

Normal feedback boundary stabilization to trajectories for 3D Navier–Stokes equations

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RICAM-Report 2014-20

NORMAL FEEDBACK BOUNDARY STABILIZATION TO TRAJECTORIES FOR 3D NAVIER–STOKES EQUATIONS

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ABSTRACT. Given a nonstationary trajectory of the Navier–Stokes system, a finite-dimensional feedback boundary controller stabilizing locally the system to the given solution is constructed. Moreover the controller is supported in a given open subset of the boundary of the domain containing the fluid and acts normal to the boundary.

In a first step a controller is constructed that stabilizes the linear Oseen–Stokes system “around the given trajectory” to zero; for that a corollary of a suitable truncated boundary observability inequality, the regularizing property for the system, and some standard techniques of the optimal control theory are used. Then it is shown that the same controller also stabilizes, locally, the Navier–Stokes system to the given trajectory.

Keywords and phrases. Navier–Stokes system, exponential stabilization, boundary feedback control.

Mathematics subject classification. 35Q30, 93D15, 93B52.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a connected bounded domain located locally on one side of its smooth boundary $\Gamma = \partial\Omega$, and let $I \subseteq \mathbb{R}$ be a nonempty open interval. The Navier–Stokes system, in $I \times \Omega$, controlled through the boundary reads

$$(1) \quad \partial_t u + \langle u \cdot \nabla \rangle u - \nu \Delta u + \nabla p_u + h = 0, \quad \operatorname{div} u = 0, \quad u|_\Gamma = \gamma + \zeta$$

where ζ is a control taking values in a suitable subspace of square-integrable functions in Γ whose support in x is contained in a given open subset $\Gamma_c \subseteq \Gamma$. Furthermore, as usual, $u = (u_1, u_2, u_3)$ and p_u , defined for $(t, x_1, x_2, x_3) \in I \times \Omega$, are the unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity, the operators ∇ and Δ are respectively the well known gradient and Laplacean in the space variables (x_1, x_2, x_3) , $\langle u \cdot \nabla \rangle v$ stands for $(u \cdot \nabla v_1, u \cdot \nabla v_2, u \cdot \nabla v_3)$, $\operatorname{div} u := \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3$, and h and γ are fixed external forces.

Suppose we are given a targeted (reference, desired) solution $\hat{u}(t) = \hat{u}(t, x)$ of (1) with $I = (0, +\infty)$ and $\zeta = 0$. If \hat{u} is stationary, $\hat{u}(t) = \hat{u}(0, x) = \hat{u}_0$, then the problem of stabilization to \hat{u}_0 is now quite well understood. Namely, it was proven that, for any initial function u_0 sufficiently close to \hat{u}_0 one can find a square integrable control $\eta \in L^2((0, +\infty), L^2(\Gamma, \mathbb{R}^3))$, such that the corresponding solution $u(t)$, supplemented with the initial condition

$$(2) \quad u(0, x) = u_0(x)$$

is defined on $[0, +\infty)$ and $u(t)$ goes to \hat{u}_0 exponentially as time t goes to ∞ ; we refer the reader to the works [Fur01, Fur04, Ray06, Ray07, RT10, BT11, Bar12].

Again for a stationary targeted trajectory $\hat{u} = \hat{u}_0$, the analogous result hold also in the case of an internal control under Dirichlet boundary conditions:

$$(3) \quad \partial_t u + \langle u \cdot \nabla \rangle u - \nu \Delta u + \nabla p_u + h + \eta = 0, \quad \operatorname{div} u = 0, \quad u|_\Gamma = 0.$$

where now η a control supported in a given open subset $w \subseteq \Omega$. For details we refer to [Bar03, BT04, BLT06].

Here, we are particularly interested in the case where the targeted trajectory \hat{u} is nonstationary (i.e., $\hat{u} = \hat{u}(t)$ depends on time), a situation that often can occur in real world applications, as in the case suitable (say non-gradient) external forces (h and γ) depend on time. Also, since they are important and often required in applications, we look for controls obeying some general constraints like to be given in feedback form, finite-dimensional, and supported in a given (small) open subset.

In [BRS11], an *internal* stabilizing finite-dimensional feedback controller was found for the case of nonstationary targeted solutions. Then, one question arises: can we find a similar *boundary* controller? The methods used in the particular case of a stationary targeted solution, use some (spectral-like) properties of the (time-independent) Oseen–Stokes operator $u \mapsto \nu \Delta u - \mathcal{B}(\hat{u}_0)u + \nabla p_u$ and/or of its “adjoint” $v \mapsto \nu \Delta v - \mathcal{B}^*(\hat{u}_0)v + \nabla p_v$, which seem to give us no hint for the nonstationary case.

Also the constraints on the boundary control, imply that the procedure in [FI99], that allow to derive suitable boundary results from internal ones is (or may be) no longer sufficient to derive the wanted boundary stabilization result.

Departing from an exact controllability result from [Rod14b], suitable truncated boundary observability inequalities have been derived for the (linear) Oseen–Stokes system in [Rod14a]. These results will enable us to follow the procedure in [BRS11] in order to construct a boundary stabilizing finite-dimensional controller to a given nonstationary targeted solution. To prove that the control can be taken in feedback form, and to

find the feedback rule, we will need to overcome some technical regularity/compatibility issues.

We will prove the following Theorem, whose exact formulation is given in Section 5.

MAIN THEOREM: *Let $(\hat{u}, p_{\hat{u}})$ be a global smooth solution for problem (1), with $\zeta = 0$ and $t \in \mathbb{R}_0 = (0, +\infty)$, such that*

$$|\hat{u}|_{L^\infty(\mathbb{R}_0 \times \Omega, \mathbb{R}^3)} + \sup_{\tau \in [0, +\infty)} |\partial_t \hat{u}|_{L^2((\tau, \tau+1), L^\sigma(\Omega, \mathbb{R}^3))} + \sup_{\tau \in [0, +\infty)} |\nabla \hat{u}|_{L^2((\tau, \tau+1), L^3(\Omega, \mathbb{R}^9))} \leq R$$

where $R > 0$ and $\sigma > \frac{6}{5}$ are constants. Then for any $\lambda > 0$ and any open subset $\Gamma_c \subseteq \Gamma = \partial\Omega$ there are an integer $M = M(R, \lambda, \Gamma_c) \geq 1$, an M -dimensional space $\mathcal{E}_M^n \subset \{f \in C^2(\Gamma, \mathbb{R}^3) \mid f \text{ is normal to } \Gamma \text{ and } f = 0 \text{ on } \Gamma \setminus \Gamma_c\}$, and a family of continuous linear operators $\mathbf{K}_{\hat{u}}^{\lambda, t}$, $t \geq 0$, from a suitable subset of $L^2(\Omega, \mathbb{R}^3)$ into \mathcal{E}_M^n , such that the following assertions hold.

- (a) *The function $t \mapsto \mathbf{K}_{\hat{u}}^{\lambda, t}$ is continuous in the weak operator topology, and its operator norm is bounded by a constant depending only on R , λ , and Γ_c .*
- (b) *For any divergence free function $u_0 \in H^1(\Omega, \mathbb{R}^3)$ such that the difference $v_0 = u_0 - \hat{u}(0)$ is sufficiently small in the $H^1(\Omega, \mathbb{R}^3)$ -norm and $v_0|_\Gamma \in \mathcal{E}_M^n$, problem (1), (2) with $\zeta(t) = v_0|_\Gamma + \int_0^t \mathbf{K}_{\hat{u}}^{\lambda, r}(u - \hat{u}(r)) dr$ has a unique global solution (u, p_u) , which satisfies the inequality*

$$|u(t) - \hat{u}(t)|_{H^1(\Omega, \mathbb{R}^3)}^2 \leq C e^{-\lambda t} |u_0 - \hat{u}(0)|_{H^1(\Omega, \mathbb{R}^3)}^2, \quad t \geq 0.$$

Notice that this Theorem remains true for the two-dimensional (2D) Navier–Stokes system, and in this case, it suffices to assume that the difference $u_0 - \hat{u}(0)$ is small in the $L^2(\Omega, \mathbb{R}^3)$ -norm and satisfies the weaker compatibility conditions $(v_0 \cdot \mathbf{n})\mathbf{n} \in \mathcal{E}_M^n$, where \mathbf{n} is the unit outward normal vector to the boundary Γ .

The fact that the control appears in integral form is meaningful from the physical point of view; indeed, since u is the velocity of the fluid then, roughly, the integral form means that we are accelerating (or forcing) the fluid particles through the boundary. From the physical and practical point of view this is more natural than instantaneously imposing the velocity of the boundary particles.

Though the integral feedback form of the controller, we will also show that the control is defined pointwise in time, that is, the control $\zeta(t)$ at time $t > 0$ depends only on $u(t) - \hat{u}(t)$, and not on $v_0 = u(0) - \hat{u}(0)$ as the integral feedback form could suggest.

We recall that, even in the case the targeted trajectory \hat{u} is stationary, the stabilization of the Navier–Stokes system by normal boundary controls is known only for the particular, however interesting, case of periodic flows in an infinite channel (cf. [BLK01, Bar07, Mun12b, Mun12a, VK05] and references therein). In the general case, stabilization to a stationary solution in the 2D case has been achieved in [Bar12], under some general conditions, by means of oblique controls, but not normal. The stabilization also works in the 3D case for the linear Oseen–Stokes system. We would like to refer also to the work [Bar13] where the idea in [Bar12] is used for boundary stabilization to a stationary solution of parabolic equations, and leads to a quite simple algorithm to construct the stabilizing controller.

Already in the case of a stationary targeted trajectory \hat{u} , there are some issues concerning the stabilization by means of nontangential controls, we refer to [Ray07], where some techniques are also proposed to treat the problem. Finally for the stabilization to a stationary trajectory by means of tangential controls we refer to [BLT06].

The rest of the paper is organized as follows. In Section 2, we introduce the functional spaces arising in the theory of the Navier–Stokes equations and recall some well-known facts. Sections 3 and 4 are devoted to studying the linearized problem, that is, the

Oseen–Stokes system; in Section 3 we prove the existence of a stabilizing control and in Section 4 we prove that the control can be taken normal to the boundary and in feedback form. Finally in Section 5 we establish the main result of the paper on local exponential stabilization of the Navier–Stokes system. The Appendix gathers a few more remarks and auxiliary results used in the main text.

Notation. We write \mathbb{R} and \mathbb{N} for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_a := (a, +\infty)$ for all $a \in \mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$. We denote by $\Omega \subset \mathbb{R}^3$ a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. Given a vector function $v: (t, x_1, x_2, x_3) \mapsto v(t, x_1, x_2, x_3) \in \mathbb{R}^k$, $k \in \mathbb{N}_0$, defined in an open subset of $\mathbb{R} \times \Omega$, its partial time derivative $\frac{\partial v}{\partial t}$ will be denoted by $\partial_t v$. Also the spatial partial derivatives $\frac{\partial v}{\partial x_i}$ will be denoted by $\partial_{x_i} v$.

Given an open interval $I \subseteq \mathbb{R}$, then we write $W(I, X, Y) := \{f \in L^2(I, X) \mid \partial_t f \in L^2(I, Y)\}$, where the derivative $\partial_t f$ is taken in the sense of distributions. This space is endowed with the natural norm $|f|_{W(I, X, Y)} := (|f|_{L^2(I, X)}^2 + |\partial_t f|_{L^2(I, Y)}^2)^{1/2}$. In the case $X = Y$ we write $H^1(I, X) := W(I, X, X)$. Again, if X and Y are endowed with a scalar product, then also $W(I, X, Y)$ is. The space of continuous linear mappings from X into Y will be denoted by $\mathcal{L}(X \rightarrow Y)$.

$\overline{C}_{[a_1, \dots, a_k]}$ denotes a function of nonnegative variables a_j that increases in each of its arguments.

$C, C_i, i = 1, 2, \dots$, stand for unessential positive constants.

2. PRELIMINARIES

2.1. Functional spaces. Let $\Omega \subset \mathbb{R}^3$ be a connected bounded domain of class C^∞ located locally on one side of its boundary $\Gamma = \partial\Omega$, with $\int_\Gamma d\Gamma < +\infty$.

We recall some spaces appearing in the study of the system (1) (cf. [Rod14b, Rod14a]). We start by the Lebesgue and Sobolev subspaces

$$\begin{aligned} L_{\text{div}}^r(\Omega, \mathbb{R}^3) &:= \{u \in L^r(\Omega, \mathbb{R}^3) \mid \text{div } u = 0 \text{ in } \Omega\}, \quad 1 \leq r \leq +\infty, \\ H_{\text{div}}^s(\Omega, \mathbb{R}^3) &:= \{u \in H^s(\Omega, \mathbb{R}^3) \mid \text{div } u = 0 \text{ in } \Omega\}, \quad s \geq 0. \end{aligned}$$

The incompressibility condition allows us to define the trace of $u \cdot \mathbf{n}$ on the boundary $\Gamma = \partial\Omega$, where \mathbf{n} is the unit outward normal vector to the boundary Γ , and then to write

$$H := \{u \in L_{\text{div}}^2(\Omega, \mathbb{R}^3) \mid u \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \quad H_c := \{u \in L_{\text{div}}^2(\Omega, \mathbb{R}^3) \mid u \cdot \mathbf{n} = 0 \text{ on } \Gamma \setminus \overline{\Gamma_c}\},$$

where Γ_c is an open subset of Γ . Some spaces of more regular vector fields we find throughout the paper are

$$(4) \quad \begin{aligned} V &:= \{u \in H_{\text{div}}^1(\Omega, \mathbb{R}^3) \mid u = 0 \text{ on } \Gamma\}, \quad V_c := \{u \in H_{\text{div}}^1(\Omega, \mathbb{R}^3) \mid u = 0 \text{ on } \Gamma \setminus \overline{\Gamma_c}\}, \\ D(L) &:= V \cap H^2(\Omega, \mathbb{R}^3). \end{aligned}$$

The spaces $H_{\text{div}}^s(\Omega, \mathbb{R}^3)$ are endowed with the scalar product inherited from $H^s(\Omega, \mathbb{R}^3)$; the spaces H and H_c with that inherited from $L^2(\Omega, \mathbb{R}^3)$; the spaces V and V_c with that inherited from $H^1(\Omega, \mathbb{R}^3)$; and $D(L)$ with that inherited from $H^2(\Omega, \mathbb{R}^3)$. Notice that if Π is the orthogonal projection in $L^2(\Omega, \mathbb{R}^3)$ onto H , it is well known that $D(L)$ coincides with the domain $\{u \in V \mid Lu \in H\}$ of the Stokes operator $L := -\nu \Pi \Delta$. That is the reason for the notation.

Next, fix a constant $\sigma > \frac{6}{5}$. For any pair of real numbers a, b , with $a < b$, we introduce the Banach spaces $\mathcal{W}^{(a, b)|\text{wk}}$ and $\mathcal{W}^{(a, b)|\text{st}}$ of the measurable vector functions

$u = (u_1, u_2, u_3)$, defined in $(a, b) \times \Omega$, satisfying

$$(5) \quad \begin{aligned} |u|_{\mathcal{W}^{(a,b)|\text{wk}}} &:= \left(|u|_{L^\infty((a,b), L^\infty_{\text{div}}(\Omega, \mathbb{R}^3))}^2 + |\partial_t u|_{L^2((a,b), L^\sigma(\Omega, \mathbb{R}^3))}^2 \right)^{\frac{1}{2}} < \infty, \\ |u|_{\mathcal{W}^{(a,b)|\text{st}}} &:= \left(|u|_{\mathcal{W}^{(a,b)|\text{wk}}}^2 + |\nabla u|_{L^2((a,b), L^3(\Omega, \mathbb{R}^9))}^2 \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

and also the Morrey-like spaces

$$(6) \quad \begin{aligned} \mathcal{W}^{\text{wk}} &:= \left\{ u \mid \sup_{\tau \in [0, +\infty)} |u|_{(\tau, \tau+1)|\text{wk}} < +\infty \right\}, \\ \mathcal{W}^{\text{st}} &:= \left\{ u \mid \sup_{\tau \in [0, +\infty)} |u|_{(\tau, \tau+1)|\text{st}} < +\infty \right\}; \end{aligned}$$

endowed with the norms $|u|_{\mathcal{W}^{\text{wk}}} := \sup_{\tau \geq 0} |u|_{\mathcal{W}^{(\tau, \tau+1)|\text{wk}}}$, and $|u|_{\mathcal{W}^{\text{st}}} := \sup_{\tau \geq 0} |u|_{\mathcal{W}^{(\tau, \tau+1)|\text{st}}}$.

Remark 2.1. The lower bound $\frac{6}{5}$ for σ is motivated from the results in [FCGIP04, Rod14a].

We recall that, in [FGH02], the set of traces $u|_\Gamma$ at the boundary Γ of the elements u in the space $W((a, b), H^s_{\text{div}}(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))$ is completely characterized, for each $s > \frac{1}{2}$, with $s \notin \{\frac{3}{2}, \frac{5}{2}\}$. Denoting that trace space by $G^s_{\text{av}}((a, b), \Gamma)$, we have that $v \mapsto v|_\Gamma$ is continuous:

$$|v|_\Gamma|_{G^s_{\text{av}}((a,b), \Gamma)} \leq C_1 |w|_{W((a,b), H^s_{\text{div}}(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))}$$

and, there is an extension $E_s: G^s_{\text{av}}((a, b), \Gamma) \rightarrow W((a, b), H^s_{\text{div}}(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))$, which is continuous:

$$(E_s w)|_\Gamma = w \text{ and } |E_s w|_{W((a,b), H^s_{\text{div}}(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))} \leq C_2 |w|_{G^s_{\text{av}}((a,b), \Gamma)}.$$

We will use only the cases $s = 1$ and $s = 2$. From [FGH02, Section 2.2] we know that

$$G^s_{\text{av}}((a, b), \Gamma) = G^s_{\mathbf{t}}((a, b), \Gamma) \oplus G^s_{\mathbf{n}, \text{av}}((a, b), \Gamma)\mathbf{n},$$

with

$$\begin{aligned} G^1_{\mathbf{t}}((a, b), \Gamma) &= L^2((a, b), H^{\frac{1}{2}}(\Gamma, T\Gamma)) \cap H^{\frac{1}{2}}((a, b), H^{-\frac{1}{2}}(\Gamma, T\Gamma)) \\ G^1_{\mathbf{n}, \text{av}}((a, b), \Gamma) &= L^2((a, b), H^{\frac{1}{2}}_{\text{av}}(\Gamma, \mathbb{R})) \cap H^{\frac{3}{4}}((a, b), H^{-1}_{\text{av}}(\Gamma, \mathbb{R})) \\ G^2_{\mathbf{t}}((a, b), \Gamma) &= L^2((a, b), H^{\frac{3}{2}}(\Gamma, T\Gamma)) \cap H^{\frac{3}{4}}((a, b), H^0(\Gamma, T\Gamma)) \\ G^2_{\mathbf{n}, \text{av}}((a, b), \Gamma) &= L^2((a, b), H^{\frac{3}{2}}_{\text{av}}(\Gamma, \mathbb{R})) \cap H^1((a, b), H^{-\frac{1}{2}}_{\text{av}}(\Gamma, \mathbb{R})) \end{aligned}$$

For technical reasons we relax a little the trace spaces: we define the superspace $G^s((a, b), \Gamma)$ of $G^s_{\text{av}}((a, b), \Gamma)$ by just omitting the average constraint:

$$(7) \quad G^s((a, b), \Gamma) := G^s_{\mathbf{t}}((a, b), \Gamma) \oplus G^s_{\mathbf{n}}((a, b), \Gamma)\mathbf{n}$$

with $G^s_{\mathbf{n}}((a, b), \Gamma) := L^2((a, b), H^{s-\frac{1}{2}}(\Gamma, \mathbb{R})) \cap H^{r_{\mathbf{n}, 1}(s)}((a, b), H^{r_{\mathbf{n}, 2}(s)}(\Gamma, \mathbb{R}))$. For a few more details see [Rod14a, Section 2.1].

2.2. The control space. Let us write $L^2(\Omega, \mathbb{R}^3) = H \oplus H^\perp$, where $H^\perp = \{\nabla \xi \mid \xi \in H^1(\Omega, \mathbb{R})\}$ denotes the orthogonal complement of H in $L^2(\Omega, \mathbb{R}^3)$, and denote by Π the orthogonal projection $\Pi: L^2(\Omega, \mathbb{R}^3) \rightarrow H$ in $L^2(\Omega, \mathbb{R}^3)$ onto H . For each positive integer N , we now define the N -dimensional space $H_N \subset H$ as follows: let $\{e_i \mid i \in \mathbb{N}_0\}$ be an orthonormal basis in H formed by eigenfunctions of the Stokes operator L , whose domain is defined by (4), and let $0 < \alpha_1 \leq \alpha_2 \leq \dots$ be the corresponding eigenvalues, $Le_i = \alpha_i e_i$, then put

$$(8) \quad H_N := \text{span}\{e_i \mid i \leq N\} \subset D(L) \subset H$$

and denote by Π_N the orthogonal projection $\Pi_N: H \rightarrow H_N$ in H onto H_N .

Let $\mathcal{O} \subseteq \Gamma$ be a connected open subset of the boundary Γ , localized on one side of its boundary. We suppose that \mathcal{O} is a C^∞ -smooth manifold, either boundaryless or with C^∞ -smooth boundary $\partial\mathcal{O}$. Let $\{\pi_i \mid i \in \mathbb{N}_0\}$ be an orthonormal basis in $L^2(\mathcal{O}, \mathbb{R})$ formed by the eigenfunctions of the Laplace–de Rham (or Laplace–Beltrami) operator $\Delta_{\mathcal{O}}$ on the smooth manifold \mathcal{O} , under Dirichlet boundary conditions, $\pi_i(p) = 0$ for all $p \in \partial\mathcal{O}$. Analogously let $\{\tau_i \mid i \in \mathbb{N}_0\}$ be an orthonormal basis in $L^2(\mathcal{O}, T\mathcal{O})$ formed by the vector fields that are eigenfunctions of $\Delta_{\mathcal{O}}$ on $T\mathcal{O}$, also under Dirichlet boundary conditions in the case $\partial\mathcal{O} \neq \emptyset$, $\tau_i(p) = 0 \in T_p\Gamma$ for all $p \in \partial\mathcal{O}$. It is known that π_i and τ_i ($i \in \mathbb{N}_0$) are smooth. Let $0 \leq \beta_1 \leq \beta_2 \leq \dots$, and $0 \leq \gamma_1 \leq \gamma_2 \leq \dots$ be the eigenvalues associated with the systems $\{\pi_i \mid i \in \mathbb{N}_0\}$ and $\{\tau_i \mid i \in \mathbb{N}_0\}$, respectively.

We may write $L^2(\mathcal{O}, \mathbb{R}^3)$ as an orthogonal sum $L^2(\mathcal{O}, \mathbb{R}^3) = L^2(\mathcal{O}, \mathbb{R})\mathbf{n} \oplus L^2(\mathcal{O}, T\mathcal{O})$. Notice that $\{\pi_i\mathbf{n} \mid i \in \mathbb{N}_0\}$ is an orthonormal basis for $L^2(\mathcal{O}, \mathbb{R})\mathbf{n} = \{f\mathbf{n} \mid f \in L^2(\mathcal{O}, \mathbb{R})\}$, and the system $\{\pi_i\mathbf{n} \mid i \in \mathbb{N}_0\} \cup \{\tau_i \mid i \in \mathbb{N}_0\}$ is an orthonormal basis in the space $L^2(\mathcal{O}, \mathbb{R}^3)$.

Define, for each $M \in \mathbb{N}_0$, the space

$$(9) \quad L_M^2(\mathcal{O}, \mathbb{R}^3) := \text{span}\{\pi_i\mathbf{n}, \tau_i \mid i \in \mathbb{N}_0, i \leq M\}$$

and, denote by $P_M^{\mathcal{O}}$ the orthogonal projection $P_M^{\mathcal{O}}: L^2(\mathcal{O}, \mathbb{R}^3) \rightarrow L_M^2(\mathcal{O}, \mathbb{R}^3)$ in $L^2(\mathcal{O}, \mathbb{R}^3)$ onto $L_M^2(\mathcal{O}, \mathbb{R}^3)$.

As in [Rod14a, Section 2.2], we suppose that the control region Γ_c and \mathcal{O} satisfy

$$(10) \quad \overline{\Gamma_c} = \text{supp}(\chi) \text{ for some } \chi \in C^2(\Gamma, \mathbb{R}); \text{ and } \Gamma_c \subseteq \overline{\Gamma_c} \subseteq \mathcal{O} \subseteq \Gamma.$$

Let us define the space

$$(11) \quad \mathcal{E}_M := \{\zeta \mid \zeta(t) = \Xi\kappa(t), \text{ and } \kappa \in H^1(\mathbb{R}_0, \mathbb{R}^{2M})\}$$

with

$$(12) \quad \begin{aligned} \Xi: \mathbb{R}^{2M} &\rightarrow E_M := \text{span}\{\chi\mathbb{E}_0^{\mathcal{O}}P_{\chi^\perp}^{\mathcal{O}}\pi_i\mathbf{n}, \chi\mathbb{E}_0^{\mathcal{O}}\tau_i \mid i \in \mathbb{N}_0, i \leq M\} \\ z &\mapsto \Xi z := \sum_{i=1}^M \chi\mathbb{E}_0^{\mathcal{O}} \left(z_i P_{\chi^\perp}^{\mathcal{O}}(\pi_i\mathbf{n}) + z_{M+i}\tau_i \right); \end{aligned}$$

where $\mathbb{E}_0^{\mathcal{O}}: L^2(\mathcal{O}, \mathbb{R}) \rightarrow L^2(\Gamma, \mathbb{R})$ is the extension by zero outside the subset \mathcal{O} , and $P_{\chi^\perp}^{\mathcal{O}}: L^2(\mathcal{O}, \mathbb{R}^3) \rightarrow \{\chi\mathbf{n}|_{\mathcal{O}}\}^\perp$ is the orthogonal projection in $L^2(\mathcal{O}, \mathbb{R}^3)$ onto $\{\chi\mathbf{n}|_{\mathcal{O}}\}^\perp = \{f \in L^2(\mathcal{O}, \mathbb{R}^3) \mid (f, \chi\mathbf{n})_{L^2(\mathcal{O}, \mathbb{R}^3)} = 0\}$:

$$\begin{cases} \mathbb{E}_0^{\mathcal{O}}\xi|_{\mathcal{O}} := \xi \\ \mathbb{E}_0^{\mathcal{O}}\xi|_{\Gamma \setminus \mathcal{O}} := 0 \end{cases}, \quad \text{and} \quad P_{\chi^\perp}^{\mathcal{O}}v := v - \frac{(v, \chi\mathbf{n})_{L^2(\mathcal{O}, \mathbb{R}^3)}}{\int_{\mathcal{O}} \chi^2 d\mathcal{O}} \chi\mathbf{n}|_{\mathcal{O}}.$$

Remark 2.2. Notice that the function $\zeta = \Xi z$ satisfies the zero-average compatibility condition: $\int_{\Gamma} \zeta(t) \cdot \mathbf{n} d\Gamma = \sum_{i=1}^M z_i(t) \int_{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}}(\pi_i\mathbf{n}) \cdot \chi\mathbf{n}|_{\mathcal{O}} d\mathcal{O} = 0$.

Our goal is to take as controls those functions in \mathcal{E}_M that are normal to the boundary, that is, we take

$$(13) \quad \mathcal{E}_M^{\mathbf{n}} := \{\zeta \mid \zeta(t) = \Xi Q_f^M \kappa(t), \text{ and } \kappa \in H^1(\mathbb{R}_0, \mathbb{R}^{2M})\}$$

as the control space, where $Q_f^M: \mathbb{R}^{2M} \rightarrow \mathbb{R}^{2M}$, $Q_f^M y = (z_1, z_2, \dots, z_{2M})$ with $z_i = y_i$ if $1 \leq i \leq M$, and $z_i = 0$ if $M+1 \leq i \leq 2M$; that is, Q_f^M is the orthogonal projection onto the first M coordinates. In particular the controls are supported in $[0, +\infty) \times \overline{\Gamma_c}$ and take their values $\zeta(t)$ in the finite-dimensional space $E_M^{\mathbf{n}} := \text{span}\{\chi\mathbb{E}_0^{\mathcal{O}}P_{\chi^\perp}^{\mathcal{O}}\pi_i\mathbf{n} \mid i \in \mathbb{N}_0, i \leq M\}$, for each $t \in [0, +\infty)$.

