www.oeaw.ac.at

ÖAW AUSTRIAN ACADEMY OF SCIENCES

Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

RICAM JOHANN-RADON-INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Detection of Multilayered Media in the Acoustic Waveguide

F. Al-Musallam, A. Boumenir, M. Sini

RICAM-Report 2014-32

Detection of Multilayered Media in the Acoustic Waveguide

Fadhel Al-Musallam

Department of Mathematics, Kuwait University, P.O. Box 13060, Safat, Kuwait. (Email: musallam@sci.kuniv.edu.kw).

Amin Boumenir

Department of Mathematics, University of West Georgia, 1601 Maple Street, Carrollton, GA 30118.
Current address: Department of Mathematics, Kuwait University, P.O. Box 13060, Safat, Kuwait. (Email: boumenir@westga.edu).

Mourad Sini

RICAM, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040, Linz, Austria. (Email:mourad.sini@oeaw.ac.at)
Partially supported by the Austrian Science Fund (FWF): P22341-N18.

Abstract

This work is concerned with the inverse problem for ocean acoustics modeled by a multilayered waveguide with a finite depth. We provide explicit formulae to locate the layers, including the seabed, and reconstruct the speed of sound and the densities in each layer from measurements collected on the surface of the waveguide. We proceed in two steps. First, we use Gaussian type excitations on the upper surface of the waveguide and then from the corresponding scattered fields, collected on the same surface, we recover the boundary spectral data of the related $1D$ spectral problem. Second, from these spectral data, we reconstruct the values of the normal derivatives of the singular solutions, of the original waveguide problem, on that upper surface. Finally, we derive formulae to reconstruct the layers from these values based on the asymptotic expansion of these singular solutions in terms of the source points.

Keywords: Inverse scattering problem, penetrable obstacle, complex spherical waves

1. Introduction

1.1. The model

The propagation of acoustic waves in the waveguide $\Omega := \mathbb{R}^2 \times (L, 0)$, $L < 0$, is governed by the following model, see [5]:

$$\begin{cases} \nabla \cdot a\nabla u + \kappa^2 n u = 0 \ \text{ in } \Omega, \\ u = f \ \text{ on } x_3 = 0, \\ a\frac{\partial}{\partial x_3} u = 0 \ \text{ on } x_3 = L \end{cases} \tag{1.1}$$

where $a(x) := \frac{1}{\rho(x)}$ and $n(x) := c^2(x)\rho(x)$. Here ρ is the density, c is the speed of sound and κ is the frequency of the propagation which is assumed to be constant and fixed.

We are interested in stratified waveguides, that is the coefficients a and n are assumed to satisfy the properties:

- $a(x) := a(x_3) := \sum_{i=1}^{M} a_i \chi_i$, where $a_i,\ i = 1, ..., M$ are positive constants and
- $n(x) := n(x_3) := \sum_{i=1}^{M} n_i \chi_i$, where $n_i,\ i = 1, ..., M$ are positive constants

with

$$\chi_i := \begin{cases} 1 \ \text{ in } (\overline{z}_{i-1}, \overline{z}_i), \\ 0 \ \text{ in } (L, 0) \setminus (\overline{z}_{i-1}, \overline{z}_i) \end{cases} \tag{1.2}$$

$i = 1, ..., M$, $\overline{z}_0 = 0$ and $\overline{z}_M = L$. Sometimes we use the notation $x := (x', x_3)$ for $x \in \mathbb{R}^3$ where $x' := (x_1, x_2)$.

The forward problem (1.1) is well posed, see section 4. Precisely, if $f \in H^2(\mathbb{R}^2)$ with compact support or with an exponential decay at infinity, then (1.1) has one and only one solution $u \in H^1_{loc}(\Omega)$ satisfying appropriate radiation conditions at infinity in the horizontal directions of the waveguide. Hence $a\frac{\partial}{\partial x_3} u(x', 0)$ is in $H^{-\frac{1}{2}}_{loc}(\mathbb{R}^2)$ and we have the following estimate

$$\|a\frac{\partial}{\partial x_3} u(x', 0)\|_{H^{-\frac{1}{2}}_{loc}(\mathbb{R}^2)} \leq C\|f\|_{H^2(\mathbb{R}^2)} \tag{1.3}$$

with a positive constant C independent on f.

1.2. The inverse problem

In this work, we are concerned with the following inverse problem: *reconstruct the vectors*

$$(a_i)_{i=1}^M, (n_i)_{i=1}^M \ \ \text{and} \ (\overline{z}_i)_{i=1}^M$$

from the measurements $a\frac{\partial}{\partial x_3} u(x', 0)$, *for* $x' \in \mathbb{R}^2$, *corresponding to finitely many excitations* f.

In other words, we aim at localizing the layers and reconstruct the density and the speed of sound in each layer from finitely many measurements on the surface.

A classical idea is to solve such inverse problems for stratified media by reducing the problem to $1d$ inverse spectral problems. There is an extensive literature on these problems. Instead of reviewing the known results, we recall some of the most popular ideas proposed to solve these problems. We mention the asymptotic expansion technique used for the first time by Borg and then simplified by Levinson at the end of the 40s, see [6, 14]. A second approach is the integral equation method by Gelfand and Levitan introduced in the 50s for solving Sturm-Liouville inverse spectral problems, see [12]. During the period from the 50s till the 80s these two approaches have been extensively studied by many authors, see the references [15, 17, 21] for more details on these methods and the related results till mid 80s. A third approach is the so called the C-property by Ramm, see [23]. A fourth approach is the boundary control method (BC method in short) by Belichev and Kurylev introduced in the mid 80s, see [4] for the original version and [13] for a different presentation. It is worth mentioning that two of them (the Gelfand-Levitan method and the boundary control method) are constructive. In addition, the three first methods are designed to solve the problem for equations of the standard or normal form, i.e. the coefficient a is taken to be smooth enough so that we can reduce the equation in (1.4) to the form $\frac{d^2}{dy^2}v + q\ v = \lambda\ v$. The boundary control method has been generalized to the general Sturm Liouville problem (1.4) with discontinuous coefficients a in [27]. So, in principle this method could be used to solve our problem. However in the present work, we will not do so for two reasons. The first one is that the BC method, which is designed for quite general models, see [4], is quite expensive and requires several steps. We wish to propose direct formulae to solve our problem. The second reason is that we wish, as a next step to our work, to solve similar problems for the Lamé system and the coupled system of connected beams, see [5] and [10, 19], where we have multiple speeds of propagation. It is known that for these systems the BC method faces the difficulty of finding local estimates for the time domain of influence of the different waves, see [4] for more details. Finally, let us mention the work [1], and the references therein, regarding the recovery of discontinuities of the lower coefficients n, when a is smooth, from only the asymptotics of the eigenvalues. However, these asymptotics are not enough to recover the discontinuities of the leading coefficients a, see for instance the point (5) of the epilogue of [1] and the cited references there on this issue.

Instead, we propose to use ideas from the multidimensional inverse problems. The main idea is to reconstruct the vectors $(a_i)_{i=1}^M, (n_i)_{i=1}^M$ and $(\overline{z}_i)_{i=1}^M$ as singularities of singular solutions of the problem (1.1). These singular solutions can be generated using our collected measurements. Hence, we derive explicit formulae linking the measurements to the unknown coefficients. This approach has its roots in the probe and the sampling methods proposed in the last 15 years to solve the inverse scattering problem by interfaces, see [22] for a review. The use of asymptotic expansions of singular solutions to detect interfaces originated from the works [16, 20]. Compared to the other approaches mentioned above, our approach is limited to reconstructing singularities (i.e. localization of the discontinuities of the coefficients and reconstructions of their jumps). However, it has the advantage that it can be applied to a wider range of models as the two systems we mentioned above. Let us finally stress that our approach also solves, with a reconstruction algorithm, the inverse spectral problem for discontinuous coefficients a and n by locating their jumps, compare [1] and the references therein.

The paper is organized as follows. We complete the current section by giving more details on our approach. In section 2, we derive the formulae for the first two layers and in section 3, we state the results for the other layers. In section 4, we discuss the forward problem.

1.3. The approach to solve the inverse problem

We divide our approach into two steps.

1.3.1. Step 1. Reconstruction of the spectral data from one measurement

In this step, we use a finite number of special excitations f, which are not δ-type functions, to reconstruct the spectral data $(\lambda_i, |a\frac{de_i}{dx_3}(0)|)_{i=1}^{\infty}$ where $(\lambda_i, e_i)_{i=1}^{\infty}$ are the eigenvalues and eigenfunctions of the corresponding Sturm-Liouville problem:

$$\begin{cases} -\frac{d}{dx_3}a\frac{d}{dx_3}v - \kappa^2 n\, v = \lambda a\, v \quad \text{in } (L,0), \\ v = 0 \quad \text{on } x_3 = 0, \\ a\frac{d}{dx_3}v = 0 \quad \text{on } x_3 = L \end{cases} \tag{1.4}$$

with the understanding that the eigenfunctions are normalized as $\|e_i\|_{L^2(L,0)} = 1,\ i = 1, ..., \infty$. This idea has been proposed for the first time in [3] and it has been applied for the recovery of the refraction index in the acoustic waveguide in [2]. In the model used in [2], the excitation is located inside the waveguide and it is of the form $\delta(x')f(x_3)$. It is shown, in particular, that for bounded refraction index, one particular excitation f is enough to recover the corresponding Gelfand-Levitan spectral data if we measure the response of the medium both on the surface and on the bottom.

In the following, we show how we can do it for our model by choosing only one excitation. We take as a candidate the Gaussian type function: $f(x') = e^{-|x'|^2}$, $x' \in \mathbb{R}^2$. Its Fourier transform is again a Gaussian type function $\hat{f}(\lambda) = \frac{1}{\sqrt{2}}e^{-\frac{1}{4}|\lambda|^2}$, $\lambda \in \mathbb{C}^2$ and vanishes nowhere in $\mathbb{C}^2$.

Let u be the solution of (1.1) corresponding to f. Then its Fourier transform with respect to x', $\hat{u}(\lambda, x_3) := \mathcal{F}_{x'}(u)(\lambda, x_3)$, is the solution of the following problem:

$$\begin{cases} \frac{d}{dx_3}a\frac{d}{dx_3}\hat{u} + \kappa^2 n\, \hat{u} - \lambda^2 a\, \hat{u} = 0 \quad \text{in } (L,0), \\ \hat{u} = \hat{f}(\lambda) \quad \text{on } x_3 = 0, \\ a\frac{d}{dx_3}\hat{u} = 0 \quad \text{on } x_3 = L. \end{cases} \tag{1.5}$$

Using a spectral expansion, we obtain the following representation of the Fourier transform of the measured data $a\frac{\partial}{\partial x_3}u(x',0)$, $x' \in \mathbb{R}^2$, i.e. $a\frac{d}{dx_3}\hat{u}(\lambda,0)$:

$$a\frac{d}{dx_3}\hat{u}(\lambda,0) = \big[\sum_{i=1}^{\infty} \frac{-\lambda^2+\alpha^2}{(\lambda_i+\alpha)(\lambda_i+\lambda^2)}|a\frac{de_i}{dx_3}(0)|^2\big]\hat{f}(\lambda) \tag{1.6}$$

for $\lambda \in \mathbb{C}^2$. Here $\alpha \in \mathbb{C}\setminus\mathbb{R}$ is taken as a fixed complex number, hence different from the eigenvalues $\lambda_j,\ j=1,...,\infty,$ see section 4.

In the one hand, if we take $\lambda = (i\xi, 0)$ in (1.6) where $\xi \in \mathbb{R}$ then we can compute

$$\frac{a\frac{d}{dx_3}\hat{u}(\lambda,0)}{(\xi^2+\alpha)\hat{f}(\lambda)} = \big[\sum_{i=1}^{\infty} \frac{1}{(\lambda_i+\alpha)(\lambda_i-\xi^2)}|a\frac{de_i}{dx_3}(0)|^2\big]. \tag{1.7}$$

This equality makes sense for every ξ in $\mathbb{R}$ since in this set $\hat{f}(\lambda)$ is not vanishing. From this equality, we can reconstruct all the poles given by the positive eigenvalues λ_j and the corresponding zeros, given by $|a\frac{de_j}{dx_3}(0)|^2$, of the function $\frac{a\frac{d}{dx_3}\hat{u}(\lambda,0)}{(\xi^2+\alpha)\hat{f}(\lambda)}$.

In the other hand, if we take $\lambda := (\xi, 0)$ with $\xi \in \mathbb{R}$ in (1.6) and arguing as above, we recover all the negative eigenvalues λ_j with the corresponding traces of the derivatives of the eigenfunctions.

Remark 1.1. To recover the positive eigenvalues, we need to evaluate (1.7) for ξ in the whole $\mathbb{R}$. However, to recover the negative eigenvalues, we need, in principle, only a finite interval for ξ since we have only a finite number of such negative eigenvalues. If we know apriori the upper bounds of a and n, then the length of this interval can be estimated by these bounds using the Lieb-Thirring Theorem.

1.3.2. Step 2. Reconstruction of the layers from the spectral data

From step 1, we can assume that we know the spectral data which we use to detect the vectors $(a_i)_{i=1}^M, (n_i)_{i=1}^M$ and $(\overline{z}_i)_{i=1}^M$, $i=1,...,M$. For this we proceed step by step. First we generate a special excitation f_1 with which we reconstruct (a_1, n_1), see the formulae (2.4), (2.5) and (2.6). Knowing these values, we generate another data f_2 with which we locate $\overline{z}_1$ and simultaneously reconstruct (a_2, n_2), see the formulas (2.10), (2.11) and (2.12). We continue the process until the last layer, see the formulas (3.5), (3.6) and (3.7). In section 2, we explain the idea for the first two layers. Then in section 3, we state the results for all the other layers by induction. In addition, we explain how one can locate the seabed $\{x=(x',x_3) \in \mathbb{R}^2; x_3 = L\}$, see the formulas in (3.8) and Remark 3.2. In our analysis, we assume that we know a lower bound of the depth L.

