$\begin{aligned} -A_k((I-P_{k-1})\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{u}}) &\lesssim m^{-1/4}h_k^{-1}\|(I-P_{k-1})\widetilde{\boldsymbol{u}}\|_{k,0}\|\boldsymbol{u}\|_{A_k} \\ &\lesssim m^{-1/4}\|\widetilde{\boldsymbol{u}}\|_{A_k}\|\boldsymbol{u}\|_{A_k} \lesssim m^{-1/4}\|\boldsymbol{u}\|_{A_k}^2. \end{aligned}$$

□

The estimate in Lemma 4.1 provides the prerequisite to apply the general theory developed in [31]. Indeed, according to [31], assumptions (A0), (A1) and (A2) and Lemma 4.1 are sufficient to show spectral equivalence for the variable V-cycle multigrid preconditioner (Theorem 4.1) and a uniform convergence result for the W-cycle multigrid method (Theorem 4.2). The first result is just a restatement of [31, Theorem 6] under full regularity.

Theorem 4.1 (Theorem 6 in [31])**.** *Assume that (A0), (A1) and (A2) hold and define $\mathbb{B}_j$ in Algorithm 4.2 with $p=1$. Further assume that the number of smoothing steps $m(k)$ satisfies $\beta_3 m(k)\le m(k-1)\le \beta_4 m(k)$ with $\beta_3\ge 1$ and $\beta_4>1$ independent of k. Then the following spectral equivalence holds*

$$\eta_0 A_k(\boldsymbol{u},\boldsymbol{u}) \le A_k(\mathbb{B}_k\mathbb{A}_k\boldsymbol{u},\boldsymbol{u}) \le \eta_1 A_k(\boldsymbol{u},\boldsymbol{u}) \textit{ for all } \boldsymbol{u}\in\boldsymbol{M}_k. \tag{4.6}$$

with constants η_0 and η_1 such that

$$\eta_0 \ge \frac{m(k)^\alpha}{M+m(k)^\alpha} \textit{ and } \eta_1 \le \frac{M+m(k)^\alpha}{m(k)^\alpha},$$

where M is independent of ϵ,ρ,λ and h, and α denotes the regularity index.

The convergence analysis of the W-cycle is also been conducted in [31].

Theorem 4.2 (Theorem 4 in [31])**.** *Assume that (A0), (A1) and (A2) hold and that the number of smoothing steps $m(k)=m$ is constant for all k. Then, for sufficiently large m, $\mathbb{B}_k$ defined via the W-cycle algorithm (p=2) satisfies*

$$|A_k((I-\mathbb{B}_k\mathbb{A}_k)\boldsymbol{u},\boldsymbol{u})| \le \frac{M}{M+m^\alpha}\|\boldsymbol{u}\|_{A_k}^2 \quad \textit{for all} \quad \boldsymbol{u}\in\boldsymbol{M}_k$$

with M independent of ϵ,ρ,λ and h, where α denotes the regularity index.

We remark here that modifying assumption (A1) one can prove the results above for the case of less than full elliptic regularity. For details we refer to Bramble, Pasciak and Xu [31].

As we have seen, the estimates in Theorems 4.1-4.2 are valid if assumptions (A0), (A1) and (A2) are verified. In the next subsections we show that these assumptions hold in our framework.

4.4. **Approximation property.** In this subsection, we verify (A1). One of the difficulties in the analysis is that the bilinear forms $A_k(\cdot,\cdot), k=1,\cdots,J$ are not nested. We now prove a simple relation between $A_k(\cdot,\cdot)$ and $A_{k-1}(\cdot,\cdot)$.

Lemma 4.2. *If $h_k=\gamma h_{k-1}$, $\gamma\in(0,1)$, then*

$$\|\boldsymbol{u}\|^2_{A_{k-1}} \le \|\boldsymbol{u}\|^2_{A_k} \lesssim \|\boldsymbol{u}\|^2_{A_{k-1}}, \quad \textit{for all} \quad \boldsymbol{u}\in \boldsymbol{M}_{k-1}. \tag{4.7}$$

Proof. Let $\boldsymbol{u}\in\boldsymbol{M}_{k-1}$. Observe that $[\boldsymbol{u}_t]_e=0$ for edges (faces) $e\in E_k$ which are interior to the elements in T_{k-1}, because $\boldsymbol{u}$ is a continuous, in fact a polynomial, function in each element from T_{k-1}. Hence,

$$\sum_{e\in E_{k-1}}\int_e \eta\gamma^{-1}h_{k-1}^{-1}|[\boldsymbol{u}_t]|^2 ds = \sum_{e\in E_k}\int_e \eta h_k^{-1}|[\boldsymbol{u}_t]|^2 ds, \quad \text{for all} \quad \boldsymbol{u}\in\boldsymbol{M}_{k-1}$$

and we have

$$\begin{aligned} A_k(\boldsymbol{u},\boldsymbol{u}) &= A_{k-1}(\boldsymbol{u},\boldsymbol{u}) + \epsilon^2\Big(\sum_{e\in E_k}\int_e \eta h_k^{-1}|[\boldsymbol{u}_t]|^2 ds - \sum_{e\in E_{k-1}}\int_e \eta h_{k-1}^{-1}|[\boldsymbol{u}_t]|^2 ds\Big) \\ &= A_{k-1}(\boldsymbol{u},\boldsymbol{u}) + \epsilon^2(\gamma^{-1}-1)\sum_{e\in E_{k-1}}\int_e \eta h_{k-1}^{-1}|[\boldsymbol{u}_t]|^2 ds. \end{aligned}$$

The estimates in (4.7) then easily follow from the identity above. □

Remark 4.1. *From Lemma 4.2, for any given $\boldsymbol{u}\in\boldsymbol{M}_k$, we also have*

$$\begin{aligned} \|P_{k-1}\boldsymbol{u}\|^2_{A_{k-1}} &\le \|P_{k-1}\boldsymbol{u}\|^2_{A_k} = A_k(\boldsymbol{u},P_{k-1}\boldsymbol{u}) \le \|\boldsymbol{u}\|_{A_k}\|P_{k-1}\boldsymbol{u}\|_{A_k} \\ &\lesssim \|\boldsymbol{u}\|_{A_k}\|P_{k-1}\boldsymbol{u}\|_{A_{k-1}}, \end{aligned}$$

namely,

$$\|P_{k-1}\boldsymbol{u}\|_{A_{k-1}} \lesssim \|\boldsymbol{u}\|_{A_k}. \tag{4.8}$$

We now introduce the dual problem for the primal problem (3.17) : Find $\boldsymbol{w}\in H_0^1(\Omega)^d$ such that

$$\epsilon^2(\varepsilon(\boldsymbol{v}):\varepsilon(\boldsymbol{w})) + \rho^2(\boldsymbol{v},\boldsymbol{w}) + \lambda(\operatorname{div}\boldsymbol{v},\operatorname{div}\boldsymbol{w}) = (\boldsymbol{G},\boldsymbol{v}), \quad \text{for all} \quad \boldsymbol{v}\in H_0^1(\Omega)^d \tag{4.9}$$

under the following regularity assumption on $\boldsymbol{w}$:

$$\epsilon^2\|\boldsymbol{w}\|_2 + \epsilon\rho\|\boldsymbol{w}\|_1 + \lambda\|\operatorname{div}\boldsymbol{w}\|_1 \lesssim \|\boldsymbol{G}\|. \tag{4.10}$$

Remark 4.2. *The regularity property* (4.10) *holds if Ω satisfies the conditions of Lemma 2.1 (see* [32, 33]*).*

From the definitions of the bilinear forms $A_{k-1}(\cdot,\cdot)$ and $A_k(\cdot,\cdot)$ we have the following simple identity for the solution $\boldsymbol{w}$ of (4.9):

$$A_k(\boldsymbol{v},\boldsymbol{w}) = A_{k-1}(\boldsymbol{v},\boldsymbol{w}), \text{ for all } \quad \boldsymbol{v}\in\boldsymbol{M}_{k-1}. \tag{4.11}$$

This follows immediately, since both $A_{k-1}(\cdot,\cdot)$ and $A_k(\cdot,\cdot)$ are consistent. Indeed, for any $\boldsymbol{v}\in\boldsymbol{M}_{k-1}\subset\boldsymbol{M}_k$ we have $A_k(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{G},\boldsymbol{v})=A_{k-1}(\boldsymbol{v},\boldsymbol{w})$, which proves (4.11).