Let us be given a constant $\lambda > 0$ and two (fixed) regular enough functions h and γ ; in addition we suppose that \hat{u} is also regular enough and solves, in $\mathbb{R}_0 \times \Omega$, the Navier–Stokes system (1) with $\zeta = 0$, and a suitable pressure function $p_u = p_{\hat{u}}$. Given $u(0)$ close enough to $\hat{u}(0)$, with $(u(0) - \hat{u}(0))|_{\Gamma} \in E_M^{\mathbf{p}}$, our goal is to find a (time-dependent) feedback linear controller $v \mapsto \mathcal{K}_{\hat{u}}^{\lambda, t} v \in \mathbb{R}^{2M}$ such that the solution of the problem (1), with $\zeta = (u(0) - \hat{u}(0))|_{\Gamma} + \Xi Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, r} (u - \hat{u}(r)) dr$, is defined for all $t \geq 0$ and converges exponentially to \hat{u} , with rate $\frac{\lambda}{2}$, that is,

$$|u(t) - \hat{u}(t)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq C e^{-\lambda t} |u(0) - \hat{u}(0)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \quad \text{for } t \geq 0,$$

where C is independent of $u(0) - \hat{u}(0)$ and time t . It will be precised later in Section 5 what we mean by “regular enough”, “close enough” and “solution”.

Notice that seeking a solution of (1), (2) in the form $u = \hat{u} + v$, formally we obtain the following equivalent problem for v :

$$(14) \quad \begin{aligned} \partial_t v + \mathcal{B}(\hat{u})v + \langle v \cdot \nabla \rangle v - \nu \Delta v + \nabla p_v &= 0, & \text{div } v &= 0, \\ v|_{\Gamma} &= \zeta, & v(0) &= u_0 - \hat{u}(0), \end{aligned}$$

with $p_v = p_u - p_{\hat{u}}$, and $\mathcal{B}(\hat{u})v$ stands for $\langle \hat{u} \cdot \nabla \rangle v + \langle v \cdot \nabla \rangle \hat{u}$. We can see that it suffices to study the problem of stabilization of system (14) to the zero solution. We shall start by deriving the (global) stabilization of the Oseen–Stokes system, in $\mathbb{R}_0 \times \Omega$:

$$(15) \quad \begin{aligned} \partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v + \nabla p_v &= 0, & \text{div } v &= 0, \\ v|_{\Gamma} &= \zeta, & v(0) &= v_0, \end{aligned}$$

to the zero solution; from which we shall derive the local result for (14).

2.3. Weak and strong solutions, and admissible initial conditions. We briefly recall some notions and results from [Rod14b, Rod14a] concerning the weak and strong solutions for the Oseen–Stokes system, in a bounded cylinder $(a, b) \times \Omega$, with a, b real numbers, $0 \leq a < b$.

$$(16) \quad \begin{aligned} \partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v + \nabla p_v + g &= 0, & \text{div } v &= 0, \\ v|_{\Gamma} &= \zeta = K\eta, & v(a) &= v_0. \end{aligned}$$

Recall the extensions E_s , $s \in \{1, 2\}$, in Section 2.1.

Definition 2.3. Given $\hat{u} \in \mathcal{W}^{(a, b)|\text{wk}}$, $v_0 \in L_{\text{div}}^2(\Omega, \mathbb{R}^3)$, $g \in L^2((a, b), H^{-1}(\Omega, \mathbb{R}^3))$, and $\zeta \in G_{\text{av}}^1((a, b), \Gamma)$; we say that v , in the space $W((a, b), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$, is a weak solution for system (16), if $v - E_1 \zeta \in L^2((a, b), V, V')$ is a weak solution for the system

$$(17) \quad \begin{aligned} \partial_t y + \mathcal{B}(\hat{u})y - \nu \Delta y + \nabla p_y + f &= 0, & \text{div } y &= 0, \\ y|_{\Gamma} &= 0, & y(a) &= y_0 \end{aligned}$$

with $f = g + \partial_t E_1 \zeta + \mathcal{B}(\hat{u})E_1 \zeta - \nu \Delta E_1 \zeta$, and $y_0 = v_0 - E_1 \zeta(a) \in H$. Here weak solution for (17) is understood in the classical sense as in [Lio69, Chapter 1, Sections 6.1 and 6.4], [Tem95, Sections 2.4 and 3.2], [Tem01, Chapter 3, Section 3].

Definition 2.4. Given $\hat{u} \in \mathcal{W}^{(a, b)|\text{st}}$, $v_0 \in H_{\text{div}}^1(\Omega, \mathbb{R}^3)$, $g \in L^2((a, b), L^2(\Omega, \mathbb{R}^3))$, and $\zeta \in G_{\text{av}}^2((a, b), \Gamma)$; we say that v , in the space $W((a, b), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$, is a strong solution for system (16), if $v - E_2 \zeta \in L^2((a, b), D(L), H)$ is a strong solution for system (17) with $f = g + \partial_t E_2 \zeta + \mathcal{B}(\hat{u})E_2 \zeta - \nu \Delta E_2 \zeta$, and $y_0 = v_0 - E_2 \zeta(a) \in V$. Again, strong solution for (17) is understood in the classical sense as in [Tem95, Section 2.4].

In the case our control ζ is in the space \mathcal{E}_M a natural question is: what are the admissible initial vector fields v_0 , if we want to guarantee the existence of a weak solution? Notice that, from (11), ζ takes the form $\zeta = \Xi\kappa$ with $\kappa \in H^1(\mathbb{R}_0, \mathbb{R}^{2M})$. It is also not hard to check that the mapping Ξ , in the definition of the control space (11), maps $H^1((a, b), \mathbb{R}^{2M})$ into $G_{\text{av}}^2((a, b), \Gamma) \subset G_{\text{av}}^1((a, b), \Gamma)$ continuously, that is, our control is of the form as in (16) with $K = \Xi$.

The set of admissible weak initial conditions for system (16), with $\zeta \in \mathcal{E}_M$, is given by $\mathcal{A}_{\Xi_1} = H + \mathcal{H}_{\Xi_1}$, with $\mathcal{H}_{\Xi_1} := E_1\Xi H^1((a, b), \mathbb{R}^{2M})(a) = \{\gamma(a) \mid \gamma = E_1\Xi\eta \text{ and } \eta \in H^1((a, b), \mathbb{R}^{2M})\}$.

Similarly, the set of admissible strong initial conditions for system (16), with $\zeta \in \mathcal{E}_M$, is $\mathcal{A}_{\Xi_2} := V + \mathcal{H}_{\Xi_2}$, with $\mathcal{H}_{\Xi_2} := E_2\Xi H^1((a, b), \mathbb{R}^{2M})(a)$.

Moreover \mathcal{H}_{Ξ_1} , \mathcal{A}_{Ξ_1} , \mathcal{H}_{Ξ_2} and \mathcal{A}_{Ξ_2} are Hilbert spaces, with associated range norms

$$(18) \quad \begin{aligned} |u|_{\mathcal{H}_{\Xi_i}} &:= \inf \left\{ |\eta|_{H^1((a, b), \mathbb{R}^{2M})} \mid u = E_i\Xi\eta(a), \eta \in H^1((a, b), \mathbb{R}^{2M}) \right\}, \\ |u|_{\mathcal{A}_{\Xi_1}} &:= \inf \left\{ |(w, z)|_{H \times \mathcal{H}_{\Xi_1}} \mid u = w + z \text{ and } (w, z) \in H \times \mathcal{H}_{\Xi_1} \right\}. \\ |u|_{\mathcal{A}_{\Xi_2}} &:= \inf \left\{ |(w, z)|_{V \times \mathcal{H}_{\Xi_2}} \mid u = w + z \text{ and } (w, z) \in V \times \mathcal{H}_{\Xi_2} \right\}. \end{aligned}$$

Theorem 2.5. *Given $\hat{u} \in \mathcal{W}^{(a, b)|\text{wk}}$, $g \in L^2((a, b), H^{-1}(\Omega, \mathbb{R}^3))$, $v_0 \in \mathcal{A}_{\Xi_1}$, and $\eta \in H^1((a, b), \mathbb{R}^{2M})$, with $v_0 - E_1\Xi\eta(a) \in H$, then there is a weak solution v in the space $W((a, b), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ for system (16), with $\zeta = \Xi\eta$. Moreover v is unique and depends continuously on the given data (v_0, g, η) :*

$$\begin{aligned} &|v|_{W((a, b), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ &\leq \bar{C}_{[\hat{u}|_{\mathcal{W}^{(a, b)|\text{wk}}}] \left(|v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |g|_{L^2((a, b), H^{-1}(\Omega, \mathbb{R}^3))}^2 + |\eta|_{H^1((a, b), \mathbb{R}^{2M})}^2 \right). \end{aligned}$$

Theorem 2.6. *Given $\hat{u} \in \mathcal{W}^{(a, b)|\text{st}}$, $g \in L^2((a, b), L^2(\Omega, \mathbb{R}^3))$, $v_0 \in \mathcal{A}_{\Xi_2}$ and $\eta \in H^1((a, b), \mathbb{R}^{2M})$, with $v_0 - E_2\Xi\eta(a) \in V$, then there is a strong solution v in the space $W((a, b), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$ for system (16), with $\zeta = \Xi\eta$. Moreover v is unique and depends continuously on the given data (v_0, g, η) :*

$$\begin{aligned} &|v|_{W((a, b), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \\ &\leq \bar{C}_{[\hat{u}|_{\mathcal{W}^{(a, b)|\text{st}}}] \left(|v_0|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |g|_{L^2((a, b), L^2(\Omega, \mathbb{R}^3))}^2 + |\eta|_{H^1((a, b), \mathbb{R}^{2M})}^2 \right). \end{aligned}$$

Remark 2.7. The weak solution given in Theorem 2.5 does not depend on the extension E_1 . Also, the set of admissible weak initial conditions is independent of E_1 (cf. [Rod14b, Rems. 3.2 and 3.4]). Analogously, the strong solution given in Theorem 2.6, and the set of admissible strong initial conditions are independent of E_2 .

2.4. Smoothing property. The following Lemma will play a key role.

Lemma 2.8. *Let us be given $\hat{u} \in \mathcal{W}^{(a, b)|\text{st}}$, $g \in L^2((a, b), L^2(\Omega, \mathbb{R}^3))$, $v_0 \in \mathcal{A}_{\Xi_1}$, and $\eta \in H^1((a, b), \mathbb{R}^{2M})$, with $v_0 - E_1\Xi\eta(a) \in H$; then for the weak solution v of system (16) with $\zeta = \Xi\eta$, we have $(t - a)v \in W((a, b), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$, and*

$$\begin{aligned} &|(t - a)v|_{W((a, b), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \\ &\leq \bar{C}_{[\hat{u}|_{\mathcal{W}^{(a, b)|\text{st}}}] \left(|v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |g|_{L^2((a, b), L^2(\Omega, \mathbb{R}^3))}^2 + |\eta|_{H^1((a, b), \mathbb{R}^{2M})}^2 \right). \end{aligned}$$

Proof. Since v solves (16), it turns out that also $w = (t - a)v$ does, with different data:

$$\begin{aligned} \partial_t w + \mathcal{B}(\hat{u})w - \nu\Delta w + \nabla(t - a)p_v + (t - a)g - v &= 0, & \text{div } w &= 0, \\ w|_{\Gamma} &= \Xi(t - a)\eta, & w(a) &= 0. \end{aligned}$$

Then, from Theorem 2.6, we obtain

$$\begin{aligned} & |w|_{W((a,b), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \\ & \leq \overline{C}_{[\hat{u}]_{\mathcal{W}((a,b), \text{st})}} \left(|(t-a)g - v|_{L^2((a,b), L^2(\Omega, \mathbb{R}^3))}^2 + |(t-a)\eta|_{H^1((a,b), \mathbb{R}^{2M})}^2 \right); \end{aligned}$$

thus Lemma 2.8 follows from $|v|_{L^2((a,b), L^2(\Omega, \mathbb{R}^3))}^2 \leq |v|_{W((a,b), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2$ and Theorem 2.5. \square

Remark 2.9. Notice that Theorems 2.5 and 2.6, and Lemma 2.8 also hold if we take controls in the subspace $\mathcal{E}_M^{\mathbf{n}} \subset \mathcal{E}_M$, that is, if we replace Ξ by ΞQ_f^M and the admissible initial conditions $\mathcal{A}_{\Xi_1} = H + \mathcal{H}_{\Xi_1}$ and $\mathcal{A}_{\Xi_2} = H + \mathcal{H}_{\Xi_2}$ by $\mathcal{A}_{\Xi_1}^{\mathbf{n}} := \mathcal{A}_{\Xi Q_f^M} = H + \mathcal{H}_{\Xi Q_f^M}$ and $\mathcal{A}_{\Xi_2}^{\mathbf{n}} := \mathcal{A}_{\Xi Q_f^M} = H + \mathcal{H}_{\Xi Q_f^M}$, respectively. Where each $\mathcal{H}_{\Xi Q_f^M}$ is defined similarly, $\mathcal{H}_{\Xi Q_f^M} := E_i \Xi Q_f^M H^1((a,b), \mathbb{R}^{2M})(a)$, $i \in \{1, 2\}$.

3. THE OSEEN–STOKES SYSTEM: EXISTENCE OF A STABILIZING CONTROL

Using a controllability result from [Rod14a], we shall construct a control, in the space \mathcal{E}_M , exponentially stabilizing the linear system (15) to the zero solution.

Definition 3.1. We say that v_0 is a weak, respectively strong, admissible initial condition for system (15), with $\zeta = \Xi \kappa \in \mathcal{E}_M$, if it is a weak, respectively strong, admissible initial condition for the same system in $(0, 1) \times \Omega$ with $\zeta = \Xi \kappa|_{(0,1)} = \Xi H^1((0, 1), \mathbb{R}^{2M})$.

Let \mathcal{A}_{Ξ_1} be the set of admissible initial conditions for controls $\zeta = \Xi \eta$ in \mathcal{E}_M , that is, $\mathcal{A}_{\Xi_1} = H + E_1 \Xi H^1((0, 1), \mathbb{R}^{2M})(0)$. In this Section we prove the following:

Theorem 3.2. *Let us be given $\lambda > 0$ and $\hat{u} \in \mathcal{W}^{\text{st}}$. Then there exists $M = \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda}} \geq 1$ with the following property: for each $v_0 \in \mathcal{A}_{\Xi_1}$, there exists a “control” vector function $\kappa^{\hat{u}, \lambda} = \kappa^{\hat{u}, \lambda}(v_0) \in H^1(\mathbb{R}_0, \mathbb{R}^{2M})$ such that the weak solution v of system (15) in $\mathbb{R}_0 \times \Omega$, with $\zeta = \Xi \kappa^{\hat{u}, \lambda}$, satisfies the inequality*

$$|v(t)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda}} e^{-\lambda t} |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2, \quad t \geq 0.$$

Moreover, for $0 \leq \hat{\lambda} < \lambda$, the mapping $v_0 \mapsto \kappa^{\hat{u}, \lambda}(v_0)$ is linear and satisfies:

$$\left| e^{\frac{\hat{\lambda}}{2} t} \kappa^{\hat{u}, \lambda}(v_0) \right|_{H^1(\mathbb{R}_0, \mathbb{R}^{2M})}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{(\lambda - \hat{\lambda})}}} |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2.$$

3.1. Auxiliary results. We start by recalling the space H_N and the orthogonal projection $\Pi_N: H \rightarrow H_N$, see Section 2.2.

Since the trace $\kappa(0)$, at time $t = 0$, is well defined for any given $\kappa \in H^1(\mathbb{R}_0, \mathbb{R}^{2M})$, we can easily obtain the explicit form of the spaces \mathcal{A}_{Ξ_1} and \mathcal{A}_{Ξ_2} , of all admissible weak and strong initial conditions, for controls in the space \mathcal{E}_M defined in (11).

Writing $v \in \mathcal{A}_{\Xi_1}$ as $v = u + (E_1 \Xi \kappa)(0)$, where $u \in H$ and $\kappa \in H^1((0, 1), \mathbb{R}^{2M})$, from the fact that $E_1 \Xi \kappa \in W((0, 1), H^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \subset C([0, 1], L^2(\Omega, \mathbb{R}^3))$, and the continuity of the mapping $u \mapsto u \cdot \mathbf{n}$ from $L_{\text{div}}^2(\Omega, \mathbb{R}^3)$ into $H^{-\frac{1}{2}}(\Gamma, \mathbb{R})$, we obtain that $(v \cdot \mathbf{n})\mathbf{n} = \lim_{t \searrow 0} ((E_1 \Xi \kappa) \cdot \mathbf{n})(t)\mathbf{n} = \lim_{t \searrow 0} ((E_1 \Xi \kappa(t)) \cdot \mathbf{n})\mathbf{n} = \lim_{t \searrow 0} ((\Xi \kappa(t)) \cdot \mathbf{n})\mathbf{n} = ((\Xi \kappa(0)) \cdot \mathbf{n})\mathbf{n}$.

It follows that

$$(19) \quad \mathcal{A}_{\Xi_1} = \left\{ u \in L_{\text{div}}^2(\Omega, \mathbb{R}^3) \mid (u \cdot \mathbf{n})\mathbf{n}|_{\Gamma} = \Xi Q_f^M z, \text{ for some } z \in \mathbb{R}^{2M} \right\}.$$

Analogously, we can conclude that the set of strong admissible conditions is given by

$$(20) \quad \mathcal{A}_{\Xi_2} = \left\{ u \in H_{\text{div}}^1(\Omega, \mathbb{R}^3) \mid u|_{\Gamma} = \Xi z, \text{ for some } z \in \mathbb{R}^{2M} \right\}.$$

Remark 3.3. Notice that for $u \in L^2_{\text{div}}(\Omega, \mathbb{R}^3)$, we may define $\langle u \cdot \mathbf{n}, \psi \rangle_{H^{-\frac{1}{2}}(\Gamma, \mathbb{R}), H^{\frac{1}{2}}(\Gamma, \mathbb{R})} := (u, \nabla \mathcal{R}_\Gamma^1 \psi)_{L^2(\Omega, \mathbb{R})}$, where $\mathcal{R}_\Gamma^1 : H^{\frac{1}{2}}(\Gamma, \mathbb{R}) \rightarrow H^1(\Omega, \mathbb{R})$ is a continuous linear right inverse of the trace mapping $\Psi \mapsto \Psi|_\Gamma$; in particular we have $(\mathcal{R}_\Gamma^1 \psi)|_\Gamma = \psi$ and the mapping $u \mapsto u \cdot \mathbf{n}|_\Gamma$ is linear and continuous: $|u \cdot \mathbf{n}|_{H^{-\frac{1}{2}}(\Gamma, \mathbb{R})} \leq |u|_H |\mathcal{R}_\Gamma^1|_{\mathcal{L}(H^{\frac{1}{2}}(\Gamma, \mathbb{R}) \rightarrow H^1(\Omega, \mathbb{R}))}$. See, for example, [Tem01, Chapter 1, Section 1.3] and [LM72, Chapter 1, Section 8.2].

Corollary 3.4. *The space of admissible weak initial conditions \mathcal{A}_{Ξ_1} is a closed subspace of $L^2_{\text{div}}(\Omega, \mathbb{R}^3)$, and the space of admissible strong initial conditions \mathcal{A}_{Ξ_2} is a closed subspace of $H^1_{\text{div}}(\Omega, \mathbb{R}^3)$.*

Recalling the orthogonal projection $\Pi : L^2(\Omega, \mathbb{R}^3) \rightarrow H$ (cf. Section 2.1); we have:

Lemma 3.5. *The norms $|u|_{L^2_{\text{div}}(\Omega, \mathbb{R}^3)}$ and $\left(|\Pi u|_H^2 + |u \cdot \mathbf{n}|_{H_{\text{av}}^{-\frac{1}{2}}(\Gamma, \mathbb{R})}^2 \right)^{\frac{1}{2}}$, are equivalent in $L^2_{\text{div}}(\Omega, \mathbb{R}^3)$.*

Proof. Writing $L^2(\Omega, \mathbb{R}^3) = H \oplus H^\perp$, each $v \in L^2(\Omega, \mathbb{R}^3)$ can be rewritten as

$$(21) \quad v = \Pi v + \nabla P_\nabla v, \quad \text{with} \quad \int_\Omega P_\nabla v \, d\Omega = 0;$$

further in this way the mapping $v \mapsto P_\nabla v \in H^1(\Omega, \mathbb{R})$ is well defined, and we see that $P_\nabla v$ solves the system $\Delta P_\nabla v = \text{div } v$, $\nabla P_\nabla v \cdot \mathbf{n} = v \cdot \mathbf{n}$. It follows that if $v \in L^2_{\text{div}}(\Omega, \mathbb{R}^3)$, then $|\nabla P_\nabla v|_{L^2(\Omega, \mathbb{R}^3)}^2 = \langle u \cdot \mathbf{n}, P_\nabla v \rangle_{H^{-\frac{1}{2}}(\Gamma, \mathbb{R}), H^{\frac{1}{2}}(\Gamma, \mathbb{R})}$; since $P_\nabla v$ is zero averaged in Ω , there is a constant $C > 0$ such that $|\nabla P_\nabla v|_{L^2(\Omega, \mathbb{R}^3)}^2 \leq C |v \cdot \mathbf{n}|_{H^{-\frac{1}{2}}(\Gamma, \mathbb{R})} |\nabla P_\nabla v|_{L^2(\Omega, \mathbb{R}^3)}$; hence $|\nabla P_\nabla v|_{L^2(\Omega, \mathbb{R}^3)}^2 \leq C^2 |v \cdot \mathbf{n}|_{H^{-\frac{1}{2}}(\Gamma, \mathbb{R})}^2 \leq C_1 |v|_{L^2_{\text{div}}(\Omega, \mathbb{R}^3)}^2$. \square

From now, for convenience, we suppose \mathcal{A}_{Ξ_1} and \mathcal{A}_{Ξ_2} endowed with the norm inherited from $L^2_{\text{div}}(\Omega, \mathbb{R}^3)$ and from $H^1_{\text{div}}(\Omega, \mathbb{R}^3)$, respectively; from Corollary 3.4 the spaces \mathcal{A}_{Ξ_1} and \mathcal{A}_{Ξ_2} are Hilbert spaces.

Remark 3.6. In Section 2.3, we have considered the spaces of admissible conditions endowed with a suitable range norm also making them Hilbert spaces. Changing the norms now to those inherited from $L^2_{\text{div}}(\Omega, \mathbb{R}^3)$ and $H^1_{\text{div}}(\Omega, \mathbb{R}^3)$ will not cause any trouble concerning continuity properties. Indeed, we have that \mathcal{A}_{Ξ_1} and \mathcal{A}_{Ξ_2} endowed with the range norm are continuously embedded in $L^2_{\text{div}}(\Omega, \mathbb{R}^3)$ and $H^1_{\text{div}}(\Omega, \mathbb{R}^3)$, respectively, (cf. [Rod14a, Section 2.4]). Thus, from the completeness of both norms they are necessary equivalent (cf. Corollary A.2 in the Appendix).

Next, we define the space

$$(22) \quad \mathcal{N} := \left\{ z \in \mathbb{R}^{2M} \mid \Xi z = \chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} \sum_{i=1}^M (z_i \pi_i \mathbf{n} + z_{M+i} \tau_i) = 0 \right\};$$

and observe that $\Xi z = \Xi P_{\mathcal{N}^\perp} z$, for any $z \in \mathbb{R}^{2M}$, where $P_{\mathcal{N}^\perp} : \mathbb{R}^{2M} \rightarrow \mathcal{N}^\perp$ is the orthogonal projection in \mathbb{R}^{2M} onto the orthogonal subspace \mathcal{N}^\perp to \mathcal{N} . Furthermore, we have $\Xi k = 0$ only if $\Xi Q_f^M k = 0$, recalling that $Q_f^M : \mathbb{R}^{2M} \mapsto \mathbb{R}^{2M}$ is the projection onto the first M coordinates (cf. the definition, in (13), of the control space $\mathcal{E}_f^{\mathbf{n}}$). Denoting $P_{\mathcal{N}} := 1 - P_{\mathcal{N}^\perp}$, we have that $P_{\mathcal{N}} k = k$ only if $P_{\mathcal{N}} Q_f^M k = Q_f^M k$; from which we conclude that $P_{\mathcal{N}} k = k$ only if $P_{\mathcal{N}} Q_f^M k = Q_f^M P_{\mathcal{N}} k$. Now given $z \in \mathbb{R}^{2M}$, we have $P_{\mathcal{N}} Q_f^M z = P_{\mathcal{N}} Q_f^M (P_{\mathcal{N}} z + P_{\mathcal{N}^\perp} z) = Q_f^M P_{\mathcal{N}} P_{\mathcal{N}} z + P_{\mathcal{N}} Q_f^M P_{\mathcal{N}^\perp} z$, that is, $P_{\mathcal{N}} Q_f^M z = Q_f^M P_{\mathcal{N}} z + P_{\mathcal{N}} Q_f^M P_{\mathcal{N}^\perp} z$; on the other hand for any given $k \in \mathcal{N}$ we find that $(P_{\mathcal{N}} Q_f^M P_{\mathcal{N}^\perp} z, k)_{\mathbb{R}^{2M}} = (P_{\mathcal{N}^\perp} z, Q_f^M P_{\mathcal{N}} k)_{\mathbb{R}^{2M}}$, and since $P_{\mathcal{N}} k = k$, we obtain that

$(P_{\mathcal{N}}Q_f^M P_{\mathcal{N}^\perp} z, k)_{\mathbb{R}^{2M}} = (P_{\mathcal{N}^\perp} z, P_{\mathcal{N}}Q_f^M k)_{\mathbb{R}^{2M}} = 0$, which implies that $P_{\mathcal{N}}Q_f^M P_{\mathcal{N}^\perp} z = 0$ and so, $P_{\mathcal{N}}Q_f^M z = Q_f^M P_{\mathcal{N}} z$; that is, $P_{\mathcal{N}}$ commutes with Q_f^M . Hence, it is straightforward to derive that, if $Q_l^M := 1 - Q_f^M$ is the orthogonal projection in \mathbb{R}^{2M} onto the last M coordinates, we also have $P_{\mathcal{N}^\perp} Q_f^M z = Q_f^M P_{\mathcal{N}^\perp} z$, $P_{\mathcal{N}^\perp} Q_l^M z = Q_l^M P_{\mathcal{N}^\perp} z$ and $P_{\mathcal{N}} Q_l^M z = Q_l^M P_{\mathcal{N}} z$, for all $z \in \mathbb{R}^{2M}$.

Notice that, for any $u \in \mathcal{A}_{\Xi_1}$ there exists at least one $z \in \mathbb{R}^{2M}$ such that $(u \cdot \mathbf{n})|_{\Gamma} = \Xi Q_f^M z = \chi \mathbb{E}_0^{\mathcal{O}} \sum_{i=1}^M (z_i P_{\chi^\perp}^{\mathcal{O}} \pi_i \mathbf{n})$; it follows that the mapping

$$(23) \quad u \mapsto z^{u \cdot \mathbf{n}} := P_{\mathcal{N}^\perp} Q_f^M z = Q_f^M P_{\mathcal{N}^\perp} z$$

is continuous from \mathcal{A}_{Ξ_1} onto $P_{\mathcal{N}^\perp} Q_f^M \mathbb{R}^{2M}$ and $(u \cdot \mathbf{n})|_{\Gamma} = \Xi z^{u \cdot \mathbf{n}}$. Indeed, to check that the mapping is well defined, we set another vector $w \in \mathbb{R}^{2M}$ such that $(u \cdot \mathbf{n})|_{\Gamma} = \Xi Q_f^M w$, then necessarily $\Xi Q_f^M (w - z) = 0$ which means that $P_{\mathcal{N}^\perp} Q_f^M (w - z) = 0$. On the other hand the continuity of the mapping $u \mapsto u \cdot \mathbf{n}$ and the fact that both $|u \cdot \mathbf{n}|_{H_{\text{av}}^{-\frac{1}{2}}(\Gamma, \mathbb{R})}$ and $|z^{u \cdot \mathbf{n}}|_{\mathbb{R}^{2M}}$ are norms in the finite dimensional space $\Xi Q_f^M \mathbb{R}^{2M} = \{u \cdot \mathbf{n} \mid u \in \mathcal{A}_{\Xi_1}\}$ (and, so necessarily equivalent) give us

$$(24) \quad |z^{u \cdot \mathbf{n}}|_{\mathbb{R}^{2M}} \leq C |u|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}.$$

Lemma 3.7. *The norms $|u|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}$ and $(|\Pi u|_H^2 + |z^{u \cdot \mathbf{n}}|_{\mathbb{R}^{2M}}^2)^{\frac{1}{2}}$ are equivalent in \mathcal{A}_{Ξ_1} ; further they make \mathcal{A}_{Ξ_1} a Hilbert space.*

Proof. From Corollary 3.4 we have that \mathcal{A}_{Ξ_1} endowed with the norm inherited from $L_{\text{div}}^2(\Omega, \mathbb{R}^3)$ is a Hilbert space. The equivalence follows from Lemma 3.5 and from the fact that $|u \cdot \mathbf{n}|_{H^{-\frac{1}{2}}(\Gamma, \mathbb{R})}$ and $|z^{u \cdot \mathbf{n}}|_{\mathbb{R}^{2M}}$ are equivalent norms in $\Xi Q_f^M \mathbb{R}^{2M}$. \square

Remark 3.8. Notice that the space \mathcal{N} defined in (22) is not necessarily trivial, that is, it may contain nonzero vectors. See the Example in Section A.2 in the appendix.