2. The reconstruction formulas for the first two layers

2.1. Reconstruction of (a_1, n_1)

Let $\Phi(x;z) := \frac{1}{4\pi|x-z|}$ be the fundamental solution of the Laplacian Δ in $\mathbb{R}^3$. Take $z := (0,0,z_3)$, where $z_3 > 0$ and set $f_1 := \chi_R(x')\frac{\partial}{\partial x_3}\Phi(x',0;z)$ where χ_R is a C^2-smooth function satisfying $\chi_R(x') = 1$ if $|x_1|^2 + |x_2|^2 \leq \frac{R}{3}$ and $\chi_R(x') = 0$ if $|x_1|^2 + |x_2|^2 \geq \frac{R}{2}$. Hence, f_1 is in $H^2_{comp}(\mathbb{R}^2)$, $\forall z_3 > 0$. From section 4, we know that the corresponding solution, $u(x; z)$, of (1.1) is in $H^1_{loc}(\Omega)$ with the normal traces on the boundary of Ω as L^2_{loc}-functions. Let $A(\lambda; z) := \hat{f}_1$ and set $\hat{u}(\lambda, x_3; z)$ to be the solution of the following problem:

$$\begin{cases} \frac{d}{dx_3} a \frac{d}{dx_3} \hat{u} + \kappa^2 n \, \hat{u} - \lambda^2 a \, \hat{u} = 0 \quad \text{in } (L, 0), \\ \hat{u} = A(\lambda; z) \quad \text{on } x_3 = 0, \\ a \frac{d}{dx_3} \hat{u} = 0 \quad \text{on } x_3 = L. \end{cases} \tag{2.1}$$

Again using a spectral expansion, we obtain the following representation

$$a \frac{d}{dx_3} \hat{u}(\lambda, 0; z) = \big[\sum_{i=1}^{\infty} \frac{-\lambda^2 + \alpha}{(\lambda_i + \alpha)(\lambda_i + \lambda^2)} |a \frac{de_i}{dx_3}(0)|^2 \big] A(\lambda; z), \tag{2.2}$$

see section 4. Hence $a \frac{\partial}{\partial x_3} u(x', 0; z)$, with $x' \in \mathbb{R}^2$, which is the inverse Fourier transform of $a \frac{d}{dx_3} \hat{u}(\lambda, 0; z)$, can be computed from the spectral data $(\lambda_i, |a \frac{de_i}{dx_3}(0)|)_{i=1}^{\infty}$ using (2.2).

To any fixed $R > 0$, we correspond the cylinder $\Omega_R := B_2(0, R) \times (L, 0)$ where $B_2(0, R)$ is the 2D ball of center 0 and radius R. We partition its boundary as follows: $\partial\Omega_R := C_0 \cup C_L \cup C_R$ where $C_0 := B_2(0, R) \times \{0\}$, $C_L := B_2(0, R) \times \{L\}$ and $C_R := \partial B_2(0, R) \times (L, 0)$.

Multiplying the first equation of (1.1) by Φ and integrating by parts, recalling the regularities of Φ and u, we obtain:

$$\int_{\partial\Omega_R} \Phi(x; z) a \frac{\partial}{\partial \nu} u(x', 0; z) dx_1 dx_2 = \int_{\Omega_R} a \nabla u(x; z) \cdot \nabla \Phi(x; z) - \kappa^2 n(x) u(x; z) \Phi(x; z) dx. \tag{2.3}$$

Now, we write

$$\int_{\Omega_R} a \nabla u(x; z) \cdot \nabla \Phi(x; z) dx$$
$$= a_1 \int_{\Omega_R} \nabla u(x; z) \cdot \nabla \Phi(x; z) dx + \int_{\Omega_R} (a - a_1) \nabla u(x; z) \cdot \nabla \Phi(x; z) dx$$

and

$$\int_{\Omega_R} \nabla u(x; z) \cdot \nabla \Phi(x; z) dx = -\int_{\Omega_R} u \Delta \Phi dx + \int_{\partial\Omega_R} \frac{\partial}{\partial \nu} \Phi(x; z) u(x; z) ds(x).$$

But $\Delta\Phi = 0$ since $z \notin \Omega$. Hence we have the representation:

$$\int_{C_0} \Phi(x; z) a \frac{\partial}{\partial x_3} u(x; z) ds(x)$$
$$= \int_{\Omega_R} (a - a_1) \nabla u(x; z) \cdot \nabla \Phi(x; z) dx - \int_{\Omega_R} \kappa^2 n(x) u(x; z) \Phi(x; z) dx$$
$$+ a_1 \int_{\partial\Omega_R} \frac{\partial}{\partial \nu} \Phi(x; z) u(x; z) ds(x) - \int_{C_R} \Phi(x; z) a \frac{\partial}{\partial \nu} u(x; z) ds(x).$$

Note that the left hand side is computable from our data. The next proposition provides the dominant part of each term of the right hand side in terms of z.

Proposition 2.1. *Recall that $z_3 > 0$. We have the following asymptotic expansions:*

1. $\int_{C_R} \Phi(x;z) a \frac{\partial}{\partial \nu} u(x;z) ds(x) = \mathcal{O}(1)$,
2. $\int_{\partial \Omega_R} \frac{\partial}{\partial \nu} \Phi(x;z) u(x;z) ds(x) = \frac{1}{8\pi^2 z_3^2} + \mathcal{O}(1)$,
3. $\int_{\Omega_R} (a - a_1) \nabla u(x;z) \cdot \nabla \Phi(x;z) dx = \mathcal{O}(1)$,
4. $\int_{\Omega_R} n(x) u(x;z) \Phi(x;z) dx = \frac{n_1}{16\pi} \ln(z_3) + \mathcal{O}(1)$,

when $z_3 \to 0$.

We deduce that the indicator function

$$I_0(z_3) := \int_{C_0} \Phi(x;z) a \frac{\partial}{\partial x_3} u(x;z) ds(x) \tag{2.4}$$

has the following asymptotic expansion

$$I_0(z_3) = \frac{a_1}{8\pi^2 z_3^2} - \frac{\kappa^2 n_1}{16\pi} \ln(z_3) + \mathcal{O}(1), \ z_3 \to 0, \tag{2.5}$$

from which we can compute a_1 and n_1 as follows

$$\begin{cases} a_1 = \lim_{z_3 \to 0} [8\pi^2 z_3^2 I_0(z_3)] \ , \\ n_1 = -\frac{16\pi}{\kappa^2} \lim_{z_3 \to 0} \frac{I_0(z_3) - \frac{a_1}{8\pi^2 z_3^2}}{\ln(z_3)}. \end{cases} \tag{2.6}$$

2.2. Localization of $\bar{z}_1$ and reconstruction of a_2 and n_2

Let us now show how we can locate $\bar{z}_1$ and reconstruct a_2 and n_2 knowing the coefficients a_1 and n_1 (which we already reconstructed). Indeed, we start with a point $z := (0, 0, z_3)$ such that $z_3 < 0$ and z located near 0. We set $\Phi_1(x;z)$ to be the fundamental solution satisfying $(a_1 \Delta + \kappa^2 n_1) \Phi_1(x;z) = -\delta(x;z)$, in $\mathbb{R}^3$. We define $\Omega_{R'} = B(0, R') \times (L', \delta)$ where $R' > R$, $L' < L$ and $\delta > 0$[1]. Let $S(g)(x) := \int_{\partial \Omega_{R'}} \Phi_1(x;z) g(z) ds(s)$. For $z := (0, 0, z_3)$ fixed such that $z_3 > \bar{z}_1$, we define the set $\Omega^z_{R'}$ such that $\overline{\Omega_{R,1}} \subset \Omega^z_{R'} \subset \Omega_{R'}$ and $z \notin \Omega^z_{R'}$ where $\Omega_{R,1} := B_2(0, R) \times (L, \bar{z}_1)$, see Figure 1.

It is known that the single layer potential $S : L^2(\partial \Omega_{R'}) \to L^2(\partial \Omega^z_{R'})$ has a dense range whenever $\kappa^2 \frac{n_1}{a_1}$ is not a Dirichlet eigenvalue of the Laplacian on $\Omega^z_{R'}$, see [8] for instance. From the monotonicity of the Dirichlet eigenvalues with respect to the domain, varying a little $\Omega^z_{R'}$ if needed, we can assume that $\kappa^2 \frac{n_1}{a_1}$ satisfies this condition. Hence there exists a sequence g_ϵ in $L^2(\partial \Omega_{R'})$ such that $S(g_\epsilon) \to \Phi_1(\cdot\,; z)$ in $L^2(\partial \Omega^z_{R'})$. Let $f_{2,\epsilon}(x') := \chi_R(x') \frac{\partial}{\partial x_3} S(g_\epsilon)(x', 0)$.

[1] Recall that we assume that we know a lower bound L' of the depth L.

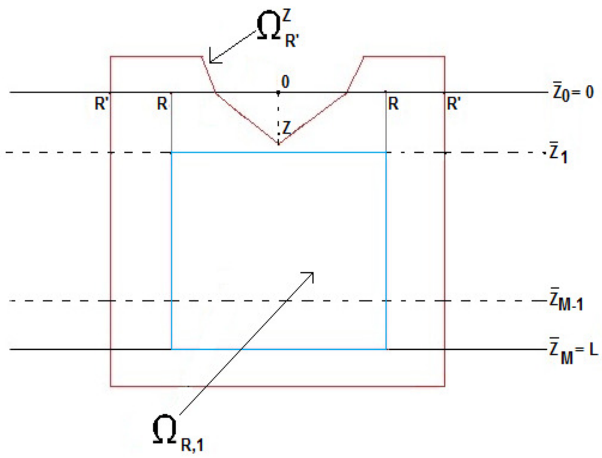

Figure 1: The figure shows the domains used to generate the exciting fields. The domain enclosed by the red curve is $\Omega^z_{R'}$ and the one enclosed by the blue curve is $\Omega_{R,1}$.

We use now as an excitation field $f_{2,\epsilon}(x')$. Let u^ϵ be the solution of (1.1) with the excitation $f_{2,\epsilon}(x')$. It enjoys then the regularity properties given in section 4. Then the trace on $x_3 = 0$ of the Fourier transform of the normal derivative of u^ϵ is nothing but $a\frac{d}{dx_3}\hat{u^\epsilon}(\lambda, 0; z)$ and it is computable using the spectral data as

$$a\frac{d}{dx_3}\hat{u^\epsilon}(\lambda, 0; z) = \Big[\sum_{i=1}^{\infty} \frac{-\lambda^2 + \alpha}{(\lambda_i + \alpha)(\lambda_i + \lambda^2)} |a\frac{de_i}{dx_3}(0)|^2\Big] A_\epsilon(\lambda; z) \tag{2.7}$$

where we set $A_\epsilon(\lambda; z) := \hat{f_{2,\epsilon}}$. Hence $\frac{\partial}{\partial x_3} u^\epsilon(x', 0; z)$ is also computable from the spectral data. Replacing in (2.3) Φ by $S(g_\epsilon)$ we obtain:

$$\begin{aligned} &\int_{\partial\Omega_R} S(g_\epsilon)(x', 0; z) a\frac{\partial}{\partial\nu} u^\epsilon(x', 0; z) dx_1 dx_2 \\ &= \int_{\Omega_R} \left[a\nabla u^\epsilon(x; z)\cdot\nabla S(g_\epsilon)(x; z) - \kappa^2 n(x) u^\epsilon(x; z) S(g_\epsilon)(x; z)\right] dx. \end{aligned}$$

But

$$\begin{aligned} &\int_{\Omega_R} \left[a\nabla u^\epsilon(x; z)\cdot\nabla S(g_\epsilon)(x; z) - \kappa^2 n(x) u^\epsilon(x; z) S(g_\epsilon)(x; z)\right] dx \\ &= \int_{\Omega_R} (a - a_1)\nabla u^\epsilon(x; z)\cdot\nabla S(g_\epsilon)(x; z) dx - \int_{\Omega_R} \kappa^2 n_1 u^\epsilon(x; z) S(g_\epsilon)(x; z)) dx \\ &+ \int_{\Omega} a_1 \nabla u^\epsilon(x; z)\cdot\nabla S(g_\epsilon)(x; z) dx - \int_{\Omega_R} \kappa^2 (n(x) - n_1) u^\epsilon(x; z) S(g_\epsilon)(x; z)) dx \\ &= \int_{\Omega_{R,1}} (a - a_1)\nabla u^\epsilon(x; z)\cdot\nabla S(g_\epsilon)(x; z) dx - \int_{\Omega_{R,1}} \kappa^2 (n(x) - n_1) u^\epsilon(x; z) S(g_\epsilon)(x; z) dx \\ &+ \int_{\partial\Omega_R} u^\epsilon(x; z)\cdot a_1 \frac{\partial}{\partial\nu} S(g_\epsilon)(x; z) ds(x) \end{aligned}$$

since

$$\int_{\Omega_R} a_1 \nabla u^\epsilon(x;z)\cdot\nabla S(g_\epsilon)(x;z) - \kappa^2 n_1 u^\epsilon(x;z) S(g_\epsilon)(x;z)dx$$
$$= \int_{\partial\Omega_R} a_1 u^\epsilon(x;z)\frac{\partial}{\partial\nu}S(g_\epsilon)(x;z)ds(x).$$

Hence

$$\begin{aligned}
&\int_{C_0\cup C_R} S(g_\epsilon)(x;z)a\frac{\partial}{\partial\nu}u^\epsilon(x;z)ds(x)\\
&= \int_{C_0\cup C_L\cup C_R} u^\epsilon(x;z)a_1\frac{\partial}{\partial\nu}S(g_\epsilon)(x;z)ds(x) + \int_{\Omega_{R,1}}(a-a_1)\nabla u^\epsilon(x;z)\cdot\nabla S(g_\epsilon)(x;z)dx\\
&- \int_{\Omega_{R,1}}\kappa^2(n(x)-n_1)u^\epsilon(x;z)S(g_\epsilon)(x;z)dx
\end{aligned}$$

recalling that $\Omega_{R,1} := B_2(0,R)\times(L,\bar{z}_1)$. From the above we get

$$\begin{aligned}
&\int_{C_0}\left(S(g_\epsilon)(x;z)a_1\frac{\partial}{\partial x_3}u^\epsilon(x;z) - u^\epsilon(x;z)a_1\frac{\partial}{\partial x_3}S(g_\epsilon)(x;z)\right)ds(x)\\
&= \int_{C_R}\left(u^\epsilon(x;z)a_1\frac{\partial}{\partial\nu}S(g_\epsilon)(x;z) - S(g_\epsilon)(x;z)a\frac{\partial}{\partial\nu}u^\epsilon(x;z)\right)ds(x)\\
&- \int_{C_L} u^\epsilon(x;z)a_1\frac{\partial}{\partial x_3}S(g_\epsilon)(x;z)ds(x) + \int_{\Omega_{R,1}}(a-a_1)\nabla u^\epsilon(x;z)\cdot\nabla S(g_\epsilon)(x;z)dx\\
&- \int_{\Omega_{R,1}}\kappa^2(n(x)-n_1)u^\epsilon(x;z)S(g_\epsilon)(x;z)dx.
\end{aligned} \tag{2.8}$$