The next lemma provides estimates on the interpolation error.

Lemma 4.3. *Let $\boldsymbol{w} \in H^{l+1}(\Omega)^d$, $l = 0, 1$, and $\Pi_{k-1}\boldsymbol{w}$ be the interpolant of $\boldsymbol{w}$ in $\boldsymbol{M}_{k-1}$, then*

$$\begin{aligned} \|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|_{A_{k-1}}^2 &\lesssim h_{k-1}^{2l}(\epsilon^2|\boldsymbol{w}|_{l+1}^2 + \rho^2|\boldsymbol{w}|_l^2 + \lambda|\operatorname{div}\boldsymbol{w}|_l^2), \\ \|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|_{A_k}^2 &\lesssim h_{k-1}^{2l}(\epsilon^2|\boldsymbol{w}|_{l+1}^2 + \rho^2|\boldsymbol{w}|_l^2 + \lambda|\operatorname{div}\boldsymbol{w}|_l^2). \end{aligned} \tag{4.12}$$

Proof. By the continuity of $a_k(\cdot,\cdot)$, the trace theorem and the interpolation error estimate (3.4), we have

$$\|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|_{a_{k-1}}^2 \lesssim \epsilon^2\|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|_{DG}^2 + \rho^2\|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|^2 \lesssim h_{k-1}^{2l}(\epsilon^2|\boldsymbol{w}|_{l+1}^2 + \rho^2|\boldsymbol{w}|_l^2).$$

Noting $\operatorname{div}\Pi_{k-1}\boldsymbol{w} = Q_{k-1}\operatorname{div}\boldsymbol{w}$, by the standard approximation error estimate of the projection Q_{k-1}, we have

$$\|\operatorname{div}(\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w})\|^2 = \|\operatorname{div}\boldsymbol{w} - Q_{k-1}\operatorname{div}\boldsymbol{w}\|^2 \lesssim h_{k-1}^{2l}|\operatorname{div}\boldsymbol{w}|_l^2.$$

Combining the above two inequalities and noting the definition of the norm $\|\cdot\|_{A_{k-1}}$, we get the first inequality in (4.12). The proof of the second inequality in (4.12) is carried out in a similar fashion. □

For the verification of the approximation property we will need the two following two-level estimates.

Theorem 4.3. *For all $\boldsymbol{u} \in \boldsymbol{M}_k$ we have*

$$\epsilon\|(I - P_{k-1})\boldsymbol{u}\| \lesssim h_k\|\boldsymbol{u}\|_{A_k}, \;\; \rho\|(I - P_{k-1})\boldsymbol{u}\| \lesssim \|\boldsymbol{u}\|_{A_k}. \tag{4.13}$$

Proof. We first estimate $\epsilon\|(I - P_{k-1})\boldsymbol{u}\|$ using a standard duality argument. Let $\boldsymbol{w} \in H_0^1(\Omega)^d$ be the solution of the dual problem (4.9) with $\boldsymbol{G} = \boldsymbol{u} - P_{k-1}\boldsymbol{u}$. Since, $A_k(\cdot,\cdot)$ is a consistent bilinear form, we have

$$A_k(\boldsymbol{w}, \boldsymbol{v}) = (\boldsymbol{u} - P_{k-1}\boldsymbol{u}, \boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \boldsymbol{M}_k.$$

Now let $\boldsymbol{v} = \boldsymbol{u} - P_{k-1}\boldsymbol{u}$ and $\Pi_{k-1}\boldsymbol{w}$ be the interpolant of $\boldsymbol{w}$ in $\boldsymbol{M}_{k-1}$. Using the fact that $A_k(\cdot,\cdot)$ is symmetric for all $k = 1, \cdots, J$, identity (4.11), and the definition of the operator P_{k-1}, we have

$$\begin{aligned} \|\boldsymbol{u} - P_{k-1}\boldsymbol{u}\|^2 &= A_k(\boldsymbol{w}, \boldsymbol{u} - P_{k-1}\boldsymbol{u}) \\ &= A_k(\boldsymbol{u}, \boldsymbol{w}) - A_k(\boldsymbol{w}, P_{k-1}\boldsymbol{u}) = A_k(\boldsymbol{u}, \boldsymbol{w}) - A_{k-1}(\boldsymbol{w}, P_{k-1}\boldsymbol{u}) \\ &= A_k(\boldsymbol{u}, \boldsymbol{w}) - A_{k-1}(P_{k-1}\boldsymbol{u}, \boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}) - A_{k-1}(P_{k-1}\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w}) \\ &= A_k(\boldsymbol{u}, \boldsymbol{w}) - A_{k-1}(P_{k-1}\boldsymbol{u}, \boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}) - A_k(\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w}) \\ &= A_k(\boldsymbol{u}, \boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}) - A_{k-1}(P_{k-1}\boldsymbol{u}, \boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}). \end{aligned} \tag{4.14}$$

Applying the Cauchy-Schwarz inequality to the right hand side of the identity above and using the approximation estimates (4.12), inequality (4.8), and the regularity estimate (4.10), leads to

$$\begin{aligned} \|\boldsymbol{u} - P_{k-1}\boldsymbol{u}\|^2 &\le \|\boldsymbol{u}\|_{A_k}\|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|_{A_k} + \|P_{k-1}\boldsymbol{u}\|_{A_{k-1}}\|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|_{A_{k-1}} \\ &\lesssim h_{k-1}(\|\boldsymbol{u}\|_{A_k} + \|P_{k-1}\boldsymbol{u}\|_{A_{k-1}})(\epsilon^2|\boldsymbol{w}|_2^2 + \rho^2|\boldsymbol{w}|_1^2 + \lambda|\operatorname{div}\boldsymbol{w}|_1^2)^{1/2} \\ &\lesssim h_{k-1}\|\boldsymbol{u}\|_{A_k}(\epsilon^2|\boldsymbol{w}|_2^2 + \rho^2|\boldsymbol{w}|_1^2 + \lambda|\operatorname{div}\boldsymbol{w}|_1^2)^{1/2} \\ &\lesssim h_{k-1}\|\boldsymbol{u}\|_{A_k}(\epsilon|\boldsymbol{w}|_2 + \rho|\boldsymbol{w}|_1 + \sqrt{\lambda}|\operatorname{div}\boldsymbol{w}|_1) \\ &\lesssim \epsilon^{-1}h_{k-1}\|\boldsymbol{u}\|_{A_k}(\epsilon^2|\boldsymbol{w}|_2 + \epsilon\rho|\boldsymbol{w}|_1 + \epsilon\sqrt{\lambda}|\operatorname{div}\boldsymbol{w}|_1) \\ &\lesssim \epsilon^{-1}h_{k-1}\|\boldsymbol{u}\|_{A_k}\|\boldsymbol{u} - P_{k-1}\boldsymbol{u}\|, \end{aligned}$$

which completes the proof of the first inequality in (4.13).

By inequality (4.8), the second inequality in (4.13) is obtained from

$$\rho^2\|\boldsymbol{u} - P_{k-1}\boldsymbol{u}\|^2 \le \rho^2\|\boldsymbol{u}\|^2 + \rho^2\|P_{k-1}\boldsymbol{u}\|^2 \le \|\boldsymbol{u}\|_{A_k}^2 + \|P_{k-1}\boldsymbol{u}\|_{A_{k-1}}^2 \lesssim \|\boldsymbol{u}\|_{A_k}^2 + \|\boldsymbol{u}\|_{A_k}^2,$$

which completes the proof. □

The next two Lemmas verify the approximation property (A1).