Now, let us be given four nonnegative constants

$$0 \leq a < b, \quad 0 < \varepsilon, \quad 0 < \delta,$$

and two functions $\varphi, \tilde{\varphi} \in C^1([a, b], \mathbb{R})$ such that $\text{supp}(\varphi) \neq \emptyset$, $\tilde{\varphi}(t) \geq \varepsilon$ for all $t \in \text{supp}(\varphi)$, and $\tilde{\varphi}(t) = 0$ for $t \in [a, a + \delta] \cup [b - \delta, b]$. Further let $\vartheta \in C^2(\Gamma, \mathbb{R})$ be a function such that $\text{supp}(\vartheta) \subseteq \overline{\mathcal{O}}$ and $\vartheta(x) \geq \varepsilon$ for all $x \in \Gamma_c$; see (10).

Given a Hilbert space X , we define the orthogonal projection P_M^t in $L^2((a, b), X)$:

$$P_M^t f(t, x) := \sum_{n=1}^M \left(\int_a^b f(\tau, x) \sigma_n(\tau) d\tau \right) \sigma_n$$

where the σ_n , $n \in \mathbb{N}_0$, are the eigenfunctions of the Dirichlet Laplacean $\Delta_t := \partial_t \partial_t$ in (a, b) : $\{\sigma_n(t) := (\frac{2}{b-a})^{\frac{1}{2}} \sin(n\pi(\frac{t-a}{b-a})) \mid n \in \mathbb{N}_0\}$, and $\{\lambda_n = -(\frac{n\pi}{b-a})^2 \mid n \in \mathbb{N}_0\}$; $\Delta_t \sigma_n = \lambda_n \sigma_n$. Inspired by an Example in [Rod14a, Section 5], we consider the auxiliary control space

$$\begin{aligned} \mathcal{G}_M &:= \varphi \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} P_M^t \tilde{\varphi} \vartheta G^2((a, b), \Gamma)|_{\mathcal{O}} \\ &:= \{\zeta \mid \zeta = \varphi \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} P_M^t (\tilde{\varphi} \vartheta \eta)|_{\mathcal{O}} \text{ and } \eta \in G^2((a, b), \Gamma)\}. \end{aligned}$$

and consider the operator

$$(25) \quad \eta \mapsto K_t^{\mathcal{O}} \eta := \varphi \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} P_M^t (\tilde{\varphi} \vartheta \eta)|_{\mathcal{O}}.$$

From [Rod14a, Section 5] we know the following controllability result:

Theorem 3.9. *Let us be given $\hat{u} \in \mathcal{W}^{(a,b)|\text{st}}$ and $N \in \mathbb{N}$, then there exists an integer $M = \overline{C}_{[N, |\hat{u}|_{\mathcal{W}^{(a,b)|\text{st}}]} \in \mathbb{N}_0$ with the following property: for every $v_0 \in H$, we can find $\eta = \eta(v_0) \in G^2((a, b), \Gamma)$, depending linearly on v_0 , such that the boundary control $\zeta = K_t^{\mathcal{O}} \eta = \varphi \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} P_M^t (\tilde{\varphi} \partial \eta|_{\mathcal{O}})$ drives the system (15) to a vector $v(b) \in V$ such that $\Pi_N v(b) = 0$. Moreover, there exists a constant $\overline{C}_{[|\hat{u}|_{\mathcal{W}^{(a,b)|\text{st}}]}$, depending on $|\hat{u}|_{\mathcal{W}^{(a,b)|\text{st}}}$, φ , $\tilde{\varphi}$, and $b - a$ but, not on the pair (N, v_0) , such that*

$$(26) \quad |\eta|_{G^2((a,b), \Gamma)}^2 \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}^{(a,b)|\text{st}}]} |v_0|_H^2.$$

3.2. Proof of Theorem 3.2. Let us fix a sufficiently large $N \geq 1$ and let M be the integer given in Theorem 3.9. We organize the proof in four main steps.

Ⓢ Step 1: *driving the system, from $v(0) = v_0$ at time $t = 0$, to a vector $v(1) = v_1 \in V$ at time $t = 1$.* Let $z^{v_0, \mathbf{n}} \in P_{N^\perp} Q_f^M \mathbb{R}^{2M}$ be the vector defined as in (23), and let $\phi \in C^1([0, 1], \mathbb{R})$ be a function taking the value 1 in a neighborhood $[0, \delta]$ of $t = 0$, and the value 0 in a neighborhood $[1 - \delta, 1]$ of $t = 1$, with $\delta < \frac{1}{2}$. Then, the function $\kappa_\phi^{\mathbf{n}} = \phi z^{v_0, \mathbf{n}} = (z_1^{v_0, \mathbf{n}} \phi, z_2^{v_0, \mathbf{n}} \phi, \dots, z_M^{v_0, \mathbf{n}} \phi, 0, 0, \dots, 0)$ is in $C^1([0, 1], \mathbb{R}^{2M})$; further $\Xi \kappa_\phi^{\mathbf{n}} = \Xi Q_f^M \kappa_\phi^{\mathbf{n}} \in \mathcal{E}_M^{\mathbf{n}} \subset \mathcal{E}_M$.

Next we consider the system (15) in $(0, 1) \times \Omega$, and the control $\zeta = \Xi \kappa_\phi^{\mathbf{n}}$:

$$\begin{aligned} \partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v + \nabla p &= 0, & \operatorname{div} v &= 0, \\ v|_\Gamma &= \Xi \kappa_\phi^{\mathbf{n}}, & v(0) &= v_0; \end{aligned}$$

since $((v_0 - E_1 \Xi \kappa_\phi^{\mathbf{n}}) \cdot \mathbf{n}) \mathbf{n}|_\Gamma = (v_0 \cdot \mathbf{n}) \mathbf{n}|_\Gamma - \Xi z^{v_0, \mathbf{n}} = 0$, we have that $v_0 - E_1 \Xi \kappa_\phi^{\mathbf{n}}$ is in H .

By Theorem 2.5 there exists a weak solution satisfying $|v|_{W((0,1), H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))} \leq \overline{C}_{[M, |\hat{u}|_{\mathcal{W}^{(0,1)|\text{wk}}}] \left(|v_0|_{L_{\operatorname{div}}^2(\Omega, \mathbb{R}^3)}^2 + |\kappa_\phi^{\mathbf{n}}|_{H^1((0,1), \mathbb{R}^{2M})}^2 \right)$, from which we can derive

$$(27) \quad |v|_{W((0,1), H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[M, |\hat{u}|_{\mathcal{W}^{(0,1)|\text{wk}}}] |v_0|_{L_{\operatorname{div}}^2(\Omega, \mathbb{R}^3)}^2.$$

Further, $v|_\Gamma$ vanishes in a neighborhood of $t = 1$, which implies that $v_1 := v(1) \in H$. Furthermore, from Lemma 2.8, since $\hat{u}|_{(0,1)} \in \mathcal{W}^{(0,1)|\text{st}}$, we actually have $v(1) \in V \subset H$.

Ⓢ Step 2: *driving the system from $v(n) = v_n \in V$ at time $t = n \in \mathbb{N}_0$ to a vector $v(n+1) = v_{n+1} \in V$ at time $t = n+1$, with $|v_{n+1}|_H^2 \leq e^{-\lambda} |v_n|_H^2$.* Now we consider the system (15) in $(n, n+1) \times \Omega$, and the control $\zeta = K_t^{\mathcal{O}} \bar{\eta}^{\hat{u}, n}(v_n)$, where $\bar{\eta}^{\hat{u}, n}(v_n) = \eta(v_n)$ is given in Theorem 3.9, with $(a, b) = (n, n+1)$ and $v(a) = v_0 = v_n$:

$$\begin{aligned} \partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v + \nabla p &= 0, & \operatorname{div} v &= 0, \\ v|_\Gamma &= K_t^{\mathcal{O}} \bar{\eta}^{\hat{u}, n}(v_n), & v(n) &= v_n. \end{aligned}$$

From Lemma 2.8, and from the continuity of the mapping $v \mapsto v(n+1)$ from $W((n, n+1), H_{\operatorname{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$ into $H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3)$, we have that

$$|v(n+1)|_{H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}^{(n, n+1)|\text{st}}}] \left(|v_n|_{L_{\operatorname{div}}^2(\Omega, \mathbb{R}^3)}^2 + |\bar{\eta}^{\hat{u}, n}|_{G^2((n, n+1), \Gamma)}^2 \right);$$

on the other hand from the definition of $K_t^{\mathcal{O}}$, in (25), we have that $v|_\Gamma$ vanishes in a neighborhood of $t = n+1$ and, since $\bar{\eta}^{\hat{u}, n}$ satisfies (26), we have that $|v(n+1)|_V^2 \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}^{(n, n+1)|\text{st}}}] |v_n|_H^2$. Now we use the fact that $\Pi_N v(n+1) = 0$ to obtain $\alpha_N |v(n+1)|_H^2 \leq |v(n+1)|_V^2$, where α_N is the N th eigenvalue of the Stokes operator (see Section 2.2), which allow us to write $|v(n+1)|_H^2 \leq \alpha_N^{-1} \overline{C}_{[|\hat{u}|_{\mathcal{W}^{(n, n+1)|\text{st}}}] |v_n|_H^2$; then, for big enough N , such that $\alpha_N \geq e^\lambda \overline{C}_{[|\hat{u}|_{\mathcal{W}^{(n, n+1)|\text{st}}}]$, we have that $v_{n+1} := v(n+1)$ satisfies

$$(28) \quad |v_{n+1}|_H^2 \leq e^{-\lambda} |v_n|_H^2.$$

Ⓢ Step 3: *concatenation; a stabilizing control*. First of all, we may fix the functions φ and $\tilde{\varphi}$ (appearing in Theorem 3.9) for the interval $(a, b) = (1, 2)$ and then set $\varphi(t) := \varphi(t - n + 1)$ and $\tilde{\varphi}(t) := \tilde{\varphi}(t - n + 1)$ for $t \in (n, n + 1)$. Since $\hat{u} \in \mathcal{W}^{\text{st}}$ (cf. (6)), the integer N in Step 2 may be taken the same in each interval $(n, n + 1)$, $n \in \mathbb{N}_0$; then, the same holds for the integer M in Theorem 3.9, with $(a, b) = (n, n + 1)$. Now we show that, given $v_0 \in \mathcal{A}_{\Xi_1}$, the control

$$(29) \quad \zeta_{\hat{u}}^\lambda = \zeta_{\hat{u}}^\lambda(v_0) := \begin{cases} \Xi_{\kappa_\phi}^{\mathbf{n}}(v_0), & \text{if } t \in [0, 1]; \\ K_t^\mathcal{O} \tilde{\eta}^{\hat{u}, n}(v_n), & \text{if } t \in [n, n + 1), \text{ with } n \in \mathbb{N}_0; \end{cases}$$

stabilizes system (15) to the zero solution. Here, for $n \in \mathbb{N}_0$, $v_n := v(n)$ where v is the solution of the system (15) in $(0, n) \times \Omega$ with control $\zeta_{\hat{u}}^\lambda|_{(0, n) \times \Gamma}$.

From (27), and the inequality $1 \leq e^\lambda e^{-\lambda t}$, for $t \in [0, 1]$, we obtain

$$(30) \quad |v(t)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \leq \bar{C}_{[\lambda, |\hat{u}|_{\mathcal{W}^{\text{st}}}]}} e^{-\lambda t} |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2, \text{ for all } t \in [0, 1];$$

on the other hand for $t \geq 1$ we also have, $|v(t)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \leq \bar{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}]}} |v(\lfloor t \rfloor)|_H^2$, where $\lfloor t \rfloor$ denotes the biggest integer that is smaller than t , $\lfloor t \rfloor + 1 > t \geq \lfloor t \rfloor \in \mathbb{N}_0$. Thus we obtain that $|v(t)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \leq \bar{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}]}} e^{-\lambda(\lfloor t \rfloor - 1)} |v(1)|_H^2 = \bar{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}]}} e^{\lambda(t - \lfloor t \rfloor)} e^{-\lambda(t - 1)} |v(1)|_H^2 \leq \bar{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}]}} e^\lambda e^{-\lambda(t - 1)} |v(1)|_H^2$. Using (30) (with $t = 1$), we can conclude that

$$(31) \quad |v(t)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \leq \bar{C}_{[\lambda, |\hat{u}|_{\mathcal{W}^{\text{st}}}]}} e^{-\lambda t} |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2, \text{ for all } t \in [0, +\infty).$$

Ⓢ Step 4: *control estimate*. Now, we observe that $\sin(m\pi(t - n)) = (-1)^{mn} \sin(m\pi t)$ and $\{\sqrt{2} \sin(m\pi t) \mid m \in \mathbb{N}_0\} \subset H_0^1((n, n + 1), \mathbb{R})$ is an orthonormal basis in $L^2((n, n + 1), \mathbb{R})$. For time $t \in (n, n + 1)$, the control $K_t^\mathcal{O} \tilde{\eta}^{\hat{u}, n}(v_n)$ can be rewritten as

$$\varphi \chi \mathbb{E}_0^\mathcal{O} \sum_{i=1}^M \sum_{m=1}^M (\eta_{i,m}^{\mathbf{n}} \sigma_m(t) P_{\chi^\perp}^\mathcal{O} \pi_i \mathbf{n} + \eta_{i,m}^{\mathbf{t}} \sigma_m(t) \tau_i)$$

where $\eta_{i,m}^{\mathbf{n}}$ and $\eta_{i,m}^{\mathbf{t}}$ are constants, and $\sigma_m(t) = \sqrt{2} \sin(m\pi t)$. Define the operator $K^{[n]}: \mathcal{M}_{2M \times M} \rightarrow H^1((n, n + 1), H^2(\Omega, \mathbb{R}^3))$, mapping a matrix $A = [A_{j,m}]$, with real entries $A_{j,m} \in \mathbb{R}$ for $j = 1, 2, \dots, 2M$ and $m = 1, 2, \dots, M$, to

$$K^{[n]} A := \varphi \chi \mathbb{E}_0^\mathcal{O} \sum_{i=1}^M \sum_{m=1}^M (A_{i,m} \sigma_m(t) P_{\chi^\perp}^\mathcal{O} \pi_i \mathbf{n} + A_{M+i,m} \sigma_m(t) \tau_i),$$

and consider the space of matrices

$$(32) \quad \mathcal{N}_\times := \left\{ A \in \mathcal{M}_{2M \times M} \mid K^{[n]} A = 0 \right\};$$

we suppose that $\mathcal{M}_{2M \times M} \sim \mathbb{R}^{2M^2}$ is endowed with the scalar product $(A, B)_\mathcal{M} := \sum_{i=1}^M \sum_{m=1}^M (A_{i,m} B_{i,m} + A_{M+i,m} B_{M+i,m})$. It follows that $A \mapsto |P_{\mathcal{N}_\times^\perp} A|_\mathcal{M}$ is a norm in the range $K_t^\mathcal{O} G^2((n, n + 1), \Gamma) = K^{[n]} \mathcal{M}_{2M \times M}$. Since the operator $K_t^\mathcal{O}$ is linear and continuous from $\eta \in G^2((a, b), \Gamma)$ into $G_{\text{av}}^2((a, b), \Gamma)$ (cf. [Rod14a, Proposition 5.1]), from the finite dimensionality of $K_t^\mathcal{O} G^2((n, n + 1), \Gamma)$, it follows that $|P_{\mathcal{N}_\times^\perp} A|_\mathcal{M} \leq C |\eta|_{G^2((n, n + 1), \Gamma)}$, where A is any matrix satisfying $K^{[n]} A = K_t^\mathcal{O} \eta$. Moreover the mapping $\eta \mapsto A^\eta := P_{\mathcal{N}_\times^\perp} A$ from $G^2((n, n + 1), \Gamma)$ into \mathcal{N}_\times^\perp is well defined, that is, A^η is the unique element in \mathcal{N}_\times^\perp that solves $K^{[n]} A^\eta := K_t^\mathcal{O} \eta$.

As a consequence of (26) we have that $|A^{\bar{\eta}^{\hat{u},n}}|_{\mathcal{M}}^2 \leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}}} |v_n|_H^2$. Defining, for each $1 \leq j \leq 2M$, the functions $\bar{\kappa}_j^{\hat{u}}(n, t) := \sum_{m=1}^M A_{j,m}^{\bar{\eta}^{\hat{u},n}} \sigma^m(n, t)$, we find that $\bar{\kappa}^{\hat{u}}(n, \cdot) = \bar{\kappa}^{\hat{u}}(n, \cdot)(v_n) := (\bar{\kappa}_1^{\hat{u}}(n, \cdot), \bar{\kappa}_2^{\hat{u}}(n, \cdot), \dots, \bar{\kappa}_{2M}^{\hat{u}}(n, \cdot))$ is in $C^1([n, n+1], \mathbb{R}^{2M})$; furthermore

$$(33) \quad K_t^{\mathcal{O}} \bar{\eta}^{\hat{u},n} = \Xi \bar{\kappa}^{\hat{u}}(n, \cdot) \quad \text{and} \quad |\bar{\kappa}^{\hat{u}}(n, \cdot)(v_n)|_{C^1([n, n+1], \mathbb{R}^{2M})}^2 \leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}}} |v_n|_H^2.$$

Now, defining the mapping

$$(34) \quad \kappa^{\hat{u},\lambda} = \kappa^{\hat{u},\lambda}(v_0) := \begin{cases} \kappa_{\phi}^{\mathbf{n}}(v_0), & \text{if } t \in [0, 1); \\ \bar{\kappa}^{\hat{u}}(n, \cdot)(v_n), & \text{if } t \in [n, n+1), \text{ with } n \in \mathbb{N}_0; \end{cases}$$

we see that the control in (29) can be rewritten as

$$(35) \quad \zeta_{\hat{u}}^{\lambda} = \Xi \kappa^{\hat{u},\lambda} = \Xi \kappa^{\hat{u},\lambda}(v_0).$$

Notice that for any given positive integer $n \in \mathbb{N}_0$, the control function $\kappa^{\hat{u},\lambda}$ vanishes in a neighborhood of n . Indeed, from Step 1, $\kappa^{\hat{u},\lambda}$ vanishes in $[1 - \delta, 1]$, and from Step 2, it also vanishes in $[n, n + \delta] \cup [n + 1 - \delta, n + 1]$, because $\text{supp}(\varphi|_{[n, n+1]}) \subset \text{supp}(\tilde{\varphi}|_{[n, n+1]}) \subseteq [n + \delta, n + 1 - \delta]$. Let us be given $\hat{\lambda} \in [0, \lambda)$; then the mapping $v_0 \mapsto e^{\hat{\lambda}t} \kappa^{\hat{u},\lambda}(v_0)$ is linear and continuous, from \mathcal{A}_{Ξ_1} into $H^1(\mathbb{R}_0, \mathbb{R}^{2M})$. Indeed, the linearity follows essentially from the linearity of system (15) and the linearity of the mappings $v_0 \mapsto \kappa_{\phi}^{\mathbf{n}}(v_0)$ and $v_n \mapsto \bar{\kappa}^{\hat{u}}(n, \cdot)(v_n)$. To show the boundedness we start by computing

$$\begin{aligned} & \left| e^{\hat{\lambda}t} \kappa^{\hat{u},\lambda} \right|_{H^1(\mathbb{R}_0, \mathbb{R}^{2M})}^2 \\ &= \left| e^{\hat{\lambda}t} \kappa_{\phi}^{\mathbf{n}}(v_0) \right|_{H^1((0,1), \mathbb{R}^{2M})}^2 + \sum_{n \in \mathbb{N}_0} \left| e^{\hat{\lambda}t} \bar{\kappa}^{\hat{u}}(n, \cdot)(v_n) \right|_{H^1((n, n+1), \mathbb{R}^{2M})}^2 \\ &\leq \bar{C}_{[\hat{\lambda}]} \left(\left| \kappa_{\phi}^{\mathbf{n}}(v_0) \right|_{H^1((0,1), \mathbb{R}^{2M})}^2 + \sum_{n \in \mathbb{N}_0} e^{\hat{\lambda}n} \left| \bar{\kappa}^{\hat{u}}(n, \cdot)(v_n) \right|_{H^1((n, n+1), \mathbb{R}^{2M})}^2 \right) \\ &\leq \bar{C}_{[\lambda, |\hat{u}|_{\mathcal{W}^{\text{st}}}] } \left(|v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + \sum_{n \in \mathbb{N}_0} e^{\hat{\lambda}n} |v_n|_H^2 \right). \end{aligned}$$

Now, using (31) (with $t = n$) and the identity $\sum_{n \in \mathbb{N}_0} e^{(\hat{\lambda} - \lambda)n} = \frac{e^{(\hat{\lambda} - \lambda)}}{1 - e^{(\hat{\lambda} - \lambda)}}$, it follows that

$$(36) \quad \left| e^{\hat{\lambda}t} \kappa^{\hat{u},\lambda}(v_0) \right|_{H^1(\mathbb{R}_0, \mathbb{R}^{2M})}^2 \leq \bar{C}_{[(\lambda - \hat{\lambda})^{-1}, \lambda, |\hat{u}|_{\mathcal{W}^{\text{st}}}] } |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2,$$

for any given $0 \leq \hat{\lambda} < \lambda$, which finishes the proof of Theorem 3.2. \square

Corollary 3.10. *The solution $v = v(v_0)$ in Theorem 3.2 satisfies the estimate*

$$(37) \quad \left| e^{\hat{\lambda}t} v \right|_{W(\mathbb{R}_0, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \leq \bar{C}_{[(\lambda - \hat{\lambda})^{-1}, \lambda, |\hat{u}|_{\mathcal{W}^{\text{st}}}] } |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2.$$

Proof. Proceeding as in the proof of Lemma 2.8, we start by noticing that $w := e^{\hat{\lambda}t} v$ solves system (16), in each interval of time $(a, b) \subset \mathbb{R}_0$, with the data $(w(a), g, K\eta) = e^{\hat{\lambda}a} v(a), -\hat{\lambda} e^{\hat{\lambda}t} v, \Xi e^{\hat{\lambda}t} \kappa^{\hat{u},\lambda}$. From Theorem 2.5, we have

$$\begin{aligned} & |w|_{W((n, n+1), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ &\leq \bar{C}_{[|\hat{u}|_{\mathcal{W}^{\text{wk}}}] } \left(e^{\hat{\lambda}n} |v(n)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + \frac{\hat{\lambda}^2}{4} |e^{\hat{\lambda}t} v|_{L^2((n, n+1), H^{-1}(\Omega, \mathbb{R}^3))}^2 \right. \\ &\quad \left. + |e^{\hat{\lambda}t} \kappa^{\hat{u},\lambda}|_{H^1((n, n+1), \mathbb{R}^{2M})}^2 \right); \end{aligned}$$

thus, from (36) and (31), and the continuity of the inclusion $L^2(\Omega, \mathbb{R}^3) \subset H^{-1}(\Omega, \mathbb{R}^3)$, it follows

$$\begin{aligned} & |w|_{W(\mathbb{R}_0, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ & \leq \overline{C}_{[(\lambda-\hat{\lambda})^{-1}, \lambda, |\hat{u}|_{\mathcal{W}^{\text{st}}}] } \left(\sum_{n \in \mathbb{N}} e^{\hat{\lambda}n} e^{-\lambda n} + \int_{\mathbb{R}_0} e^{\hat{\lambda}t} e^{-\lambda t} dt + 1 \right) |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2, \end{aligned}$$

which implies (37), because $\sum_{n \in \mathbb{N}} e^{\hat{\lambda}n} e^{-\lambda n} + \int_{\mathbb{R}_0} e^{\hat{\lambda}t} e^{-\lambda t} dt + 1 = \frac{1}{(1-e^{\hat{\lambda}-\lambda})} + \frac{1}{(\lambda-\hat{\lambda})} + 1$, since $\hat{\lambda} < \lambda$. \square

4. THE OSEEN–STOKES SYSTEM: NORMAL INTEGRAL FEEDBACK CONTROL

In this Section, we show that the finite-dimensional exponentially stabilizing control can be chosen in integral feedback form and normal to the boundary. More precisely we prove the following:

Theorem 4.1. *Given $\hat{u} \in \mathcal{W}^{\text{st}}$ and $\lambda > 0$, let $M = \overline{C}_{[|\hat{u}|_{\mathcal{W}}, \lambda]} \in \mathbb{N}$ be the integer constructed in Theorem 3.2. Then there is a family of continuous operators $\mathcal{K}_{\hat{u}}^{\lambda, s} : \mathcal{A}_{\Xi_1} \rightarrow \mathbb{R}^{2M}$ such that the following properties hold:*

- (i). *The function $s \mapsto \mathcal{K}_{\hat{u}}^{\lambda, s}$, $s \in [0, +\infty)$, is continuous in the weak operator topology, and its operator norm is bounded by $\overline{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} e^{\frac{\lambda}{2}s}$.*
- (ii). *For any $v_0 \in \mathcal{A}_{\Xi_1}$, the solution of the system*

$$(38) \quad \begin{aligned} & \partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v + \nabla p = 0, & \text{div } v &= 0, \\ & v|_{\Gamma} = e^{-\frac{\lambda}{2}t} \left((v_0 \cdot \mathbf{n})\mathbf{n} + \Xi Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} v(s) ds \right), & v(0) &= v_0, \end{aligned}$$

exists, in $\mathbb{R}_0 \times \Omega$, and satisfies the estimate

$$(39) \quad |e^{\frac{\lambda}{2}(t-a)} v|_{W(\mathbb{R}_a, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |v(a)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2.$$

for all $a \geq 0$.

Remark 4.2. Notice that the stabilizing control $\zeta_{\hat{u}}^{\lambda}$, in (35), is normal to the boundary for time $t \in [0, 1]$, but it is not yet known whether it can be taken normal to the boundary for all time $t > 1$.

4.1. Some auxiliary results. In order to make the proof of Theorem 4.1 easier to follow, we gather here some auxiliary results. Once more we recall the orthogonal projection $\Pi : L_{\text{div}}^2(\Omega, \mathbb{R}^3) \rightarrow H$.