In addition,

$$\int_{\Omega_{R,1}} (a-a_1)\nabla u^\epsilon(x;z)\cdot\nabla S(g_\epsilon)(x;z)dx - \int_{\Omega_{R,1}} \kappa^2(n(x)-n_1)u^\epsilon(x;z)S(g_\epsilon)(x;z)dx$$

$$= (a_2-a_1)\int_{\Omega_{R,1,2}} \nabla u^\epsilon(x;z)\cdot\nabla S(g_\epsilon)(x;z)dx$$

$$- \int_{\Omega_{R,1,2}} \kappa^2(n_2-n_1)u^\epsilon(x;z)S(g_\epsilon)(x;z)dx + L_{2,\epsilon}(z)$$

$$= -(a_2-a_1)\left[\int_{\Omega_{R,1,2}} u^\epsilon(x;z)\Delta S(g_\epsilon)(x;z)dx - \int_{\partial\Omega_{R,1,2}} u^\epsilon(x;z)\frac{\partial}{\partial\nu}S(g_\epsilon)(x;z)dx_1dx_2\right]$$

$$- \int_{\Omega_{R,1,2}} \kappa^2(n_2-n_1)u^\epsilon(x;z)S(g_\epsilon)(x;z)dx + L_{1,\epsilon}(z)$$

[use: $\Delta S(g_\epsilon) = -\frac{\kappa^2 n_1}{a_1}S(g_\epsilon)$]

$$= (a_2-a_1)\left[\int_{\partial\Omega_{R,1,2}} u^\epsilon(x;z)\frac{\partial}{\partial\nu}S(g_\epsilon)(x;z)dx_1dx_2 + \int_{\Omega_{R,1,2}} u^\epsilon(x;z)\frac{\kappa^2 n_1}{a_1}S(g_\epsilon)(x;z)dx\right]$$

$$- \int_{\Omega_{R,1,2}} \kappa^2(n_2-n_1)u^\epsilon(x;z)S(g_\epsilon)(x;z)dx + L_{2,\epsilon}(z)$$

$$= (a_2-a_1)\int_{\partial\Omega_{R,1,2}} u^\epsilon(x;z)\frac{\partial}{\partial\nu}S(g_\epsilon)(x;z)ds(x)$$

$$+ [-\kappa^2 n_2 + \frac{a_2}{a_1}\kappa^2 n_1]\int_{\Omega_{R,1,2}} u^\epsilon(x;z)S(g_\epsilon)(x;z)dx + L_{1,\epsilon}(z)$$

where $\Omega_{R,1,2} := B_2(0,R)\times(\bar{z}_2,\bar{z}_1)$ and

$$L_{1,\epsilon}(z) := \int_{\Omega_{R,2}} (a-a_1)\nabla u^\epsilon(x;z)\cdot\nabla S(g_\epsilon)(x;z)dx - \int_{\Omega_{R,2}} \kappa^2(n(x)-n_1)u^\epsilon(x;z)S(g_\epsilon)(x;z)dx$$

with $\Omega_{R,2} := B_2(0,R)\times(L,\bar{z}_2)$.
Summing up, we deduce the representation

$$\int_{C_0}\left(S(g_\epsilon)(x;z)a\frac{\partial}{\partial x_3}u^\epsilon(x;z) - u^\epsilon(x;z)a\frac{\partial}{\partial x_3}S(g_\epsilon)(x;z)\right)ds(x)$$

$$= \int_{C_R}\left(u^\epsilon(x;z)a_1\frac{\partial}{\partial\nu}S(g_\epsilon)(x;z) - S(g_\epsilon)(x;z)a\frac{\partial}{\partial\nu}u^\epsilon(x;z)\right)ds(x)$$

$$- \int_{C_L} u^\epsilon(x;z)a_1\frac{\partial}{\partial x_3}S(g_\epsilon)(x;z)ds(x) + L_{2,\epsilon}(z) + (a_2-a_1)\int_{\partial\Omega_{R,1,2}} u^\epsilon(x;z)\frac{\partial}{\partial\nu}S(g_\epsilon)(x;z)ds(x)$$

$$+ [-\kappa^2 n_2 + \frac{a_2}{a_1}\kappa^2 n_1]\int_{\Omega_{R,1,2}} u^\epsilon(x;z)S(g_\epsilon)(x;z)dx \tag{2.9}$$

Note that the left-hand side is computable using our data. The following proposition provides the dominant part of each term of the right hand side in terms of z.

Proposition 2.2. *For z_3 near $\bar{z}_1$, we have*

1. $\lim_{\epsilon\to 0}\int_{C_L} u^\epsilon(x;z)\frac{\partial}{\partial x_3}S(g_\epsilon)(x;z)ds(x) = \mathcal{O}(1)$,
2. $\lim_{\epsilon\to 0}\int_{C_R} u^\epsilon(x;z)a_1\frac{\partial}{\partial \nu}S(g_\epsilon)(x;z) - S(g_\epsilon)(x;z)a\frac{\partial}{\partial \nu}u^\epsilon(x;z)ds(x) = \mathcal{O}(1)$,
3. $\lim_{\epsilon\to 0} L_{1,\epsilon}(z) = \mathcal{O}(1)$,
4. $\lim_{\epsilon\to 0}\int_{\partial\Omega_{R,1,2}} u^\epsilon(x;z)\frac{\partial}{\partial \nu}S(g_\epsilon)(x;z)ds(x)$
$$= \frac{1}{8\pi^2 a_1(a_1+a_2)}\frac{1}{|z_3-\bar{z}_1|^2} - \frac{\kappa^2(n_2-n_1)}{16\pi a_1(a_1+a_2)^2}\ln(|z_3-\bar{z}_1|) + \mathcal{O}(1),$$
5. $\lim_{\epsilon\to 0}\int_{\Omega_{R,1,2}} u^\epsilon(x;z)S(g_\epsilon)(x;z)ds(x) = \frac{1}{8\pi a_1(a_1+a_2)}\ln(|z_3-\bar{z}_1|) + \mathcal{O}(1)$.

Hence, the indicator function

$$I_1(z_3) := \lim_{\epsilon\to 0}\int_{C_0} S(g_\epsilon)(x;z)\big[a\frac{\partial}{\partial x_3}u^\epsilon(x,z) - a\frac{\partial}{\partial x_3}S(g_\epsilon)(x;z)\big]ds(x) \tag{2.10}$$

has the following asymptotic expansion as $z_3 \to \bar{z}_1$,

$$I_1(z_3) = \frac{(a_2-a_1)a_1^{-1}(a_1+a_2)^{-1}}{8\pi^2|z_3-\bar{z}_1|^2} - \frac{-2a_1^2\kappa^2 n_2 + (a_1^2+a_2^2)\kappa^2 n_1}{16\pi a_1^2(a_1+a_2)^2}\ln|z_3-\bar{z}_1| + \mathcal{O}(1). \tag{2.11}$$

Remember that $I_1(z_3)$ is computable from our data. Hence using the previous behavior of $I_1(z_3)$, we can locate $\bar{z}_1$. Indeed, plotting $I_1(z_3)$ for z_3 starting from $z_3 = 0$, then $\bar{z}_1$ is located where $I_1(z_3)$ becomes very large. After localizing $\bar{z}_1$, we can compute a_2 and c_2 by the formulas:

$$\begin{cases} \frac{(a_2-a_1)}{a_1(a_1+a_2)} = \lim_{z_3\to\bar{z}_1}[8\pi^2|z_3-\bar{z}_1|^2 I_1(z_3)] \text{ (} a_1 \text{ is already computed)}, \\ n_2 = \frac{1}{2}(1+(\frac{a_2}{a_1})^2)n_1 - \frac{16\pi}{\kappa^2}(a_1+a_2)^2\lim_{z_3\to\bar{z}_1}\frac{[\frac{(a_1-a_2)a_1^{-1}(a_1+a_2)^{-1}}{8\pi^2|z_3-\bar{z}_1|^2} - I_1(z_3)]}{\ln|z_3-\bar{z}_1|}. \end{cases} \tag{2.12}$$

2.3. Proof of the propositions 2.1 and 2.2

2.3.1. Proof of Proposition 2.1

Proof. Let G_M be the Green function of the waveguide with $(a,c) := (\sum_{i=1}^M a_i\chi_i, \sum_{i=1}^M c_i\chi_i)$, where $\chi_i := (z_{i-1}, z_i), i = 1, ..., M$, see [5] for an explicit form of G_M. The solution u can be written as :

$$u(x;z) = -\int_{C_0} f_1(t;z)a_1\frac{\partial}{\partial x_3}G_M(t;x)ds(t) + \int_{C_R}\Big(a_1\frac{\partial}{\partial \nu}u(t;z)G_M(t;x) - \frac{\partial}{\partial \nu}G_M(t;x)u(t;z)\Big)ds(t). \tag{2.13}$$

for $x \in \Omega_R$. By a standard use of the radiation conditions [2], see [5], we know that

$$\int_{C_R} \left(a_1 \frac{\partial}{\partial \nu} u(t;z) G_M(t;x) - \frac{\partial}{\partial \nu} G_M(t;x) u(t;z) \right) ds(t) \to 0, \ R \to \infty.$$

Hence

$$u(x;z) = -\int_{C_0} f_1(t;z) a_1 \frac{\partial}{\partial x_3} G_M(t;x) ds(t). \tag{2.14}$$

for $x \in \Omega$.

(1). We have

$$\left| \int_{C_R} \Phi(x;z) a \frac{\partial}{\partial x_3} u(x;z) ds(x) \right| \leq C \int_{C_R} |\Phi(x;z)|^2 ds(x) \int_{C_R} \left|\frac{\partial}{\partial x_3} u(x;z)\right|^2 ds(x).$$

From (2.14), we obtain

$$\frac{\partial}{\partial x_3} u(x;z) = -\int_{C_0} \chi_R(t) \frac{\partial}{\partial t_3} \Phi(t;z) \frac{\partial^2}{\partial^2 x_3} G_M(t;x) ds(t).$$

We know that $\frac{\partial^2}{\partial x_3^2} G_M(t;x) = O(|t-x|^{-3}) = \mathcal{O}(1)$, for $t \in \tilde{C}_0$ and $x \in C_R$ where $\tilde{C}_0 := \{(x',0), \ |x'| < \frac{R}{2}\}$. In addition, $\chi_R(t) = 0$ if $|t| \geq \frac{R}{2}$. Then $|\frac{\partial}{\partial x_3} u(x;z)| \leq C \int_{\tilde{C}_0} |\frac{\partial}{\partial t_3} \Phi(t;z)| ds(t)$ for $x \in C_R$ and z near 0. Since $\Phi(t;z) := \frac{1}{4\pi |t-z|}$ then $\frac{\partial}{\partial t_3} \Phi(t;z)$ $= \frac{z_3}{4\pi |t-z|^3}$ for $t_3 = 0$ and

$$\int_{t_3=0} \left|\frac{\partial}{\partial t_3} \Phi(t;z)\right| dt = \frac{z_3}{4\pi} \int_{t_3=0} \frac{1}{|t-z|^3} dt = \frac{z_3}{4\pi} \mathcal{O}(\frac{1}{z_3}) = \mathcal{O}(1), \ z_3 \to 0.$$

Also, due to the singularity of the fundamental solution Φ, $\int_{C_R} |\Phi(x;z)|^2 ds(x)$ is bounded for z near 0. This ends the proof of the point (1).

(2).

(2.a). Again from (2.14), we have

$$|u(x;z)_{x_3=L}| \leq c \int_{C_0} |\Phi(t;z)| dt, \text{ since } \left|\frac{\partial}{\partial x_3} G_M(t;x)\right| < \infty, \text{ for } t_3 = 0 \text{ and } x_3 = L$$

and then, $|u(x;z)_{x_3=L}| \approx \mathcal{O}(1)$. Hence, as in the point (1),

$$\int_{C_L} \frac{\partial}{\partial x_3} \Phi(x;z) u(x;z) ds(x) = \mathcal{O}(1).$$

[2]By radiation conditions, we mean the ones stated in (4.30)-(4.32).

(2.b). With similar considerations as in (2.a), we show that

$$\int_{C_R} \frac{\partial}{\partial \nu}\Phi(x;z)u(x;z)ds(x) = \mathcal{O}(1).$$

(2.c). From the boundary conditions, we have

$$\int_{C_0} \frac{\partial}{\partial x_3}\Phi(x;z)u(x;z)ds(x) = \int_{C_0} \chi_R(x')\Big(\frac{\partial}{\partial x_3}\Phi(x;z)\Big)^2 ds(x).$$

Hence using the explicit form of Φ, we deduce that

$$\int_{C_0} \frac{\partial}{\partial x_3}\Phi(x;z)u(x;z)ds(x) = \frac{1}{8\pi^2 z_3^2} + \mathcal{O}(1), \quad \text{for } x \text{ near } 0.$$

Summing up (2.a), (2.b) and (2.c), we deduce (2).

(3). Since $a = a_1$ for x_3 in $(\bar{z}_1, 0)$, we have

$$\int_{\Omega_R} (a - a_1)\nabla u(t;z) \cdot \nabla\Phi(t;z)dt = \int_{\Omega_{R,1}} (a - a_1)\nabla u(t;z) \cdot \nabla\Phi(t;z)dt.$$

We know that, $\nabla\Phi(t;z) = \mathcal{O}(1)$ for $z_3 \to 0$, $(z \notin \Omega_{R,1}$ and $t \in \Omega_{R,1})$. Using again (2.14), we obtain

$$\nabla u(t;z) = -\int_{s_3=0} a_1 f_1(s;z)\nabla\frac{\partial}{\partial t_3}G_M(s;t)ds.$$

For $t \in \Omega_{R,1}$ and $s_3 = 0$ $(s \notin \Omega_{R,1})$, we know that $\nabla\frac{\partial}{\partial t_3}G_M(s;t) \approx \mathcal{O}(1)$. Hence,

$$|\nabla u(t;z)| \leq \int_{C_0} |f_1(s;z)|ds \approx z_3 \int_{s_3=0} \frac{1}{|s-z|^3} + \mathcal{O}(1) = \mathcal{O}(1)$$

which implies that

$$\int_{\Omega_R} (a - a_1)\nabla u(t;z) \cdot \nabla\Phi(t;z)dt \sim \mathcal{O}(1), \ z_3 \to 0.$$

(4). We have

$$\int_{\Omega_R} nu\Phi = \int_{\Omega_{R,1}} (n - n_1)u\Phi + \int_{\Omega_R} n_1 u\Phi.$$

As in (3), we can show that $\int_{\Omega_{R,1}}(n-n_1)u\Phi \sim \mathcal{O}(1)$. Let us now estimate, $\int_{\Omega_R} n_1 u\Phi$. We decompose it as

$$\int_{\Omega_R} n_1 u\Phi = \int_{B(0,R)\times(\bar{z}_1,0)} n_1 u\Phi + \int_{\Omega_R\setminus B(0,R)\times(\bar{z}_1,0)} n_1 u\Phi.$$