Lemma 4.4. *For all $\boldsymbol{u} \in \boldsymbol{M}_k$ we have the estimate*

$$\lambda\epsilon\|Q_{k-1}\operatorname{div}(\boldsymbol{u} - P_{k-1}\boldsymbol{u})\| \lesssim \max\{\epsilon^2, \min\{h_k, \epsilon\}\}\|\boldsymbol{u}\|_{A_k}. \tag{4.15}$$

Proof. Let $q = Q_{k-1}\operatorname{div}(\boldsymbol{u} - P_{k-1}\boldsymbol{u})$, then there is a $\boldsymbol{w} \in H_0^1(\Omega)^d$ such that [18, 34]

$$\operatorname{div}\boldsymbol{w} = q \text{ and } \|\boldsymbol{w}\|_1 \lesssim \|q\|. \tag{4.16}$$

By the properties of the L^2-projections on S_k and S_{k-1} and the fact that $S_{k-1} \subset S_k$ we have $Q_{k-1}Q_k = Q_{k-1}$ and $Q_{k-1}^2 = Q_{k-1}$. Hence,

$$\begin{aligned}\|q\|^2 &= (q, \operatorname{div}\boldsymbol{w}) = (Q_{k-1}\operatorname{div}(\boldsymbol{u} - P_{k-1}\boldsymbol{u}), Q_{k-1}\operatorname{div}\boldsymbol{w})\\ &= (Q_{k-1}Q_k\operatorname{div}(\boldsymbol{u} - P_{k-1}\boldsymbol{u}), Q_{k-1}\operatorname{div}\boldsymbol{w})\\ &= (Q_k\operatorname{div}(\boldsymbol{u} - P_{k-1}\boldsymbol{u}), Q_{k-1}^2\operatorname{div}\boldsymbol{w}) = (\operatorname{div}(\boldsymbol{u} - P_{k-1}\boldsymbol{u}), Q_{k-1}\operatorname{div}\boldsymbol{w}).\end{aligned} \tag{4.17}$$

Noting that $Q_{k-1}\operatorname{div}\boldsymbol{w} = \operatorname{div}\Pi_{k-1}\boldsymbol{w}$, we have

$$\|q\|^2 = (\operatorname{div}(\boldsymbol{u} - P_{k-1}\boldsymbol{u}), \operatorname{div}\Pi_{k-1}\boldsymbol{w}). \tag{4.18}$$

Now from the definition of P_{k-1} in (4.2), we have

$$\begin{aligned}&\lambda(\operatorname{div}\boldsymbol{u}, \operatorname{div}\Pi_{k-1}\boldsymbol{w}) - \lambda(\operatorname{div}(P_{k-1}\boldsymbol{u}), \operatorname{div}\Pi_{k-1}\boldsymbol{w})\\ &= -\epsilon^2 a_k^s(\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w}) + \epsilon^2 a_{k-1}^s(P_{k-1}\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w}) - \rho^2(\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w}) + \rho^2(P_{k-1}\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w}).\end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19), we get

$$\begin{aligned}\|q\|^2 &= \lambda^{-1}\left(-\epsilon^2 a_k^s(\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w}) + \epsilon^2 a_{k-1}^s(P_{k-1}\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w})\right)\\ &- \lambda^{-1}\rho^2(\boldsymbol{u} - P_{k-1}\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w} - \boldsymbol{w}) - \lambda^{-1}\rho^2(\boldsymbol{u} - P_{k-1}\boldsymbol{u}, \boldsymbol{w})\\ &= I_1 + I_2 + I_3,\end{aligned} \tag{4.20}$$

where

$$\begin{aligned}I_1 &= \lambda^{-1}\left(-\epsilon^2 a_k^s(\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w}) + \epsilon^2 a_{k-1}^s(P_{k-1}\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w})\right);\\ I_2 &= -\lambda^{-1}\rho^2(\boldsymbol{u} - P_{k-1}\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w} - \boldsymbol{w});\\ I_3 &= -\lambda^{-1}\rho^2(\boldsymbol{u} - P_{k-1}\boldsymbol{u}, \boldsymbol{w}).\end{aligned}$$

First, by the boundedness of both $a_k^s(\cdot,\cdot)$ and $a_{k-1}^s(\cdot,\cdot)$, the the H^1-stability of Π_{k-1} given in (3.30), and (4.8), we can estimate I_1 as follows

$$\begin{aligned}I_1 &\lesssim \lambda^{-1}\epsilon(\epsilon\|\boldsymbol{u}\|_{1,k}\|\Pi_{k-1}\boldsymbol{w}\|_{1,k} + \epsilon\|P_{k-1}\boldsymbol{u}\|_{1,k-1}\|\Pi_{k-1}\boldsymbol{w}\|_{1,k-1})\\ &\lesssim \lambda^{-1}\epsilon(\|\boldsymbol{u}\|_{A_k}\|\boldsymbol{w}\|_{1,k} + \|P_{k-1}\boldsymbol{u}\|_{A_{k-1}}\|\boldsymbol{w}\|_{1,k})\\ &\lesssim \lambda^{-1}\epsilon\|\boldsymbol{u}\|_{A_k}\|\boldsymbol{w}\|_{1,k}.\end{aligned}$$

Noting that $\boldsymbol{w} \in H_0^1(\Omega)^d$ implies $\|\boldsymbol{w}\|_{1,k} = |\boldsymbol{w}|_1$, and by (4.16), we obtain

$$I_1 \lesssim \lambda^{-1}\epsilon\|\boldsymbol{u}\|_{A_k}\|\boldsymbol{w}\|_{1,k} = \lambda^{-1}\epsilon\|\boldsymbol{u}\|_{A_k}|\boldsymbol{w}|_1 \lesssim \lambda^{-1}\epsilon\|\boldsymbol{u}\|_{A_k}\|q\|. \tag{4.21}$$

Next, using the Cauchy-Schwarz inequality, the approximation estimates of Π_{k-1} given in (4.12), $\rho \le 1$, (4.8) and (4.16), we obtain the following estimate for I_2:

$$\begin{aligned}I_2 &= -\lambda^{-1}\rho^2(\boldsymbol{u} - P_{k-1}\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w} - \boldsymbol{w})\\ &\le \lambda^{-1}\rho^2\|\boldsymbol{u} - P_{k-1}\boldsymbol{u}\|\|\Pi_{k-1}\boldsymbol{w} - \boldsymbol{w}\|\\ &\lesssim \lambda^{-1}\rho(\rho\|\boldsymbol{u}\| + \rho\|P_{k-1}\boldsymbol{u}\|)h_{k-1}|\boldsymbol{w}|_1\\ &\lesssim \lambda^{-1}\rho h_{k-1}(\|\boldsymbol{u}\|_{A_k} + \|P_{k-1}\boldsymbol{u}\|_{A_{k-1}})|\boldsymbol{w}|_1\\ &\lesssim \lambda^{-1}h_{k-1}\|\boldsymbol{u}\|_{A_k}|\boldsymbol{w}|_1 \lesssim \lambda^{-1}h_{k-1}\|\boldsymbol{u}\|_{A_k}\|q\|.\end{aligned} \tag{4.22}$$