Lemma 4.3. *Let v solve system (15), with $\zeta = \Xi \kappa$. If $e^{\frac{\lambda}{2}t} \Pi v \in L^2(\mathbb{R}_0, H)$ and $e^{\frac{\lambda}{2}t} \kappa \in H^1(\mathbb{R}_0, \mathcal{N}^{\perp})$, then $e^{\frac{\lambda}{2}t} v \in W(\mathbb{R}_0, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$, with*

$$(40) \quad \begin{aligned} & |e^{\frac{\lambda}{2}t} v|_{W(\mathbb{R}_0, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ & \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda]} \left(|v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |e^{\frac{\lambda}{2}t} \Pi v|_{L^2(\mathbb{R}_0, H)}^2 + |e^{\frac{\lambda}{2}t} \kappa|_{H^1(\mathbb{R}_0, \mathcal{N}^{\perp})}^2 \right). \end{aligned}$$

Proof. We may write $|e^{\frac{\lambda}{2}t} v|_{W(\mathbb{R}_0, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 = |e^{\frac{\lambda}{2}t} v|_{W((0, 1), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 + |e^{\frac{\lambda}{2}t} v|_{W(\mathbb{R}_1, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2$; from Theorem 2.5 we can derive that

$$(41) \quad \begin{aligned} & |e^{\frac{\lambda}{2}t} v|_{W((0, 1), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \leq C |e^{\frac{\lambda}{2}t}|_{C^1([0, 1], \mathbb{R})}^2 |v|_{W((0, 1), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ & \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda]} \left(|v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |\kappa|_{H^1((0, 1), \mathcal{N}^{\perp})}^2 \right) \end{aligned}$$

and, from Lemma 2.8 we have that for all $t \geq 1$

$$|v(t)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}]} \left(|v(t-1)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |\kappa|_{H^1((t-1, t), \mathcal{N}^\perp)}^2 \right),$$

which allow us to obtain

$$\begin{aligned} |e^{\frac{\lambda}{2}t} v|_{L^2(\mathbb{R}_1, H_{\text{div}}^1(\Omega, \mathbb{R}^3))}^2 &\leq \sum_{n=1}^{+\infty} e^{(n+1)\lambda} |v|_{L^2((n, n+1), H_{\text{div}}^1(\Omega, \mathbb{R}^3))}^2 \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}]} \sum_{n=1}^{+\infty} e^{\lambda(n+1)} \int_n^{n+1} |v(t-1)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |\kappa|_{H^1((t-1, t), \mathcal{N}^\perp)}^2 dt \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}]} \sum_{n=1}^{+\infty} \left(e^{2\lambda} \int_n^{n+1} |e^{\frac{\lambda}{2}(t-1)} v(t-1)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 dt + e^{\lambda(n+1)} |\kappa|_{H^1((n-1, n+1), \mathcal{N}^\perp)}^2 \right) \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}]} e^{2\lambda} \int_0^{+\infty} |e^{\frac{\lambda}{2}t} v(t)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 dt \\ &\quad + \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}]} e^{2\lambda} \sum_{n=1}^{+\infty} \left(|e^{\frac{\lambda}{2}t} \kappa|_{L^2((n-1, n+1), \mathcal{N}^\perp)}^2 + |e^{\frac{\lambda}{2}t} \partial_t \kappa|_{L^2((n-1, n+1), \mathcal{N}^\perp)}^2 \right); \end{aligned}$$

since $e^{\frac{\lambda}{2}t} \partial_t \kappa = \partial_t(e^{\frac{\lambda}{2}t} \kappa) - \frac{\lambda}{2} e^{\frac{\lambda}{2}t} \kappa$, we can derive that

$$\begin{aligned} &|e^{\frac{\lambda}{2}t} v|_{L^2(\mathbb{R}_1, H_{\text{div}}^1(\Omega, \mathbb{R}^3))}^2 \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}}, \lambda} \left(|e^{\frac{\lambda}{2}t} v(t)|_{L^2(\mathbb{R}_0, L_{\text{div}}^2(\Omega, \mathbb{R}^3))}^2 + \sum_{n=1}^{+\infty} |e^{\frac{\lambda}{2}t} \kappa|_{H^1((n-1, n+1), \mathcal{N}^\perp)}^2 \right) \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}}, \lambda} \left(|e^{\frac{\lambda}{2}t} v(t)|_{L^2(\mathbb{R}_0, L_{\text{div}}^2(\Omega, \mathbb{R}^3))}^2 + 2|e^{\frac{\lambda}{2}t} \kappa|_{H^1(\mathbb{R}_0, \mathcal{N}^\perp)}^2 \right). \end{aligned}$$

Thus, from $\partial_t(e^{\frac{\lambda}{2}t} v) = \frac{\lambda}{2} e^{\frac{\lambda}{2}t} v + e^{\frac{\lambda}{2}t} \partial_t v$, since v solves system (15), we can derive that $|\partial_t(e^{\frac{\lambda}{2}t} v)|_{L^2(\mathbb{R}_1, H^{-1}(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}}, \lambda} |e^{\frac{\lambda}{2}t} v|_{L^2(\mathbb{R}_1, H_{\text{div}}^1(\Omega, \mathbb{R}^3))}^2$, which allow us to obtain

$$\begin{aligned} &|e^{\frac{\lambda}{2}t} v|_{W(\mathbb{R}_1, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H_{\text{div}}^{-1}(\Omega, \mathbb{R}^3))}^2 \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}}, \lambda} \left(|e^{\frac{\lambda}{2}t} v(t)|_{L^2(\mathbb{R}_0, L_{\text{div}}^2(\Omega, \mathbb{R}^3))}^2 + |e^{\frac{\lambda}{2}t} \kappa|_{H^1(\mathbb{R}_0, \mathcal{N}^\perp)}^2 \right); \end{aligned}$$

and then, using (41), we obtain

$$\begin{aligned} &|e^{\frac{\lambda}{2}t} v|_{W(\mathbb{R}_0, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}}, \lambda} \left(|v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |e^{\frac{\lambda}{2}t} v(t)|_{L^2(\mathbb{R}_0, L_{\text{div}}^2(\Omega, \mathbb{R}^3))}^2 + |e^{\frac{\lambda}{2}t} \kappa|_{H^1(\mathbb{R}_0, \mathcal{N}^\perp)}^2 \right). \end{aligned}$$

Finally, from $\kappa(t) \in \mathcal{N}^\perp$, we have $Q_f^M \kappa(t) = Q_f^M P_{\mathcal{N}^\perp} \kappa(t) = z^{v \cdot \mathbf{n}(t)}$; from Lemma 3.7, it follows that $|v(t)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \leq C(|\Pi v(t)|_H^2 + |Q_f^M \kappa(t)|_{\mathcal{N}^\perp}^2)$, which allow us to derive the estimate (40). \square

Corollary 4.4. *Let $s \geq 0$ and let v solve system (15), in $\mathbb{R}_s \times \Omega$, with $\zeta = \Xi \kappa$ and $v(s) = v_s$. If $e^{\frac{\lambda}{2}t} \Pi v \in L^2(\mathbb{R}_s, H)$ and $e^{\frac{\lambda}{2}t} \kappa \in H^1(\mathbb{R}_s, \mathcal{N}^\perp)$, then*

$$(42) \quad \begin{aligned} &|e^{\frac{\lambda}{2}t} v|_{W(\mathbb{R}_s, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}}, \lambda} \left(e^{\lambda s} |v_s|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |e^{\frac{\lambda}{2}t} \Pi v|_{L^2(\mathbb{R}_s, H)}^2 + |e^{\frac{\lambda}{2}t} \kappa|_{H^1(\mathbb{R}_s, \mathcal{N}^\perp)}^2 \right). \end{aligned}$$

Proof. Since v solve system (15), in $\mathbb{R}_s \times \Omega$, we have that $w(r) := v(r + s)$ solves system (15), in $\mathbb{R}_0 \times \Omega$, with $w(0) = v_s$, $\zeta = \Xi\kappa_s(r) := \Xi\kappa(r + s)$, and $\hat{u}_s(r) := \hat{u}(r + s)$ in the place of \hat{u} . Since $|\hat{u}_s|_{\mathcal{W}^{\text{st}}} \leq |\hat{u}|_{\mathcal{W}^{\text{st}}}$, by Lemma 4.3, we have that

$$\begin{aligned} & |e^{\frac{\lambda}{2}r} w|_{W(\mathbb{R}_0, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ & \leq \bar{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda]} \left(|v_s|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |e^{\frac{\lambda}{2}r} \Pi w|_{L^2(\mathbb{R}_0, H)}^2 + |e^{\frac{\lambda}{2}r} \kappa_s|_{H^1(\mathbb{R}_0, \mathcal{N}^\perp)}^2 \right), \end{aligned}$$

that is,

$$(43) \quad \begin{aligned} & |e^{\frac{\lambda}{2}(t-s)} v(t)|_{W(\mathbb{R}_s, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ & \leq \bar{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda]} \left(|v_s|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |e^{\frac{\lambda}{2}(t-s)} \Pi v|_{L^2(\mathbb{R}_s, H)}^2 + |e^{\frac{\lambda}{2}(t-s)} \kappa|_{H^1(\mathbb{R}_s, \mathcal{N}^\perp)}^2 \right), \end{aligned}$$

which implies (42). \square

From Theorem 3.2, it makes sense to consider the following problem:

Problem 4.5. *Let us be given $s \geq 0$, $\lambda > 0$, $\hat{u} \in \mathcal{W}^{\text{st}}$, and let $M \in \mathbb{N}$ be given by Theorem 3.2. Then for given $w \in \mathcal{A}_{\Xi_1}$, find the minimum of the functional*

$$M_s^\lambda(v, \kappa) := \left| e^{\frac{\lambda}{2}t} \Pi v \right|_{L^2(\mathbb{R}_s, H)}^2 + \left| e^{\frac{\lambda}{2}t} \kappa \right|_{H^1(\mathbb{R}_s, \mathcal{N}^\perp)}^2$$

on the set of functions

$$(v, \kappa) \in \mathcal{X}_s^{1,1} := \left\{ (v, \kappa) \in \mathcal{Z}_s^{1,1} \mid \begin{array}{l} e^{\frac{\lambda}{2}t}(v, \kappa) \in \mathcal{Z}_s^{1,1}; \text{ and } v \text{ solves (15),} \\ \text{in } \mathbb{R}_s \times \Omega, \text{ with } v(s) \in \mathcal{A}_{\Xi_1} \text{ and } \zeta = \Xi\kappa \end{array} \right\}$$

that satisfy $A(v, \kappa) := v(s) = w$; with $\mathcal{Z}_s^{1,1} := W(\mathbb{R}_s, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \times H^1(\mathbb{R}_s, \mathcal{N}^\perp)$.

Remark 4.6. Of course, it makes also sense to consider cost functionals other than M_s^λ ; however, a suitable choice of the (bounded) cost functional can make the task of finding a feedback rule easier. On the other hand, different choices may lead to feedback controllers with different properties.

Lemma 4.7. *Let us be given $s \geq 0$, $\lambda > 0$, $\hat{u} \in \mathcal{W}^{\text{st}}$, and let $M \in \mathbb{N}$ be given by Theorem 3.2. Then Problem 4.5 has a unique minimizer (v_s^*, κ_s^*) . Moreover, there exists a continuous linear self-adjoint operator $R_{\hat{u}}^{\lambda, s}: \mathcal{A}_{\Xi_1} \rightarrow \mathcal{A}_{\Xi_1}$ such that*

$$(44) \quad (R_{\hat{u}}^{\lambda, s} w, w)_{\mathcal{A}_{\Xi_1}} = M_s^\lambda(v_s^*, \kappa_s^*)$$

$$(45) \quad |R_{\hat{u}}^{\lambda, s}|_{\mathcal{L}(\mathcal{A}_{\Xi_1} \rightarrow \mathcal{A}_{\Xi_1})} \leq \bar{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} e^{\lambda s}.$$

Proof. We suppose $\mathcal{X}_s^{1,1}$ endowed with the norm inherited from $\mathcal{Z}_s^{1,1}$ and we start by observing that the set $A_{w,c} := \{(v, \kappa) \in \mathcal{X}_s^{1,1} \mid Av = w \text{ and } M_s^\lambda(v, \kappa) \leq c\}$ is bounded, for any $(w, c) \in \mathcal{A}_{\Xi_1} \times \mathbb{R}_0$. Indeed, since the initial condition w and initial time $t = s$ are fixed, the boundedness follows from Corollary 4.4. On the other hand, the set $\mathcal{X}_s^{1,1}$ is nonempty for any given $w \in \mathcal{A}_{\Xi_1}$, because we can set $\tilde{\lambda} > \lambda$ and use Theorem 3.2 (with the initial point moved to time $t = s$ and with $(\tilde{\lambda}, \lambda)$ in the role of $(\lambda, \hat{\lambda})$), to guarantee the existence of an element $(v, \kappa) = (v, \kappa^{\hat{u}, \tilde{\lambda}}) \in \mathcal{X}_s^{1,1}$. From Theorem 3.2 we also have that the mapping $A: \mathcal{X}_s^{1,1} \rightarrow \mathcal{A}_{\Xi_1}$ taking (v, κ) to $v(s)$ is surjective. Further, we observe that $M_s^\lambda(v, \kappa)$ induces a scalar product in $\mathcal{Z}_s^{1,1}$:

$$\left((v, \kappa), (u, \eta) \right)_{M_s^\lambda} := \left(e^{\frac{\lambda}{2}t} \Pi v, e^{\frac{\lambda}{2}t} \Pi u \right)_{L^2(\mathbb{R}_s, V)} + \left(e^{\frac{\lambda}{2}t} \kappa, e^{\frac{\lambda}{2}t} \eta \right)_{H^1(\mathbb{R}_s, \mathcal{N}^\perp)}.$$

We can conclude that Problem 4.5 has a unique minimizer $(v_s^*, \kappa_s^*) = (v_s^*, \kappa_s^*)(w)$, which linearly depends on w (cf. [Rod14a, Appendix, Lemma A.14 and Remark A.15]).

Now, setting for example $\tilde{\lambda} = 2\lambda$ above it follows, again from Theorem 3.2, that the mapping

$$(w^1, w^2) \mapsto \left((v_s^*, \kappa_s^*)(w^1), (v_s^*, \kappa_s^*)(w^2) \right)_{M_s^\lambda}$$

is a symmetric continuous bilinear form on \mathcal{A}_{Ξ_1} which is bounded by $\overline{C}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} e^{\lambda s}$ on the unit ball; thus, the optimal cost can be written as (44), where $R_{\hat{u}}^{\lambda, s}$ is a bounded and self-adjoint operator, which norm satisfy (45). \square

Next we consider another minimization problem related to Problem 4.5.

Problem 4.8. *Let us be given $s > s_0 \geq 0$, $\lambda > 0$, $\hat{u} \in \mathcal{W}^{\text{st}}$, and let $M \in \mathbb{N}$ be given by Theorem 3.2. Given $w \in \mathcal{A}_{\Xi_1}$, find the minimum of the functional*

$$N_{s_0, s}^\lambda(v, \kappa) := \left| e^{\frac{\lambda}{2}t} \Pi v \right|_{L^2((s_0, s), H)}^2 + \left| e^{\frac{\lambda}{2}t} \kappa \right|_{H^1((s_0, s), \mathcal{N}^\perp)}^2 + \left(R_{\hat{u}}^{\lambda, s} v(s), v(s) \right)$$

on the set of functions

$$(v, \kappa) \in \widehat{\mathcal{X}} := \left\{ (v, \kappa) \in \mathcal{Z}_{(s_0, s)}^{1,1} \mid \begin{array}{l} v \text{ solves (15), in } (s_0, s) \times \Omega, \\ \text{with } v(s_0) = w \text{ and } \zeta = \Xi \kappa \end{array} \right\}$$

that satisfy $A(v, \kappa) := v(s_0) = w$; where $\mathcal{Z}_{(s_0, s)}^{1,1} := W((s_0, s), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \times H^1((s_0, s), \mathcal{N}^\perp)$.

Proceeding as in the proof of Lemma 4.7, we can derive that Problem 4.8 has a unique minimizer $(v_{s_0, s}^\bullet, \kappa_{s_0, s}^\bullet)(w)$, which is a linear function of $w \in \mathcal{A}_{\Xi_1}$. The following Lemma is the *dynamic programming principle* for Problem 4.5 (with $s = s_0$). The proof is analogous to that of Theorem 3.10 in [BRS11]), and so we skip it.

Lemma 4.9. *The minimizers of Problems 4.5 and 4.8 have the following properties: the restriction of $(v_{s_0, s}^*, \kappa_{s_0, s}^*)(w)$ to the interval (s_0, s) coincides with $(v_{s_0, s}^\bullet, \kappa_{s_0, s}^\bullet)(w)$, and the restriction of $(v_{s_0, s}^*, \kappa_{s_0, s}^*)(w)$ to the half-line \mathbb{R}_s coincides with $(v_s^*, \kappa_s^*)(v_{s_0}^*(s))$.*

4.2. Proof of Theorem 4.1. We organize the proof into five main steps. In Step 1 we use a Lagrange multiplier approach to derive two key optimality conditions for the minimizer $(v_{s_0, s}^\bullet, \kappa_{s_0, s}^\bullet)(w)$ of Problem 4.8; in Step 2, we use those conditions and the dynamic programming principle to find appropriate properties of the minimizer $(v_{s_0, s}^*, \kappa_{s_0, s}^*)(w)$ of Problem 4.5, with $s = s_0$. In Step 3 we give the integral feedback rule, and observe that the optimal control is normal to the boundary. In Step 4 we show the uniqueness of the solution under the feedback controller, prove the bound, for each instant of time, of the feedback operator norm, and prove estimate (39). Finally in Step 5 we prove the continuity of the time-dependent family of feedback operators in the weak operator topology.

Ⓢ Step 1: *Karush–Kuhn–Tucker Theorem.* First we notice that, considering as usual H as a pivot space, we can extend the projection $\Pi: L^2(\Omega, \mathbb{R}^3) \rightarrow H$ to a mapping $\Pi: H^{-1}(\Omega, \mathbb{R}^3) \rightarrow V'$ by simply setting $\langle \Pi f, u \rangle_{V', V} := \langle f, u \rangle_{H^{-1}(\Omega, \mathbb{R}^3), H_0^1(\Omega, \mathbb{R}^3)}$ for all $u \in V$ (cf. beginning of Proof of Theorem 5.3 in [Rod14a]). Then, we define the spaces

$$\begin{aligned} \mathcal{X} &:= W_{\mathcal{A}_{\Xi_1}}((s_0, s), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \times H^1((s_0, s), \mathcal{N}^\perp), \\ \mathcal{Y} &:= \mathcal{A}_{\Xi_1} \times L^2((s_0, s), V') \times G_{\text{av}, \mathcal{A}_{\Xi_1}}^1((s_0, s), \Gamma), \end{aligned}$$

where we denote

$$\begin{aligned} W_{\mathcal{A}_{\Xi_1}}((s_0, s), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \\ := \{u \in W((s_0, s), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \mid u(s_0) \in \mathcal{A}_{\Xi_1}\}, \\ G_{\text{av}, \mathcal{A}_{\Xi_1}}^1((s_0, s), \Gamma) := \{u|_{\Gamma} \mid u \in W_{\mathcal{A}_{\Xi_1}}((s_0, s), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))\}. \end{aligned}$$

Next we define the affine operator $F: \mathcal{X} \rightarrow \mathcal{Y}$, by

$$F(v, \kappa) := (v(s_0) - w, \Pi(\partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v), v|_{\Gamma} - \Xi \kappa).$$

We show now that the derivative dF of F , $dF(v, \kappa) = F(v, \kappa) - (w, 0, 0)$, is surjective. Indeed let us be given $(w_0, g_0, \zeta_0) \in \mathcal{Y}$; from (19) and (23) there are $z^{w_0 \cdot \mathbf{n}} \in \mathbb{R}^{2M}$ and $z^{E_1 \zeta_0(s_0) \cdot \mathbf{n}} \in \mathbb{R}^{2M}$ such that $(w_0 \cdot \mathbf{n})\mathbf{n}|_{\Gamma} = \Xi z^{w_0 \cdot \mathbf{n}}$ and $(E_1 \zeta_0(s_0) \cdot \mathbf{n})\mathbf{n}|_{\Gamma} = \Xi z^{E_1 \zeta_0(s_0) \cdot \mathbf{n}}$ and then we write $\zeta_1 = \zeta_0 - \Xi z^{E_1 \zeta_0(s_0) \cdot \mathbf{n}}$. Taking K in system (16) to be the inclusion from $G_{\text{av}, \mathcal{A}_{\Xi_1}}^1((s_0, s), \Gamma)$ into $G_{\text{av}}^1((s_0, s), \Gamma)$: $\eta \mapsto \zeta = K\eta = \eta$, then by Theorem 2.5, we have that system (16), in $(s_0, s) \times \Omega$, has a weak solution v for the data $(v_0, g, \zeta) = (w_0, g_0, \zeta_1 + \Xi z^{w_0 \cdot \mathbf{n}})$, because $(w_0 - \Xi z^{w_0 \cdot \mathbf{n}} - \zeta_1(s_0)) \cdot \mathbf{n} = 0$, that is, $w_0 - E_1(\zeta_1 + \Xi z^{w_0 \cdot \mathbf{n}})(s_0) \in H$, and because $\zeta_1 + \Xi z^{w_0 \cdot \mathbf{n}} \in G_{\text{av}}^1((s_0, s), \Gamma)$; notice that, recalling (12), both $\Xi z^{w_0 \cdot \mathbf{n}}$ and $\Xi z^{E_1 \zeta_0(s_0) \cdot \mathbf{n}}$ are in $H_{\text{av}}^1(\Gamma, \mathbb{R})\mathbf{n} \subset H_{\text{av}}^{\frac{1}{2}}(\Gamma, \mathbb{R})\mathbf{n}$, and so we can say that both are in $G_{\mathbf{n}, \text{av}}^1((s_0, s), \Gamma)\mathbf{n} \subset G_{\text{av}}^1((s_0, s), \Gamma)$, see Section 2.1. Therefore, if we set $\kappa = z^{w_0 \cdot \mathbf{n}} - z^{E_1 \zeta_0(s_0) \cdot \mathbf{n}}$ we find $dF(v, \kappa) = (v(s_0), g, \zeta_1 + \Xi z^{w_0 \cdot \mathbf{n}} - \Xi \kappa) = (w_0, g_0, \zeta_0)$, and we can conclude that dF is surjective.

We see that the minimizer of the cost $N_{s_0, s}^{\lambda}$, in Problem 4.8, is a minimizer in \mathcal{X} , $(v_{s_0, s}^{\bullet}, \kappa_{s_0, s}^{\bullet}) \in \mathcal{X}$, and satisfies $F(v_{s_0, s}^{\bullet}, \kappa_{s_0, s}^{\bullet}) = 0$. By the Karush–Kuhn–Tucker Theorem (e.g., see [BRS11, Theorem A.1]), there exists a Lagrange multiplier $(\mu_s, q_s, \gamma_s) \in \mathcal{Y}' = \mathcal{A}_{\Xi_1} \times L^2((s_0, s), V) \times G_{\text{av}, \mathcal{A}_{\Xi_1}}^1((s_0, s), \Gamma)'$ such that

$$dN_{s_0, s}^{\lambda}(v_{s_0, s}^{\bullet}, \kappa_{s_0, s}^{\bullet}) + (\mu_s, q_s, \gamma_s) \circ dF(v_{s_0, s}^{\bullet}, \kappa_{s_0, s}^{\bullet}) = 0.$$

Hence, for all $z \in W_{\mathcal{A}_{\Xi_1}}((s_0, s), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ and $\xi \in H^1((s_0, s), \mathcal{N}^{\perp})$

$$(46) \quad 0 = 2 \int_{s_0}^s e^{\lambda t} (\Pi v_{s_0, s}^{\bullet}, \Pi z)_H dt + 2 \left(R_{\hat{u}}^{\lambda, s} v_{s_0, s}^{\bullet}(s), z(s) \right)_{\mathcal{A}_{\Xi_1}} + (z(s_0), \mu_s)_{\mathcal{A}_{\Xi_1}} \\ + \int_{s_0}^s \langle \Pi(\partial_t z + \mathcal{B}(\hat{u})z - \nu \Delta z), q_s \rangle_{V', V} dt + \langle \gamma_s, z|_{\Gamma} \rangle_{(G_{\Xi}^{1, s})', G_{\Xi}^{1, s}},$$

$$(47) \quad 0 = 2 \left(e^{\frac{\lambda}{2} t} \kappa_{s_0, s}^{\bullet}, e^{\frac{\lambda}{2} t} \xi \right)_{H^1((s_0, s), \mathcal{N}^{\perp})} + \langle \gamma_s, -\Xi \xi \rangle_{(G_{\Xi}^{1, s})', G_{\Xi}^{1, s}}$$

where for simplicity we denote $G_{\Xi}^{1, s} := G_{\text{av}, \mathcal{A}_{\Xi_1}}^1((s_0, s), \Gamma)$.

Ⓢ Step 2: *properties of the optimal pair.* Letting z run over all $z \in W((s_0, s), V, V')$, with $z(s_0) = z(s) = 0$, from (46) and from $H^{-1}(\Omega, \mathbb{R}^3) = V' \oplus \{\nabla p \mid p \in L^2(\Omega, \mathbb{R})\}$ (see, e.g., [Tem01, Chapter 1, Section 1.4]), we can see that for some $p_{q_s} \in L^2((s_0, s), L^2(\Omega, \mathbb{R}))$,

$$(48) \quad -\partial_t q_s - \nu \Delta q_s + \mathcal{B}^*(\hat{u})q_s + \nabla p_{q_s} + 2e^{\lambda t} \Pi v_{s_0, s}^{\bullet}(t) = 0$$

where $\mathcal{B}^*(\hat{u})$ is the formal adjoint to $\mathcal{B}(\hat{u})$: defined for $q \in V$ and $v \in H_{\text{div}}^1(\Omega, \mathbb{R}^3)$ by $(v, \mathcal{B}^*(\hat{u})q)_{L^2(\Omega, \mathbb{R}^3)} := \langle \mathcal{B}(\hat{u})v, q \rangle_{H^{-1}(\Omega, \mathbb{R}^3), H_0^1(\Omega, \mathbb{R}^3)}$.

From $\Pi v_{s_0, s}^{\bullet} \in L^2((s_0, s), H)$ and $q_s \in L^2((s_0, s), V)$, it follows $\partial_t q_s \in L^2((s_0, s), V')$, in particular $q_s \in C([s_0, s], H)$ (cf. [LM72, Chapter 1, Theorem 3.1]). Using again (46), with arbitrary $z \in W((s_0, s), V, V')$, we derive that $q_s(s_0) = \Pi \mu_s$ and

$$(49) \quad q_s(s) = -2\Pi R_{\hat{u}}^{\lambda, s} v_{s_0, s}^{\bullet}(s) \in H.$$

Now we observe that since $\Pi v_{s_0, s}^\bullet \in L^2((s_0, s), H)$, by standard arguments we can prove that $q_s \in W((s_0, s - 2\epsilon), D(L), H)$ for any $0 < \epsilon < \frac{s-s_0}{2}$; taking, again in (46), arbitrary $z \in W_{\mathcal{A}_{\Xi_1}}((s_0, s), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ supported in $[s_0 + \epsilon, s - \epsilon]$, then

$$(50) \quad 0 = (\langle \mathbf{n} \cdot \nabla \rangle q_s - p_{q_s} \mathbf{n}, z|_\Gamma)_{L^2((s_0, s), L^2(\Gamma, \mathbb{R}^3))} + \langle \gamma_s, z|_\Gamma \rangle_{(G_{\Xi}^{1, s})', G_{\Xi}^{1, s}}.$$

On the other hand, relation (47) implies that for any $\xi \in H^1((s_0, s), \mathcal{N}^\perp)$ supported in $[s_0 + \epsilon, s - \epsilon]$, we have

$$(51) \quad 0 = 2 \left\langle (1 - \Delta_t)(e^{\frac{\lambda}{2}t} \kappa_{s_0, s}^\bullet), e^{\frac{\lambda}{2}t} \xi \right\rangle_{H^{-1}((s_0, s), \mathcal{N}^\perp), H_0^1((s_0, s), \mathcal{N}^\perp)} + \langle \gamma_s, -\Xi \xi \rangle_{(G_{\Xi}^{1, s})', G_{\Xi}^{1, s}},$$

with $\Delta_t := \partial_t \partial_t$, which shows us that γ_s is independent of s in $[s_0 + \epsilon, s - \epsilon]$, because by Lemma 4.9 we have that $(v_{s_0, s}^\bullet, \kappa_{s_0, s}^\bullet)(w) = (v_{s_0}^*, \kappa_{s_0}^*)(w)|_{(s_0, s)}$. Thus, since the space $\{f \in G_{\Xi}^{1, s} \mid \text{supp}(f) \subset (s_0, s)\}$ is dense in $L^2((s_0, s), L^2(\Gamma, \mathbb{R}))$, from (50) we can derive that $\langle \mathbf{n} \cdot \nabla \rangle q_s - p_{q_s} \mathbf{n} = \gamma_s$ is independent of s .

Suppose now that we are given another s_1 with $s_0 < s < s_1$. Then, the difference $(\bar{q}, p_{\bar{q}}) := (q_s - q_{s_1}, p_{q_s} - p_{q_{s_1}})$ satisfies

$$-\partial_t \bar{q} - \nu \Delta \bar{q} + \mathcal{B}^*(\hat{u}) \bar{q} + \nabla p_{\bar{q}} = 0$$

in $(s_0, s) \times \Omega$; from an observability inequality we can find in [Rod14a, Inequality (3.4)], we obtain

$$|\bar{q}(s - 2\epsilon)|_H^2 \leq C |p_{\bar{q}} \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle \bar{q}|_{G_{\text{av}}^1((s-2\epsilon, s), \Gamma)'}^2 = 0 \text{ for all } \epsilon < \frac{s-s_0}{2}.$$

Since $\bar{q} \in C([s_0, s], H)$, it follows that $\bar{q} = 0$ in $[s_0, s]$. In particular, we can conclude that

$$(52) \quad -2\Pi R_{\hat{u}}^{\lambda, s} v_{s_0, s}^\bullet(s) = q_s(s) = q_{s_1}(s) \in V, \text{ for all } s \in [s_0, s_1]$$

which, in turn, implies that $q_s \in W((s_0, s), D(L), H)$.