As in (1), we can show that the second term is bounded for $z = (0, 0, z_3)$ near 0. Let us estimate the first term. We set $w := u - \frac{\partial}{\partial x_3}\Phi$. Hence w satisfies:

$$
\begin{cases}
\nabla \cdot a\nabla w + \kappa^2 n w = -\nabla \cdot a\nabla \frac{\partial}{\partial x_3}\Phi - \kappa^2 n \frac{\partial}{\partial x_3}\Phi, \text{ in } B(0,R) \times (\bar{z}_1, 0), \\
w = 0 \text{ on } (B(0,R) \times \{0\})), \\
w = \mathcal{O}(1) \text{ on } \partial[B(0,R) \times (\bar{z}_1, 0)] \backslash \{x_3 = 0\}.
\end{cases}
\tag{2.15}
$$

Recall that $z \notin \Omega_R$ ($z_3 > 0$). Then $\nabla \cdot a\nabla \frac{\partial}{\partial x_3}\Phi = 0$ in $\mathbb{R}^2 \times (\bar{z}_1, 0)$ and the right hand side of the first equation in (2.15) is reduced to $-\kappa^2 n \frac{\partial}{\partial x_3}\Phi$. Note that $\frac{\partial}{\partial x_3}\Phi$ is bounded in $L^p(B(0,R) \times (\bar{z}_1, 0))$ for $p < \frac{3}{2}$. From (2.14) with some simple estimates, we show also that u (and then w) is bounded in $L^p(B(0,R) \times (\bar{z}_1, 0))$. Hence, the problem (2.15) can be read as :

$$
\begin{cases}
\Delta w = \mathcal{O}(1); \quad \text{in } L^p(B(0,R) \times (\bar{z}_1, 0)), \\
w|_{\partial(B(0,r) \times (\bar{z}_1, 0))} = \mathcal{O}(1); \quad \text{in } C^\infty(\partial(B(0,R) \times (\bar{z}_1, 0)))
\end{cases}
\tag{2.16}
$$

for $p < \frac{3}{2}$. By elliptic regularity, w is bounded in $W^{2,p}(B(0,R) \times (\bar{z}_1, 0))$ for $p < \frac{3}{2}$. By Sobolev embedding, w is bounded in $L^2(B(0,R) \times (\bar{z}_1, 0))$, in particular. Using the Cauchy-Schwartz inequality, $\int_{B(0,R)\times(\bar{z}_1,0)} w(x; z)\Phi(x; z)dx$ is also bounded in terms of z. This implies that

$$
\begin{aligned}
\int_{\Omega_R} n_1 u \Phi &= \int_{B(0,R)\times(\bar{z}_1,0)} n_1 \frac{\partial}{\partial x_3}\Phi \; \Phi + \mathcal{O}(1) \\
&= \frac{n_1}{16\pi} \ln|z_3| + \mathcal{O}(1),
\end{aligned}
$$

where we have used the estimate

$$
\int_{B(0,R)\times(\bar{z}_1,0)} n_1 \frac{\partial}{\partial x_3}\Phi(x; z) \; \Phi(x; z) = \frac{n_1}{16\pi^2} \int_{B(0,R)\times(\bar{z}_1,0)} \frac{x_3 - z_3}{|x - z|^4} dx = \frac{n_1}{16\pi} \ln(z_3) + \mathcal{O}(1),
\tag{2.17}
$$

for z near 0. □

2.3.2. Proof of Proposition 2.2

Proof. Let $v_z^\epsilon := u^\epsilon - \tilde{\chi}_R \frac{\partial}{\partial x_3} S(g_\epsilon)$. The truncating function $\tilde{\chi}_R$ is of the form $\tilde{\chi}_R(x) = \chi_R(x') l_1(x_3)$ with χ_R as defined in section 2.1 and $l(x_3)$ is a smooth function in $[L, 0]$ such that $l(x_3) = 1$ near $x_3 = 0$ and $l(x_3) = 0$ near L. Then v_z^ϵ satisfies:

$$
\begin{cases}
\nabla \cdot a\nabla v_z^\epsilon + \kappa^2 n v_z^\epsilon = -\nabla \cdot \big(a - a_1\big)\nabla(\tilde{\chi}_R \frac{\partial}{\partial x_3} S(g_\epsilon)) - \kappa^2 (n - n_1)\tilde{\chi}_R \frac{\partial}{\partial x_3} S(g_\epsilon) - R^\epsilon, \text{ in } \Omega, \\
v_z^\epsilon \text{ satisfies the radiation conditions for } |x'| \to \infty, \\
\frac{\partial}{\partial x_3} v_z^\epsilon = 0 \text{ on } C_L, \\
v_z^\epsilon = 0 \text{ on } C_0,
\end{cases}
\tag{2.18}
$$

where $R^\epsilon(x;z) := \nabla\cdot a_1\nabla\big[\tilde\chi_R\frac{\partial}{\partial x_3}S(g_\epsilon)\big] + \kappa^2 n_1\tilde\chi_R\frac{\partial}{\partial x_3}S(g_\epsilon)$ which we can develop as

$$R^\epsilon(x;z) := \tilde\chi_R\nabla\cdot(a_1\nabla\frac{\partial}{\partial x_3}S(g_\epsilon)) + \nabla\tilde\chi_R\cdot a_1\nabla\frac{\partial}{\partial x_3}S(g_\epsilon)$$
$$+\nabla\cdot\big[a_1(\nabla\tilde\chi)\frac{\partial}{\partial x_3}S(g_\epsilon)\big] + \kappa^2 n_1\tilde\chi_R\frac{\partial}{\partial x_3}S(g_\epsilon).$$

However

$$\tilde\chi_R\nabla\cdot(a_1\nabla\frac{\partial}{\partial x_3}S(g_\epsilon))+\kappa^2 n_1\tilde\chi_R\frac{\partial}{\partial x_3}S(g_\epsilon) = \tilde\chi_R\big[\nabla\cdot(a_1\nabla\frac{\partial}{\partial x_3}S(g_\epsilon))+\kappa^2 n_1\frac{\partial}{\partial x_3}S(g_\epsilon)\big] = 0$$

hence $R^\epsilon(x;z) = \nabla\tilde\chi_R\cdot a_1\nabla\frac{\partial}{\partial x_3}S(g_\epsilon) + \nabla\cdot\big[a_1(\nabla\tilde\chi)\frac{\partial}{\partial x_3}S(g_\epsilon)\big]$. We observe then that the left hand side of the first equation in (2.18) can be written as $\frac{\partial}{\partial x_3}F_\epsilon + G_\epsilon$ where

$$F_\epsilon := -\big(a-a_1\big)\frac{\partial}{\partial x_3}(\tilde\chi_R\frac{\partial}{\partial x_3}S(g_\epsilon)) - a_1(\frac{\partial}{\partial x_3}\tilde\chi)\frac{\partial}{\partial x_3}S(g_\epsilon)$$

and

$$G_\epsilon := -\,\big(a-a_1\big)\Delta'(\tilde\chi_R\frac{\partial}{\partial x_3}S(g_\epsilon)) - a_1\nabla'\cdot\big((\nabla'\tilde\chi_R)\frac{\partial}{\partial x_3}S(g_\epsilon)\big)$$
$$-\,\kappa^2(n-n_1)\tilde\chi_R\frac{\partial}{\partial x_3}S(g_\epsilon) - \nabla\tilde\chi_R\cdot a_1\nabla\frac{\partial}{\partial x_3}S(g_\epsilon).$$

Here we used the notaion $\Delta' := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ and $\nabla' := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T$ and the fact that a depends only on the variable x_3. In addition, due to the properties of $\tilde\chi_R$, the functions F_ϵ and G_ϵ are smooth and supported in the interior of $\Omega^z_{R'}$.

Since $S(g_\epsilon)\to\Phi_1(\cdot;z)$ in $L^2(\partial\Omega^z_{R'})$, then by the well posedness of the problem $\nabla\cdot a_1\nabla u+\kappa^2 nu = 0,$ in $\Omega^z_{R'}$ and $u = f$ on $\partial\Omega^z_{R'}$, and the interior estimates, recalling that a_1 is constant here, we deduce that

$$S(g_\epsilon)\to\Phi_1(\cdot;z) \text{ in } H^m(K),\ \forall K,\ \overline{K}\subset\Omega^z_{R'},\ m\in\mathbb{N}. \tag{2.19}$$

From (2.19), we see that F_ϵ converges in $L^2(\Omega)$ to

$$F := -\big(a-a_1\big)\frac{\partial}{\partial x_3}(\tilde\chi_R\frac{\partial}{\partial x_3}\Phi_1(\cdot;z)) - a_1(\frac{\partial}{\partial x_3}\tilde\chi)\frac{\partial}{\partial x_3}\Phi_1(\cdot;z).$$

We have also the convergence in $L^2(\Omega)$ of their $2D$-Laplacian Δ'. Similarly, G_ϵ converges in $L^2(\Omega)$ to

$$G := -\big(a-a_1\big)\Delta'(\tilde\chi_R\frac{\partial}{\partial x_3}\Phi_1(\cdot;z) - a_1\nabla'\cdot\big((\nabla'\tilde\chi_R)\frac{\partial}{\partial x_3}\Phi_1(\cdot;z)$$

$$-\kappa^2(n-n_1)\tilde\chi_R\frac{\partial}{\partial x_3}\Phi_1(\cdot;z) - \nabla\tilde\chi_R\cdot a_1\nabla\frac{\partial}{\partial x_3}\Phi_1(\cdot;z).$$

Then from the part 2 of Theorem 4.1, we deduce that

$$v_z^\epsilon \to v_z \text{ in } H^1_{loc}(\Omega) \tag{2.20}$$

where v_z satisfies

$$\begin{cases} \nabla\cdot a\nabla v_z + \kappa^2 n v_z = -\nabla\cdot\big(a-a_1\big)\nabla(\tilde{\chi}_R\frac{\partial}{\partial x_3}\Phi_1) - \kappa^2(n-n_1)\tilde{\chi}_R\frac{\partial}{\partial x_3}\Phi_1 - R, \ \text{ in } \Omega,\\ v_z \text{ satisfies the radiation conditions for } |x'|\to\infty,\\ \frac{\partial}{\partial x_3}v_z = 0 \ \text{ on } C_L,\\ v_z = 0 \ \text{ on } C_0, \end{cases} \tag{2.21}$$

and $R(x;z) := \nabla\tilde{\chi}_R\cdot a_1\nabla\frac{\partial}{\partial x_3}\Phi_1(\cdot;z) + \nabla\cdot\big[a_1(\nabla\tilde{\chi})\frac{\partial}{\partial x_3}\Phi_1(\cdot;z)\big]$. Here, we used the fact that $\frac{\partial}{\partial x_3}F + G$ is equal to the right hand side of the first equation in (2.21). This can be seen by similar computations as we did just after (2.18).

Now, we observe that $+\nabla\cdot\big(a-a_1\big)\nabla(\tilde{\chi}_R\frac{\partial}{\partial x_3}\Phi_1) + \kappa^2(n-n_1)\tilde{\chi}_R\frac{\partial}{\partial x_3}\Phi_1 + R = \tilde{\chi}_R(x)\frac{\partial}{\partial x_3}\delta(x;z) + \nabla\cdot a_1\nabla(\tilde{\chi}_R\frac{\partial}{\partial x_3}\Phi_1) + \kappa^2 n_1\tilde{\chi}_R\frac{\partial}{\partial x_3}\Phi_1$. In addition, $\tilde{\chi}_R(x)\frac{\partial}{\partial x_3}\delta(x;z) = \frac{\partial}{\partial x_3}\delta(x;z)$ since $\tilde{\chi}_R = 1$ near z. Hence $w_z := v_z + \tilde{\chi}_R\frac{\partial}{\partial x_3}\Phi_1$ satisfies

$$\begin{cases} \nabla\cdot a\nabla w_z + \kappa^2 n w_z = -\frac{\partial}{\partial x_3}\delta \ \text{ in } \Omega,\\ w_z \text{ satisfies the radiation conditions}, |x'|\to\infty,\\ \frac{\partial}{\partial x_3}w_z = 0 \ \text{ on } C_L,\\ w_z = \tilde{\chi}_R\frac{\partial}{\partial x_3}\Phi_1 \ \text{ on } C_0. \end{cases} \tag{2.22}$$

From (2.19) and (2.20), we deduce that $u^\epsilon := v_z^\epsilon + \tilde{\chi}_R\frac{\partial}{\partial x_3}S(g_\epsilon) \to w_z := v_z + \tilde{\chi}_R\frac{\partial}{\partial x_3}\Phi_1$ in $H^1(K)$, $\forall K, \overline{K}\subset\Omega^z_{R'}$. Note that C_L, C_R and $\Omega_{R,1}$ are strictly included in $\Omega^z_{R'}$.

Since for z near $(0,0,\overline{z_1})$ $\frac{\partial}{\partial x_3}\Phi_1|_{C_0}$ is bounded with respect to z with all the Sobolev norms on C_0, then w_z behaves as the derivative with respect to x_3 of the Green function, G_M, of the PDE problem in (2.22), i.e. $w_z \approx \frac{\partial}{\partial x_3}G_M$. Using these properties, we obtain the following estimates:

(A) $$\lim_{\epsilon\to 0}\int_{C_L} u^\epsilon(x;z)\frac{\partial}{\partial x_3}S(g_\epsilon)(x;z)ds(x)$$
$$= \int_{C_L} w_z(x;z)\frac{\partial}{\partial x_3}\Phi_1(x;z)ds(x) = \mathcal{O}(1), \ \text{ for } z \text{ near } \bar{z}_1,$$

since $w_z(x;z)\approx|x-z|^{-1}$ and $\frac{\partial}{\partial x_3}\Phi_1(x;z)\approx|x-z|^{-2}$ as $(0,0,\bar{z}_1)$ is away from C_L.

(B) $\lim_{\epsilon\to 0}\int_{C_R} u^\epsilon(x;z)a_1\frac{\partial}{\partial\nu}S(g_\epsilon)(x;z) - S(g_\epsilon)(x;z)a\frac{\partial}{\partial\nu}u^\epsilon(x;z)ds(x)$

$$= \int_{C_R} w_z(x;z)a_1\frac{\partial}{\partial\nu}\Phi_1(x;z) - \Phi_1(x;z)a\frac{\partial}{\partial\nu}w_z(x;z)s(x) = \mathcal{O}(1)$$

for z near $\bar{z}_1$ since $(0,0,\bar{z}_1)$ is away from C_R.