Finally, we have two estimates for I_3. First, by Cauchy-Schwarz inequality, the L^2-error estimate for $\boldsymbol{u} - P_{k-1}\boldsymbol{u}$ given in (4.13), $\rho \le 1$ and (4.16), we get

$$\begin{aligned} I_3 &= -\lambda^{-1}\rho^2(\boldsymbol{u} - P_{k-1}\boldsymbol{u}, \boldsymbol{w}) \le \lambda^{-1}\rho^2\|\boldsymbol{u} - P_{k-1}\boldsymbol{u}\|\|\boldsymbol{w}\| \lesssim \lambda^{-1}\rho^2\epsilon^{-1}h_k\|\boldsymbol{u}\|_{A_k}\|\boldsymbol{w}\| \\ &\le \lambda^{-1}\epsilon^{-1}h_k\|\boldsymbol{u}\|_{A_k}\|\boldsymbol{w}\| \lesssim \lambda^{-1}\epsilon^{-1}h_k\|\boldsymbol{u}\|_{A_k}\|q\|. \end{aligned} \tag{4.23}$$

Second, by Cauchy-Schwarz inequality, $\rho \le 1$, (4.8) and (4.16), we obtain

$$\begin{aligned} I_3 &= -\lambda^{-1}\rho^2(\boldsymbol{u} - P_{k-1}\boldsymbol{u}, \boldsymbol{w}) \le \lambda^{-1}\rho^2\|\boldsymbol{u} - P_{k-1}\boldsymbol{u}\|\|\boldsymbol{w}\| \lesssim \lambda^{-1}\rho(\rho\|\boldsymbol{u}\| + \rho\|P_{k-1}\boldsymbol{u}\|)\|\boldsymbol{w}\| \\ &\lesssim \lambda^{-1}\rho(\|\boldsymbol{u}\|_{A_k} + \|P_{k-1}\boldsymbol{u}\|_{A_{k-1}})\|\boldsymbol{w}\| \lesssim \lambda^{-1}\|\boldsymbol{u}\|_{A_k}\|\boldsymbol{w}\| \lesssim \lambda^{-1}\|\boldsymbol{u}\|_{A_k}\|q\|. \end{aligned} \tag{4.24}$$

Finally, in view of (4.20), by combining the estimates for I_1, I_2 and I_3, namely (4.21), (4.22), (4.23) and (4.24), we get

$$\begin{aligned} \|q\|^2 &= I_1 + I_2 + I_3 \lesssim \lambda^{-1}\epsilon\|\boldsymbol{u}\|_{A_k}\|q\| + \lambda^{-1}h_{k-1}\|\boldsymbol{u}\|_{A_k}\|q\| + \lambda^{-1}\min\{\epsilon^{-1}h_k, 1\}\|\boldsymbol{u}\|_{A_k}\|q\| \\ &\lesssim \lambda^{-1}\max\{\epsilon, h_k, \min\{\epsilon^{-1}h_k, 1\}\}\|\boldsymbol{u}\|_{A_k}\|q\|, \end{aligned}$$

which implies

$$\|Q_{k-1}\operatorname{div}(\boldsymbol{u} - P_{k-1}\boldsymbol{u})\| \lesssim \lambda^{-1}\max\{\epsilon, h_k, \min\{\epsilon^{-1}h_k, 1\}\}\|\boldsymbol{u}\|_{A_k}.$$

□

Lemma 4.5. *For all $\boldsymbol{u} \in \boldsymbol{M}_k$ we have the estimate*

$$\lambda\|\operatorname{div}(\boldsymbol{u} - P_{k-1}\boldsymbol{u})\|^2 \lesssim \|\boldsymbol{u}\|_{A_k}^2. \tag{4.25}$$

Proof. Using (4.8), the inequality can be obtained as follows

$$\lambda\|\operatorname{div}(\boldsymbol{u} - P_{k-1}\boldsymbol{u})\|^2 \le \lambda\|\operatorname{div}\boldsymbol{u}\|^2 + \lambda\|\operatorname{div}(P_{k-1}\boldsymbol{u})\|^2 \lesssim \|\boldsymbol{u}\|_{A_k}^2 + \|P_{k-1}\boldsymbol{u}\|_{A_{k-1}}^2 \lesssim \|\boldsymbol{u}\|_{A_k}^2.$$

□

Combining the L^2-estimate (4.13), and the estimates given in Lemma 4.4 and Lemma 4.5, we obtain the following theorem, which verifies (A1).

Theorem 4.4. *The following approximation estimate holds for $\lambda \gtrsim 1$ and for all $\boldsymbol{u} \in \boldsymbol{M}_k$.*

$$\|(I - P_{k-1})\boldsymbol{u}\|_{k,0} \lesssim h_k\|\boldsymbol{u}\|_{A_k}.$$

4.5. **Smoothing property.** In this subsection, we verify the smoothing property (A2). We consider only the 3D case because the 2D case is similar and simpler. By $\mathcal{V}_k$ and $\mathcal{E}_k$ we denote the sets of vertices and edges of the partition T_k, respectively. For $\nu \in \mathcal{V}_k \cup \mathcal{E}_k$ we define

$$T_k^\nu = \{K \in T_k : \nu \subset K\},\ \bar{\Omega}_k^\nu = \cup_{K \in T_k^\nu}\bar{K},\ \Omega_k^\nu = \text{interior}(\ \bar{\Omega}_k^\nu).$$

Thus Ω_k^ν is the subdomain of Ω formed by the patch of elements meeting at ν, and T_k^ν is the restriction of the mesh partition T_k to Ω_k^ν.

We now consider the decomposition of the spaces $\boldsymbol{M}_k$ into sums of spaces supported in small patches of elements. Define

$$\boldsymbol{M}_k^\nu = \{\boldsymbol{r} \in \boldsymbol{M}_k : \operatorname{supp}\boldsymbol{r} \subset \bar{\Omega}_k^\nu\},\ \nu \in \mathcal{V}_k \cup \mathcal{E}_k.$$

Then

$$\boldsymbol{M}_k = \sum_{i \in \mathcal{V}_k}\boldsymbol{M}_k^i = \sum_{e \in \mathcal{E}_k}\boldsymbol{M}_k^e.$$

For each of these decompositions there is a corresponding estimate on the sum of the squares of the L^2-norms of the components. For example, we can decompose an arbitrary element $\boldsymbol{u} \in \boldsymbol{M}_k$ as $\boldsymbol{u} = \sum_{i \in \mathcal{V}_k}\boldsymbol{u}^i$ with $\boldsymbol{u}^i \in \boldsymbol{M}_k^i$ such that the estimate

$$\sum_{i \in \mathcal{V}_k}\|\boldsymbol{u}^i\|^2 \lesssim \|\boldsymbol{u}\|^2 \tag{4.26}$$

holds with a constant that depends only on the shape regularity of the mesh.

Since the kernel basis functions of the divergence operator are captured by the above subspaces $\boldsymbol{M}_k^i$, we must use a block damped Jacobi smoother or a block Gauss-Seidel smoother where the blocks correspond to one of the above L^2-decompositions in order to preserve the structure of the kernel. For example, we can use a vertex block damped Jacobi smoother, a vertex block Gauss-Seidel smoother, an edge block damped Jacobi smoother, or an edge block Gauss-Seidel smoother.