Now from (50) and (51), taking z with $z|_\Gamma = \Xi \xi$, it follows that

$$(53) \quad 0 = \left\langle 2e^{\frac{\lambda}{2}t}(1 - \Delta_t)(e^{\frac{\lambda}{2}t} \kappa_{s_0, s}^\bullet) - \Xi^\diamond(p_{q_s} \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q_s), \xi \right\rangle_{H^{-1}((s_0, s), \mathcal{N}^\perp), H_0^1((s_0, s), \mathcal{N}^\perp)},$$

where $\Xi^\diamond: L^2(\Gamma, \mathbb{R}^3) \rightarrow \mathcal{N}^\perp$ is defined by:

$$(54) \quad (\Xi^\diamond \gamma, k)_{\mathcal{N}^\perp} := (\gamma, \Xi k)_{L^2(\Gamma, \mathbb{R}^3)}, \text{ for all } k \in \mathcal{N}^\perp.$$

We show that Ξ^\diamond is well defined and continuous: notice that

$$\begin{aligned} (\gamma, \Xi \kappa)_{L^2(\Gamma, \mathbb{R}^3)} &= \left(P_{\chi^\perp}^\mathcal{O}(\chi \gamma|_\mathcal{O}), \sum_{i=1}^M (k_i \pi_i \mathbf{n} + k_{M+i} \tau_i) \right)_{L^2(\mathcal{O}, \mathbb{R}^3)} \\ &= \left(P_M^\mathcal{O} P_{\chi^\perp}^\mathcal{O}(\chi \gamma|_\mathcal{O}), \sum_{i=1}^M (k_i \pi_i \mathbf{n} + k_{M+i} \tau_i) \right)_{L^2(\mathcal{O}, \mathbb{R}^3)}, \end{aligned}$$

and observe that if $g \in \mathbb{R}^{2M}$ is such that

$$(55) \quad P_M^\mathcal{O} P_{\chi^\perp}^\mathcal{O}(\chi \gamma|_\mathcal{O}) = \sum_{i=1}^M (g_i \pi_i \mathbf{n} + g_{M+i} \tau_i),$$

then we find that $(\gamma, \Xi \kappa)_{L^2(\Gamma, \mathbb{R}^3)} = (g, k)_{\mathbb{R}^{2M}} = (P_{\mathcal{N}^\perp} g, k)_{\mathcal{N}^\perp}$; recall that $k \in \mathcal{N}^\perp$. Therefore we can set $\Xi^\diamond \gamma = P_{\mathcal{N}^\perp} g$, and conclude that $\Xi^\diamond \gamma$ is well defined; notice that $P_{\mathcal{N}^\perp} g \in \mathcal{N}^\perp$ and if $w \in \mathcal{N}^\perp$ and $(w, k)_{\mathcal{N}^\perp} = (\gamma, \Xi k)_{L^2(\Gamma, \mathbb{R}^3)}$ for all $k \in \mathcal{N}^\perp$, then $(w - P_{\mathcal{N}^\perp} g, k)_{\mathcal{N}^\perp} = 0$, for all $k \in \mathcal{N}^\perp$; which implies $w - P_{\mathcal{N}^\perp} g \in \mathcal{N} \cap \mathcal{N}^\perp = \{0\}$. Further, we

observe that Ξ^\diamond is continuous, because both $|P_{\mathcal{N}^\perp}g|_{\mathbb{R}^{2M}}$ and $\left|\sum_{i=1}^M(g_i\pi_i\mathbf{n} + g_{M+i}\tau_i)\right|_{L^2(\mathcal{O}, \mathbb{R}^3)}$ are norms in the finite-dimensional space $\text{span}\{\pi_i\mathbf{n}, \tau_i \mid 1 \leq i \leq M\}$, which implies that they are equivalent and $|P_{\mathcal{N}^\perp}g|_{\mathbb{R}^{2M}} \leq \bar{C}_{[M]} \left|\sum_{i=1}^M(g_i\pi_i\mathbf{n} + g_{M+i}\tau_i)\right|_{L^2(\mathcal{O}, \mathbb{R}^3)}$; then from (55) we can conclude that $|\Xi^\diamond\gamma|_{\mathcal{N}^\perp} = |P_{\mathcal{N}^\perp}g|_{\mathbb{R}^{2M}} \leq \bar{C}_{[M]}|\gamma|_{L^2(\Gamma, \mathbb{R}^3)}$.

Now, the function $\eta := e^{\frac{\lambda}{2}t}\kappa_{s_0, s}^\bullet(t) + \frac{1}{s-s_0} \left((t-s)e^{\frac{\lambda}{2}s_0}\kappa_{s_0, s}^\bullet(s_0) - (t-s_0)e^{\frac{\lambda}{2}s}\kappa_{s_0, s}^\bullet(s) \right)$ is in the space $H_0^1((s_0, s), \mathcal{N}^\perp)$ and, using (53), solves the Poisson equation

$$(56) \quad (1 - \Delta_t)\eta = 2^{-1}e^{-\frac{\lambda}{2}t}\Xi^\diamond(p_{q_s}(t)\mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q_s(t)) \\ + \frac{1}{s-s_0} \left((t-s)e^{\frac{\lambda}{2}s_0}\kappa_{s_0, s}^\bullet(s_0) - (t-s_0)e^{\frac{\lambda}{2}s}\kappa_{s_0, s}^\bullet(s) \right)$$

since the right hand side is in $L^2((s_0, s), \mathcal{N}^\perp)$, by standard arguments it follows that $\eta \in H_0^1((s_0, s), \mathcal{N}^\perp) \cap H^2((s_0, s), \mathcal{N}^\perp)$, which implies that $e^{\frac{\lambda}{2}t}\kappa_s^\bullet(t) \in H^2((s_0, s), \mathcal{N}^\perp)$. Whence, taking an arbitrary $\xi \in H^1((s_0, s), \mathcal{N}^\perp)$ in (47), we can now write

$$(57) \quad 0 = 2(\partial_t|_{t=s}(e^{\frac{\lambda}{2}t}\kappa_{s_0, s}^\bullet), e^{\frac{\lambda}{2}s}\xi(s))_{\mathcal{N}^\perp} - 2(\partial_t|_{t=s_0}(e^{\frac{\lambda}{2}t}\kappa_{s_0, s}^\bullet), e^{\frac{\lambda}{2}s_0}\xi(s_0))_{\mathcal{N}^\perp} \\ + 2 \left((1 - \Delta_t)(e^{\frac{\lambda}{2}t}\kappa_{s_0, s}^\bullet), e^{\frac{\lambda}{2}t}\xi \right)_{L^2((s_0, s), \mathcal{N}^\perp)} + \langle \gamma_s, -\Xi\xi \rangle_{(G_\Xi^{1, s})', G_\Xi^{1, s}},$$

and taking arbitrary $z \in W_{\mathcal{A}_{\Xi_1}}((s_0, s), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ in (46), we find

$$0 = (\langle \mathbf{n} \cdot \nabla \rangle q_s - p_{q_s}\mathbf{n}, z|_\Gamma)_{L^2((s_0, s), L^2(\Gamma, \mathbb{R}^3))} + \langle \gamma_s, z|_\Gamma \rangle_{(G_\Xi^{1, s})', G_\Xi^{1, s}} \\ + ((1 - \Pi)\mu_s, z(s_0))_{\mathcal{A}_{\Xi_1}} + 2((1 - \Pi)R_u^{\lambda, s}v_{s_0, s}^\bullet(s), z(s))_{\mathcal{A}_{\Xi_1}}.$$

Recall that, for given $u \in L_{\text{div}}^2(\Omega, \mathbb{R}^3)$ and $v \in L_{\text{div}}^2(\Omega, \mathbb{R}^3)$, writing $u = \Pi u + \nabla P_\nabla u$, where $P_\nabla u \in H^1(\Omega, \mathbb{R})$ is defined as in (21), then the identity $((1 - \Pi)u, v)_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)} = \langle v \cdot \mathbf{n}, P_\nabla u \rangle_{H^{-\frac{1}{2}}(\Gamma, \mathbb{R}), H^{\frac{1}{2}}(\Gamma, \mathbb{R})}$ is well defined. In particular we see that $(1 - \Pi)u$ is supported on the boundary Γ , in the sense that the value $((1 - \Pi)u, v)_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}$ at v , depends only on $(v \cdot \mathbf{n})|_\Gamma$. We can write

$$0 = (\langle \mathbf{n} \cdot \nabla \rangle q_s - p_{q_s}\mathbf{n}, z|_\Gamma)_{L^2((s_0, s), L^2(\Gamma, \mathbb{R}^3))} + \langle \gamma_s, z|_\Gamma \rangle_{(G_\Xi^{1, s})', G_\Xi^{1, s}} \\ + \langle (z \cdot \mathbf{n})(s_0), P_\nabla \mu_s \rangle_{H^{-\frac{1}{2}}(\Gamma, \mathbb{R}), H^{\frac{1}{2}}(\Gamma, \mathbb{R})} + \left\langle (z \cdot \mathbf{n})(s), P_\nabla 2R_u^{\lambda, s}v_{s_0, s}^\bullet(s) \right\rangle_{H^{-\frac{1}{2}}(\Gamma, \mathbb{R}), H^{\frac{1}{2}}(\Gamma, \mathbb{R})}.$$

Therefore, using (57), and setting (any) $z \in W_{\mathcal{A}_{\Xi_1}}((s_0, s), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ with $z|_\Gamma = \Xi\xi$ we find that

$$0 = (\langle \mathbf{n} \cdot \nabla \rangle q_s - p_{q_s}\mathbf{n}, z|_\Gamma)_{L^2((s_0, s), L^2(\Gamma, \mathbb{R}^3))} \\ + \int_\Gamma P_\nabla \mu_s(\Xi\xi \cdot \mathbf{n})(s_0) \, d\Gamma + \int_\Gamma P_\nabla 2R_u^{\lambda, s}v_{s_0, s}^\bullet(s)(\Xi\xi \cdot \mathbf{n})(s) \, d\Gamma \\ + 2(\partial_t|_{t=s}(e^{\frac{\lambda}{2}t}\kappa_{s_0, s}^\bullet), e^{\frac{\lambda}{2}s}\xi(s))_{\mathcal{N}^\perp} - 2(\partial_t|_{t=s_0}(e^{\frac{\lambda}{2}t}\kappa_{s_0, s}^\bullet), e^{\frac{\lambda}{2}s_0}\xi(s_0))_{\mathcal{N}^\perp} \\ + 2 \left((1 - \Delta_t)(e^{\frac{\lambda}{2}t}\kappa_{s_0, s}^\bullet), e^{\frac{\lambda}{2}t}\xi \right)_{L^2((s_0, s), \mathcal{N}^\perp)},$$

and from (53), we arrive to

$$\begin{aligned}
0 &= \int_{\Gamma} P_{\nabla} \mu_s (\Xi \xi \cdot \mathbf{n})(s_0) \, d\Gamma + \int_{\Gamma} P_{\nabla} 2R_{\hat{u}}^{\lambda, s} v_{s_0, s}^{\bullet}(s) (\Xi \xi \cdot \mathbf{n})(s) \, d\Gamma \\
&\quad + 2(\partial_t|_{t=s} (e^{\frac{\lambda}{2}t} \kappa_{s_0, s}^{\bullet}), e^{\frac{\lambda}{2}s} \xi(s))_{\mathcal{N}^{\perp}} - 2(\partial_t|_{t=s_0} (e^{\frac{\lambda}{2}t} \kappa_{s_0, s}^{\bullet}), e^{\frac{\lambda}{2}s_0} \xi(s_0))_{\mathcal{N}^{\perp}} \\
&= (P_{\nabla} \mu_s \mathbf{n}, \Xi Q_f^M \xi(s_0))_{L^2((s_0, s), L^2(\Gamma, \mathbb{R}^3))} + (P_{\nabla} 2R_{\hat{u}}^{\lambda, s} v_{s_0, s}^{\bullet}(s) \mathbf{n}, \Xi Q_f^M \xi(s))_{L^2((s_0, s), L^2(\Gamma, \mathbb{R}^3))} \\
&\quad + 2(\partial_t|_{t=s} (e^{\frac{\lambda}{2}t} \kappa_{s_0, s}^{\bullet}), e^{\frac{\lambda}{2}s} \xi(s))_{\mathcal{N}^{\perp}} - 2(\partial_t|_{t=s_0} (e^{\frac{\lambda}{2}t} \kappa_{s_0, s}^{\bullet}), e^{\frac{\lambda}{2}s_0} \xi(s_0))_{\mathcal{N}^{\perp}},
\end{aligned}$$

that is,

$$\begin{aligned}
0 &= (Q_f^M \Xi^{\diamond} (P_{\nabla} \mu_s \mathbf{n}), \xi(s_0))_{\mathcal{N}^{\perp}} + (2Q_f^M \Xi^{\diamond} (P_{\nabla} R_{\hat{u}}^{\lambda, s} v_{s_0, s}^{\bullet}(s) \mathbf{n}), \xi(s))_{\mathcal{N}^{\perp}} \\
&\quad + 2(\partial_t|_{t=s} (e^{\frac{\lambda}{2}t} \kappa_{s_0, s}^{\bullet}), e^{\frac{\lambda}{2}s} \xi(s))_{\mathcal{N}^{\perp}} - 2(\partial_t|_{t=s_0} (e^{\frac{\lambda}{2}t} \kappa_{s_0, s}^{\bullet}), e^{\frac{\lambda}{2}s_0} \xi(s_0))_{\mathcal{N}^{\perp}},
\end{aligned}$$

from which we can derive, recalling Lemma 4.9 and setting ξ such that $\xi(s_0) = 0$,

$$(58) \quad \partial_t|_{t=s} (e^{\frac{\lambda}{2}t} \kappa_{s_0}^*(t)) = -e^{-\frac{\lambda}{2}s} Q_f^M \Xi^{\diamond} (P_{\nabla} R_{\hat{u}}^{\lambda, s} v_{s_0}^*(s) \mathbf{n}), \text{ for all } s > s_0.$$

Notice that since $\bar{\gamma} := P_{\nabla} R_{\hat{u}}^{\lambda, s} v_{s_0}^*(s) \mathbf{n}$ is normal to the boundary, from (54), we necessarily have $(\Xi^{\diamond} \bar{\gamma}, k)_{\mathcal{N}^{\perp}} = (\bar{\gamma}, \Xi k)_{L^2(\Gamma, \mathbb{R}^3)} = (\bar{\gamma}, \Xi Q_f^M k)_{L^2(\Gamma, \mathbb{R}^3)} = (Q_f^M \Xi^{\diamond} \bar{\gamma}, k)_{L^2(\Gamma, \mathbb{R}^3)}$, for all $k \in \mathcal{N}^{\perp}$, that is, $\Xi^{\diamond} (P_{\nabla} R_{\hat{u}}^{\lambda, s} v_{s_0}^*(s) \mathbf{n}) = Q_f^M \Xi^{\diamond} (P_{\nabla} R_{\hat{u}}^{\lambda, s} v_{s_0}^*(s) \mathbf{n})$. Therefore, we obtain the following integral rule

$$(59) \quad \kappa_{s_0}^*(t) = e^{-\frac{\lambda}{2}t} \left(e^{\frac{\lambda}{2}s_0} \kappa_{s_0}^*(s_0) - \int_{s_0}^t e^{-\frac{\lambda}{2}s} \Xi^{\diamond} (P_{\nabla} R_{\hat{u}}^{\lambda, s} v_{s_0}^*(s) \mathbf{n}) \, ds \right).$$

⑤ *Step 3: normality of the control and the integral feedback rule.* For all $z \in \mathbb{R}^{2M}$ we have that $\Xi Q_f^M z = (\Xi z \cdot \mathbf{n}) \mathbf{n}$; on the other hand, writing $\kappa_{s_0}^*(s_0) = Q_f^M \kappa_{s_0}^*(s_0) + Q_l^M \kappa_{s_0}^*(s_0)$, we can write (59) as

$$\kappa_{s_0}^*(t) = e^{-\frac{\lambda}{2}(t-s_0)} Q_l^M \kappa_{s_0}^*(s_0) + e^{-\frac{\lambda}{2}t} Q_f^M \left(e^{\frac{\lambda}{2}s_0} \kappa_{s_0}^*(s_0) - \int_{s_0}^t e^{-\frac{\lambda}{2}s} \Xi^{\diamond} (P_{\nabla} R_{\hat{u}}^{\lambda, s} v_{s_0}^*(s) \mathbf{n}) \, ds \right)$$

and, since by Lemma 4.7, we know that $M_{s_0}^{\lambda} (v_{s_0}^*, \kappa_{s_0}^*)(w) \leq e^{\lambda s_0} \bar{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |w|_{\mathcal{A}_{\Xi_1}}^2$, in particular we have $|e^{\frac{\lambda}{2}t} Q_l^M \kappa_{s_0}^*(t)|_{H^1(\mathbb{R}_{s_0}, \mathcal{N}^{\perp})}^2 \leq e^{\lambda s_0} \bar{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |w|_{\mathcal{A}_{\Xi_1}}^2$; now, we have that $e^{\frac{\lambda}{2}t} Q_l^M \kappa_{s_0}^*(t) = e^{\frac{\lambda}{2}s_0} Q_l^M \kappa_{s_0}^*(s_0)$ remains constant, which implies that $Q_l^M \kappa_{s_0}^*(s_0)$ must vanish. Hence $\kappa_{s_0}^* = Q_f^M \kappa_{s_0}^*$, and we can write $\kappa_{s_0}^*(s_0) = z^{v_{s_0}^*(s_0) \cdot \mathbf{n}}$ (see (23)). It follows that (59) can be rewritten as

$$(60) \quad \kappa_{s_0}^*(t) = e^{-\frac{\lambda}{2}t} \left(e^{\frac{\lambda}{2}s_0} z^{v_{s_0}^*(s_0) \cdot \mathbf{n}} - \int_{s_0}^t e^{-\frac{\lambda}{2}s} \Xi^{\diamond} (P_{\nabla} R_{\hat{u}}^{\lambda, s} v_{s_0}^*(s) \mathbf{n}) \, ds \right).$$

Taking $s_0 = 0$, the optimal control is given by the following integral feedback rule

$$(61) \quad \begin{aligned} v_0^*|_{\Gamma}(t) &= \Xi \kappa_0^*(t) = \Xi Q_f^M e^{-\frac{\lambda}{2}t} \left(z^{v_0^*(0) \cdot \mathbf{n}} + \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} v_0^*(s) \, ds \right) \\ &= e^{-\frac{\lambda}{2}t} \left((v_0^*(0) \cdot \mathbf{n}) \mathbf{n} + \Xi \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} v_0^*(s) \, ds \right), \quad t \geq 0, \end{aligned}$$

with $\mathcal{K}_{\hat{u}}^{\lambda, s} v_0^*(s) = e^{-\frac{\lambda}{2}s} \Xi^{\diamond} (P_{\nabla} R_{\hat{u}}^{\lambda, s} v_0^*(s) \mathbf{n})$, for $s \in \mathbb{R}_0$. In particular, the control $v_0^*|_{\Gamma}$ is in the space $\mathcal{E}_{M}^{\mathbf{n}}$, defined in (13), and so it is normal to the boundary Γ .

Ⓢ Step 4: *uniqueness, operator norm, and exponential decay estimates.* We know that given $v_0 \in \mathcal{A}_{\Xi_1}$ there exists a solution for system (38), this solution is $v_0^*(v_0)$, where $(v_0^*, \kappa_0^*)(v_0)$ is the minimizer of Problem 4.5 taking $(s, w) = (0, v_0)$. This solution is unique: if z is another solution, then $d = v_0^*(v_0) - z$ solves

$$\begin{aligned} \partial_t d + \mathcal{B}(\hat{u})d - \nu \Delta d + \nabla p_d &= 0, & \operatorname{div} d &= 0, \\ d|_{\Gamma} = \Xi Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} d(s) ds, & & d(0) &= 0 \end{aligned}$$

and, from the continuity of the mappings Ξ^\diamond and $u \mapsto P_{\nabla} u$, and from (45), we can find that for all $0 < r \leq 1$,

$$(62) \quad |\mathcal{K}_{\hat{u}}^{\lambda, r} d(r)|_{\mathbb{R}^{2M}}^2 \leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}} e^{\lambda r} |d(r)|_{L_{\operatorname{div}}^2(\Omega, \mathbb{R}^3)}^2$$

and, from Theorem 2.5 and $d(0) = 0$,

$$(63) \quad |\mathcal{K}_{\hat{u}}^{\lambda, r} d(r)|_{\mathbb{R}^{2M}}^2 \leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}} e^{\lambda r} \left| \int_0^r \mathcal{K}_{\hat{u}}^{\lambda, s} d(s) ds \right|_{H^1((0, r), \mathbb{R}^{2M})}^2;$$

note that $|\hat{u}|_{\mathcal{W}^{(0, r)}|_{\text{wk}}} \leq |\hat{u}|_{\mathcal{W}^{\text{st}}}$. It follows that

$$\begin{aligned} (64) \quad |\mathcal{K}_{\hat{u}}^{\lambda, r} d(r)|_{\mathbb{R}^{2M}}^2 &\leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}} e^{\lambda r} \int_0^r \left| \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} d(s) ds \right|_{\mathbb{R}^{2M}}^2 + \left| \mathcal{K}_{\hat{u}}^{\lambda, t} d(t) \right|_{\mathbb{R}^{2M}}^2 dt; \\ &\leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}} e^{\lambda r} \int_0^r \left(\int_0^t \left| \mathcal{K}_{\hat{u}}^{\lambda, s} d(s) \right|_{\mathbb{R}^{2M}}^2 ds \int_0^t 1 ds \right) + \left| \mathcal{K}_{\hat{u}}^{\lambda, t} d(t) \right|_{\mathbb{R}^{2M}}^2 dt; \\ &\leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}} e^{\lambda r} \left(\frac{r^2}{2} + 1 \right) \int_0^r \left| \mathcal{K}_{\hat{u}}^{\lambda, t} d(t) \right|_{\mathbb{R}^{2M}}^2 dt; \end{aligned}$$

thus, $\iota(r) := \int_0^r |\mathcal{K}_{\hat{u}}^{\lambda, t} d(t)|_{\mathbb{R}^{2M}}^2 dt$ satisfies $\frac{d\iota}{dr}(r) \leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}} e^{\lambda r} \left(\frac{r^2}{2} + 1 \right) \iota(r)$. By the Gronwall Lemma, we have $\iota(r) \leq \iota(0) \exp\left(\bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}} \int_0^r \frac{\rho^2 + 2}{2} e^{\lambda \rho} d\rho\right)$; since $\iota(0) = 0$, $\mathcal{K}_{\hat{u}}^{\lambda, t} d(t)$ vanishes in $[0, 1]$, which implies that d solves the Oseen–Stokes system under homogeneous Dirichlet boundary conditions, in $(0, 1) \times \Omega$; by standard arguments, since $d(0) = 0$, we can conclude that $d = 0$ in $[0, 1]$. Of course, just repeating the argument we can conclude that d vanishes in $[n, n+1]$ if $d(n) = 0$, for all $n \in \mathbb{N}$; notice that $d|_{\Gamma}(t) = \Xi \int_n^t \mathcal{K}_{\hat{u}}^{\lambda, s} d(s) ds$ for all $t \geq n$, if $d(n) = 0$. Therefore, d vanishes in \mathbb{R}_0 .

The uniqueness of the solution of system (38), implies that we need to prove the estimate (39), for the optimal trajectory $v_0^*(v_0)$ solving Problem 4.5 (taking $(s, w) = (0, v_0)$). In this case, from (44) and for any $a \geq 0$, we already know that $M_a^\lambda(v_0^*|_{\mathbb{R}_a}, \kappa_0^*|_{\mathbb{R}_a}) = (R_{\hat{u}}^{\lambda, a} v_0^*(a), v_0^*(a))_{\mathcal{A}_{\Xi_1}} \leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}} e^{\lambda a} |v_0^*(a)|_{\mathcal{A}_{\Xi_1}}^2$. Hence from (43),

$$\begin{aligned} &|e^{\frac{\lambda}{2}(t-a)} v_0^*(t)|_{W(\mathbb{R}_a, H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ &\leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda}} \left(|v_0^*(a)|_{L_{\operatorname{div}}^2(\Omega, \mathbb{R}^3)}^2 + |e^{\frac{\lambda}{2}(t-a)} \Pi v_0^*|_{L^2(\mathbb{R}_s, H)}^2 + |e^{\frac{\lambda}{2}(t-a)} \kappa_0^*|_{H^1(\mathbb{R}_s, \mathcal{N}^\perp)}^2 \right) \\ &\leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda}} \left(|v_0^*(a)|_{L_{\operatorname{div}}^2(\Omega, \mathbb{R}^3)}^2 + e^{-\lambda a} M_a^\lambda(v_0^*|_{\mathbb{R}_a}, \kappa_0^*|_{\mathbb{R}_a}) \right) \leq \bar{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}} |v_0^*(a)|_{\mathcal{A}_{\Xi_1}}^2. \end{aligned}$$

Ⓢ Step 5: *continuity in the weak operator topology.* By definition, the family of operators $\left\{ \mathcal{K}_{\hat{u}}^{\lambda, s} \mid s \in [0, +\infty) \right\} \subseteq \mathcal{L}(\mathcal{A}_{\Xi_1} \rightarrow \mathcal{A}_{\Xi_1})$ does depend continuously on s in the weak operator topology if for all $w^1, w^2 \in \mathcal{A}_{\Xi_1}$ and $s_0 \in [0, +\infty)$, $(\mathcal{K}_{\hat{u}}^{\lambda, s} w^1, w^2)_{\mathcal{A}_{\Xi_1}}$ goes to $(\mathcal{K}_{\hat{u}}^{\lambda, s_0} w^1, w^2)_{\mathcal{A}_{\Xi_1}}$ as s goes to s_0 ; with $s \in [0, +\infty)$. Form $\mathcal{K}_{\hat{u}}^{\lambda, s} v = e^{-\frac{\lambda}{2}s} \Xi^\diamond(P_{\nabla} R_{\hat{u}}^{\lambda, s} v \mathbf{n})$, again from the continuity of the mappings Ξ^\diamond and $u \mapsto P_{\nabla} u$, it suffices to prove that the

same property hold true for the family $\left\{R_{\hat{u}}^{\lambda, s} \mid s \in [0, +\infty)\right\}$. Writing $2(R_{\hat{u}}^{\lambda, s} w^1, w^2)_{\mathcal{A}_{\Xi_1}} = (R_{\hat{u}}^{\lambda, s}(w^1 + w^2), w^1 + w^2)_{\mathcal{A}_{\Xi_1}} - (R_{\hat{u}}^{\lambda, s} w^1, w^1)_{\mathcal{A}_{\Xi_1}} - (R_{\hat{u}}^{\lambda, s} w^2, w^2)_{\mathcal{A}_{\Xi_1}}$, we also see that it suffices to prove that

$$(65) \quad (R_{\hat{u}}^{\lambda, s} w, w)_{\mathcal{A}_{\Xi_1}} \rightarrow (R_{\hat{u}}^{\lambda, s_0} w, w)_{\mathcal{A}_{\Xi_1}} \quad \text{as } s \rightarrow s_0 \text{ for any } w \in \mathcal{A}_{\Xi_1},$$

or equivalently that

$$(66) \quad (R_{\hat{u}}^{\lambda, s_0 \pm \delta_n} w, w)_{\mathcal{A}_{\Xi_1}} \rightarrow (R_{\hat{u}}^{\lambda, s_0} w, w)_{\mathcal{A}_{\Xi_1}} \text{ as } n \rightarrow +\infty,$$

for any sequence $(\delta_n)_{n \in \mathbb{N}}$ of real numbers, with $0 < \delta_n < 1$, $\delta_n \rightarrow 0$, and any $w \in \mathcal{A}_{\Xi_1}$ (still, with $s = s_0 \pm \delta_n \geq 0$). We consider separately two cases: $s \searrow s_0$ and $s \nearrow s_0$.