(C) $\lim_{\epsilon\to 0} L_{2,\epsilon}(z)$

$$= \int_{\Omega_{R,2}} (a - a_1)\nabla w_z(x;z)\cdot\nabla\Phi_1(x;z)dx - \int_{\Omega_{R,2}} \kappa^2(n(x)-n_1)w_z(x;z)\Phi_1(x;z)dx$$
$$= \mathcal{O}(1)$$

for z_3 near $\bar{z}_1$ since $(0,0,\bar{z}_1)$ is away from $\Omega_{R,2}$.
(D)

$$\lim_{\epsilon\to 0}\int_{\partial\Omega_{R,1,2}} u^\epsilon(x;z)\frac{\partial}{\partial\nu}S(g_\epsilon)(x;z)ds(x) = \int_{\partial\Omega_{R,1,2}} w_z(x;z)\frac{\partial}{\partial\nu}\Phi_1(x;z)ds(x).$$

(E)

$$\lim_{\epsilon\to 0}\int_{\Omega_{R,1,2}} u^\epsilon(x;z)S(g_\epsilon)(x;z)ds(x) = \int_{\Omega_{R,1,2}} w_z(x;z)\Phi_1(x;z)ds(x).$$

Let us estimate $\int_{\partial\Omega_{R,1,2}} w_z(x;z)\frac{\partial}{\partial\nu}\Phi_1(x;z)ds(x)$. We write it as

$$\int_{\partial\Omega_{R,1,2}} w_z(x;z)\frac{\partial}{\partial\nu}\Phi_1(x;z)ds(x) = \int_{\partial\Omega_{R,1,2}\setminus\{x_3=\bar{z}_1\}} w_z(x;z)\frac{\partial}{\partial x_3}\Phi_1(x;z)ds(x)+$$
$$\int_{\partial\Omega_{R,1,2}\cap\{x_3=\bar{z}_1\}} w_z(x;z)\frac{\partial}{\partial\nu}\Phi_1(x;z)ds(x).$$

From (2.22), using interior estimates, we deduce that $w_z \approx \partial_{x_3}\Phi_2$, where Φ_2 is the fundamental solution of the stratified medium given by (a_1,n_1) if $x_3 > \bar{z}_1$ and (a_2,n_2) if $x_3 < \bar{z}_1$, since $G_M \approx \Phi_2$, with their derivatives, in the interior of $B_2(0,R)\times(\bar{z}_1,0)$. Similar estimates are also true for the derivatives. Hence $\int_{\partial\Omega_{R,1,2}\setminus\{x_3=\bar{z}_1\}} w_z(x;z)\frac{\partial}{\partial\nu}\Phi_1(x;z)ds(x) = \mathcal{O}(1)$ as z_3 tends to $\bar{z}_1$ since $(0,0,\bar{z}_1)$ is away from $\partial\Omega_{R,1,2}\setminus\{x_3=\bar{z}_1\}$. In addition, we have $\int_{\partial\Omega_{R,1,2}\cap\{x_3=\bar{z}_1\}} w_z(x;z)\frac{\partial}{\partial x_3}\Phi_1(x;z)ds(x)$ $= \int_{\partial\Omega_{R,1,2}\cap\{x_3=\bar{z}_1\}} \frac{\partial}{\partial x_3}\Phi_2(x;z)\frac{\partial}{\partial x_3}\Phi_1(x;z)ds(x) + \mathcal{O}(1)$ for z_3 tending to $\bar{z}_1$. Summing up, we obtain

$$\int_{\partial\Omega_{R,1,2}} w_z(x;z)\frac{\partial}{\partial\nu}\Phi_1(x;z)ds(x) = \int_{\partial\Omega_{R,1,2}\cap\{x_3=\bar{z}_1\}}\frac{\partial}{\partial x_3}\Phi_2(x;z)\frac{\partial}{\partial x_3}\Phi_1(x;z)ds(x)+\mathcal{O}(1)$$

for z_3 tending to $\bar{z}_1$. Similarly, we obtain

$$\int_{\Omega_{R,1,2}} w_z(x;z)\Phi_1(x;z)ds(x) = \int_{\Omega_{R,1,2}}\frac{\partial}{\partial x_3}\Phi_2(x;z)\Phi_1(x;z)ds(x) + \mathcal{O}(1)$$

for z_3 tending to $\bar{z}_1$.

We have the following lemma which, lengthy but simple, proof is based on the explicit forms of $\tilde{\Phi}_i, i = 1,2$, given in section 2.4, and the fact that $\tilde{\Phi}_i, i = 1,2$ are the dominant parts of $\Phi_i, i = 1,2$.

Lemma 2.3. *The following asymptotic expansions hold as z_3 tends to $\bar{z}_1$:*

1. $\int_{\partial\Omega_{R,1,2}\cap\{x_3=\bar{z}_1\}} \frac{\partial}{\partial x_3}\Phi_2(x;z)\frac{\partial}{\partial x_3}\Phi_1(x;z)ds(x) = \frac{1}{8\pi^2 a_1(a_1+a_2)}\frac{1}{|z_3-\bar{z}_1|^2} - \frac{\kappa^2(n_2-n_1)}{16\pi(a_1+a_2)^2}$ $\ln(|z_3-\bar{z}_1|) + \mathcal{O}(1)$.
2. $\int_{\Omega_{R,1,2}} \frac{\partial}{\partial x_3}\Phi_2(x;z)\Phi_1(x;z)ds(x) = \frac{1}{8\pi a_1(a_1+a_2)}\ln(|z_3-\bar{z}_1|) + \mathcal{O}(1)$.

Using the properties $(A), (B), (C), (D), (E)$, we obtain the sough formulas of Proposition 2.2.

□

2.4. Explicit forms of $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$

2.4.1. Explicit form of $\tilde{\Phi}_1$

Since $\tilde{\Phi}_1$ satisfies $a_1\Delta\tilde{\Phi}_1 = -\delta$ in $\mathbb{R}^3$, then we see that $\tilde{\Phi}_1(x;z) = \frac{1}{a_1}\Phi(x;z)$ $= \frac{1}{4\pi a_1}\frac{1}{|x-z|}$.

2.4.2. Explicit form of $\tilde{\Phi}_2$

Recall that $\tilde{\Phi}_2$ satisfies

$$\nabla\cdot a\nabla\tilde{\Phi}_2 = -\delta(x-z), \quad \text{in } \mathbb{R}^3$$

I. If $z_3 > \bar{z}_1$ we set $W^1 := \tilde{\Phi}_2 - \tilde{\Phi}_1$ and $W^1 := \begin{cases} W^1_+ \text{ if } x_3 > \bar{z}_1, \\ W^1_- \text{ if } x_3 < \bar{z}_1. \end{cases}$

Then W^1 satisfies

$$\begin{cases} \Delta W^1_+ = 0, \quad \text{if } x_3 > \bar{z}_1, \\ \Delta W^1_- = 0, \quad \text{if } x_3 < \bar{z}_1, \\ W^1_+ = W^1_- \quad \text{if } x_3 = \bar{z}_1, \\ a_1\frac{\partial}{\partial x_3}W^1_+ - a_2\frac{\partial}{\partial x_3}W^1_- = (a_2-a_1)\frac{\partial}{\partial x_3}\tilde{\Phi}_1 \quad \text{if } x_3 = \bar{z}_1. \end{cases} \tag{2.23}$$

II. If $z_3 < \bar{z}_1$, we set $W^2 := \tilde{\Phi}_2 - \tilde{\Phi}_{1,2}$ where $\tilde{\Phi}_{1,2}(x;z) = \frac{1}{a_2}\Phi(x;z)$. Then W^2 satisfies

$$\begin{cases} \Delta W^2_+ = 0, \quad \text{if } x_3 > \bar{z}_1, \\ \Delta W^2_- = 0, \quad \text{if } x_3 < \bar{z}_1, \\ W^2_+ = W^2_- \quad \text{if } x_3 = \bar{z}_1, \\ a_1\frac{\partial}{\partial x_3}W^2_+ - a_2\frac{\partial}{\partial x_3}W^2_- = (a_2-a_1)\frac{\partial}{\partial x_3}\tilde{\Phi}_{1,2} \quad \text{if } x_3 = \bar{z}_1. \end{cases} \tag{2.24}$$

We do the calculations for **I**. We represent $W^1_\pm$ as:

$$W^1_\pm(x;z) := \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{ix'\cdot\xi' \mp (x_3-\bar{z}_1)|\xi'|}\hat{\Psi}_\pm(\xi';z')d\xi'.$$

These functions satisfy the first two equations of (2.23). From the 3rd equation, we obtain $\hat{\Psi}_+ = \hat{\Psi}_+ =: \hat{\Psi}$. Let us consider the fourth equation of (2.23). We obtain

$$-a_1 \int_{\mathbb{R}^2} e^{ix'\cdot\xi'}|\xi'|\hat{\Psi}(\xi';z')d\xi' - a_2 \int_{\mathbb{R}^2} e^{ix'\cdot\xi'}|\xi'|\hat{\Psi}(\xi';z')d\xi' = (a_2 - a_1)\int_{\mathbb{R}^2} \frac{1}{2a_1} e^{i(x'-z')\cdot\xi' + (\bar{z}_1 - z_3)|\xi'|}d\xi',$$

hence

$$-(a_1 + A_2)|\xi'|\Psi = \frac{a_2 - a_1}{2a_1} e^{-z'\cdot\xi' - (\bar{z}_1 - z_3)|\xi'|}$$

and

$$\Psi(\xi'; z) = -\frac{a_2 - a_1}{2a_1(a_1 + a_2)} \frac{e^{-z'\cdot\xi' + (\bar{z}_1 - z_3)|\xi'|}}{|\xi'|}$$

which gives

$$W^1_-(x;z) = -\frac{a_2 - a_1}{(a_1 + a_2)} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{1}{2a_1} \frac{e^{i(x'-z')\cdot\xi' + (x_3 - z_3)|\xi'|}}{|\xi'|} d\xi'$$

and

$$W^1_+(x;z) = -\frac{a_2 - a_1}{(a_1 + a_2)} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{1}{2a_1} \frac{e^{i(x'-z')\cdot\xi' - (x_3 - 2\bar{z}_1 + z_3)|\xi'|}}{|\xi'|} d\xi'.$$

We set $z_3^* := -z_3 + 2\bar{z}_1$. Note that $z^* := (z_1, z_2, z_3^*)$ is nothing but the symmetric of $z := (z_1, z_2, z_3)$ with respect to the plan $z_3 = \bar{z}_1$ i,e $z_3 - \bar{z}_1 = \bar{z}_1 - z_3^*$. Then

$$W^1_-(x;z) = -\frac{a_2 - a_1}{(a_1 + a_2)} \tilde{\Phi}_1(x;z),$$
$$W^1_+(x;z) = -\frac{a_2 - a_1}{(a_1 + a_2)} \tilde{\Phi}_1(x;z^*).$$

In a similar way, we obtain, taking into account the differences in the 4th equations of (2.23) and (2.24),

$$W^2_-(x;z) = +\frac{a_2 - a_1}{(a_1 + a_2)} \tilde{\Phi}_{1,2}(x;z^*),$$
$$W^2_+(x;z) = +\frac{a_2 - a_1}{(a_1 + a_2)} \tilde{\Phi}_{1,2}(x;z).$$

Finally, we have the close form of $\tilde{\Phi}_2$ as follows:

$$\begin{cases} \tilde{\Phi}_2(x;z) = \frac{1}{a_1}\Phi(x;z) - \frac{a_2 - a_1}{a_1(a_1 + a_2)}\Phi(x;z^*), & \text{if } z_3 < \bar{z}_1 \text{ and } x_3 > \bar{z}_1, \\ \tilde{\Phi}_2(x;z) = \frac{2}{(a_1 + a_2)}\Phi(x;z), & \text{if } z_3 < \bar{z}_1, (\text{or } z_3 > \bar{z}_1) \text{ and } x_3 < \bar{z}_1, \\ \tilde{\Phi}_2(x;z) = \frac{1}{a_2}\Phi(x;z) + \frac{a_2 - a_1}{a_2(a_1 + a_2)}\Phi(x;z^*), & \text{if } x_3, z_3 > \bar{z}_1. \end{cases} \tag{2.25}$$

3. The reconstruction formulae for the other layers

3.1. The reconstruction formulae for $(a_j, c_j, \bar{z}_{j-1})$, $3 \leq j \leq M$

Following the same way as for reconstructing $(a_2, n_2, \bar{z}_1)$, we reconstruct $(a_j, n_j, \bar{z}_{j-1})$, $3 \leq j \leq M$. Assume that we have already reconstructed the coefficients (a_i, n_i), $i = 1, ..., j-1$ and $\bar{z}_0, ..., \bar{z}_{j-2}$. For $j = 3$, we have already shown in the previous section how to reconstruct $(a_1, n_1, \bar{z}_0)$, recalling that $\bar{z}_0 = 0$, and $(a_2, n_2, \bar{z}_1)$. Let Φ_{j-1} be the fundamental solution of $\nabla \cdot \tilde{a} \nabla \Phi_{j-1} + \kappa^2 \tilde{n} \Phi_{j-1} = -\delta$ in $\mathbb{R}^3$ with

$$(\tilde{a}, \tilde{n}) = \begin{cases} (a_1, n_1) \text{ if } z_3 \in (\bar{z}_1, +\infty), \\ (a_2, n_2) \text{ if } z_3 \in (\bar{z}_2, \bar{z}_1), \\ . \\ . \\ (a_{j-1}, n_{j-1}) \text{ if } z_3 \in (-\infty, \bar{z}_{j-2}). \end{cases} \tag{3.1}$$

Similar to section 2, we start with z_3 just below $\bar{z}_{j-2}$. Let us define $S_{j-1}(g)(x) := \int_{\partial\Omega_R} \Phi_{j-1}(x; z) g(z) ds(s)$. For $z := (0, 0, z_3)$ fixed such that $z_3 < \bar{z}_{j-2}$, we define again the set $\Omega^z_{R'}$ such that $\overline{\Omega_{R,j-1}} \subset \Omega^z_{R'} \subset \Omega_{R'}$ and $z \notin \Omega^z_{R'}$ where $\Omega_{R,j-1,j} := B_2(0, R) \times (\bar{z}_j, \bar{z}_{j-1})$, With similar arguments as in the previous section, there exists a sequence g_ϵ in $L^2(\partial\Omega_{R'})$ such that $S_{j-1}(g_\epsilon) \to \Phi_{j-1}(.\,; z)$ in $L^2(\partial\Omega^z_{R'})$. Let $f_{j-1,\epsilon}(x') := \chi_R(x') \frac{\partial}{\partial x_3} S_{j-1}(g_\epsilon)(x')$.

We use now as an excitation field $f_{j-1,\epsilon}(x')$. Let u^ϵ_{j-1} be the solution of (1.1) with the excitation $f_{j-1,\epsilon}(x')$. From section 4, the trace on $x_3 = 0$ of the Fourier transform of the normal derivative of u^ϵ_{j-1} is no thing but $a \frac{d}{dx_3} \hat{u^\epsilon_{j-1}}(\lambda, 0; z)$ and it is computable using the spectral data as

$$a \frac{d}{dx_3} \hat{u^\epsilon_{j-1}}(\lambda, 0; z) = \Big[\sum_{i=1}^\infty \frac{-\lambda^2 + \alpha}{(\lambda_i + \alpha)(\lambda_i + \lambda^2)} |a \frac{de_i}{dx_3}(0)|^2 \Big] A_{j-1}(\lambda; z) \tag{3.2}$$

where we set $A_{j-1}(\lambda; z) := \hat{f_{j-1,\epsilon}}$. Here also $\frac{\partial}{\partial x_3} u^\epsilon_{j-1}(x', 0; z)$ is nothing but the inverse Fourier transform of $a \frac{d}{dx_3} \hat{u^\epsilon_{j-1}}(\lambda, 0; z)$. Hence it is computable from the spectral data.