Remark 4.3. *We should point out that the block Gauss-Seidel smoother satisfies the assumption (A0). But for the block damped Jacobi smoother, we need to choose the damping parameter such that the basic assumption (A0) is satisfied. A damped Richardson smoother $I - \tau A_k$ would need a damping parameter τ proportional to λ^{-1}. Thus the components of the error in the kernel of A_k would be smoothed out very slowly as for large λ. We should also point out that in the 2-dimensional case, we can only use vertex block smoothers.*

In the rest of this subsection, we consider the vertex block damped Jacobi smoother since the others are similar, and define the operator $P_{k,i} : \boldsymbol{M}_k \to \boldsymbol{M}_k^i$ for $i \in \mathcal{V}_k$ by

$$A_k(P_{k,i}\boldsymbol{u}, \boldsymbol{v}_i) = A_k(\boldsymbol{u}, \boldsymbol{v}_i) \text{ for all } \quad \boldsymbol{u} \in \boldsymbol{M}_k, \boldsymbol{v}_i \in \boldsymbol{M}_k^i.$$

We use exact local solves and hence the block damped Jacobi smoother $R_k = \tau \sum_{i\in\mathcal{V}_k} P_{k,i}A_k^{-1} := \tau D_k^{-1}$, where τ is the damping parameter such that (A0) is satisfied. In this case, $K_k^* = K_k$ and $\widetilde{K}_k^{(m)} = K_k^m$. By the assumption (A0), the estimate

$$\|K_k^m \boldsymbol{u}\|_{A_k}^2 = (D_k^{-1} A_k K_k^{2m} \boldsymbol{u}, \boldsymbol{u})_{D_k} \lesssim m^{-1} \|\boldsymbol{u}\|_{D_k}^2 \tag{4.27}$$

holds, which is well known in multigrid theory (see e.g. Hackbusch [35]). By additive Schwarz techniques [36, 37] the induced norm $\|\boldsymbol{u}\|_{D_k} = (D_k\boldsymbol{u}, \boldsymbol{u})^{1/2}$ can be written as

$$\|\boldsymbol{u}\|_{D_k}^2 = \inf_{\boldsymbol{u}=\sum \boldsymbol{u}_k^i} \sum_{i\in\mathcal{V}_k} \|\boldsymbol{u}_k^i\|_{A_k}^2. \tag{4.28}$$

On the other hand, choosing τ sufficiently small it is obvious that $\|K_k^m\boldsymbol{u}\|_{A_k} \le \|\boldsymbol{u}\|_{A_k}$ (the assumption (A0) holds). Then an interpolation between this estimate and the estimate (4.27) gives

$$\|K_k^m \boldsymbol{u}\|_{A_k} \lesssim m^{-1/4} \|\boldsymbol{u}\|_{[D_k, A_k]},$$

where $\|\boldsymbol{u}\|_{[D_k,A_k]}$ is the interpolation norm between $\|\cdot\|_{D_k}$ and $\|\cdot\|_{A_k}$ with parameter 1/2. Thus, one way to verify assumption (A2), is to show that

$$\|\boldsymbol{u}\|_{[D_k,A_k]} \lesssim h_k^{-1} \|\boldsymbol{u}\|_{k,0}, \tag{4.29}$$

and the rest of this section is devoted to this. We now define a decomposition of $\boldsymbol{u} \in \boldsymbol{M}_k$ which is stable in the norm $\|\cdot\|_{k,0}$ and then show the estimates for the components of this decomposition.

We consider three solutions of problem (3.51) defined as follows:

$$(\boldsymbol{u}_1, p_1) \text{ is the solution of (3.51) with } \boldsymbol{w}_1 = \boldsymbol{u},\ \boldsymbol{w}_2 = 0. \tag{4.30}$$

$$(\boldsymbol{u}_2, p_2) \text{ is the solution of (3.51) with } \boldsymbol{w}_1 = 0,\ \boldsymbol{w}_2 = \boldsymbol{u} - \Pi_{k-1}\boldsymbol{u}. \tag{4.31}$$

$$(\boldsymbol{u}_3, p_3) \text{ is defined as the solution of (3.51) with } \boldsymbol{w}_1 = 0,\ \boldsymbol{w}_2 = \Pi_{k-1}\boldsymbol{u}. \tag{4.32}$$

It is straightforward to check that $\boldsymbol{u} - \boldsymbol{u}_1 - \boldsymbol{u}_2 - \boldsymbol{u}_3$ and $p_1 + p_2 + p_3$ satisfy the equation (3.51) with $\boldsymbol{w}_1 = 0$ and $\boldsymbol{w}_2 = 0$ and therefore $\boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3 = \boldsymbol{u}$. With these settings in hand, we have the following stability result.

Lemma 4.6. *For the decomposition given in* (4.30)–(4.32) *we have*

$$\|\boldsymbol{u}_1\|_{k,0} + \|\boldsymbol{u}_2\|_{k,0} + \|\boldsymbol{u}_3\|_{k,0} \lesssim \|\boldsymbol{u}\|_{k,0}, \tag{4.33}$$

$$\|\boldsymbol{u}_2\| \lesssim \lambda^{-\frac{1}{2}} \|\boldsymbol{u}\|_{k,0}. \tag{4.34}$$

Proof. Computing $\|\cdot\|_{k,0}$ for all the components shows that

$$\|\boldsymbol{u}_1\|_{k,0} \le \epsilon\|\boldsymbol{u}_1\| + h_k\|\boldsymbol{u}_1\|, \tag{4.35}$$

$$\|\boldsymbol{u}_2\|_{k,0} \le \epsilon\|\boldsymbol{u}_2\| + h_k\|\boldsymbol{u}_2\| + \lambda^{\frac{1}{2}} h_k\|\operatorname{div}(\boldsymbol{u} - \Pi_{k-1}\boldsymbol{u})\|. \tag{4.36}$$

The rest of the proof is immediate from the definitions of the components (4.30)–(4.31), the definition of the $\|\cdot\|_{k,0}$ norm, Lemma 3.8 and Lemma 3.4. Finally, using the triangle inequality, it is easily seen that $\|\boldsymbol{u}_3\|_{k,0} \lesssim \|\boldsymbol{u}\|_{k,0}$. □

4.6. **Smoothing property via interpolation.** Define the $H(\mathrm{curl};\Omega)$-conforming finite element space at level k (see, e.g., [10])

$$\boldsymbol{W}_k = \{\boldsymbol{w} \in H(\mathrm{curl},\Omega) : \boldsymbol{w}|_K \in \boldsymbol{W}(K), K \in T_k, \boldsymbol{w}\times\boldsymbol{n}|_{\partial\Omega} = 0\}.$$

Then the three spaces $\boldsymbol{M}_k, S_k$ and $\boldsymbol{W}_k$ are related by the exact sequence ([10])

$$0 \longrightarrow \boldsymbol{W}_k \xrightarrow{\mathrm{curl}} \boldsymbol{M}_k \xrightarrow{\mathrm{div}} S_k \longrightarrow 0.$$

Furthermore, we define

$$\boldsymbol{W}_k^{\nu} = \{\boldsymbol{r} \in \boldsymbol{W}_k : \operatorname{supp}\boldsymbol{r} \subset \bar{\Omega}_k^{\nu}\},\ \nu \in \mathcal{V}_k \cup \mathcal{E}_k.$$

Then

$$\boldsymbol{W}_k = \sum_{i\in\mathcal{V}_k} \boldsymbol{W}_k^i = \sum_{e\in\mathcal{E}_k} \boldsymbol{W}_k^e.$$

Note that for any $\boldsymbol{v} \in \boldsymbol{M}_k$, we have that $\|\boldsymbol{v}\|_{A_k} \lesssim \|\boldsymbol{v}\|_{D_k}$ and $\|\boldsymbol{v}\|_{D_k} \le \|\boldsymbol{v}\|_{D_k}$ and this implies that

$$\|\boldsymbol{v}\|_{[D_k,A_k]} \lesssim \|\boldsymbol{v}\|_{D_k}. \tag{4.37}$$

The next two lemmas bound only the $\|\cdot\|_{D_k}$-norm, which is sufficient in view of (4.37).

Lemma 4.7. *Let* $\boldsymbol{u}_1$ *be defined as in* (4.30). *Then*

$$\|\boldsymbol{u}_1\|_{D_k} \lesssim h_k^{-1}\|\boldsymbol{u}_1\|_{k,0}. \tag{4.38}$$

Proof. Since $\operatorname{div}\boldsymbol{u}_1 = 0$, we have $\boldsymbol{u}_1 = \operatorname{curl}\boldsymbol{w}_k$ (see [10]), where $\boldsymbol{w}_k \in \boldsymbol{W}_k$.