(a) *The case $s \searrow s_0$.* If $s = s_0 + \delta_n$, we write

$$(67) \quad \begin{aligned} & (R_{\hat{u}}^{\lambda, s_0 + \delta_n} w, w)_{\mathcal{A}_{\Xi_1}} \\ &= \left(R_{\hat{u}}^{\lambda, s_0 + \delta_n} (w - v_{s_0}^*(w)(s_0 + \delta_n)), w \right)_{\mathcal{A}_{\Xi_1}} + \left(R_{\hat{u}}^{\lambda, s_0 + \delta_n} v_{s_0}^*(w)(s_0 + \delta_n), w \right)_{\mathcal{A}_{\Xi_1}} \end{aligned}$$

where we recall $v_{s_0}^*(w)$ is the optimal trajectory for Problem 4.5 given by Lemma 4.7. Rewriting the last term, in (67), as

$$\begin{aligned} & \left(R_{\hat{u}}^{\lambda, s_0 + \delta_n} v_{s_0}^*(w)(s_0 + \delta_n), w - v_{s_0}^*(w)(s_0 + \delta_n) \right)_{\mathcal{A}_{\Xi_1}} \\ &+ \left(R_{\hat{u}}^{\lambda, s_0 + \delta_n} v_{s_0}^*(w)(s_0 + \delta_n), v_{s_0}^*(w)(s_0 + \delta_n) \right)_{\mathcal{A}_{\Xi_1}} \end{aligned}$$

which, recalling the dynamical programming principle (cf. Lemma 4.9), can be written as

$$\begin{aligned} & \left(R_{\hat{u}}^{\lambda, s_0 + \delta_n} v_{s_0}^*(w)(s_0 + \delta_n), w - v_{s_0}^*(w)(s_0 + \delta_n) \right)_{\mathcal{A}_{\Xi_1}} + \left(R_{\hat{u}}^{\lambda, s_0} w, w \right)_{\mathcal{A}_{\Xi_1}} \\ & - |e^{\frac{\lambda}{2}t} \Pi v_{s_0}^*(w)|_{L^2((s_0, s_0 + \delta_n), H)}^2 - |e^{\frac{\lambda}{2}t} \kappa_{s_0}^*(w)|_{H^1((s_0, s_0 + \delta_n), \mathcal{N}^\perp)}^2. \end{aligned}$$

Using the self-adjointness of $R_{\hat{u}}^{\lambda, s_0 + \delta_n}$, we arrive to

$$(68) \quad \begin{aligned} & \left(R_{\hat{u}}^{\lambda, s_0 + \delta_n} w, w \right)_{\mathcal{A}_{\Xi_1}} - \left(R_{\hat{u}}^{\lambda, s_0} w, w \right)_{\mathcal{A}_{\Xi_1}} \\ &= \left(R_{\hat{u}}^{\lambda, s_0 + \delta_n} (w - v_{s_0}^*(w)(s_0 + \delta_n)), w + v_{s_0}^*(w)(s_0 + \delta_n) \right)_{\mathcal{A}_{\Xi_1}} \\ & - |e^{\frac{\lambda}{2}t} \Pi v_{s_0}^*(w)|_{L^2((s_0, s_0 + \delta_n), H)}^2 - |e^{\frac{\lambda}{2}t} \kappa_{s_0}^*(w)|_{H^1((s_0, s_0 + \delta_n), \mathcal{N}^\perp)}^2, \end{aligned}$$

Now, for the first term on the right-hand side of (68), we have

$$\begin{aligned} & \left(R_{\hat{u}}^{\lambda, s_0 + \delta_n} (w - v_{s_0}^*(w)(s_0 + \delta_n)), w + v_{s_0}^*(w)(s_0 + \delta_n) \right)_{\mathcal{A}_{\Xi_1}} \\ & \leq \overline{C} [|\hat{u}|_{\text{wst}}, \lambda, \frac{1}{\lambda}] e^{\lambda(s_0 + \delta_n)} |w - v_{s_0}^*(w)(s_0 + \delta_n)|_{\mathcal{A}_{\Xi_1}} |w + v_{s_0}^*(w)(s_0 + \delta_n)|_{\mathcal{A}_{\Xi_1}} \end{aligned}$$

and, from the continuity of $v_{s_0}^*(w)(t)$ in the time variable t , we can conclude that this term goes to zero with δ_n . On the other hand, the terms $|e^{\frac{\lambda}{2}t} \Pi v_{s_0}^*(w)|_{L^2((s_0, s_0 + \delta_n), H)}^2$ and $|e^{\frac{\lambda}{2}t} \kappa_{s_0}^*(w)|_{H^1((s_0, s_0 + \delta_n), \mathcal{N}^\perp)}^2$ also go to zero with δ_n , because the functions $v_{s_0}^*(w)$ and $\kappa_{s_0}^*(w)$ do not depend on δ_n . Therefore we can conclude that

$$(69) \quad \lim_{n \rightarrow +\infty} \left(R_{\hat{u}}^{\lambda, s_0 + \delta_n} w, w \right)_{\mathcal{A}_{\Xi_1}} = \left(R_{\hat{u}}^{\lambda, s_0} w, w \right)_{\mathcal{A}_{\Xi_1}}.$$

(b) *The case $s \nearrow s_0$.* Though we will follow the same idea, this case carries additional difficulties. Moreover, we can check that we cannot just repeat the idea in [BRS11, proof

of Lemma 3.8] due to boundary compatibility issues. We propose a slightly variation. If $s = s_0 - \delta_n$, we write

$$\begin{aligned} & \left(R_{\hat{u}}^{\lambda, s_0 - \delta_n} w, w \right)_{\mathcal{A}_{\Xi_1}} \\ &= |e^{\frac{\lambda}{2}t} \Pi v_{s_0 - \delta_n}^*(w)|_{L^2((s_0 - \delta_n, s_0), H)}^2 + |e^{\frac{\lambda}{2}t} \kappa_{s_0 - \delta_n}^*(w)|_{H^1((s_0 - \delta_n, s_0), \mathcal{N}^\perp)}^2 \\ & \quad + \left(R_{\hat{u}}^{\lambda, s_0} v_{s_0 - \delta_n}^*(w)(s_0), v_{s_0 - \delta_n}^*(w)(s_0) \right)_{\mathcal{A}_{\Xi_1}} \end{aligned}$$

and, writing $v_{s_0 - \delta_n}^*(w)(s_0) = v_{s_0 - \delta_n}^*(w)(s_0) - w + w$ in the last term, we obtain

$$\begin{aligned} (70) \quad & \left(R_{\hat{u}}^{\lambda, s_0 - \delta_n} w, w \right)_{\mathcal{A}_{\Xi_1}} - \left(R_{\hat{u}}^{\lambda, s_0} w, w \right)_{\mathcal{A}_{\Xi_1}} \\ &= |e^{\frac{\lambda}{2}t} \Pi v_{s_0 - \delta_n}^*(w)|_{L^2((s_0 - \delta_n, s_0), H)}^2 + |e^{\frac{\lambda}{2}t} \kappa_{s_0 - \delta_n}^*(w)|_{H^1((s_0 - \delta_n, s_0), \mathcal{N}^\perp)}^2 \\ & \quad + \left(R_{\hat{u}}^{\lambda, s_0} (v_{s_0 - \delta_n}^*(w)(s_0) - w), v_{s_0 - \delta_n}^*(w)(s_0) - w \right)_{\mathcal{A}_{\Xi_1}} \\ & \quad + 2 \left(R_{\hat{u}}^{\lambda, s_0} w, v_{s_0 - \delta_n}^*(w)(s_0) - w \right)_{\mathcal{A}_{\Xi_1}} \end{aligned}$$

Now by Corollary 4.4 we have, in particular, that

$$\begin{aligned} (71) \quad & |e^{\frac{\lambda}{2} \cdot} v_{s_0 - \delta_n}^*(w)(\cdot)|_{C([s_0 - \delta_n, s_0 - \delta_n + 1], \mathcal{A}_{\Xi_1})}^2 \\ & \leq \bar{C}_{[\hat{u}|_{\mathcal{V}^{\text{st}}}, \lambda]} \left(e^{\lambda(s_0 - \delta_n)} |w|_{\mathcal{A}_{\Xi_1}}^2 + \left(R_{\hat{u}}^{\lambda, s_0 - \delta_n} w, w \right)_{\mathcal{A}_{\Xi_1}} \right) \leq \bar{C}_{[\hat{u}|_{\mathcal{V}^{\text{st}}}, \lambda, \frac{1}{\lambda}, s_0]} |w|_{\mathcal{A}_{\Xi_1}}^2 \end{aligned}$$

which implies, since $\delta_n < 1$,

$$(72) \quad |e^{\frac{\lambda}{2} \cdot} \Pi v_{s_0 - \delta_n}^*(w)(\cdot)|_{L^2((s_0 - \delta_n, s_0), H)}^2 \leq \bar{C}_{[\hat{u}|_{\mathcal{V}^{\text{st}}}, \lambda, \frac{1}{\lambda}, s_0]} |w|_{\mathcal{A}_{\Xi_1}}^2 \delta_n.$$

On the other hand, from (60) and $v_{s_0 - \delta_n}^*(w)(s_0 - \delta_n) = w$, it follows that

$$\kappa_{s_0 - \delta_n}^*(w)(t) = e^{-\frac{\lambda}{2}t} \left(e^{\frac{\lambda}{2}(s_0 - \delta_n)} z^{w \cdot \mathbf{n}} - \int_{s_0 - \delta_n}^t e^{-\frac{\lambda}{2}s} \Xi^\diamond (P_\nabla R_{\hat{u}}^{\lambda, s} v_{s_0 - \delta_n}^*(w)(s) \mathbf{n}) ds \right).$$

and so

$$\begin{aligned} & \left| e^{\frac{\lambda}{2} \cdot} \kappa_{s_0 - \delta_n}^*(w)(\cdot) \right|_{H^1((s_0 - \delta_n, s_0), \mathcal{N}^\perp)}^2 \\ &= \left| e^{\frac{\lambda}{2}(s_0 - \delta_n)} z^{w \cdot \mathbf{n}} - \int_{s_0 - \delta_n}^{\cdot} e^{-\frac{\lambda}{2}\tau} \Xi^\diamond (P_\nabla R_{\hat{u}}^{\lambda, s} v_{s_0 - \delta_n}^*(w)(s) \mathbf{n}) ds \right|_{L^2((s_0 - \delta_n, s_0), \mathcal{N}^\perp)}^2 \\ & \quad + \left| e^{-\frac{\lambda}{2} \cdot} \Xi^\diamond (P_\nabla R_{\hat{u}}^{\lambda, \cdot} v_{s_0 - \delta_n}^*(w)(\cdot) \mathbf{n}) \right|_{L^2((s_0 - \delta_n, s_0), \mathcal{N}^\perp)}^2 \end{aligned}$$

and, using (71) we can derive, since $\delta_n < 1$, that

$$(73) \quad \left| e^{\frac{\lambda}{2} \cdot} \kappa_{s_0 - \delta_n}^*(w)(\cdot) \right|_{H^1((s_0 - \delta_n, s_0), \mathcal{N}^\perp)}^2 \leq \bar{C}_{[\hat{u}|_{\mathcal{V}^{\text{st}}}, \lambda, \frac{1}{\lambda}, s_0]} |w|_{\mathcal{A}_{\Xi_1}}^2 \delta_n.$$

Now, from (70), (72) and (73) we arrive to

$$\begin{aligned} (74) \quad & \lim_{n \rightarrow +\infty} \left(\left(R_{\hat{u}}^{\lambda, s_0 - \delta_n} w, w \right)_{\mathcal{A}_{\Xi_1}} - \left(R_{\hat{u}}^{\lambda, s_0} (v_{s_0 - \delta_n}^*(w)(s_0) + w), v_{s_0 - \delta_n}^*(w)(s_0) - w \right)_{\mathcal{A}_{\Xi_1}} \right) \\ &= \left(R_{\hat{u}}^{\lambda, s_0} w, w \right)_{\mathcal{A}_{\Xi_1}}. \end{aligned}$$

Since \mathcal{A}_{Ξ_2} is dense in \mathcal{A}_{Ξ_1} , there exists a sequence $(w_n)_{n \in \mathbb{N}}$, with $w_n \in \mathcal{A}_{\Xi_2}$, such that $|w_n - w|_{\mathcal{A}_{\Xi_1}}$ goes to zero as n goes to $+\infty$. Thus, writing $w = w - w_n + w_n$ and using (71) we obtain that

$$\begin{aligned} |v_{s_0 - \delta_n}^*(w)(s_0) - w|_{\mathcal{A}_{\Xi_1}} &\leq |v_{s_0 - \delta_n}^*(w - w_n)(s_0) - w + w_n|_{\mathcal{A}_{\Xi_1}} + |v_{s_0 - \delta_n}^*(w_n)(s_0) - w_n|_{\mathcal{A}_{\Xi_1}} \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}, s_0}} |w - w_n|_{\mathcal{A}_{\Xi_1}} + |v_{s_0 - \delta_n}^*(w_n)(s_0) - w_n|_{\mathcal{A}_{\Xi_1}}, \end{aligned}$$

and

$$\begin{aligned} &\left| \left(R_{\hat{u}}^{\lambda, s_0} (v_{s_0 - \delta_n}^*(w)(s_0) + w), v_{s_0 - \delta_n}^*(w)(s_0) - w \right)_{\mathcal{A}_{\Xi_1}} \right|_{\mathbb{R}} \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}, s_0}} |w|_{\mathcal{A}_{\Xi_1}} (|w - w_n|_{\mathcal{A}_{\Xi_1}} + |v_{s_0 - \delta_n}^*(w_n)(s_0) - w_n|_{\mathcal{A}_{\Xi_1}}), \end{aligned}$$

which implies that

$$(75) \quad \begin{aligned} &\lim_{n \rightarrow +\infty} \left| \left(R_{\hat{u}}^{\lambda, s_0} (v_{s_0 - \delta_n}^*(w)(s_0) + w), v_{s_0 - \delta_n}^*(w)(s_0) - w \right)_{\mathcal{A}_{\Xi_1}} \right|_{\mathbb{R}} \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}, s_0}} |w|_{\mathcal{A}_{\Xi_1}} \lim_{n \rightarrow +\infty} |v_{s_0 - \delta_n}^*(w_n)(s_0) - w_n|_{\mathcal{A}_{\Xi_1}}. \end{aligned}$$

It turns out that $v_n = v_n(t) := v_{s_0 - \delta_n}^*(w_n)(t) - w_n$ solves (16), in $(s_0 - \delta_n, s_0) \times \Omega$, with $g = \mathcal{B}(\hat{u})w_n - \nu \Delta w_n$, $K = \Xi$, $\eta = \kappa_{s_0 - \delta_n}^*(w_n)(t) - z^{w_n \cdot \mathbf{n}}$, and $v(s_0 - \delta_n) = 0$. From Theorem 2.5, and $|g(t)|_{H^{-1}(\Omega, \mathbb{R}^3)}^2 \leq C(|\hat{u}(t)|_{L^\infty(\Omega, \mathbb{R}^3)}^2 + 1)|w_n|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2$, we obtain that

$$\begin{aligned} &|v_{s_0 - \delta_n}^*(w_n)(s_0) - w_n|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 = |v_n(s_0)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{(s_0 - \delta_n, s_0)}|_{\text{wk}}}} \left(|w_n|_{L^2((s_0 - \delta_n, s_0), H_{\text{div}}^1(\Omega, \mathbb{R}^3))}^2 + |\kappa_{s_0 - \delta_n}^*(w_n) - z^{w_n \cdot \mathbf{n}}|_{H^1((s_0 - \delta_n, s_0), \mathcal{N}^\perp)}^2 \right); \end{aligned}$$

and by (75) and (73), we obtain

$$(76) \quad \begin{aligned} &\lim_{n \rightarrow +\infty} \left| \left(R_{\hat{u}}^{\lambda, s_0} (v_{s_0 - \delta_n}^*(w)(s_0) + w), v_{s_0 - \delta_n}^*(w)(s_0) - w \right)_{\mathcal{A}_{\Xi_1}} \right|_{\mathbb{R}} \\ &\leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}, s_0}} |w|_{\mathcal{A}_{\Xi_1}} \lim_{n \rightarrow +\infty} \delta_n |w_n|_{\mathcal{A}_{\Xi_2}}^2. \end{aligned}$$

Recall that \mathcal{A}_{Ξ_2} is supposed to be endowed with the norm inherited from $H_{\text{div}}^1(\Omega, \mathbb{R}^3)$ (cf. Corollary 3.4), that is, $|w_n|_{\mathcal{A}_{\Xi_2}}$ will go to $+\infty$ with n , if $w \in \mathcal{A}_{\Xi_1} \setminus \mathcal{A}_{\Xi_2}$. Next we, show that the sequence $(w_n)_{n \in \mathbb{N}}$ can be chosen such that $\delta_n |w_n|_{\mathcal{A}_{\Xi_2}}^2 \rightarrow 0$, as $n \rightarrow +\infty$.

For that, we start by recalling that we can extend a given function $f \in H_{\text{av}}^{\frac{3}{2}}(\Gamma, \mathbb{R}^3)$ to the solution $\widehat{F}f \in H_{\text{div}}^2(\Omega, \mathbb{R}^3)$ of the Stokes system

$$(77) \quad -\Delta \widehat{F}f + \nabla p_f = 0, \quad \text{div } \widehat{F}f = 0, \quad \widehat{F}f|_{\Gamma} = f$$

and the mapping $f \mapsto \widehat{F}f$ is continuous (cf. [Tem01, Chapter 1, Proposition 2.3]). Thus writing $w = w - \widehat{F}\Xi z^{w \cdot \mathbf{n}} + \widehat{F}\Xi z^{w \cdot \mathbf{n}}$, if we find $\bar{w}_n \in V$ such that $|\bar{w}_n - w + \widehat{F}\Xi z^{w \cdot \mathbf{n}}|_H \rightarrow 0$ and $\delta_n |\bar{w}_n|_V^2 \rightarrow 0$, then we can set $w_n := \bar{w}_n + \widehat{F}\Xi z^{w \cdot \mathbf{n}} \rightarrow w$, in \mathcal{A}_{Ξ_1} , because in that case we will have $w_n \in \mathcal{A}_{\Xi_2}$ and $\delta_n |w_n|_{\mathcal{A}_{\Xi_2}}^2 \leq 2\delta_n \left(|\bar{w}_n|_V^2 + |\widehat{F}\Xi z^{w \cdot \mathbf{n}}|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \right) \rightarrow 0$. Let us denote $\bar{w} := w - \widehat{F}\Xi z^{w \cdot \mathbf{n}}$; we know that the sequence $\Pi_{n+1}\bar{w} \in H_{n+1} \subset V$ converges to \bar{w} , in H , where $\Pi_N: H \rightarrow H_N$ is the orthogonal projection in H onto the space H_N spanned by the first N eigenfunctions of the Stokes operator (see (8)) but, we have no guarantee that $\delta_n |\Pi_{n+1}\bar{w}|_V^2 \rightarrow 0$, that is, $\bar{w}_n = \Pi_{n+1}\bar{w}$ may be not a good choice. Next we show

that this issue can be overcome by somehow “slowing down” the convergence of $\Pi_{n+1}\bar{w}$ to \bar{w} , in H : we define the sequence $N: \mathbb{N} \rightarrow \mathbb{N}_0$, by

$$\begin{cases} N_n = 1 & \text{if } \frac{1}{2\alpha_2} < \delta_n \\ N_n = j & \text{if } j \geq 2 \text{ and } \frac{1}{(j+1)\alpha_{j+1}} < \delta_n \leq \frac{1}{j\alpha_j} \end{cases};$$

where $(\alpha_j)_{j \in \mathbb{N}_0}$ is the nondecreasing sequence of (repeated) eigenvalues of the Stokes operator (cf. Section 2.2). We can see that N_n goes to $+\infty$ with n , because δ_n goes to 0. Hence, setting $\bar{w}_n := \Pi_{N_n}\bar{w}$, we have that $\bar{w}_n \rightarrow \bar{w}$, in H , and $|\bar{w}_n|_V^2 \leq C|\nabla\bar{w}_n|_{L^2(\Omega, \mathbb{R}^9)}^2 = C(L\bar{w}_n, \bar{w}_n)_H \leq C\alpha_{N_n}|\bar{w}|_H^2$ and, then $\delta_n|\bar{w}_n|_V^2 \leq C\delta_n\alpha_{N_n}|\bar{w}|_H^2 \leq \frac{1}{N_n}C|\bar{w}|_H^2$, for all $N_n \geq 2$; thus $\delta_n|\bar{w}_n|_V^2 \rightarrow 0$ as $n \rightarrow +\infty$. Setting $w_n = \bar{w}_n + \widehat{F}\Xi z^{w_n}$ in (76), and using (74), we arrive to

$$(78) \quad \lim_{n \rightarrow +\infty} \left(R_{\hat{u}}^{\lambda, s_0 - \delta_n} w, w \right)_{\mathcal{A}_{\Xi_1}} = \left(R_{\hat{u}}^{\lambda, s_0} w, w \right)_{\mathcal{A}_{\Xi_1}}.$$

We can conclude, from (69) and (78), that the family $R_{\hat{u}}^{\lambda, s}$ depends continuously on s , in the weak operator topology, which ends the proof of Theorem 4.1. \square

4.3. Miscellaneous remarks. As we said in the Introduction, looking for feedback finite-dimensional controllers supported in a small subset of the boundary, is motivated by the importance of such controllers in applications. We give a few remarks concerning this point.

4.3.1. Dependence on the Present state only. Though given in integral form, the feedback control is defined pointwise in time. Indeed, Lemma 4.9 tells us that once we arrive at $w \in \mathcal{A}_{\Xi_1}$ at time $t = s_0$, then the control for time $t \geq s_0$ can be found by solving Problem 4.5, with $s = s_0$, which is independent of the Past $t < s_0$. On the other side from (61) we know that if we start at v_0 at time $t = 0$, then $v_0^*|_{\Gamma}(s_0) = e^{-\frac{\lambda}{2}s_0} \left((v_0 \cdot \mathbf{n})\mathbf{n} + \Xi \int_0^{s_0} \mathcal{K}_{\hat{u}}^{\lambda, s} v_0^*(s) ds \right)$, that is, we already know the control when we arrive at time $t = s_0$; so the control is independent of the Future $t > s_0$. Except the supposed knowledge of the targeted solution $\hat{u} = \hat{u}(t)$, for all time $t > s_0$, of course.

4.3.2. Dimension of the controller. The range E_M of the controller depends only on the norm $|\hat{u}|_{\mathcal{W}^{\text{st}}}$ of the targeted solution \hat{u} , and the feedback rule depends on time. We do not address here the problem of finding an estimate for M , that is of crucial importance for application purposes (e.g., numerical simulations). For internal controls, this problem has been started in [KR14] in the simpler case of the 1D Burgers system in a bounded interval $(0, L)$, and estimates on the number M of needed controls is given that depend exponentially in $M_{\text{ref}} := (\nu^{-2}|\hat{u}|_{\mathcal{W}}^2 + \nu^{-1}\lambda)^{\frac{1}{2}}$ in the general case, and that are proportional to M_{ref} in the case of no constraint on the support of the control, with $\mathcal{W} = L^\infty(\mathbb{R}_0, L^\infty((0, L), \mathbb{R}))$. However also in [KR14] the results of numerical simulations suggest that it might be sufficient to take M proportional to M_{ref} also in the general case. Following [KR14], we see that in the particular case, the estimate for M is derived from an inequality like $\alpha_M \geq \nu C M_{\text{ref}}^2$, where α_M is the M th eigenvalue of the Laplacean operator. Hence we may (perhaps) conjecture that in the d D case, $d \in \{2, 3\}$, it should be enough to take a number M of internal controls proportional to $(\nu^{-2}|\hat{u}|_{\mathcal{W}^{\text{st}}}^2 + \nu^{-1}\lambda)^{\frac{d}{2}}$, because $\alpha_M \sim \nu C_1 M^{\frac{2}{d}}$ (cf. [Ily09]). Does the conjecture hold true? Can we derive similar estimates for the case of the boundary controls we treat here? These questions will be addressed in future works.

Recall that, in the case of stationary \hat{u} , for example in [BT04, BT11, RT10], we can find rather sharp estimates, though M depend on \hat{u} and not only in the norm $|\hat{u}|_{\mathcal{W}^{\text{st}}}$. The method cannot be (at least not straightforwardly) used in the nonstationary case.

4.3.3. *Lyapunov function.* Once a feedback control is constructed, it is easy to find a time-dependent Lyapunov function for the problem in question. Indeed, the functional

$$\Phi(r, w) = \int_r^\infty |\mathcal{S}_{r,t} w|_{\mathcal{A}_{\Xi_1}}^2 dt$$

decays along the trajectories of (38), where $\mathcal{S}_{r,t} w$ denotes the solution $v_r^* = v_r^*(t)$ of (38), in $\mathbb{R}_r \times \Omega$, with the initial condition $v_r^*(r) = w$: we may write

$$\Phi(r, v_0^*(r)) = \int_r^\infty |\mathcal{S}_{r,\tau} v_r^*(r)|_{\mathcal{A}_{\Xi_1}}^2 d\tau = \int_r^\infty |\mathcal{S}_{0,\tau} v_0^*(0)|_{\mathcal{A}_{\Xi_1}}^2 d\tau = \int_r^\infty |v_0^*(\tau)|_{\mathcal{A}_{\Xi_1}}^2 d\tau,$$

from which, together with (39), we can obtain

$$\begin{aligned} \frac{d}{ds} \Big|_{s=r} \Phi(s, v_0^*(s)) &= -|v_0^*(r)|_{\mathcal{A}_{\Xi_1}}^2 \leq -\bar{C}^{-1}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |e^{\frac{\lambda}{2}(t-r)} v_0^*(t)|_{W(\mathbb{R}_r, H^1_{\text{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \\ &\leq -\bar{C}^{-1}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} C |e^{\frac{\lambda}{2}(t-r)} v_0^*(t)|_{L^2(\mathbb{R}_r, L^2_{\text{div}}(\Omega, \mathbb{R}^3))}^2 \leq -\bar{C}^{-1}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} C \Phi(r, v_0^*(r)). \end{aligned}$$

Another Lyapunov function is the ‘‘cost to go’’ from time $t = r$ onwards, that is $\Psi(r, w) = (R^r w, w)_{\mathcal{A}_{\Xi_1}}$, where $R^r := R_{\hat{u}}^{\lambda, r}$ is the operator defining the optimal cost. Indeed, from the dynamical principle we have

$$\Psi(r, v_0^*(r)) = \Psi(0, v_0^*(0)) - \left| e^{\frac{\lambda}{2}t} \Pi v_0^* \right|_{L^2((0,r), H)}^2 - \left| e^{\frac{\lambda}{2}t} \kappa_0^* \right|_{H^1((0,r), \mathcal{N}^\perp)}^2$$

which implies

$$\begin{aligned} \frac{d}{ds} \Big|_{s=r} \Psi(s, v_0^*(s)) &= -e^{\lambda r} |\Pi v_0^*(r)|_H^2 - e^{\lambda r} |\kappa_0^*(r)|_{\mathcal{N}^\perp}^2 - e^{\lambda r} \left| \frac{\lambda}{2} \kappa_0^*(r) + \frac{d}{ds} \Big|_{s=r} \kappa_0^*(s) \right|_{\mathcal{N}^\perp}^2 \\ &\leq -e^{\lambda r} |\Pi v_0^*(r)|_H^2 - e^{\lambda r} |z^{v_0^*(s), \mathbf{n}}|_{\mathcal{N}^\perp}^2 \leq -e^{\lambda r} C |v_0^*(r)|_{\mathcal{A}_{\Xi_1}}^2 \\ &\leq -C \bar{C}^{-1}_{[|\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} \Psi(r, v_0^*(r)). \end{aligned}$$

It is, however, difficult to write down the functions Φ and Ψ in a more explicit form.

4.3.4. *Riccati equation.* In the case of internal controls it was shown in [BRS11, Remark 3.11(b)] that the operator defining the optimal cost, and from which we can obtain the feedback law, satisfies a suitable differential Riccati equation. For applications (e.g., simulations) it is important to have such equation at our disposal; for example, we refer to [KR14] where the optimal feedback rule has been obtained by solving (numerically) a similar differential Riccati equation, in the simpler case of the 1D Burgers system and internal controls.

It turns out that also in our case the operator $R := R^t := R_{\hat{u}}^{\lambda, t}$, from which we can obtain the boundary integral feedback rule, satisfies a differential Riccati equation. Namely, for $t \in \mathbb{R}_0$

$$(79) \quad \dot{R} + RA + \mathbf{A}^* R - RBB^* R + \mathbf{C} = 0$$

with

$$\mathbf{A} := \Pi L_{\hat{u}(t)} - \frac{\lambda}{2} (P_{\mathcal{A}_{\Xi_1}} - \Pi); \quad \mathbf{B} := e^{-\frac{\lambda}{2}t} (P_{\mathcal{A}_{\Xi_1}} - \Pi) \hat{F} \Xi; \quad \mathbf{C} := e^{\lambda t} (\Pi + \mathcal{Z}_{\Xi});$$

where $(\mathcal{Z}_{\Xi} u, v)_{\mathcal{A}_{\Xi_1}} := (z^{u, \mathbf{n}}, z^{v, \mathbf{n}})_{\mathcal{N}^\perp} = (u, \mathcal{Z}_{\Xi} v)_{\mathcal{A}_{\Xi_1}}$, for all $(u, v) \in \mathcal{A}_{\Xi_1}^2$, \hat{F} is defined by (77), and $L_{\hat{u}(t)}$ is the Oseen–Stokes operator $L_{\hat{u}(t)}: H^1_{\text{div}}(\Omega, \mathbb{R}^3) \rightarrow H^{-1}(\Omega, \mathbb{R}^3)$, $v \mapsto$

$\nu \Delta v - \mathcal{B}(\hat{u}(t))v - \nabla p_v \in \mathcal{A}_{\Xi_1}^{[-1]}$, where $\mathcal{A}_{\Xi_1}^{[-1]}$ stands for the closure of \mathcal{A}_{Ξ_1} in $H^{-1}(\Omega, \mathbb{R}^3)$, and $P_{\mathcal{A}_{\Xi_1}} : L^2(\Omega, \mathbb{R}^3) \rightarrow \mathcal{A}_{\Xi_1}$ for the the orthogonal projection in $L^2(\Omega, \mathbb{R}^3)$ onto \mathcal{A}_{Ξ_1} .