We state the corresponding relations to (2.8).

$$\begin{aligned} &\int_{C_0} \left(S_{j-1}(g_\epsilon)(x; z) a \frac{\partial}{\partial x_3} u^\epsilon_{j-1}(x; z) - u^\epsilon_{j-1} a \frac{\partial}{\partial x_3} S_{j-1}(g_\epsilon)(x; z) \right) ds(x) \\ &= \int_{C_R} \left(u^\epsilon_{j-1}(x; z) \tilde{a} \frac{\partial}{\partial \nu} S_{j-1}(g_\epsilon)(x; z) - S_{j-1}(g_\epsilon)(x; z) a \frac{\partial}{\partial \nu} u^\epsilon_{j-1}(x; z) \right) ds(x) \\ &- \int_{C_L} u^\epsilon_{j-1}(x; z) \tilde{a} \frac{\partial}{\partial x_3} S_{j-1}(g_\epsilon)(x; z) ds(x) + \int_{\Omega_{R,j-1}} (a - \tilde{a}) \nabla u^\epsilon_{j-1}(x; z) \cdot \nabla S_{j-1}(g_\epsilon)(x; z) dx \\ &- \int_{\Omega_{R,j-1}} \kappa^2 (n(x) - \tilde{n}) u^\epsilon_{j-1}(x; z) S_{j-1}(g_\epsilon)(x; z) dx. \end{aligned} \tag{3.3}$$

Arguing as after (2.8), we obtain the corresponding representation to (2.9)

$$\begin{aligned}
&\int_{C_0}\left(S_{j-1}(g_\epsilon)(x;z)a\frac{\partial}{\partial x_3}u^\epsilon_{j-1}(x;z)-u^\epsilon_{j-1}a\frac{\partial}{\partial x_3}S_{j-1}(g_\epsilon)(x;z))\right)ds(x)\\
&=\int_{C_R}\left(u^\epsilon_{j-1}(x;z)\tilde{a}\frac{\partial}{\partial\nu}S_{j-1}(g_\epsilon)(x;z)-S_{j-1}(g_\epsilon)(x;z)a\frac{\partial}{\partial\nu}u^\epsilon_{j-1}(x;z)\right)ds(x)\\
&-\int_{C_L}u^\epsilon_{j-1}(x;z)a_{j-1}\frac{\partial}{\partial x_3}S_{j-1}(g_\epsilon)(x;z)ds(x)+L_{j-1,\epsilon}(z)\\
&+(a_j-a_{j-1})\int_{\partial\Omega_{R,j-1,j}}u^\epsilon_{j-1}(x;z)\frac{\partial}{\partial x_3}S_{j-1}(g_\epsilon)(x;z)ds(x)\\
&+[-\kappa^2n_j+\frac{a_j}{a_{j-1}}\kappa^2n_{j-1}]\int_{\Omega_{R,j-1,j}}u^\epsilon_{j-1}(x;z)S_{j-1}(g_\epsilon)(x;z)dx
\end{aligned}\tag{3.4}$$

where $\Omega_{R,j-1,j}:=B_2(0,R)\times(\bar{z}_j,\bar{z}_{j-1})$ and

$$\begin{aligned}
L_{j-1,\epsilon}(z):=&\int_{\Omega_{R,j}}(a-a_{j-1})\nabla u^\epsilon_{j-1}(x;z)\cdot\nabla S_{j-1}(g_\epsilon)(x;z)dx\\
&-\int_{\Omega_{R,j}}\kappa^2(n(x)-n_{j-1})u^\epsilon_{j-1}(x;z)S_{j-1}(g_\epsilon)(x;z)dx
\end{aligned}$$

with $\Omega_{R,j}:=B_2(0,R)\times(L,\bar{z}_j)$.

Note again that the left-hand side of (3.4) is computable using our data. The following proposition provides the dominant part of each term of the right hand side in terms of z.

Proposition 3.1. *For z_3 near $\bar{z}_{j-1}$, we have*

1. $\lim_{\epsilon\to0}\int_{C_L}u^\epsilon_{j-1}(x;z)\frac{\partial}{\partial x_3}S_{j-1}(g_\epsilon)(x;z)ds(x)=\mathcal{O}(1)$,
2. $\lim_{\epsilon\to0}\int_{C_R}\left[u^\epsilon_{j-1}(x;z)\tilde{a}\frac{\partial}{\partial\nu}S_{j-1}(g_\epsilon)(x;z)-S_{j-1}(g_\epsilon)(x;z)a\frac{\partial}{\partial\nu}u^\epsilon_{j-1}(x;z)\right]ds(x)=\mathcal{O}(1)$,
3. $\lim_{\epsilon\to0}L_{j-1,\epsilon}(z)=\mathcal{O}(1)$,
4. $\lim_{\epsilon\to0}\int_{\partial\Omega_{R,j-1}}u^\epsilon_{j-1}(x;z)\frac{\partial}{\partial\nu}S_{j-1}(g_\epsilon)(x;z)ds(x)$

$$=\frac{1}{8\pi^2a_{j-1}(a_{j-1}+a_j)}\frac{1}{|z_3-\bar{z}_j|^2}-\frac{\kappa^2(n_j-n_{j-1})}{16\pi a_{j-1}(a_{j-1}+a_j)^2}\ln(|z_3-\bar{z}_j|)+\mathcal{O}(1),$$

5. $\lim_{\epsilon\to0}\int_{\Omega_{R,j-1}}u^\epsilon_{j-1}(x;z)S_{j-1}(g_\epsilon)(x;z)ds(x)=\frac{1}{8\pi a_{j-1}(a_{j-1}+a_j)}\ln(|z_3-\bar{z}_j|)+\mathcal{O}(1)$.

Hence, the indicator function

$$I_{j-1}(z_3):=\lim_{\epsilon\to0}\int_{C_0}S_{j-1}(g_\epsilon)(x;z)\big[a\frac{\partial}{\partial x_3}u^\epsilon_{j-1}(x;z)-a\frac{\partial}{\partial x_3}S_{j-1}(g_\epsilon)(x;z)\big]ds(x)\tag{3.5}$$

has the following asymptotic expansion

$$
\begin{aligned}
I_{j-1}(z_3) =& \frac{(a_j - a_{j-1})a_{j-1}^{-1}(a_{j-1}+a_j)^{-1}}{8\pi^2|z_3 - \bar{z}_{j-1}|^2} \\
&- \frac{-2a_{j-1}^2\kappa^2 n_j + (a_{j-1}^2 + a_j^2)\kappa^2 n_{j-1}}{16\pi a_{j-1}^2(a_{j-1}+a_j)^2} \ln|z_3 - \bar{z}_{j-1}| + \mathcal{O}(1) \text{ as } z_3 \to \bar{z}_{j-1}.
\end{aligned} \tag{3.6}
$$

Using the previous behavior of $I_{j-1}(z_3)$, we can locate $\bar{z}_{j-1}$. After localizing $\bar{z}_{j-1}$, we can compute a_j and c_j by the formulas:

$$
\begin{cases}
\frac{(a_j - a_{j-1})}{a_{j-1}(a_{j-1}+a_j)} = \lim_{z_3 \to \bar{z}_j}[8\pi^2|z_3 - \bar{z}_{j-1}|^2 I_{j-1}(z_3)] \; (\; a_{j-1} \text{ is already computed }), \\
n_j = \frac{1}{2}(1 + (\frac{a_j}{a_{j-1}})^2)n_{j-1} - \frac{16\pi}{\kappa^2}(a_{j-1}+a_j)^2 \lim_{z_3 \to \bar{z}_j} \frac{[\frac{(a_{j-1}-a_j)a_{j-1}^{-1}(a_{j-1}+a_j)^{-1}}{8\pi^2|z_3 - \bar{z}_j|^2} - I_{j-1}(z_3)]}{\ln|z_3 - \bar{z}_{j-1}|}.
\end{cases} \tag{3.7}
$$

3.2. Proof of Proposition 3.1

Proof. We proceed as in the proof of Proposition 2.2 using the fact that $\tilde{\Phi}_{j-1}$ is the fundamental solution of $a_{j-1}\Delta U = -\delta$ i,e $\tilde{\Phi}_{j-1}(x,z) = \frac{1}{a_{j-1}}\Phi(x;z)$ and $\tilde{\Phi}_j$ is the one of $\nabla \cdot \tilde{a}_j \nabla U = -\delta$, where now $\tilde{a}_j(z', z_3) = \begin{cases} a_{j-1} \text{ in } z_3 > \bar{z}_j \\ a_j \text{ in } z_3 < \bar{z}_j \end{cases}$, which has the same explicit form as in (2.25) replacing (a_1, a_2) by (a_{j-1}, a_j). As a truncating function we use $\tilde{\chi}_R(x) := \chi_R(x')l_j(x_3)$ where l_j is a smooth function such that $l_j(x_3) = 1$ near $x_3 = 0$ and $x_3 = \overline{z}_j$ with $l_j(x_3) = 0$ near $x_3 = \overline{z}_m, \; m := 1, ..., j-1$, and $x_3 = L$. In addition, the property (2.19) is now satisfied in each layer. This property should be combined with the form of the truncating function $\tilde{\chi}_R(x)$ to justify the steps of the proof. □

3.3. Localization of the seabed $\{x_3 = L\}$

Note that for $\bar{z}_M = L$, $\tilde{a} = a$ and $\tilde{c} = c$ in $(L, 0)$. Hence taking $j = M+1$ in (3.1) and in the paragraph after (3.1), then from (3.3), we obtain

$$
\begin{aligned}
I_M(z_3) :=& \lim_{\epsilon \to 0} \int_{C_0} \left(S_M(g_\epsilon)(x;z) a \frac{\partial}{\partial x_3} u_M^\epsilon(x;z) - u_M^\epsilon(x;z) a \frac{\partial}{\partial x_3} S_M(g_\epsilon)(x;z) \right) ds(x) \\
=& \lim_{\epsilon \to 0} [\int_{C_R} \left(u_M^\epsilon(x;z) \tilde{a} \frac{\partial}{\partial \nu} S_M(g_\epsilon)(x;z) \right. \\
& \left. - S_M(g_\epsilon)(x;z) a \frac{\partial}{\partial \nu} u_M^\epsilon(x;z) \right) ds(x) - \int_{C_L} u_M^\epsilon(x;z) a \frac{\partial}{\partial x_3} S_M(g_\epsilon)(x;z) ds(x)]
\end{aligned}
$$

As in the propositions (2.1), (2.2) and (3.1), we show that

$$
\lim_{\epsilon \to 0} \int_{C_R} \left(u_M^\epsilon(x;z) \tilde{a} \frac{\partial}{\partial \nu} S_M(g_\epsilon)(x;z) - S_M(g_\epsilon)(x;z) a \frac{\partial}{\partial \nu} u_M^\epsilon(x;z) \right) ds(x) = \mathcal{O}(1), \; z_3 \to L
$$

while

$$\lim_{\epsilon\to 0}\int_{C_L} u_M^\epsilon(x;z)a\frac{\partial}{\partial x_3}S_M(g_\epsilon)(x;z)ds(x) = \int_{C_L} a_M(\frac{\partial}{\partial x_3}\Phi_1(x;z))^2 ds(x) + \mathcal{O}(1)$$
$$= \frac{1}{8\pi^2 a_M}\frac{1}{|z_3-L|^2} + \mathcal{O}(1)$$

from which we obtain the asymptotic behavior

$$I_M(z_3) = -\frac{1}{8\pi^2 a_M}\frac{1}{|z_3-L|^2} + \mathcal{O}(1)$$

for z_3 near L. As a conclusion, we can locate L by the behavior of $I_{j-1}(z_3)$, for $j \geq 2$, given as follows:

$$\begin{cases} \text{If } \lim_{z_3\to\bar{z}_{j-1}} 8\pi^2|z_3-\bar{z}_{j-1}|^2 I_{j-1}(z_3) = \frac{a_j-a_{j-1}}{a_{j-1}(a_{j-1}+a_j)}, \text{ then } j < M+1, \text{i.e. } \bar{z}_{j-1} < L, \\ \text{If } \lim_{z_3\to\bar{z}_{j-1}} 8\pi^2|z_3-\bar{z}_{j-1}|^2 I_{j-1}(z_3) = -\frac{1}{a_{j-1}}, \text{ then } j = M+1, \text{i.e. } \bar{z}_{j-1} = \bar{z}_M = L, \end{cases} \tag{3.8}$$

observing that a_{j-1} and a_j are computed before estimating L.

Remark 3.2. Observe that $\frac{a_j-a_{j-1}}{a_{j-1}(a_{j-1}+a_j)} \neq -\frac{1}{a_{j-1}}$ since otherwise $\frac{a_j-a_{j-1}}{a_{j-1}+a_j} = -1$ and then $a_j = 0$ which contradicts our hypothesis on the positivity of the coefficient a. We remark that $a_j = 0$ (or $a_j \approx 0$) means that the density ρ of the corresponding layer is very large since $a = \frac{1}{\rho}$. Hence in this case, the upper surface of this layer will play the role of the bottom surface of the waveguide.

4. The forward problem

The object of this section is to justify the following theorem

Theorem 4.1. *Let $a := a(x_3)$ and $n := n(x_3)$ be $L^\infty(L,0)$-coefficients where a is bounded from below by a positive constant. We have the following two results:*

1. *Let $f \in H^2_{comp}(\mathbb{R}^2) := \{f \in H^2(\mathbb{R}^2);\ \text{with compact support}\}$ or satisfies $e^{b|\cdot|}f(\cdot) \in H^2(\mathbb{R}^2)$ with some positive constant b. Then the problem*

$$\begin{cases} \nabla\cdot a\nabla u + \kappa^2 n u = 0 \quad in\ \Omega, \\ u = f \quad on\ x_3 = 0, \\ a\frac{\partial}{\partial x_3}u = 0 \quad on\ x_3 = L \end{cases} \tag{4.1}$$

has one and only one weak solution in $H^1_{loc}(\Omega)$ satisfying appropriate radiation conditions, see (4.32). This solution enjoys the estimate

$$\|u\|_{H^1_{loc}(\Omega)} \leq C\|f\|_{H^2(\mathbb{R}^2)}, \quad \textit{with a positive constant } C \textit{ independent on } f. \tag{4.2}$$

In particular, we have the estimate

$$\|\frac{\partial}{\partial x_3}u(\cdot,0)\|_{H^{-\frac{1}{2}}_{loc}(\mathbb{R}^2)} \leq C\|f\|_{H^2(\mathbb{R}^2)},. \tag{4.3}$$

with a positive constant C independent on f

2. *Let $F \in L^2_{comp}(\Omega) := \{f \in L^2(\Omega);\ \ with\ compact\ support\}$ such that $\Delta' F \in L^2_{comp}(\Omega)$. Let also $G \in L^2_{comp}(\Omega)$. Then the problem*

$$\begin{cases} \nabla\cdot a\nabla u + \kappa^2 n u = \frac{\partial}{\partial x_3}F + G \quad in\ \Omega, \\ u = 0 \quad on\ x_3 = 0, \\ a\frac{\partial}{\partial x_3}u = 0 \quad on\ x_3 = L \end{cases} \tag{4.4}$$

has one and only one weak solution in $H^1_{loc}(\Omega)$ satisfying the radiation conditions in (4.32). This solution enjoys the estimate

$$\|u\|_{H^1_{loc}(\Omega)} \leq C\big[\|F\|_{L^2(\Omega)} + \|\Delta' F\|_{L^2(\Omega)} + \|G\|_{L^2(\Omega)}\big], \tag{4.5}$$

with a positive constant C independent on f.