Noting that $\boldsymbol{w}_k = \sum_{i\in\mathcal{V}_k}\boldsymbol{w}_k^i$, where $\boldsymbol{w}_k^i \in \boldsymbol{W}_k^i$ and $\operatorname{curl}\boldsymbol{w}_k^i \in \boldsymbol{M}_k^i$, by identity (4.28) and inequality (4.26), we have

$$\begin{aligned}
\|\boldsymbol{u}_1\|_{D_k}^2 &= \inf_{\boldsymbol{u}_1=\sum \boldsymbol{u}_1^i}\sum_{i\in\mathcal{V}_k}\|\boldsymbol{u}_1^i\|_{A_k}^2 \le \sum_{i\in\mathcal{V}_k}\|\operatorname{curl}\boldsymbol{w}_k^i\|_{A_k}^2 = \sum_{i\in\mathcal{V}_k}\|\operatorname{curl}\boldsymbol{w}_k^i\|_{a_k}^2 \\
&= \sum_{i\in\mathcal{V}_k}\|\boldsymbol{u}_1^i\|_{a_k}^2 \lesssim h_k^{-2}\sum_{i\in\mathcal{V}_k}(\epsilon^2\|\boldsymbol{u}_1^i\|^2 + h_k^2\rho^2\|\boldsymbol{u}_1^i\|^2) \\
&\lesssim h_k^{-2}(\epsilon^2\|\boldsymbol{u}_1\|^2 + h_k^2\rho^2\|\boldsymbol{u}_1\|^2) = h_k^{-2}\|\boldsymbol{u}_1\|_{k,0}^2.
\end{aligned}$$

□

Lemma 4.8. *Let* $\boldsymbol{u}_2$ *be defined as in* (4.31). *Then*

$$\|\boldsymbol{u}_2\|_{D_k} \lesssim h_k^{-1}\|\boldsymbol{u}\|_{k,0}. \tag{4.39}$$

Proof. By the identity (4.28) and Lemma 4.6, we have

$$\|\boldsymbol{u}_2\|_{D_k}^2 = \inf_{\boldsymbol{u}_2=\sum \boldsymbol{u}_2^i}\sum_{i\in\mathcal{V}_k}\|\boldsymbol{u}_2^i\|_{A_k}^2 \lesssim \sum_{i\in\mathcal{V}_k} h_k^{-2}\lambda\|\boldsymbol{u}_2^i\|^2 \lesssim h_k^{-2}\lambda\|\boldsymbol{u}_2\|^2 \lesssim h_k^{-2}\|\boldsymbol{u}\|_{k,0}^2.$$

□

Corollary 4.1. *From the inequality* (4.37) *and the Lemmas 4.7 and 4.8, we immediately have*

$$\begin{aligned}\|\boldsymbol{u}_1\|_{[D_k,A_k]} &\lesssim h_k^{-1}\|\boldsymbol{u}_1\|_{k,0},\\ \|\boldsymbol{u}_2\|_{[D_k,A_k]} &\lesssim h_k^{-1}\|\boldsymbol{u}\|_{k,0}.\end{aligned} \tag{4.40}$$

Lemma 4.9. *Let $\boldsymbol{u}_3$ be defined as in* (4.32), *$h_k \le \epsilon^2$ or $\lambda \gtrsim \min\{\epsilon^{-1}, h_k^{-1}\}$. Then*

$$\|\boldsymbol{u}_3\|_{[D_k,A_k]} \lesssim h_k^{-1}\|\boldsymbol{u}\|_{k,0}. \tag{4.41}$$

Proof. By the inf-sup condition (3.52) we have $\|\boldsymbol{u}_3\|_{1,k}+\|p_3\| \lesssim \|Q_{k-1}\operatorname{div}\boldsymbol{u}\|$. Furthermore, from the definition of $\boldsymbol{u}_3$, we have $\operatorname{div}\boldsymbol{u}_3 = Q_{k-1}\operatorname{div}\boldsymbol{u}$. These together with $\epsilon \le 1, \rho \le 1$ and $\|\boldsymbol{u}_3\| \lesssim \|\boldsymbol{u}_3\|_{1,k}$ give

$$\begin{aligned}\|\boldsymbol{u}_3\|_{A_k}^2 &\lesssim (\epsilon^2\|\boldsymbol{u}_3\|_{1,k}^2 + \rho^2\|\boldsymbol{u}_3\|^2 + \lambda\|\operatorname{div}\boldsymbol{u}_3\|^2)\\ &\lesssim \|Q_{k-1}\operatorname{div}\boldsymbol{u}\|^2 + \lambda\|Q_{k-1}\operatorname{div}\boldsymbol{u}\|^2 \lesssim \lambda\|Q_{k-1}\operatorname{div}\boldsymbol{u}\|^2\end{aligned}$$

which implies

$$\|\boldsymbol{u}_3\|_{A_k}^2 \lesssim \lambda^{-1}\epsilon^2 h_k^{-2}\|\boldsymbol{u}\|_{k,0}^2 \text{ if } h_k \le \epsilon^2 \text{ and } \|\boldsymbol{u}_3\|_{A_k}^2 \lesssim \lambda^{-1}h_k^{-2}\|\boldsymbol{u}\|_{k,0}^2 \text{ if } h_k \ge \epsilon^2.$$

On the other hand, by the identity (4.28) and inverse inequality, we have

$$\|\boldsymbol{u}_3\|_{D_k}^2 = \inf_{\boldsymbol{u}_3=\sum \boldsymbol{u}_3^i}\sum_{i\in\mathcal{V}_k}\|\boldsymbol{u}_3^i\|_{A_k}^2 \lesssim \sum_{i\in\mathcal{V}_k} h_k^{-2}\lambda\|\boldsymbol{u}_3^i\|^2 \lesssim h_k^{-2}\lambda\|\boldsymbol{u}_3\|^2.$$

If $h_k \le \epsilon^2$, by noting that $\|\boldsymbol{u}_3\|_{k,0} \lesssim \|\boldsymbol{u}\|_{k,0}$, we have

$$\|\boldsymbol{u}_3\|_{D_k}^2 \lesssim h_k^{-2}\lambda\|\boldsymbol{u}_3\|^2 \lesssim h_k^{-2}\lambda\epsilon^{-2}\|\boldsymbol{u}_3\|_{k,0}^2 \lesssim h_k^{-2}\lambda\epsilon^{-2}\|\boldsymbol{u}\|_{k,0}^2.$$

If $h_k \ge \epsilon^2$, by Lemma 3.8, it follows that

$$\|\boldsymbol{u}_3\|_{D_k}^2 \lesssim h_k^{-2}\lambda\|\boldsymbol{u}_3\|^2 \lesssim h_k^{-2}\lambda\|Q_{k-1}\operatorname{div}\boldsymbol{u}\|_{-1}^2 \le h_k^{-2}\lambda\|Q_{k-1}\operatorname{div}\boldsymbol{u}\|^2 \lesssim h_k^{-2}\lambda^{-1}\min\{\epsilon^{-2},h_k^{-2}\}\|\boldsymbol{u}\|_{k,0}^2.$$

A standard interpolation argument, see, e.g., [38], completes the proof. □

We close this subsection by the following theorem which verifies (A2).

Theorem 4.5. *If $h_k \le \epsilon^2$ or $\lambda \gtrsim \min\{\epsilon^{-1}, h_k^{-1}\}$, the following estimate holds for all $\boldsymbol{u} \in \boldsymbol{M}_k$*

$$\|(\widetilde{K}_k^{(m)})^*\boldsymbol{u}\|_{A_k} \lesssim m^{-1/4}h_k^{-1}\|\boldsymbol{u}\|_{k,0}. \tag{4.42}$$

Proof. By Lemma 4.6, inequalities (4.40) and (4.41), we obtain the smoothing property (4.42). □

5. Numerical results

To test the performance of the multigrid algorithms that we have proposed we present three sets of numerical tests solving equation (3.17).