Observe that, from Lemmas 4.7 and 4.9, we have

$$(R^s v_0^*(s), v_0^*(s))_{\mathcal{A}_{\Xi_1}} = \left| e^{\frac{\lambda}{2}t} \Pi v_0^*(t) \right|_{L^2(\mathbb{R}_s, H)}^2 + \left| e^{\frac{\lambda}{2}t} \kappa_0^*(t) \right|_{H^1(\mathbb{R}_s, \mathcal{N}^\perp)}^2$$

from which we can derive that, formally,

$$\begin{aligned} & (\partial_s|_{s=t} R^s v_0^*(t), v_0^*(t))_{\mathcal{A}_{\Xi_1}} \\ & + (R^t \partial_s|_{s=t} v_0^*(s), v_0^*(t))_{\mathcal{A}_{\Xi_1}} + (R^t v_0^*(t), \partial_s|_{s=t} v_0^*(s))_{\mathcal{A}_{\Xi_1}} \\ & = - \left| e^{\frac{\lambda}{2}t} \Pi v_0^*(t) \right|_H^2 - \left| e^{\frac{\lambda}{2}t} \kappa_0^*(t) \right|_{\mathcal{N}^\perp}^2 - \left| \partial_s|_{s=t} (e^{\frac{\lambda}{2}s} \kappa_0^*(s)) \right|_{\mathcal{N}^\perp}^2. \end{aligned}$$

Recalling (58), it follows that for $t > 0$,

$$\begin{aligned} & (\partial_s|_{s=t} R^s v_0^*(t), v_0^*(t))_{\mathcal{A}_{\Xi_1}} \\ & + (R^t \Pi L_{\hat{u}(t)}(v_0^*(t)), v_0^*(t))_{\mathcal{A}_{\Xi_1}} + (R^t v_0^*(t), \Pi L_{\hat{u}(t)} v_0^*(t))_{\mathcal{A}_{\Xi_1}} \\ & + (R^t (1 - \Pi) \partial_s|_{s=t} v_0^*(s), v_0^*(t))_{\mathcal{A}_{\Xi_1}} + (R^t v_0^*(t), (1 - \Pi) \partial_s|_{s=t} v_0^*(s))_{\mathcal{A}_{\Xi_1}} \\ (80) \quad & = -e^{\lambda t} (\Pi v_0^*(t), v_0^*(t))_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)} - e^{\lambda t} |z^{v(t) \cdot \mathbf{n}}|_{\mathcal{N}^\perp}^2 - e^{-\lambda t} |\Xi^\diamond (P_\nabla R^t v_0^*(t) \mathbf{n})|_{\mathcal{N}^\perp}^2. \end{aligned}$$

Next, observe that

$$\begin{aligned} & (R^t (1 - \Pi) \partial_s|_{s=t} v_0^*(s), v_0^*(t))_{\mathcal{A}_{\Xi_1}} = (\partial_s|_{s=t} v_0^*(s), (1 - \Pi) R^t v_0^*(t))_{\mathcal{A}_{\Xi_1}} \\ & = (\Xi \partial_s|_{s=t} \kappa_0^*(s), P_\nabla R^t v_0^*(t) \mathbf{n})_{L^2(\Gamma, \mathbb{R}^3)} \\ & = \left(e^{-\frac{\lambda}{2}t} \partial_s|_{s=t} (e^{\frac{\lambda}{2}s} \kappa_0^*(s)) - \frac{\lambda}{2} \kappa_0^*(t), \Xi^\diamond P_\nabla R^t v_0^*(t) \mathbf{n} \right)_{\mathcal{N}^\perp} \\ & = \left(-e^{-\lambda t} \Xi^\diamond P_\nabla R^t v_0^*(t) - \frac{\lambda}{2} \kappa_0^*(t), \Xi^\diamond P_\nabla R^t v_0^*(t) \mathbf{n} \right)_{\mathcal{N}^\perp} \\ & = -e^{-\lambda t} |\Xi^\diamond P_\nabla R^t v_0^*(t)|_{\mathcal{N}^\perp}^2 - \left(\frac{\lambda}{2} \kappa_0^*(t), \Xi^\diamond P_\nabla R^t v_0^*(t) \mathbf{n} \right)_{\mathcal{N}^\perp} \\ (81) \quad & = -e^{-\lambda t} |\Xi^\diamond P_\nabla R^t v_0^*(t)|_{\mathcal{N}^\perp}^2 - \frac{\lambda}{2} (R^t (1 - \Pi) v_0^*(t), v_0^*(t))_{\mathcal{A}_{\Xi_1}} \end{aligned}$$

and, similarly

$$\begin{aligned} & (R^t v_0^*(t), (1 - \Pi) \partial_s|_{s=t} v_0^*(s))_{\mathcal{A}_{\Xi_1}} = ((1 - \Pi) R^t v_0^*(t), \partial_s|_{s=t} v_0^*(s))_{\mathcal{A}_{\Xi_1}} \\ & = (\Xi^\diamond P_\nabla R^t v_0^*(t) \mathbf{n}, \partial_s|_{s=t} \kappa_0^*(s))_{\mathcal{N}^\perp} \\ & = -e^{-\lambda t} |\Xi^\diamond P_\nabla R^t v_0^*(t)|_{\mathcal{N}^\perp}^2 - \frac{\lambda}{2} (\Xi^\diamond P_\nabla R^t v_0^*(t) \mathbf{n}, \kappa_0^*(s))_{\mathcal{N}^\perp} \\ (82) \quad & = -e^{-\lambda t} |\Xi^\diamond P_\nabla R^t v_0^*(t)|_{\mathcal{N}^\perp}^2 - \frac{\lambda}{2} ((1 - \Pi) R^t v_0^*(t), v_0^*(s))_{\mathcal{N}^\perp}. \end{aligned}$$

Hence, from (80), (81), and (82), we obtain that

$$\begin{aligned} & (\partial_s|_{s=t} R^s v_0^*(t), v_0^*(t))_{\mathcal{A}_{\Xi_1}} \\ & + (R^t \mathbf{A} v_0^*(t), v_0^*(t))_{\mathcal{A}_{\Xi_1}} + (\mathbf{A}^* R^t v_0^*(t), v_0^*(t))_{\mathcal{A}_{\Xi_1}} - e^{-\lambda t} |\Xi^\diamond P_\nabla R^t v_0^*(t)|_{\mathcal{N}^\perp}^2 \\ (83) \quad & + (e^{\lambda t} \Pi v_0^*(t), v_0^*(t))_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)} + e^{\lambda t} |z^{v(t) \cdot \mathbf{n}}|_{\mathcal{N}^\perp}^2 = 0. \end{aligned}$$

with $\mathbf{A} = \Pi L_{\hat{u}(t)} - \frac{\lambda}{2} (P_{\mathcal{A}_{\Xi_1}} - \Pi)$; notice that $(1 - \Pi)|_{\mathcal{A}_{\Xi_1}} = (P_{\mathcal{A}_{\Xi_1}} - \Pi)|_{\mathcal{A}_{\Xi_1}}$.

Next, from

$$e^{-\lambda t} |\Xi^\diamond P_\nabla R^t v_0^*(t)|_{\mathcal{N}^\perp}^2 = \left(e^{-\lambda t} R^t (1 - \Pi) \hat{F} \Xi^\diamond P_\nabla R^t v_0^*(t), v_0^*(t) \right)_{\mathcal{A}_{\Xi_1}}$$

and noticing in particular that $(1 - \Pi)\hat{F}\Xi = (\Xi^\diamond P_\nabla)^*$, or equivalently, $(1 - \Pi)\hat{F}\Xi)^* = \Xi^\diamond P_\nabla$, we arrive to

$$\begin{aligned} & (\partial_s|_{s=t} R^s v_0^*(t), v_0^*(t))_{\mathcal{A}_{\Xi_1}} + ((R^t \mathbf{A} + \mathbf{A}^* R^t) v_0^*(t), v_0^*(t))_{\mathcal{A}_{\Xi_1}} \\ & - (R^t \mathbf{B} \mathbf{B}^* R^t v_0^*(t), v_0^*(t))_{\mathcal{A}_{\Xi_1}} + (e^{\lambda t} \Pi v_0^*(t), v_0^*(t))_{L^2_{\text{div}}(\Omega, \mathbb{R}^3)} + e^{\lambda t} |z^{v(t) \cdot \mathbf{n}}|_{\mathcal{N}^\perp}^2 = 0. \end{aligned}$$

with $\mathbf{B} = e^{-\frac{\lambda}{2}t} (P_{\mathcal{A}_{\Xi_1}} - \Pi) \hat{F} \Xi$.

Now, notice that the mapping $(u, v) \mapsto (z^{u \cdot \mathbf{n}}, z^{v \cdot \mathbf{n}})_{\mathcal{N}^\perp}$ is bilinear and continuous in \mathcal{A}_{Ξ_1} . Moreover we can see that $(z^{u \cdot \mathbf{n}}, z^{v \cdot \mathbf{n}})_{\mathcal{N}^\perp} \neq 0$ only if $(u, v) \in H^\perp \times \perp$. If we set $\bar{u} \in H^\perp \cap \mathcal{A}_{\Xi_1}$ we also find that $(z^{\bar{u} \cdot \mathbf{n}}, z^{\bar{u} \cdot \mathbf{n}})_{\mathcal{N}^\perp} \geq C |u|_{\mathcal{A}_{\Xi_1}}^2$ (cf. Lemma 3.7). By the Lax-Milgram Lemma it follows that There is $\mathcal{Z}_\Xi^\perp \bar{u} \in H^\perp \cap \mathcal{A}_{\Xi_1}$ such that $(z^{\bar{u} \cdot \mathbf{n}}, z^{\bar{v} \cdot \mathbf{n}})_{\mathcal{N}^\perp} = (\mathcal{Z}_\Xi^\perp \bar{u}, \bar{v})_{\mathcal{A}_{\Xi_1}}$, for all $\bar{v} \in H^\perp \cap \mathcal{A}_{\Xi_1}$, that is, $(z^{u \cdot \mathbf{n}}, z^{v \cdot \mathbf{n}})_{\mathcal{N}^\perp} = ((P_{\mathcal{A}_{\Xi_1}} - \Pi) \mathcal{Z}_\Xi^\perp (P_{\mathcal{A}_{\Xi_1}} - \Pi) u, v)_{\mathcal{A}_{\Xi_1}}$ for all $(u, v) \in \mathcal{A}_{\Xi_1}^2$. Therefore, with $\mathcal{Z}_\Xi = (P_{\mathcal{A}_{\Xi_1}} - \Pi) \mathcal{Z}_\Xi^\perp (P_{\mathcal{A}_{\Xi_1}} - \Pi)$, and $\mathbf{C} = e^{\lambda t} (\Pi + \mathcal{Z}_\Xi)$ we can write

$$((\partial_s|_{s=t} R^s + R^t \mathbf{A} + \mathbf{A}^* R^t - R^t \mathbf{B} \mathbf{B}^* R^t + \mathbf{C}) v_0^*(t), v_0^*(t))_{\mathcal{A}_{\Xi_1}} = 0.$$

Finally, by the dynamic programming principle we can restrict ourselves to the interval of time $t > \tau > 0$ and set the initial condition $v(\tau) = w$, and should obtain again (83). Essentially this means that we may replace $v_0^*(t)$ by an arbitrary $w \in \mathcal{A}_{\Xi_1}$. Therefore we conclude that (79) holds.

Remark 4.10. Notice that from (77), we have that $(1 - \Pi)\hat{F}f = 0$ only if $f \cdot \mathbf{n} = 0$, which implies that $\mathbf{B} \mathbf{B}^* Rv = 0$ only if $Rv \in H$, which is equivalent to $P_\nabla Rv = 0$. Notice also that $(P_{\mathcal{A}_{\Xi_1}} - \Pi)\hat{F}\Xi = (\Xi^\diamond P_\nabla)^*$ implies that $(P_{\mathcal{A}_{\Xi_1}} - \Pi)\hat{F}\Xi$ is independent of the extension \hat{F} , which also reflects the fact that, for $u \in L^2_{\text{div}}(\Omega, \mathbb{R}^3)$, $(1 - \Pi)u$ is “supported” on the boundary (as we have seen in Section 4.2, Step 2). That is, we are free to choose/construct the extension \hat{F} , and not necessarily by solving the M Stokes systems in (77).

5. STABILIZATION OF THE NAVIER–STOKES SYSTEM

In this Section we prove that the feedback controller constructed in Section 4, to stabilize the linear Oseen–Stokes system (15) to zero, also stabilizes locally the nonlinear system (14) to zero. The main result of the paper is given in Section 5.2; before, due to some well known issues related with the existence and uniqueness of solutions we need to recall some definitions (cf. [Rod14b, Section 5]).

5.1. Solutions for the nonlinear systems. Let $a, b \in \mathbb{R}$ be two real numbers with $0 \leq a < b$.

Definition 5.1. Given $f \in L^2((a, b), H^{-1}(\Omega, \mathbb{R}^3))$, $k \in L^4((a, b), L^6(\Omega, \mathbb{R}^3))$, and $y_0 \in H$, we say that $y \in L^2((a, b), V) \cap L^\infty((a, b), H)$, with $\partial_t y \in L^1((a, b), V')$, is a weak solution for system

$$(84) \quad \begin{aligned} \partial_t y + \langle y \cdot \nabla \rangle y + \mathcal{B}(k)y - \nu \Delta y + \nabla p + f &= 0, & \text{div } y &= 0, \\ y|_\Gamma &= 0, & y(a) &= y_0, \end{aligned}$$

in $(a, b) \times \Omega$, if it is a weak solution in the classical sense of [Tem01].

For simplicity, and for $r > 1$, $k > 1$, we define the subspace

$$\Theta_{(a,b)}^{r,k} := W((a, b), H^1_{\text{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \cap L^r((a, b), L^k(\Omega, \mathbb{R}^3)).$$

Definition 5.2. Given $h \in L^2((a, b), H^{-1}(\Omega, \mathbb{R}^3))$, $z \in \Theta_{(a,b)}^{4,6}$, and $u_0 \in L_{\text{div}}^2(\Omega, \mathbb{R}^3)$, we say that u satisfying

$$u \in L^2((a, b), H_{\text{div}}^1(\Omega, \mathbb{R}^3)) \cap L^\infty((a, b), L_{\text{div}}^2(\Omega, \mathbb{R}^3)), \quad \partial_t u \in L^1((a, b), H^{-1}(\Omega, \mathbb{R}^3)),$$

is a weak solution for system (1), in $(a, b) \times \Omega$, with $\gamma + \zeta = z|_\Gamma$ and $u(a) = u_0$, if $y = u - z$ is a weak solution for the system (84) with $f = h + \partial_t z + \langle z \cdot \nabla \rangle z - \nu \Delta z$, $k = z$, and $y_0 = u_0 - z(a) \in H$.

Definition 5.3. Given $\hat{u} \in L_{\text{div}}^\infty((a, b) \times \Omega, \mathbb{R}^3)$, $z \in \Theta_{(a,b)}^{4,6}$, and $v_0 \in L_{\text{div}}^2(\Omega, \mathbb{R}^3)$, we say that v satisfying

$$v \in L^2((a, b), H_{\text{div}}^1(\Omega, \mathbb{R}^3)) \cap L^\infty((a, b), L_{\text{div}}^2(\Omega, \mathbb{R}^3)), \quad \partial_t v \in L^1((a, b), H^{-1}(\Omega, \mathbb{R}^3))$$

is a weak solution for system (14), in $(a, b) \times \Omega$, with $\zeta = z|_\Gamma$ and $v(a) = v_0$, if $y = v - z$ is a weak solution for the system (84) with $f = \partial_t z + \langle z \cdot \nabla \rangle z - \nu \Delta z + \mathcal{B}(\hat{u})z$, $k = \hat{u} + z$ and $y_0 = v_0 - z(a) \in H$.

Analogously, we define the global solutions in $\mathbb{R}_0 \times \Omega$: let us denote, for simplicity, $\mathcal{X}_{\text{loc}}(\mathbb{R}_0) := \{f \mid f|_{(0,T)} \in \mathcal{X}((0, T)) \text{ for all } T > 0\}$, where $\mathcal{X}((0, T))$ is a suitable space of functions defined in $(0, T) \subset \mathbb{R}_0$.

Definition 5.4. Given $h \in L_{\text{loc}}^2(\mathbb{R}_0, H^{-1}(\Omega, \mathbb{R}^3))$, $z \in \Theta_{\mathbb{R}_0, \text{loc}}^{4,6}$, and $u_0 \in L_{\text{div}}^2(\Omega, \mathbb{R}^3)$, we say that u is a weak solution for system (1)–(2), in $\mathbb{R}_0 \times \Omega$, if $u|_{(0,T)}$ is a weak solution in $(0, T) \times \Omega$, for all $T > 0$, for the same system with the data $h|_{(0,T)}$, $k|_{(0,T)}$.

Definition 5.5. Given $\hat{u} \in L_{\text{div}, \text{loc}}^\infty((0, T) \times \Omega, \mathbb{R}^3)$, $z \in \Theta_{(a,b), \text{loc}}^{4,6}$, and $v_0 \in L_{\text{div}, \text{loc}}^2(\Omega, \mathbb{R}^3)$, we say that v is a weak solution for system (14), in $\mathbb{R}_0 \times \Omega$, if $v|_{(0,T)}$ is a weak solution in $(0, T) \times \Omega$, for all $T > 0$, for the same system with the data $h|_{(0,T)}$, $k|_{(0,T)}$.

The existence of the solution in Definitions 5.1, 5.2 and 5.3 can be proved following classical arguments, and some continuity observations as mentioned in [Rod14b, Remarks 5.1 and 5.2]. Then it also follow the existence of the solutions in Definitions 5.4 and 5.5. Concerning the uniqueness, following an argument as in the proof of Lemma 5.3 in [Rod14b], we have the following:

Lemma 5.6. *The solution in Definitions 5.1, 5.2 and 5.3, is unique if it is in $\Theta_{(a,b)}^{4,6}$. The solution in Definitions 5.4, and 5.5, is unique if it is in $\Theta_{\mathbb{R}_0, \text{loc}}^{4,6}$.*

5.2. Main Theorem. In order to have enough regularity to deal with the nonlinear problems, we take strong admissible initial conditions in the subspace $\mathcal{A}_{\Xi_2}^{\mathbf{n}} := \{u \in H_{\text{div}}^1(\Omega, \mathbb{R}^3) \mid u|_\Gamma = \Xi Q_f^M z \text{ for some } z \in \mathbb{R}^{2M}\} \subset \mathcal{A}_{\Xi_2} \subset \mathcal{A}_{\Xi_1}$ of the set of admissible strong initial conditions for the Oseen-Stokes system (see Section 2.3 and Remark 2.9). That is, for simplicity we will work with so-called strong solutions, though the uniqueness issue could be overcome with less regular solutions. The main result of the paper is:

Theorem 5.7. *For each $\hat{u} \in \mathcal{W}^{\text{st}}$ and $\lambda > 0$, there exists an integer $M = \overline{C}_{[[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda}]} \geq 1$ such that, given $v_0 \in \mathcal{A}_{\Xi_2}^{\mathbf{n}}$ the weak solution v of system (14) in $\mathbb{R}_0 \times \Omega$, with $\zeta = e^{-\frac{\lambda}{2}t} \left(v_0|_\Gamma + \Xi Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} v(s) ds \right)$, is unique and satisfies the inequality*

$$(85) \quad |v(t)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}_{[[\hat{u}]_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}]} e^{-\lambda t} |v_0|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2, \quad t \geq 0,$$

where $\mathcal{K}_{\hat{u}}^{\lambda, s} : \mathcal{A}_{\Xi_1} \mapsto \mathbb{R}^{2M}$ is the feedback rule in Theorem 4.1, that is, in (61).

Before the proof we make some observations. First of all notice that for $v_0 \in \mathcal{A}_{\Xi_2}^{\mathbf{n}}$ we have $v_0|_{\Gamma} = (v_0 \cdot \mathbf{n})\mathbf{n}|_{\Gamma}$, so that the feedback integral controller in Theorem 5.7 coincides with the one in Theorem 4.1; next, we derive the following:

Lemma 5.8. *If $v_0 \in \mathcal{A}_{\Xi_2}^{\mathbf{n}}$, the solution v in Theorem 4.1 satisfies*

$$\sup_{r \geq 0} |e^{\frac{\lambda}{2}t} v|_{W((r, r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |v_0|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2.$$

Proof. We know that, taking $a = 0$ in Theorem 4.1,

$$(86) \quad |e^{\frac{\lambda}{2}t} v|_{W(\mathbb{R}_0, H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2;$$

since $w(t) := e^{\frac{\lambda}{2}t} v(t)$ solves system (16) with $g = -\frac{\lambda}{2}v$, $K = \Xi$, and $\eta(t) = e^{\frac{\lambda}{2}t} \kappa = \left(z^{v_0 \cdot \mathbf{n}} + Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} v(s) ds \right)$, then from Theorem 2.6 we have that

$$\begin{aligned} & |w|_{W((r, r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \\ & \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}] \left(|w(r)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + \frac{\lambda^2}{4} |v|_{L^2((r, r+1), L_{\text{div}}^2(\Omega, \mathbb{R}^3))}^2 + |\eta|_{H^1((r, r+1), \mathcal{N}^\perp)}^2 \right). \end{aligned}$$

On the other hand, by construction, in the proof of Theorem 4.1 we know that the solution coincides with the minimizer of Problem 4.5, with $s = 0$; and, from (86) it follows that $|v|_{L^2((r, r+1), L_{\text{div}}^2(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2$. Therefore we have

$$\begin{aligned} & |w|_{W((r, r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \\ & \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} \left(|w(r)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + M_0^\lambda(v, \kappa) \right); \end{aligned}$$

and from Lemma 4.7 we have $M_0^\lambda(v, \kappa) \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2$, which implies

$$\begin{aligned} & |w|_{W((r, r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} \left(|w(r)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \right) \\ & \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} \left(e^{\lambda r} |v(r)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \right). \end{aligned}$$

In particular, if $[r] \in \mathbb{N}$, is the biggest integer that is smaller than r , $[r] \leq r < [r] + 1$, we have $|w([r] + 1)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} \left(e^{\lambda [r]} |v([r])|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \right)$, and

$$\begin{aligned} & |w|_{W((r, r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \leq |w|_{W((\lfloor r \rfloor, \lfloor r \rfloor + 2), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \\ & \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} \left(|w(\lfloor r \rfloor)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |w(\lfloor r \rfloor + 1)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + 2|v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \right) \\ (87) \quad & \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} \left(e^{\lambda \lfloor r \rfloor} |v(\lfloor r \rfloor)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \right). \end{aligned}$$

If $[r] > 0$, that is, if $r \geq 1$ by Lemma 2.8 we also have

$$(88) \quad |v[r]|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}] \left(|v(\lfloor r \rfloor - 1)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |\kappa|_{H^1(\lfloor r \rfloor - 1, \lfloor r \rfloor), \mathcal{N}^\perp)}^2 \right),$$

and from $\eta(t) = e^{\frac{\lambda}{2}t} \kappa(t)$ and $\partial_t \eta = \frac{\lambda}{2} \eta + e^{\frac{\lambda}{2}t} \partial_t \kappa$, we can conclude that

$$\begin{aligned} & |\kappa|_{H^1(\lfloor r \rfloor - 1, \lfloor r \rfloor), \mathcal{N}^\perp)}^2 = |\kappa|_{L^2(\lfloor r \rfloor - 1, \lfloor r \rfloor), \mathcal{N}^\perp)}^2 + |\partial_t \kappa|_{L^2(\lfloor r \rfloor - 1, \lfloor r \rfloor), \mathcal{N}^\perp)}^2 \\ & \leq e^{-\lambda(\lfloor r \rfloor - 1)} |\eta|_{L^2(\lfloor r \rfloor - 1, \lfloor r \rfloor), \mathcal{N}^\perp)}^2 + |e^{-\frac{\lambda}{2}t} (\partial_t \eta - \frac{\lambda}{2} \eta)|_{L^2(\lfloor r \rfloor - 1, \lfloor r \rfloor), \mathcal{N}^\perp)}^2 \\ (89) \quad & \leq \overline{C}_{[\lambda]} e^{-\lambda(\lfloor r \rfloor - 1)} |\eta|_{H^1(\lfloor r \rfloor - 1, \lfloor r \rfloor), \mathcal{N}^\perp)}^2 \leq \overline{C}_{[\lambda]} e^{-\lambda(\lfloor r \rfloor - 1)} M_0^\lambda(v, \kappa); \end{aligned}$$

thus, from (87), (88), and (89), and recalling (44) and (45), it follows that if $r \geq 1$, then

$$\begin{aligned} & |e^{\frac{\lambda}{2}t}v(t)|_{W((r,r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 = |w|_{W((r,r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \\ & \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} \left(e^{\lambda \lfloor r \rfloor} |v(\lfloor r \rfloor - 1)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 + |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \right). \end{aligned}$$

Now, (86) implies $|v(\lfloor r \rfloor - 1)|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} e^{-\lambda(\lfloor r \rfloor - 1)} |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2$, thus

$$|e^{\frac{\lambda}{2}t}v(t)|_{W((r,r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |v_0|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2, \text{ for } r \geq 1.$$

From (87) we also have

$$|e^{\frac{\lambda}{2}t}v(t)w|_{W((r,r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |v_0|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2, \text{ for } r \in [0, 1),$$

which ends the proof. \square

Inspired by Theorem 3.2 and Lemma 5.8, we define the Banach space

$$\mathcal{Z}^\lambda := \left\{ z \in L_{\text{loc}}^2(\mathbb{R}_0, H_{\text{div}}^2(\Omega, \mathbb{R}^3)) \mid \sup_{t \geq 0} \left| e^{\frac{\lambda}{2}t}v \right|_{W((r,r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 < \infty \right\}$$

endowed with the norm $|z|_{\mathcal{Z}^\lambda} := \sup_{r \geq 0} \left| e^{\frac{\lambda}{2}t}v \right|_{W((r,r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}$.

For a given constant $\rho > 0$ and a vector function $v_0 \in \mathcal{A}_{\Xi_2}^{\mathbf{n}}$, we define the subset

$$\mathcal{Z}_\rho^\lambda := \{ z \in \mathcal{Z}^\lambda \mid z(0) = v_0, |z|_{\mathcal{Z}^\lambda}^2 \leq \rho |v_0|_{\mathcal{A}_{\Xi_2}^{\mathbf{n}}}^2 \},$$

and the mapping $\Psi: \mathcal{Z}_\rho^\lambda \rightarrow \mathcal{Z}_{\text{loc}}^\lambda$ that takes a function $\bar{z} \in \mathcal{Z}^\lambda$ to the solution z of

$$(90) \quad \begin{aligned} & \partial_t z + \mathcal{B}(\hat{u})z - \nu \Delta z + \nabla p_{z, \bar{z}} = -P_{\mathcal{A}_{\Xi_1}} \langle \bar{z} \cdot \nabla \rangle \bar{z}, \quad \text{div } z = 0, \\ & z|_\Gamma(t) = e^{-\frac{\lambda}{2}t} \left(v_0|_\Gamma + \Xi Q_f^M \int_0^t \mathcal{K}_u^{\lambda, s} z(s) ds \right), \quad z(0) = v_0, \end{aligned}$$

where, we recall, $P_{\mathcal{A}_{\Xi_1}}$ stands for the orthogonal projection in $L^2(\Omega, \mathbb{R}^3)$ onto \mathcal{A}_{Ξ_1} .