One way to study the forward problem (1.1) is to use the limiting absorption principle, see for instance [9] and [25]. However, to apply this principle, we need to assume that 0 is not an eigenvalue of the spectral problem (1.4). This assumption is not appropriate since the coefficients a and n are unknowns of our inverse problem. To avoid this eigenvalue assumption, we proceed in a straightforward way based on the Fourier analysis.

Let $f \in H^2_{comp}(\mathbb{R}^2)$ and set $A(\lambda) := \hat{f}(\lambda)$. To $A(\lambda)$, we correspond $\hat{u}(\lambda, x_3)$, the solution of the problem

$$\begin{cases} \frac{d}{dx_3}a\frac{d}{dx_3}\hat{u} + \kappa^2 n\,\hat{u} - \lambda^2 a\hat{u} = 0 \quad \text{in } (L,0), \\ \hat{u} = A(\lambda) \quad \text{on } x_3 = 0, \\ a\frac{d}{dx_3}\hat{u} = 0 \quad \text{on } x_3 = L. \end{cases} \tag{4.6}$$

This problem is well posed for every $\lambda := (\lambda^1, \lambda^2) \in \mathbb{R}^2$ such that $\lambda^2 := (\lambda^1)^2 + (\lambda^2)^2 \neq \lambda_j, j = 1, ..., \infty$. where $\lambda_j,\ j = 1, ..., \infty$ are the eigenvalues of the spectral problem corresponding to (1.4). Recall that (1.4) admits a countable set of eigenvalues accumulating at $+\infty$ and only a finite number of them is negative. We set N to be the number of these negative or zero eigenvalues.

Let also v be the solution of the problem:

$$\begin{cases} \frac{d}{dx_3}a\frac{d}{dx_3}v + \kappa^2 n\,v - \alpha a v = 0 \quad \text{in } (L,0), \\ v = A(\lambda) \quad \text{on } x_3 = 0, \\ a\frac{d}{dx_3}v = 0 \quad \text{on } x_3 = L. \end{cases} \tag{4.7}$$

We introduced $\alpha \in \mathbb{C} \setminus \mathbb{R}$ to make the problem (4.7) well posed for every κ real. If we write $\hat{u} = v + w$, then w satisfies

$$\begin{cases} \frac{d}{dx_3} a \frac{d}{dx_3} w + \kappa^2 n\, w - \alpha a w = (\lambda^2 - \alpha a)\hat{u} \quad \text{in } (L,0), \\ w = 0 \ \text{ on } x_3 = 0, \\ a\frac{d}{dx_3} w = 0 \ \text{ on } x_3 = L. \end{cases} \tag{4.8}$$

Since the family of eigenfunctions $e_j, j = 1, ..., \infty$ is an orthonormal basis in $L^2(L,0)$, then we can write

$$w = \sum_{j=1}^{\infty} (w, e_j) e_j$$

where the series converges in $H^1(L,0)$. Here $(w, e_j) := \int_L^0 w(x) e_j(x) a(x) dx$. Multiplying the first equation of (4.8) by e_j and integrating by parts, we find

$$(w, e_j) = \frac{-\lambda^2 + \alpha}{\lambda^2 + \lambda_j} (v, e_j).$$

Multiplying the first equation in (4.7) by e_j and integrating by parts, we obtain $(v, e_j) = \frac{a e_j'(0) A(\lambda)}{\lambda_j + \alpha}$. Hence

$$w(\lambda, x_3) = \Big[\sum_{j=1}^{\infty} \frac{(-\lambda^2 + \alpha) a e_j'(0)}{(\lambda_j + \alpha)(\lambda^2 + \lambda_j)} e_j(x_3) \Big] A(\lambda). \tag{4.9}$$

Finally, we deduce that

$$\hat{u}(\lambda, x_3) = \Big[\Big(\sum_{j=1}^{\infty} \frac{(-\lambda^2 + \alpha) a e_j'(0)}{(\lambda_j + \alpha)(\lambda^2 + \lambda_j)} e_j(x_3) \Big) + \overline{v}(x_3) \Big] A(\lambda) \tag{4.10}$$

where $\overline{v}$ is the solution of (4.7) with $A(\lambda)$ replaced by 1. We write

$$\hat{u}(\lambda, x_3) := \hat{u}_1(\lambda, x_3) + \hat{u}_2(\lambda, x_3) \tag{4.11}$$

where

$$\hat{u}_1(\lambda, x_3) := \Big[\sum_{j=1}^{N} \frac{(-\lambda^2 + \alpha) a e_j'(0)}{(\lambda_j + \alpha)(\lambda^2 + \lambda_j)} e_j(x_3) \Big] A(\lambda) \tag{4.12}$$

and

$$\hat{u}_2(\lambda, x_3) = \Big[\Big(\sum_{j=N+1}^{\infty} \frac{(-\lambda^2 + \alpha) a e_j'(0)}{(\lambda_j + \alpha)(\lambda^2 + \lambda_j)} e_j(x_3) \Big) + \overline{v}(x_3) \Big] A(\lambda). \tag{4.13}$$

Since the series defining w is converging in $H^1(L,0)$ (and actually in the domain of the selfadjoint operator corresponding to the problem (4.6)), then we can evaluate term by term to compute

$$\frac{d}{dx_3} \hat{u}(\lambda, x_3) := \frac{d}{dx_3} \hat{u}_1(\lambda, x_3) + \frac{d}{dx_3} \hat{u}_2(\lambda, x_3).$$

We recall the estimates

$$\|e_j\|_{L^2(L,0)} = 1, \qquad j = 1, ..., \infty, \tag{4.14}$$

$$\|\frac{d}{dx_3}e_j\|_{L^2(L,0)} \leq C\sqrt{|\lambda_j|}, \qquad j = 1, ..., \infty \tag{4.15}$$

and

$$\|\frac{d}{dx_3}\big[a\frac{d}{dx_3}e_j\big]\|_{L^2(L,0)} \leq C|\lambda_j|, \qquad j = 1, ..., \infty \tag{4.16}$$

with a positive constant C. The estimate (4.15) is obtained from the variational formulation of the spectral problem satisfied by (λ_j, e_j) and (4.14). The estimate (4.16) is obtained by taking the $L^2(L,0)$ norm in the differential equation satisfied by (λ_j, e_j) and using (4.14).

By Sobolev imbedding and interpolation we obtain:

$$|e_j(x_3)| \leq \|e_j\|_{C[L,0]} \leq C_t\|e_j\|_{H^t(L,0)} \leq C_t\|e_j\|^{1-t}_{L^2(L,0)}\|e_j\|^t_{H^1(L,0)}, \quad t > \frac{1}{2}.$$

We deduce from (4.14) and (4.15) the following point-wise estimate: [3]

$$|e_j(x_3)| \leq C_t|\lambda_j|^{\frac{t}{4}}, \qquad j = 1, ..., \infty, \quad t > \frac{1}{2}. \tag{4.17}$$

Similarly, we have

$$\begin{aligned} |\frac{d}{dx_3}e_j(x_3)| \leq \|\frac{d}{dx_3}e_j\|_{C[L,0]} &\leq C_t\|\frac{d}{dx_3}e_j\|_{H^t(L,0)} \\ &\leq C_t\|\frac{d}{dx_3}e_j\|^{1-t}_{L^2(L,0)}\|\frac{d}{dx_3}e_j\|^t_{H^1(L,0)}, \quad t > \frac{1}{2}. \end{aligned}$$

We deduce from (4.14) and (4.15) the following point-wise estimate:

$$|\frac{d}{dx_3}e_j(x_3)| \leq C_t|\lambda_j|^{\frac{1+t}{2}}, \qquad j = 1, ..., \infty, \quad t > \frac{1}{2}. \tag{4.18}$$

I. Evaluation of $\hat{u}_2$

[3]In the case $a = 1$, i.e. the Sturm Liouville problem is of a normal form, then based on the Gelfand-Levitan integral equation method one can derive better estimate than (4.17) and (4.18). Namely $|e_j(x_3)| \leq C|\lambda_j|$, and $|\frac{d}{dx_3}e_j(x_3)| \leq C|\lambda_j|^{\frac{1}{2}}$, $j = 1, ..., \infty$, with a positive contact C. These last estimates can also be generalized to the case where a is piecewise constant in $(L, 0)$, see [7]. To our Knowledge, we do not know if these estimates are also justified for any L^∞-coefficient a. However, the weaker estimates (4.17) and (4.18) are enough for our purpose.

From (4.13), (4.15) and (4.18), we have

$$\|\frac{d}{dx_3}\hat{u}_2(\lambda,\cdot)\|_{L^2(L,0)} \leq \Big[C_{p,q}\sum_{j=N+1}^{\infty}\frac{(\lambda^2+\alpha)\lambda_j^{\frac{2+t}{2}}}{(\lambda_j+\alpha)(\lambda^{\frac{2}{q}}\lambda_j^{\frac{1}{p}})}+\mathcal{O}(1)\Big]|A(\lambda)|$$

for every $p, q > 1$ such that $p^{-1}+q^{-1}=1$. Here we used the inequality $\alpha^{\frac{2}{q}}\beta^{\frac{2}{p}} \leq \frac{1}{p}\alpha^2+\frac{1}{q}\beta^2$ for $\alpha, \beta \geq 0$. Then

$$\|\frac{d}{dx_3}\hat{u}_2(\lambda,\cdot)\|_{L^2(L,0)} \leq C_{p,q}\Big(\sum_{j=1}^{\infty}\frac{1}{\lambda_j^{(\frac{1}{p}-\frac{t}{2})}}\Big)|\lambda|^{2-\frac{2}{q}}|A(\lambda)|.$$

We know that $\lambda_j \approx j^2,\ j >> 1$, hence $\lambda_j^{\frac{1}{p}-t} \approx j^{\frac{2}{p}-t},\ t > \frac{1}{2}$. If we take any p such that $1 < p < \frac{4}{3}$, then there exists $t > \frac{1}{2}$ satisfying $\frac{2}{p}-t > 1$ and then the series above converges. We set $s := 1-\frac{2}{q}$, $s \in (\frac{1}{3}, 1)$ since $q > 3$ (as $p < \frac{4}{3}$), then

$$\|\frac{d}{dx_3}\hat{u}_2(\lambda,\cdot)\|_{L^2(L,0)} \leq C_s|\lambda|^{1+s}|A(\lambda)|. \tag{4.19}$$

This means that if the function $\lambda \rightarrow |\lambda|^{1+s}|A(\lambda)|$ is in $L^2(\mathbb{R}^2)$, then also the function $\lambda \rightarrow \|\frac{d}{dx_3}\hat{u}_2(\lambda,\cdot)\|_{L^2(L,0)}$ is in $L^2(\mathbb{R}^2)$ and by the inverse Fourier transform we deduce that

$$\frac{\partial}{\partial x_3}u_2 \text{ is in } L^2(\Omega) \tag{4.20}$$

where u_2 is the inverse Fourier transform of $\hat{u}_2$. In addition, with similar estimates we obtain

$$\|\hat{u}_2(\lambda,\cdot)\|_{L^2(L,0)} \leq C|\lambda|^s|A(\lambda)|$$

and then

$$u_2 \text{ is in } L^2(\Omega) \tag{4.21}$$

This implies that if the function $\lambda \rightarrow |\lambda|^{1+s}|A(\lambda)|$ is in $L^2(\mathbb{R}^2)$ then u_2 and $\frac{\partial}{\partial x_3}u_2$ are also in $L^2(\Omega)$ with the corresponding estimates in terms of f.

Once more, multiplying the identity (4.13) by λ_1 or λ_2 where $\lambda := (\lambda_1, \lambda_2)$ and arguing as above, we deduce that if the function $\lambda \rightarrow |\lambda|^{1+s}|A(\lambda)|$ is in $L^2(\mathbb{R}^2)$ then $\frac{\partial}{\partial x_1}u_2$ and $\frac{\partial}{\partial x_2}u_2$ are in $L^2(\Omega)$ with the corresponding estimates in terms of f.

This means that if we take $f \in H^{1+s}(\mathbb{R}^2)$ then the function $\lambda \rightarrow |\lambda|^{1+s}|A(\lambda)|$ is in $L^2(\mathbb{R}^2)$ and then u_2 is in $H^1(\Omega)$ with the corresponding estimates:

$$\|u_2\|_{H^1(\Omega)} \leq C\|f\|_{H^{1+s}(\mathbb{R}^2)}, \tag{4.22}$$

with a positive constant C independent on f.