For simplicity, we take as computational domain $\Omega = [0,1]\times[0,1]$ and discretize equation (2.2) and (3.17) by $H(\operatorname{div},\Omega)$-conforming BDM_1 finite elements ($BDM_1(K)/P_0(K)$ pair for Brinkman equation) on a uniform mesh using the DG method described in Section 3. Our tests are aimed at verifying the theoretical results on the convergence of the multigrid algrithms for the linear system (3.15). We have tabulated the results obtained with the multigrid method for meshes with mesh sizes $h_J = 2^{-J}$ where $J = 2, \dots, 6$. In addition, we have varied the parameter λ. For the $V(1,1)$- and $W(1,1)$-cycles of the multigrid (MG) algorithm we have used a vertex block Gauss-Seidel smoother. In order to approximate the error reduction factor of the MG iteration, i.e. the number $\varrho = \|\mathbb{E}_J\|_{\mathbb{A}_J} := \|I - \mathbb{B}_J\mathbb{A}_J\|_{\mathbb{A}_J}$, we have set $\boldsymbol{e}_i = \mathbb{E}_J\boldsymbol{e}_{i-1}$ with a random initial guess $\boldsymbol{e}_0$ and computed the ratio $\varrho_i := (A_J\boldsymbol{e}_i,\boldsymbol{e}_i)/(A_J\boldsymbol{e}_{i-1},\boldsymbol{e}_{i-1})$ for large enough i.

In all tables J denotes the level of the finest discretization and N denotes the number of degrees of freedom for the displacement component (for BDM_1 elements, N is twice the number of edges).

		$\lambda = 5 \times 10^\ell$						
J	N	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$
2	112	0.009	0.026	0.033	0.033	0.033	0.033	0.033
3	416	0.067	0.139	0.166	0.169	0.169	0.169	0.170
4	1600	0.101	0.198	0.237	0.242	0.242	0.242	0.242
5	6272	0.108	0.219	0.262	0.267	0.267	0.267	0.267
6	24832	0.110	0.227	0.270	0.275	0.276	0.276	0.276

TABLE 5.1. Convergence rate $\|\mathbb{E}_J\|_{\mathbb{A}_J}$ of $V(1,1)$-cycle MG method (confirming Theorem 4.1) in case of $\epsilon^2 = 1, \rho^2 = 0$.

The data in Table 5.1 verifies the convergence result in Theorem 4.1 and the data in Tables 5.2–5.7 verifies the result shown in Theorem 4.2. We want to emphasize that although Theorem 4.2 requires that the number of smoothing steps is sufficiently large, the results shown in Tables 5.2-5.7 indicate that one smoothing step is sufficient for a uniform convergence of the W-cycle MG method.

		$\lambda = 5 \times 10^\ell$						
J	N	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$
2	112	0.009	0.026	0.033	0.033	0.033	0.033	0.033
3	416	0.074	0.122	0.142	0.144	0.144	0.144	0.144
4	1600	0.104	0.131	0.150	0.152	0.152	0.152	0.152
5	6272	0.108	0.134	0.153	0.155	0.155	0.155	0.155
6	24832	0.110	0.128	0.141	0.143	0.143	0.143	0.143

TABLE 5.2. Convergence rate $\|\mathbb{E}_J\|_{\mathbb{A}_J}$ of $W(1,1)$-cycle MG method (confirming Theorem 4.2) in case of $\epsilon^2 = 1, \rho^2 = 0$.

		$\lambda = 5 \times 10^\ell$						
J	N	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$
2	112	0.009	0.026	0.033	0.033	0.033	0.033	0.033
3	416	0.074	0.122	0.142	0.144	0.144	0.144	0.144
4	1600	0.104	0.131	0.150	0.152	0.152	0.152	0.152
5	6272	0.108	0.134	0.153	0.155	0.155	0.155	0.155
6	24832	0.110	0.128	0.141	0.143	0.143	0.143	0.143

TABLE 5.3. Convergence rate $\|\mathbb{E}_J\|_{\mathbb{A}_J}$ of $W(1,1)$-cycle MG method (confirming Theorem 4.2) in case of $\epsilon^2 = 1, \rho^2 = 10^{-6}$.

		$\lambda = 5\times 10^{\ell}$						
J	N	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
2	112	0.008	0.026	0.032	0.033	0.033	0.033	0.033
3	416	0.074	0.122	0.141	0.144	0.144	0.144	0.144
4	1600	0.104	0.131	0.150	0.152	0.152	0.152	0.152
5	6272	0.108	0.134	0.153	0.155	0.155	0.155	0.155
6	24832	0.110	0.128	0.141	0.143	0.143	0.143	0.143

TABLE 5.4. Convergence rate $\|\mathbb{E}_J\|_{\mathbb{A}_J}$ of $W(1,1)$-cycle MG method (confirming Theorem 4.2) in case of $\epsilon^2 = 1, \rho^2 = 10^{-3}$.

		$\lambda = 5\times 10^{\ell}$						
J	N	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
2	112	0.001	0.002	0.003	0.003	0.003	0.003	0.003
3	416	0.024	0.035	0.045	0.047	0.047	0.047	0.047
4	1600	0.065	0.109	0.128	0.130	0.130	0.130	0.130
5	6272	0.097	0.130	0.148	0.151	0.151	0.151	0.151
6	24832	0.108	0.130	0.146	0.148	0.148	0.150	0.150

TABLE 5.5. Convergence rate $\|\mathbb{E}_J\|_{\mathbb{A}_J}$ of $W(1,1)$-cycle MG method (confirming Theorem 4.2) in case of $\epsilon^2 = 1, \rho^2 = 1$.

		$\lambda = 5\times 10^{\ell}$						
J	N	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
2	112	0.0003	0.0004	0.0007	0.0008	0.0008	0.0008	0.0008
3	416	0.002	0.002	0.002	0.002	0.002	0.002	0.002
4	1600	0.007	0.009	0.009	0.009	0.009	0.009	0.009
5	6272	0.011	0.012	0.012	0.012	0.012	0.012	0.012
6	24832	0.014	0.014	0.014	0.014	0.014	0.014	0.014

TABLE 5.6. Convergence rate $\|\mathbb{E}_J\|_{\mathbb{A}_J}$ of $W(1,1)$-cycle MG method (confirming Theorem 4.2) in case of $\epsilon^2 = 10^{-3}, \rho^2 = 1$.

As predicted by the theory, the results presented in Tables 5.1–5.7 show uniform convergence independent of ϵ, ρ, λ and h.

Finally, to test the augmented Uzawa iteration we have set the right hand side of equation (2.2) to

$$\boldsymbol{f} = \begin{pmatrix} (1-6x+6x^2)(y-3y^2+2y^3)+(x^2-2x^3+x^4)(-3+6y) \\ -(1-6y+6y^2)(x-3x^2+2x^3)-(y^2-2y^3+y^4)(-3+6x) \end{pmatrix}, \; g = 0,$$

		$\lambda = 5 \times 10^{\ell}$						
J	N	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$
2	112	0.0003	0.0004	0.0007	0.0008	0.0008	0.0008	0.0008
3	416	0.0006	0.002	0.0008	0.0008	0.0008	0.0008	0.0008
4	1600	0.006	0.009	0.009	0.009	0.009	0.009	0.009
5	6272	0.011	0.012	0.012	0.012	0.012	0.012	0.012
6	24832	0.014	0.014	0.014	0.014	0.014	0.014	0.014

TABLE 5.7. Convergence rate $\|\mathbb{E}_J\|_{\mathbb{A}_J}$ of $W(1,1)$-cycle MG method (confirming Theorem 4.2) in case of $\epsilon^2 = 10^{-6}, \rho^2 = 1$.

and used the corresponding exact solution of the sub-problem for the displacement $\boldsymbol{u}$ (see (3.14)). The iteration has been initialized with $\boldsymbol{u}_h^0 = \mathbf{0}$ and $p_h^0 = 0$ and terminated after a reduction of the error of the velocity in energy norm by a factor of 10^8. In Table 5.8 we verify the convergence results of the augmented Uzawa iteration which is given in (3.14) (see also [20]).

		$\lambda = 5 \times 10^{\ell}$					
J	N	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
2	112	11	5	3	3	3	2
3	416	11	5	3	2	2	2
4	1600	10	5	3	2	2	2
5	6272	10	5	3	2	2	2
6	24832	10	4	3	2	2	2

TABLE 5.8. Iteration count for the augmented Uzawa method.

6. Conclusions

We have proved the stability of a family of discontinuous Galerkin $H(\mathrm{div};\Omega)$-conforming discretizations of the Brinkman equations and the corresponding linear elasticity type equation with mass term which appears in the Uzawa iteration for the Brinkman problem. Further, we have presented a robust iterative solution method for the arising discrete problems. Variable V-cycle and W-cycle multigrid algorithms have been analyzed in the present situation of nonnested bilinear forms. The convergence rate of the iterative methods have been proved to be independent of the parameters in the Brinkman problem, i.e., ϵ and ρ and of the mesh size h, which shows that the iterative algorithm is robust and optimal. The presented numerical results support the theoretical findings.

References

[1] O. Iliev, R. Lazarov, and J. Willems. Variational multiscale finite element method for flows in highly porous media. *Multiscale Model. Simul.*, 9(4):1350–1372, 2011.

[2] Xiaoping Xie, Jinchao Xu, and Guangri Xue. Uniformly-stable finite element methods for Darcy-Stokes-Brinkman models. *J. Comput. Math.*, 26(3):437–455, 2008.

[3] Johnny Guzmán and Michael Neilan. A family of nonconforming elements for the Brinkman problem. *IMA J. Numer. Anal.*, 32(4):1484–1508, 2012.

[4] Juho Könnö and Rolf Stenberg. $H(\text{div})$-conforming finite elements for the Brinkman problem. *Math. Models Methods Appl. Sci.*, 21(11):2227–2248, 2011.

[5] Panayot S. Vassilevski and Umberto Villa. A mixed formulation for the Brinkman problem. *SIAM J. Numer. Anal.*, 52(1):258–281, 2014.

[6] Panayot S. Vassilevski and Umberto Villa. A block-diagonal algebraic multigrid preconditioner for the Brinkman problem. *SIAM J. Sci. Comput.*, 35(5):S3–S17, 2013.

[7] C. W. Oosterlee and F. J. Lorenz. Multigrid methods for the Stokes system. *Computing in Science* & *Engineering*, 8(6):34–43, 2006.

[8] S. P. Vanka. Block-implicit multigrid solution of Navier-Stokes equations in primitive variables. *Journal of Computational Physics*, 65(1):138–158, 1986.

[9] M. Fortin and R. Glowinski. *Augmented Lagrangian Methods: Applications to the Numerical Solution of Boundary-value Problems*, volume 15. North Holland, 1983.

[10] D.N. Arnold, R.S. Falk, and R. Winther. Multigrid in H(div) and H(curl). *Numerische Mathematik*, 85(2):197–217, 2000.

[11] Y.J. Lee. Uniform stability analysis of austin, manteuffel and mccormick finite elements and fast and robust iterative methods for the Stokes-like equations. *Numerical Linear Algebra with Applications*, 17(1):109–138, 2010.

[12] B. Cockburn, G. Kanschat, and D. Schötzau. A note on discontinuous Galerkin divergence-free solutions of the Navier–Stokes equations. *Journal of Scientific Computing*, 31(1):61–73, 2007.

[13] G.. Chen, D. Li, D. Schötzau, and X. Wei. A mixed finite element method with exactly divergence-free velocities for incompressible magnetohydrodynamics. *Computer Methods in Applied Mechanics and Engineering*, 199(45):2840–2855, 2010.

[14] B. Ayuso, F. Brezzi, L.D. Marini, J. Xu, and L. Zikatanov. A simple preconditioner for a discontinuous Galerkin method for the Stokes problem. *arXiv preprint arXiv:1209.5223*, 2012.

[15] Q.G. Hong, J. Kraus, J.C. Xu, and L. Zikatanov. Robust multirid methods for discontinuous Galerkin discretization of linear elasticity problem. 2013.

[16] Richard S Falk and Ragnar Winther. Local bounded cochain projection. *arXiv preprint arXiv:1211.5893*, 2012.

[17] Joachim Schöberl. Multigrid methods for a parameter dependent problem in primal variables. *Numerische Mathematik*, 84(1):97–119, 1999.

[18] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, volume 87. Springer-Verlag, Berlin, 1986.

[19] R. B. Kellogg and J. E. Osborn. A regularity result for the Stokes problem in a convex polygon. *J. Functional Analysis*, 21(4):397–431, 1976.

[20] Y. J. Lee, J. Wu, and L. Xu, J. Zikatanov. Robust subspace correction methods for nearly singular systems. *Mathematical Models and Methods in Applied Sciences*, 17(11):1937–1963, 2007.

[21] S. C. Brenner and R. Scott. *The Mathematical Theory of Finite Element Methods*, volume 15. Springer, 2007.

[22] U. Langer and W. Queck. On the convergence factor of Uzawa's algorithm. *Journal of Computational and Applied Mathematics*, 15(2):191–202, 1986.

[23] S.C. Brenner. Korn's inequalities for piecewise H^1 vector fields. *Mathematics of Computation*, 73:1067–1088, 2004.

[24] D. Schötzau, C. Schwab, and A. Toselli. Mixed hp-DGFEM for incompressible flows. *SIAM Journal on Numerical Analysis*, 40(6):2171–2194, 2002.

[25] D.N. Arnold, F. Brezzi, B. Cockburn, and L.D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM Journal on Numerical Analysis*, 39:1749–1779, 2002.

[26] Daniele Boffi, Franco Brezzi, and Michel Fortin. *Mixed and hybrid finite element methods*, volume 44 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 2013.

[27] X.C. Tai and R. Winther. A discrete de rham complex with enhanced smoothness. *Calcolo*, 43(4):287–306, 2006.

[28] P. Monk. *Finite Element Methods for Maxwell's Equations*. Oxford University Press, USA, 2003.

[29] I. Babuška and A. K. Aziz. *Lectures on the mathematical foundations of the finite element method*. University of Maryland, College Park, Washington DC, 1972. Technical Note BN-748.

[30] Ivo Babuška. The finite element method with Lagrangian multipliers. *Numer. Math.*, 20:179–192, 1972/73.

[31] J.H. Bramble, J.E. Pasciak, and J. Xu. The analysis of multigrid algorithms with nonnested spaces or noninherited quadratic forms. *Mathematics of Computation*, 56(193):1–34, 1991.

[32] S.C. Brenner and L.Y. Sung. Linear finite element methods for planar linear elasticity. *Mathematics of Computation*, 59:321–321, 1992.

[33] Constantin Bacuta and James H Bramble. Regularity estimates for solutions of the equations of linear elasticity in convex plane polygonal domains. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 54(5):874–878, 2003.

[34] Douglas N Arnold, L Ridgway Scott, and Michael Vogelius. *Regular inversion of the divergence operator with Dirichlet boundary conditions on a polygonal*. Defense Technical Information Center, 1987.

[35] W. Hackbusch. *Multigrid Methods and Applications*, volume 4. Springer-Verlag Berlin, 1985.

[36] J. Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review*, 34:581–613, 1992.

[37] J. Xu and L. Zikatanov. The method of alternating projections and the method of subspace corrections in Hilbert space. *Journal of the American Mathematical Society*, 15(3):573–598, 2002.

[38] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

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