On the other hand, from (4.13), we deduce that $\hat{u}_2(\lambda, x_3)$, and also $\frac{d}{dx_3}\hat{u}_2(\lambda, x_3)$, is analytic in the set $\{\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2;\ |\Im\ \lambda| < \sqrt{\lambda_{N+1}}\}$. Recall that λ_{N+1} is positive. Since f is compactly supported, with support in $B_2(0,R)$, $R>0$, then $A(\lambda)$ is analytic in $\mathbb{C}^2$ and has the behavior

$$|A(\lambda)| \leq C_N \frac{e^{R|\Im\lambda|}}{(1+|\lambda|)^N}, \qquad \forall N \in \mathbb{N}. \tag{4.23}$$

Writing $|\lambda^2+\lambda_j|^2 \geq |\Re\lambda|^2 + (\lambda_j - (\Im\,\lambda)^2)$ and doing similar estimates as before (4.19), using (4.17), (4.18) and (4.23), we deduce that

$$|\hat{u}_2(\lambda, x_3)| \leq C_{p,q,N} \frac{e^{R|\Im\lambda|}}{(1+|\lambda|)^N}, \qquad \forall N \in \mathbb{N}. \tag{4.24}$$

Here we used the fact that $\frac{|\lambda|^{1+s}}{(1+|\lambda|)^N} \leq \frac{1}{(1+|\lambda|)^{N-1-s}}$ and the fact that (4.23) is valid for every N in $\mathbb{N}$. The estimate (4.24), with $\Im\lambda = 0$, implies that for every $x_3 \in [L,0]$, the function u_2. In addition, it means that its Fourier transform $F(\lambda) := \frac{d}{dx_3}u_2(\lambda, x_3)$ is analytic in the set $\{\lambda=(\lambda_1,\lambda_2)\in\mathbb{C}^2;\ |\Im\,\lambda| < \sqrt{\lambda_{N+1}}\}$, and $F(\Re\lambda + i\Im\,\lambda) \in L^1(\mathbb{R}^2)$, for $|\Im\,\lambda| < \sqrt{\lambda_{N+1}}$ with $\sup_{|\Im\,\lambda|<b}\int_{\mathbb{R}^2}|F(\Re\lambda+i\Im\,\lambda)|d\Re\lambda < \infty$; $\forall b < \sqrt{\lambda_{N+1}}$. Then by the Paley-Wiener type theorem, we deduce that

$$|u_2(x', x_3)| \leq C_b e^{-b|x'|}, \qquad \text{in } \Omega. \tag{4.25}$$

With a similar way, we can show that

$$|\nabla' u_2(x', x_3)| \leq C_b e^{-b|x'|}, \qquad \text{in } \Omega. \tag{4.26}$$

These two last estimates show in particular that u_2 satisfies the radiation conditions in the waveguide directions (x', x_3), $|x'|=1$ for an arbitrary but fixed depth x_3.

II. Evaluation of $\hat{u}_1$

Let us now deal with $\hat{u}_1(\lambda, x_3)$. From (4.12), we observe that

$$\hat{u}_1(\lambda, x_3) = (2\pi)^{-1}\mathcal{F}_{x'}\Big(\sum_{j=1}^{N}\frac{ae_j'(0)}{\lambda_j+\alpha}K_0^j * \big[(\Delta'+\alpha)f\big](x')e_j(x_3)\Big) \tag{4.27}$$

with $K^j(|x'|) := K_0(|x'|\sqrt{\lambda_j})$ if $\lambda_j \neq 0$ where K_0 is the modified Bessel function and $K_0(s) := \ln(|s|)$ if $\lambda_j = 0$. We used above the following relations $\mathcal{F}_{x'}\Big(K_0^j(|x'|)\Big) = \frac{2\pi}{\lambda^2+\lambda_j}$. Hence

$$u_1(x', x_3) := \mathcal{F}_{x'}^{-1}\hat{u}_1(\lambda, x_3) = \sum_{j=1}^{N} v_j(x')e_j(x_3) \tag{4.28}$$

where

$$v_j(x') := (2\pi)^{-1}\frac{ae_j'(0)}{\lambda_j+\alpha}K_0^j * \big[(\Delta'+\alpha)f\big](x') \tag{4.29}$$

which makes sense knowing that f is in $H^2(\mathbb{R}^2)$. Since f is compactly supported then from the known asymptotic expansions $(\partial_r - \sqrt{\lambda_j})K_0^j = o(\frac{1}{\sqrt{r}})$, we deduce the following radiation conditions satisfied by $v_j, j = 1, ..., N$:

$$(\partial_r - \sqrt{\lambda_j})v_j = o(\frac{1}{\sqrt{r}}). \tag{4.30}$$

Using the regularity properties of Newtonian potentials, see for instance ([18], chapter II), then from (4.28) and (4.29) we deduce similar, but local, estimates as (4.22) for u_1, i.e.

$$\|u_1\|_{H^1_{loc}(\Omega)} \leq C\|f\|_{H^2(\mathbb{R}^2)}, \tag{4.31}$$

with a positive constant C independent on f.

Recall that $\hat{u}(\lambda, x)$ satisfies the problem (4.6) in the weak sense then, by using Plancheral's theorem, we can show that its inverse Fourier transform u satisfies also (1.1) in the weak sense i.e

$$\forall \varphi \in H^1_{comp}(\Omega), \qquad \int_\Omega a\nabla u \cdot \nabla\varphi - \kappa^2 nu\varphi dx =< f, \varphi >_{-\frac{1}{2},\frac{1}{2}} \text{ since } u \in H^1_{loc}(\Omega).$$

So far, we have proved the existence of a solution $u \in H^1_{loc}(\Omega)$ of the following problem

$$\begin{cases} \int_\Omega a\nabla u \cdot \nabla\varphi - \kappa^2 nu\varphi dx =< f, \varphi >_{-\frac{1}{2},\frac{1}{2}}, \ \forall\varphi \in H^1_{comp}(\Omega), \\ u(x', x_3) = \sum_{j=1}^\infty v_j(x')e_j(x_3) \text{ and} \\ \text{ every } v_j \in H^1_{loc}(\mathbb{R}^2) \text{ satisfies the following radiation conditions} \\ (\partial_r - \sqrt{\lambda_j})v_j = o(\frac{1}{\sqrt{r}}). \end{cases} \tag{4.32}$$

This solution is also unique. Indeed if $u(x', x_3) = \sum_{j=1}^\infty v_j(x')e_j(x_3)$ satisfies (4.32) with $f = 0$, then by a separation of variables each component v_j satisfies the homogeneous scattering problem $(\Delta' - \lambda_j)v_j = 0$, in $\mathbb{R}^2$ coupled with the radiation conditions $(\partial_r - \sqrt{\lambda_j})v_j = o(\frac{1}{\sqrt{r}})$. From the uniqueness of this last scattering problem, we deduce that $v_j = 0, \ \forall j = 1, 2, ...$ Hence $u = 0$.

In the previous calculation, we took f to be compactly supported. Actually, the same conclusions are also true taking f such that $e^{b|\cdot|}f(\cdot) \in H^2(\mathbb{R}^2)$, with some positive constant b. This end the proof of the first part of Theorem 4.1.

Let us deal with the second part of the theorem. Since the problem (4.4) is linear, we ca study it with a source given by $\frac{\partial}{\partial x_3}F$ or G separately. Since the proof is the same we give it only for the first source. We first assume that $F(x', x_3) = F_1(x')F_2(x_3)$ and then conclude by a density argument.

The corresponding $1D$ problem to the problem (4.4) is:

$$\begin{cases} \frac{d}{dx_3}a\frac{d}{dx_3}\hat{u} + \kappa^2 n\ \hat{u} - \lambda^2 a\hat{u} = \frac{d}{dx_3}\hat{F}(\lambda, x_3) \quad \text{in } (L, 0), \\ \hat{u} = 0 \ \text{ on } x_3 = 0, \\ a\frac{d}{dx_3}\hat{u} = 0 \ \text{ on } x_3 = L \end{cases} \tag{4.33}$$

where here $\hat{F}(\lambda,\cdot)$ the Fourier transforms of F in the two variables x_1 and x_2. We write $\hat{F}(\lambda,\cdot) := \hat{F}_1(\lambda)F_2(x_3)$. We define v as the solution of the problem:

$$\begin{cases} \frac{d}{dx_3}a\frac{d}{dx_3}v + \kappa^2 n\, v - \alpha a v = \frac{d}{dx_3}\hat{F}(\lambda, x_3) \quad \text{in } (L,0), \\ v = 0 \ \text{ on } x_3 = 0, \\ a\frac{d}{dx_3}v = 0 \ \text{ on } x_3 = L. \end{cases} \tag{4.34}$$

where, as for the first part of the proof, we introduce $\alpha \in \mathbb{C}\setminus\mathbb{R}$ to make the problem (4.34) well posed for every κ real. Finally, if we write $\hat{u} = v + w$, then w satisfies

$$\begin{cases} \frac{d}{dx_3}a\frac{d}{dx_3}w + \kappa^2 n\, w - \alpha a w = (\lambda^2 - \alpha a)\hat{u} \quad \text{in } (L,0), \\ w = 0 \ \text{ on } x_3 = 0, \\ a\frac{d}{dx_3}w = 0 \ \text{ on } x_3 = L. \end{cases} \tag{4.35}$$

Hence arguing as in the first part of the proof, we can write

$$\hat{u}(\lambda, x_3) = \Big[\big(\sum_{j=1}^{\infty}\frac{(-\lambda^2+\alpha)(F_2, \frac{d}{dx_3}e_j)_{L^2(L,0)}}{(\lambda_j+\alpha)(\lambda^2+\lambda_j)}e_j(x_3)\big) + \overline{v}(x_3)\Big]\hat{F}_1(\lambda) \tag{4.36}$$

where $\overline{v}$ is the solution of (4.7) with $\hat{F}_1(\lambda)$ replaced by 1.

This form is reminiscent to the form (4.10). Hence arguing as in the steps after (4.10), we obtain the desired result. In case the source is G, then in place of (4.36), we obtain

$$\hat{u}(\lambda, x_3) = \Big[\big(\sum_{j=1}^{\infty}\frac{(-\lambda^2+\alpha)(G_2, e_j)_{L^2(L,0)}}{(\lambda_j+\alpha)(\lambda^2+\lambda_j)}e_j(x_3)\big) + \overline{v}(x_3)\Big]\hat{F}_1(\lambda), \tag{4.37}$$

with the corresponding function $\overline{v}$, where we first assume $G(x', x_3) = G_1(x')G_2(x_3)$ and then conclude by density.

5. Conclusion and future works

We have justified reconstruction formulae to detect and estimate the various layers and their speeds of sound as well as their densities from a single measurement collected on the surface of the waveguide. This concerns the stratified acoustic waveguide. This is the first step towards studying the elastic waveguide which is of a fundamental importance in geophysics (i.e. Mineral prospect ions) and interesting from the mathematical point of view as well. Recall that, in the Lamé model, we have two types of body waves: the shear waves (S-waves) and the pressure waves (P-waves). It would be interesting to see if only one of them is enough to be used to reconstruct the stratified coefficients. Our guess is that we need both waves. Using shear waves, we reconstruct the shear coefficient μ and the density ρ and using the pressure waves, we reconstruct the pressure coefficient $\lambda + 2\mu$ (and also the density ρ). We can deduce then the two Lamé coefficients λ and μ and the density ρ (which are the 3 coefficients modeling the Lamé model). Another problem we can tackle using the techniques developed in this paper is the connected beams problem, see [19] and the section 5 in [10] where a more elaborated model is described.

References

[1] G. A. Athanassoulis and V. G. Papanicolaou. Eigenvalue asymptotics of layered media and their applications to the inverse problem. SIAM J. Appl. Math. 57 (1997), no. 2, 453–471.

[2] F. Al-Musallam and A. Boumenir. Reconstruction of the refraction index in stratified ocean. Siam. J. Appl. Math, Vol. 71, No. 4, (2011), pp. 972-982.

[3] A. Boumenir and V. K. Tuan, An inverse problem for the heat equation. Proc. Amer. Math. Soc. 138 (2010), no. 11, 3911-3921.

[4] M. I. Belishev. Recent progress in the boundary control method. Inverse Problems 23 (2007), no. 5, R1-R67.

[5] J. L. Buchanan, R. P. Gilbert, A. Wirgin, and Y. S. Xu. *Marine Acoustics: Direct and Inverse Problems.* Society for Industrial and Applied Mathematics, Philadelphia, 2004.

[6] G. Borg. Eine Umkerung der Sturm-Liouville Eigenwertarfgabe Acta Math. 78 1-96, 1946.

[7] R. Carlson. An inverse spectral problem for Sturm-Liouville operators with discontinuous coefficients. Proc. Amer. Math. Soc. 120 (1994), no. 2, 475-484.

[8] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory* (Berlin: Springer), 1998.

[9] E. Croc; Y. Dermenjian. Analyse spectrale d'une bande acoustique multistratifiée. I. Principe d'absorption limite pour une stratification simple. SIAM J. Math. Anal. 26 (1995), no. 4, 880-924.

[10] M. Dilena and A. Morassi, Vibrations of steel-concrete composite beams with partially degraded connection and applications to damage detection. Journal of Sound and Vibration, 320, (2009), 101-124

[11] O. H. Hald and J. R. McLaughlin. Solutions of inverse nodal problems. Inverse Problems 5 (1989), no. 3, 307-347.

[12] I. M. Gel'fand and B. M. Levitan, On the determination of a differential equation from its special function, Izv. Akad. Nauk SSR. Ser. Mat. 15 (1951), 309–360 (Russian); English transl. in Amer. Math. Soc. Transl. Ser. 2 1 (1955), 253-304.

[13] A. Katchalov; Y. Kurylev and M. Lassas, Inverse boundary spectral problems. Monographs and Surveys in Pure and Applied Mathematics, 123. Chapman/Hall/CRC, Boca Raton, FL, 2001.

[14] N. Levinson. The inverse Sturm-Liouville problem Math. Tidsskr. B 25 25-30. 1949.

[15] B. M. Levitan. Inverse Strum-Liouville problems. VNU Science Press, Ultrecht, 1987.

[16] J. Liu, G. Nakamura and M. Sini, Reconstruction of the shape and surface impedance from acoustic scattering data for an arbitrary cylinder. Siam J. Appl. Math. 67 (2007), no. 4, 1124-1146

[17] J. R. McLaughlin. Analytical methods for recovering coefficients in differential equations from spectral data. SIAM Rev. 28 (1986), no. 1, 53-72.

[18] C. Miranda. Partial differential equations of elliptic type. Second revised edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2. Springer-Verlag, New York-Berlin 1970 xii+370 pp.

[19] Morassi, A.; Nakamura, G.; Sini, M. An inverse dynamical problem for connected beams. European J. Appl. Math. 16 (2005), no. 1, 83-109.

[20] G. Nakamura and M. Sini, Obstacle and boundary determination from scattering data. SIAM J. Math. Anal. 39 (2007), no. 3, 819-837

[21] J. Poschel and E. Trubowitz. Inverse spectral theory. Pure and Applied Mathematics, 130. Academic Press, Inc., Boston, MA, 1987. x+192 pp.

[22] R. Potthast, A survey on sampling and probe methods for inverse problems. Inverse Problems 22 (2006), no. 2, R1-R47.

[23] A. G. Ramm. *Inverse Problems*, Mathematical and Analytical Techniques with Applications to Engineering, Springer, (2004).

[24] A. G. Ramm, An inverse problem of ocean acoustics, J. Inverse Ill-Posed Probl., 9 (2001), pp. 95-102.

[25] A. G. Ramm; P. Werner. On the limit amplitude principle for a layer. J. Reine Angew. Math. 360 (1985), 19-46.

[26] Z. G. Seftel. Estimates in Lp of solutions of elliptic equations with discontinuous coefficients and satisfying general boundary conditions and conjugacy conditions. Soviet Math. Dokl. 4 (1963):321-324.

[27] M. Sini, On the one-dimensional Gelfand and Borg-Levinson spectral problems for discontinuous coefficients. Inverse Problems 20 (2004), no. 5, 1371-1386.