Lemma 5.9. *Under the hypotheses of Theorem 5.7, there exists $\rho > 0$ such that the following property holds: for any $\gamma \in (0, 1)$ one can find a constant $\epsilon = \epsilon_\gamma > 0$ such that for any $v_0 \in \mathcal{A}_{\Xi_2}^{\mathbf{n}}$ with $|v_0|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)} \leq \epsilon$ the mapping Ψ takes the set \mathcal{Z}_ρ^λ into itself and satisfies the inequality*

$$(91) \quad |\Psi(\bar{z}_1) - \Psi(\bar{z}_2)|_{\mathcal{Z}^\lambda} \leq \gamma |\bar{z}_1 - \bar{z}_2|_{\mathcal{Z}^\lambda} \quad \text{for all } \bar{z}_1, \bar{z}_2 \in \mathcal{Z}_\rho^\lambda.$$

Proof. We divide the proof into three main steps:

③ Step 1: *a preliminary estimate.* Consider the system

$$(92) \quad \begin{aligned} & \partial_t z + \mathcal{B}(\hat{u})z - \nu \Delta z + \nabla p_z = f, \quad \text{div } z = 0, \\ & z|_\Gamma(t) = e^{-\frac{\lambda}{2}t} \left((v_0 \cdot \mathbf{n}) \mathbf{n}|_\Gamma + \Xi Q_f^M \int_0^t \mathcal{K}_u^{\lambda, s} z(s) ds \right), \quad z(0) = v_0. \end{aligned}$$

where $v_0 \in \mathcal{A}_{\Xi_1}$ and $f \in L_{\text{loc}}^2(\mathbb{R}_0, \mathcal{A}_{\Xi_1})$. Now, if v is the solution of system (92) with $f = 0$, and $v_0 \in \mathcal{A}_{\Xi_2}^{\mathbf{n}} \subset \mathcal{A}_{\Xi_1}$, we know by Lemma 5.8 that

$$(93) \quad \sup_{r \geq 0} |e^{\frac{\lambda}{2}t}v|_{W((r,r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |v_0|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2.$$

We want a version of this estimate for suitable nonzero f s; for that we denote by $\mathcal{S}_{0,t}^f(v_0)$ the solution z of (92). In the case $f = 0$, the operator $\mathcal{S}_{0,t}^0$ is linear; by the Duhamel formula we can write

$$(94) \quad z(t) = \mathcal{S}_{0,t}^f(v_0) = \mathcal{S}_{0,t}^0 v_0 + \int_0^t \mathcal{S}_{s,t}^0 f(s) ds$$

where $\mathcal{S}_{s,t}^f(z_s)$ denotes the solution of the system (92), with the initial time moved to $t = s$, and the initial condition $z(s) = z_s$. On the other hand, from (39), it follows in particular that $|e^{\frac{\lambda}{2}(t-s)} \mathcal{S}_{s,t}^0 w|_{\mathcal{A}_{\Xi_1}}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} |w|_{\mathcal{A}_{\Xi_1}}^2$; then

$$(95) \quad |z(t)|_{\mathcal{A}_{\Xi_1}}^2 \leq 2 |\mathcal{S}_{0,t}^0 v_0|_{\mathcal{A}_{\Xi_1}}^2 + 2 \left| \int_0^t \mathcal{S}_{s,t}^0 f(s) ds \right|_{\mathcal{A}_{\Xi_1}}^2 \\ \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} e^{-\lambda t} \left(|z(0)|_{\mathcal{A}_{\Xi_1}}^2 + \left(\int_0^t e^{\frac{\lambda}{2}s} |f(s)|_{\mathcal{A}_{\Xi_1}} ds \right)^2 \right)$$

Now we can find, denoting again by $[t] \in \mathbb{N}$ the integer satisfying $[t] \leq t < [t] + 1$,

$$\int_0^t e^{\frac{\lambda}{2}s} |f(s)|_{\mathcal{A}_{\Xi_1}} ds \leq \sum_{k=0}^{[t]} \int_k^{k+1} e^{-\frac{\lambda}{2}s} e^{\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}} ds \\ \leq \sum_{k=0}^{[t]} \left(\int_k^{k+1} e^{-\lambda s} ds \right)^{\frac{1}{2}} \left(\int_k^{k+1} e^{2\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 ds \right)^{\frac{1}{2}} \\ \leq \sup_{\substack{k \in \mathbb{N} \\ 0 \leq k \leq [t]}} \left(\int_k^{k+1} e^{2\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 ds \right)^{\frac{1}{2}} \sum_{k=0}^{[t]} \left(\int_k^{k+1} e^{-\lambda s} ds \right)^{\frac{1}{2}}$$

and for the sum of the series, we have

$$\sum_{k=0}^{[t]} \left(\int_k^{k+1} e^{-\lambda s} ds \right)^{\frac{1}{2}} \leq \sum_{k=0}^{\infty} \left(\int_k^{k+1} e^{-\lambda s} ds \right)^{\frac{1}{2}} = \left(-\frac{1}{\lambda} (e^{-\lambda} - 1) \right)^{\frac{1}{2}} \sum_{k=0}^{\infty} e^{-\frac{\lambda}{2}k} \\ = \frac{(1 - e^{-\lambda})^{\frac{1}{2}}}{\lambda^{\frac{1}{2}} (1 - e^{-\frac{\lambda}{2}})} = \overline{C}_{[\frac{1}{\lambda}]}.$$

Hence $\int_0^t e^{\frac{\lambda}{2}s} |f(s)|_{\mathcal{A}_{\Xi_1}} ds \leq \overline{C}_{[\frac{1}{\lambda}]} \sup_{\substack{k \in \mathbb{N} \\ 0 \leq k \leq [t]}} \left(\int_k^{k+1} e^{2\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 ds \right)^{\frac{1}{2}}$ and, recalling (95),

$$(96) \quad \sup_{t \geq 0} \left(e^{\lambda t} |z(t)|_{\mathcal{A}_{\Xi_1}}^2 \right) \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}]} \left(|z(0)|_{\mathcal{A}_{\Xi_1}}^2 + \sup_{\substack{k \in \mathbb{N} \\ 0 \leq k \leq [t]}} \int_k^{k+1} e^{2\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 ds \right)$$

Now we use Lemma 2.8 to obtain

$$(97) \quad |z(r+1)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}] } \left(|z(r)|_{\mathcal{A}_{\Xi_1}}^2 + |f|_{L^2((r, r+1), \mathcal{A}_{\Xi_1})}^2 \right) \\ + \overline{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}}] } \left| e^{-\frac{\lambda}{2}t} \left(z^{v_0 \cdot \mathbf{n}} + Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} z(s) ds \right) \right|_{H^1((r, r+1), \mathcal{N}^\perp)}^2.$$

Observe that, by $z|_{\Gamma} = \Xi e^{-\frac{\lambda}{2}t} \left(z^{v_0 \cdot \mathbf{n}} + Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} z(s) ds \right)$ and by the definition of $z^{u \cdot \mathbf{n}}$ in (23), we necessarily have $e^{-\frac{\lambda}{2}t} \left(z^{v_0 \cdot \mathbf{n}} + Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} z(s) ds \right) = z^{v(t) \cdot \mathbf{n}}$. We may write

$$\left| e^{-\frac{\lambda}{2}t} \left(z^{v_0 \cdot \mathbf{n}} + Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} z(s) ds \right) \right|_{H^1((r, r+1), \mathcal{N}^\perp)}^2 \\ \leq (1 + 2\frac{\lambda^2}{4}) |z^{v(t) \cdot \mathbf{n}}|_{L^2((r, r+1), \mathcal{N}^\perp)}^2 + 2 \left| e^{-\frac{\lambda}{2}t} Q_f^M \mathcal{K}_{\hat{u}}^{\lambda, t} z(t) \right|_{L^2((r, r+1), \mathcal{N}^\perp)}^2$$

Recalling the bound in (62) and using the continuity of $u \mapsto z^{u \cdot \mathbf{n}}$, we find

$$\begin{aligned}
& \left| e^{-\frac{\lambda}{2}t} \left(z^{v_0 \cdot \mathbf{n}} + Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} z(s) \, ds \right) \right|_{H^1((r, r+1), \mathcal{N}^\perp)}^2 \\
& \leq \overline{C}[\lambda] |v(t)|_{L^2((r, r+1), \mathcal{A}_{\Xi_1})}^2 + \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}] e^{-\lambda r} e^{\lambda(r+1)} |z(t)|_{L^2((r, r+1), \mathcal{A}_{\Xi_1})}^2 \\
(98) \quad & \leq \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}] \sup_{t \in (r, r+1)} |z(t)|_{\mathcal{A}_{\Xi_1}}^2.
\end{aligned}$$

Thus, from (96) and (97), we can obtain

$$\begin{aligned}
(99) \quad & |z(r+1)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}] \left(\sup_{t \in (r, r+1)} |z(t)|_{\mathcal{A}_{\Xi_1}}^2 + |f|_{L^2((r, r+1), \mathcal{A}_{\Xi_1})}^2 \right) \\
& \leq \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}] e^{-\lambda r} \left(|z(0)|_{\mathcal{A}_{\Xi_1}}^2 + \sup_{\substack{k \in \mathbb{N} \\ 0 \leq k \leq \lfloor r+1 \rfloor}} \int_k^{k+1} e^{2\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 \, ds \right);
\end{aligned}$$

notice that $|f|_{L^2((r, r+1), \mathcal{A}_{\Xi_1})}^2 \leq e^{-\lambda r} \int_r^{r+1} e^{\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 \, ds \leq e^{-\lambda r} \int_{\lfloor r \rfloor}^{\lfloor r+1 \rfloor + 1} e^{2\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 \, ds \leq 2e^{-\lambda r} \sup_{\substack{k \in \mathbb{N} \\ 0 \leq k \leq \lfloor r+1 \rfloor}} \int_k^{k+1} e^{2\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 \, ds$.

For $t \in (0, 1)$ we also have, from Theorem 2.6 that

$$\begin{aligned}
(100) \quad & |z(t)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}] \left(|z(0)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |f|_{L^2((0, 1), \mathcal{A}_{\Xi_1})}^2 \right) \\
& \quad + \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}] \left| e^{-\frac{\lambda}{2}t} \left(z^{v_0 \cdot \mathbf{n}} + Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} z(s) \, ds \right) \right|_{H^1((0, 1), \mathcal{N}^\perp)}^2,
\end{aligned}$$

thus from (98) and (86), we find that for all $t \in (0, 1)$

$$|z(t)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}] \left(|z(0)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + \int_0^1 |f(s)|_{\mathcal{A}_{\Xi_1}}^2 \, ds \right);$$

which, together with (99), gives for all $t \geq 0$:

$$|z(t)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}] e^{-\lambda t} \left(|z(0)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + \sup_{\substack{k \in \mathbb{N} \\ 0 \leq k \leq \lfloor t \rfloor}} \int_k^{k+1} e^{2\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 \, ds \right);$$

where we have used $0 < e^{\lambda r}$, $e^{-\lambda r} = e^\lambda e^{-\lambda(r+1)}$, and $1 < e^\lambda e^{-\lambda t}$ for $r \geq 0$ and $t \in (0, 1)$.

Using again Theorem 2.6 and proceeding as above we can derive

$$\begin{aligned}
& |z|_{W((r, r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \\
& \leq \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}] \left(|z(r)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |f|_{L^2((r, r+1), \mathcal{A}_{\Xi_1})}^2 \right) \\
& \quad + \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}] \left| e^{-\frac{\lambda}{2}t} \left(z^{v_0 \cdot \mathbf{n}} + Q_f^M \int_0^t \mathcal{K}_{\hat{u}}^{\lambda, s} z(s) \, ds \right) \right|_{H^1((r, r+1), \mathcal{N}^\perp)}^2 \\
& \leq \overline{C}[\hat{u}|_{\mathcal{W}^{\text{st}}}, \lambda, \frac{1}{\lambda}] e^{-\lambda r} \left(|z(0)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + \sup_{\substack{k \in \mathbb{N} \\ 0 \leq k \leq \lfloor r \rfloor}} \int_k^{k+1} e^{2\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 \, ds \right);
\end{aligned}$$

which implies, since $\partial_t(e^{\frac{\lambda}{2}t}z) = \frac{\lambda}{2}e^{\frac{\lambda}{2}t}z + e^{\frac{\lambda}{2}t}\partial_t z$, that

$$(101) \quad \sup_{r \geq 0} |e^{\frac{\lambda}{2}t}z(t)|_{W((r, r+1), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \\ \leq \bar{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}]} \left(|z(0)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + \sup_{k \in \mathbb{N}} \int_k^{k+1} e^{2\lambda s} |f(s)|_{\mathcal{A}_{\Xi_1}}^2 ds \right)$$

which is the wanted version of (93) for nonzero f s.

Ⓢ Step 2: Ψ maps \mathcal{Z}_ρ^λ into itself, if $|v_0|_{\mathcal{A}_{\Xi_2}}$ is small. We will replace f by $-P_{\mathcal{A}_{\Xi_1}} \langle \bar{z} \cdot \nabla \rangle \bar{z}$ in (101). First we recall some standard estimates for the nonlinear term: from the Agmon inequality (see [Tem97, Chapter II, Section 1.4]), $|u|_{L^\infty(\Omega, \mathbb{R}^3)} \leq C_1 |u|_{H^1(\Omega, \mathbb{R}^3)}^{\frac{1}{2}} |u|_{H^2(\Omega, \mathbb{R}^3)}^{\frac{1}{2}} \leq C_2 |u|_{H^2(\Omega, \mathbb{R}^3)}$, we can obtain $|\langle \bar{z} \cdot \nabla \rangle \bar{w}|_{L^2(\Omega, \mathbb{R}^3)} \leq C |\bar{z}|_{H_{\text{div}}^2(\Omega, \mathbb{R}^3)} |\bar{w}|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}$ and

$$\sup_{\substack{k \in \mathbb{N} \\ 0 \leq k \leq [t]}} \int_k^{k+1} e^{2\lambda s} |\langle \bar{z} \cdot \nabla \rangle \bar{z}|_{L^2(\Omega, \mathbb{R}^3)}^2 ds \\ \leq \sup_{s \in (0, [t]+1)} |e^{\frac{\lambda}{2}s} \bar{z}(s)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \sup_{\substack{k \in \mathbb{N} \\ 0 \leq k \leq [t]}} \int_k^{k+1} |e^{\frac{\lambda}{2}s} \bar{z}|_{H_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 ds.$$

Thus, inequality (101) with $f = -P_{\mathcal{A}_{\Xi_1}} \langle \bar{z} \cdot \nabla \rangle \bar{z}$ gives us

$$(102) \quad |\Psi(\bar{z})|_{\mathcal{Z}^\lambda}^2 \leq \bar{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}]} \left(|z(0)|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |\bar{z}|_{\mathcal{Z}^\lambda}^4 \right).$$

If $\bar{z} \in \mathcal{Z}_\rho^\lambda$, then

$$(103) \quad |\Psi(\bar{z})|_{\mathcal{Z}^\lambda}^2 \leq \bar{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}]} (1 + \rho^2 |v_0|_{\mathcal{A}_{\Xi_2}}^2) |v_0|_{\mathcal{A}_{\Xi_2}}^2$$

and if we set $\rho = 2\bar{C}$ and $\epsilon < \frac{1}{\rho}$, where $\bar{C} = \bar{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}]}$ is the constant in (103), then we obtain $\bar{C}(1 + \rho^2 \epsilon^2) \leq \rho$ if $|v_0|_{\mathcal{A}_{\Xi_2}} \leq \epsilon$, which means that $\Psi(\bar{z}) \in \mathcal{Z}_\rho^\lambda$.

Ⓢ Step 3: Ψ is a contraction, if $|v_0|_{\mathcal{A}_{\Xi_2}}$ is smaller. It remains to prove (91). Let us take two functions $\bar{z}_1, \bar{z}_2 \in \mathcal{Z}_\rho^\lambda$ and set $e = \bar{z}_1 - \bar{z}_2$ and $d = \Psi(\bar{z}_1) - \Psi(\bar{z}_2)$. Then the function d solves (92) with the initial condition $d(0) = 0$ and $f = P_{\mathcal{A}_{\Xi_1}} \langle \bar{z}_2 \cdot \nabla \rangle \bar{z}_2 - P_{\mathcal{A}_{\Xi_1}} \langle \bar{z}_1 \cdot \nabla \rangle \bar{z}_1$. Therefore, by inequality (101), we have

$$(104) \quad |\Psi(\bar{z}_1) - \Psi(\bar{z}_2)|_{\mathcal{Z}^\lambda}^2 \leq \bar{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}]} \sup_{t \geq 0} \int_t^{t+1} e^{2\lambda s} |\langle \bar{z}_2 \cdot \nabla \rangle \bar{z}_2 - \langle \bar{z}_1 \cdot \nabla \rangle \bar{z}_1|_{L^2(\Omega, \mathbb{R}^3)}^2 ds.$$

Straightforward computations give us

$$|\langle \bar{z}_2 \cdot \nabla \rangle \bar{z}_2 - \langle \bar{z}_1 \cdot \nabla \rangle \bar{z}_1|_{L^2(\Omega, \mathbb{R}^3)}^2 = |\langle e \cdot \nabla \rangle \bar{z}_2 + \langle \bar{z}_1 \cdot \nabla \rangle e|_{L^2(\Omega, \mathbb{R}^3)}^2 \\ \leq C \left(|e|_{H_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 |\bar{z}_2|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |\bar{z}_1|_{H_{\text{div}}^2(\Omega, \mathbb{R}^3)}^2 |e|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 \right)^2$$

and, from (104), it follows

$$(105) \quad |\Psi(\bar{z}_1) - \Psi(\bar{z}_2)|_{\mathcal{Z}^\lambda}^2 \leq \bar{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}]} \left(|\bar{z}_1|_{\mathcal{Z}^\lambda}^2 + |\bar{z}_2|_{\mathcal{Z}^\lambda}^2 \right) |z|_{\mathcal{Z}^\lambda}^2 \\ \leq \bar{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}]} 2\rho |v_0|_{\mathcal{A}_{\Xi_2}}^2 |\bar{z}_1 - \bar{z}_2|_{\mathcal{Z}^\lambda}^2$$

Choosing $\epsilon > 0$ (smaller than the one chosen in Step 2 and) such that $2\bar{C}_2 \rho \epsilon^2 \leq \gamma^2$, where $\bar{C}_2 = \bar{C}_{[\hat{u}|_{\mathcal{W}^{\text{st}}, \lambda, \frac{1}{\lambda}}]}$ is the constant in (105), we see that if $|v_0|_{\mathcal{A}_{\Xi_2}} \leq \epsilon$, then (91) holds.

The proof of Lemma 5.9 is complete. \square

Proof of Theorem 5.7. From Lemma 5.9 and the contraction mapping principle it follows that, if $v_0 \in \mathcal{A}_{\Xi_2}^{\mathbf{n}}$ is sufficiently small, $|v_0|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)} < \epsilon$, then there exists a unique fixed point $v \in \mathcal{Z}_\rho^\lambda$ for Ψ . It follows from the definitions of Ψ and \mathcal{Z}_ρ^λ that v solves the system

$$(106) \quad \begin{aligned} \partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v + \nabla p_{v,v} &= -P_{\mathcal{A}_{\Xi_1}} \langle v \cdot \nabla \rangle v, & \operatorname{div} v &= 0, \\ v|_\Gamma(t) &= e^{-\frac{\lambda}{2}t} \left(v_0|_\Gamma + \Xi Q_f^M \int_0^t \mathcal{K}_u^{\lambda,s} v(s) ds \right), & v(0) &= v_0. \end{aligned}$$

and satisfies the inequality (85).

Notice that, since $\mathcal{A}_{\Xi_1}^\perp \subset H^1$, we can write $\langle v \cdot \nabla \rangle v = P_{\mathcal{A}_{\Xi_1}} \langle v \cdot \nabla \rangle v + \nabla p_{\bar{v}}$, for a suitable function $p_{\bar{v}} \in H^1(\Omega, \mathbb{R})$. Hence, if we set $p_v := p_{v,v} + p_{\bar{v}}$ then we can conclude that v solves (14).

Finally, the uniqueness of v follows from Lemma 5.6, because from the continuity of the inclusion $H^1(\Omega, \mathbb{R}) \subset L^6(\Omega, \mathbb{R})$ (see [Neč67, Chapter 2, Thm 3.6]), it follows that

$W((0, T), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3)) \subset C([0, T], H_{\text{div}}^1(\Omega, \mathbb{R}^3)) \subset L^4((0, T), L_{\text{div}}^6(\Omega, \mathbb{R}^3))$ are continuous inclusions; in particular $W((0, T), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3)) \subset \Theta_{(0,T)}^{4,6}$. \square

— APPENDIX —

A.1. Complete norms in a vector space. We say that a given norm $|\cdot|_1$ in a vector space X is complete if $(X, |\cdot|_1)$ is a Banach space. We say that two norms $|\cdot|_1$ and $|\cdot|_2$ in the vector space X are equivalent if there is $(C_1, C_2) \in \mathbb{R}_0^2$ such that $C_1 |\cdot|_1 \leq |\cdot|_2 \leq C_2 |\cdot|_1$. Here we remark some relations between complete and equivalent norms.

Theorem A.1. *Let $|\cdot|_1$ and $|\cdot|_2$ be two norms in the vector space X , and let (W, \mathcal{T}) be a Hausdorff topological vector space. Suppose that the inclusions $(X, |\cdot|_1) \subseteq (W, \mathcal{T})$ and $(X, |\cdot|_2) \subseteq (W, \mathcal{T})$ are both continuous. If the norms $|\cdot|_1$ and $|\cdot|_2$ are both complete, then they are equivalent.*

Proof. Consider the graph $G_{1 \rightarrow 2} = \{(x, x) \mid x \in X\} \subseteq (X \times X, |\cdot|_1 \times |\cdot|_2)$ of the inclusion mapping $(X, |\cdot|_1) \subseteq (X, |\cdot|_2)$. If $(x^n, x^n)_{n \in \mathbb{N}}$ is a sequence in $G_{1 \rightarrow 2}$ converging to $(a, b) \in (X \times X, |\cdot|_1 \times |\cdot|_2)$, then from the continuity of the inclusions $(X, |\cdot|_i) \subseteq (W, \mathcal{T})$, $i \in \{1, 2\}$, it follows that necessarily $a = b$, because (W, \mathcal{T}) is Hausdorff. Hence $G_{1 \rightarrow 2}$ is closed and by the Closed Graph Theorem (see e.g. [Con85, section III.12]) it follows the continuity of the inclusion $(X, |\cdot|_1) \subseteq (X, |\cdot|_2)$. The same argument shows the continuity of the reverse inclusion $(X, |\cdot|_2) \subseteq (X, |\cdot|_1)$. \square

Corollary A.2. *Let $|\cdot|_1$ and $|\cdot|_2$ be two complete norms in the vector space X . The norms are equivalent if the inclusion $(X, |\cdot|_1) \subseteq (X, |\cdot|_2)$ is continuous.*

Proof. Set $W = X$ and let \mathcal{T} be the topology induced by $|\cdot|_2$. Then, it follows that the inclusions $(X, |\cdot|_i) \subseteq (W, \mathcal{T})$, $i \in \{1, 2\}$, are both continuous. \square

A.2. An Example concerning Remark 3.8. We illustrate the fact that the space \mathcal{N} defined in (22) is not necessarily trivial.

Let $\mathcal{O} = (0, \pi) \times \mathbb{T}^1$; the family $\{\pi_i \mid i \in \mathbb{N}_0\}$ contains the family $\{\sigma_n \mid n \in \mathbb{N}_0\}$, with $\sigma_n(r, s) := \frac{1}{\pi} \sin(nr)$, $(r, s) \in (0, \pi) \times \mathbb{T}^1 \sim (0, \pi) \times [0, 2\pi)$. Define the indicator operator $\mathcal{I}_{(\frac{\pi}{3}, \frac{2\pi}{3})}: C(\mathcal{O}, \mathbb{R}) \rightarrow L^2(\mathcal{O}, \mathbb{R})$, sending f to $\mathcal{I}_{(\frac{\pi}{3}, \frac{2\pi}{3})}f$ defined by $\mathcal{I}_{(\frac{\pi}{3}, \frac{2\pi}{3})}f(r, s) := \begin{cases} f(r, s) & \text{if } r \in (\frac{\pi}{3}, \frac{2\pi}{3}) \\ 0 & \text{if } r \in (0, \pi) \setminus [\frac{\pi}{3}, \frac{2\pi}{3}] \end{cases}$. Now, set the mapping $\chi := \mathcal{I}_{(\frac{\pi}{3}, \frac{2\pi}{3})}(3\sigma_3 - \sigma_9)$, from direct computations we obtain that $\frac{\partial \chi}{\partial s} = 0$, $\frac{\partial \chi}{\partial r} = \mathcal{I}_{(\frac{\pi}{3}, \frac{2\pi}{3})} \frac{\partial}{\partial r}(3\sigma_3 - \sigma_9)$, and $\frac{\partial^2 \chi}{\partial r^2} = \mathcal{I}_{(\frac{\pi}{3}, \frac{2\pi}{3})} \frac{\partial^2}{\partial r^2}(3\sigma_3 - \sigma_9)$; we can check that $\chi \in C^2(\mathcal{O}, \mathbb{R})$ and $\operatorname{supp}(\chi) = [\frac{\pi}{3}, \frac{2\pi}{3}] \times \mathbb{T}^1 \subset \mathcal{O}$.

Now we can show that the functions $\chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O}(\sigma_3 \mathbf{n})$ and $\chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O}(\sigma_9 \mathbf{n})$ are linearly dependent: we find

$$\begin{aligned} P_{\chi^\perp}^\mathcal{O}(\sigma_3 \mathbf{n}) &= \sigma_3 \mathbf{n} - \frac{(\sigma_3 \mathbf{n}, \chi \mathbf{n})_{L^2(\mathcal{O}, R)}}{|\chi \mathbf{n}|_{L^2(\mathcal{O}, R)}^2} \chi \mathbf{n} = \sigma_3 \mathbf{n} - \frac{3}{10} \chi \mathbf{n}, \\ P_{\chi^\perp}^\mathcal{O}(\sigma_9 \mathbf{n}) &= \sigma_9 \mathbf{n} - \frac{(\sigma_9 \mathbf{n}, \chi \mathbf{n})_{L^2(\mathcal{O}, R)}}{|\chi \mathbf{n}|_{L^2(\mathcal{O}, R)}^2} \chi \mathbf{n} = \sigma_9 \mathbf{n} + \frac{1}{10} \chi \mathbf{n}, \end{aligned}$$

from which it follows $3P_{\chi^\perp}^\mathcal{O}(\sigma_3 \mathbf{n}) - P_{\chi^\perp}^\mathcal{O}(\sigma_9 \mathbf{n}) = 3\sigma_3 \mathbf{n} - \frac{9}{10} \chi \mathbf{n} - \sigma_9 \mathbf{n} - \frac{1}{10} \chi \mathbf{n} = 0$. Therefore, if the functions $\chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O}(\sigma_3 \mathbf{n})$ and $\chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O}(\sigma_9 \mathbf{n})$ are in the family $\{\chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O}(\pi_i \mathbf{n}) \mid i \leq M\}$, then the family is linearly dependent; it follows that $Q_f^M \mathcal{N} \subset \mathcal{N}$ contains nonzero vectors.

A.3. Linear feedback rule. Since u stands for the velocity of the fluid in system (1) the integral form of the feedback rule in (61) is natural from the physical point of view. Indeed, $v = u - \hat{u}$ standing for a difference of velocities, the time integral in (61) would correspond to an acceleration, that is, to a real forcing.

Though we find the integral form in rules (60) and (61) more natural, we show here that they can be “formally” rewritten as linear feedback rules; which in particular underlines once more the fact that the feedback control is defined pointwise in time (cf. Section 4.3.1), and do not depend on past events as the integral form could suggest.

Let us set $\tilde{\kappa}(t) := e^{\frac{\lambda}{2}t} \kappa_0^*(t)$. From (58), and (53), we find that for all $s_1 > t > 0$

$$\begin{aligned} \text{(A.1)} \quad \tilde{\kappa}(t) &= \Delta_t \tilde{\kappa}(t) + (1 - \Delta_t) \tilde{\kappa}(t) \\ &= -\partial_s|_{s=t} \Xi^\diamond e^{-\frac{\lambda}{2}s} P_\nabla R_{\hat{u}}^{\lambda, s} v_0^*(s) \mathbf{n} + 2^{-1} e^{-\frac{\lambda}{2}t} \Xi^\diamond (p_{q_{s_1}}(t) \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q_{s_1}(t)). \end{aligned}$$

Now we observe that, formally,

$$\text{(A.2)} \quad \partial_s|_{s=t} P_\nabla R_{\hat{u}}^{\lambda, s} v_0^*(s) = P_\nabla \left(\partial_s|_{s=t} R_{\hat{u}}^{\lambda, s} v_0^*(t) + R_{\hat{u}}^{\lambda, t} \partial_s|_{s=t} v_0^*(s) \right).$$

Since v_0^* solves (38), we have $\partial_s|_{s=t} v_0^*(s) = L_{O_{\hat{u}}}^t v_0^* := -\mathcal{B}(\hat{u}(t)) v_0^*(t) + \nu \Delta v_0^*(t) - \nabla p^{L_{O_{\hat{u}}}^t(v_0^*)}$, and $p^{L_{O_{\hat{u}}}^t(v_0^*)}$ satisfies

$$\begin{aligned} \text{(A.3)} \quad \Delta p^{L_{O_{\hat{u}}}^t(v_0^*)} &= \operatorname{div}(\nu \Delta v_0^*(t) - \mathcal{B}(\hat{u}(t)) v_0^*(t) - \partial_s|_{s=t} v_0^*(s)), \text{ in } \Omega, \\ \nabla p^{L_{O_{\hat{u}}}^t(v_0^*)} \cdot \mathbf{n} &= (\nu \Delta v_0^*(t) - \mathcal{B}(\hat{u}(t)) v_0^*(t) - \partial_s|_{s=t} v_0^*(s)) \cdot \mathbf{n}, \text{ on } \Gamma, \end{aligned}$$

and $\partial_s|_{s=t} v_0^*(s) \cdot \mathbf{n} = \partial_s|_{s=t} (v_0^*(s) \cdot \mathbf{n}) = (\partial_s|_{s=t} \Xi \kappa_0^*(s)) \cdot \mathbf{n} = \Xi(\partial_s|_{s=t} \kappa_0^*(s)) \cdot \mathbf{n}$, recalling $\kappa_0^*(s) = e^{-\frac{\lambda}{2}s} \tilde{\kappa}(s)$ we obtain that

$$\begin{aligned} \text{(A.4)} \quad \partial_s|_{s=t} v_0^*(s) \cdot \mathbf{n} &= \Xi \left(-\frac{\lambda}{2} \kappa_0^*(t) - e^{-\lambda t} \Xi^\diamond P_\nabla R_{\hat{u}}^{\lambda, t} v_0^*(t) \mathbf{n} \right) \cdot \mathbf{n} \\ &= -\frac{\lambda}{2} v_0^*(t) \cdot \mathbf{n} - e^{-\lambda t} \left(\Xi \Xi^\diamond P_\nabla R_{\hat{u}}^{\lambda, t} v_0^*(t) \mathbf{n} \right) \cdot \mathbf{n} \end{aligned}$$

On the other hand, from $\operatorname{div}(\nu \Delta v_0^*(t) - \partial_s|_{s=t} v_0^*(s)) = \nu \Delta \operatorname{div} v_0^*(t) - \partial_s|_{s=t} \operatorname{div} v_0^*(s)$, we can rewrite system (A.3) as

$$\begin{aligned} \text{(A.5)} \quad \Delta p^{L_{O_{\hat{u}}}^t(v_0^*)}(t) &= -\operatorname{div} \mathcal{B}(\hat{u}(t)) v_0^*(t), \\ \nabla p^{L_{O_{\hat{u}}}^t(v_0^*)}(t) \cdot \mathbf{n} &= (\nu \Delta - \mathcal{B}(\hat{u}(t)) + \Upsilon) v_0^*(t) \cdot \mathbf{n}, \end{aligned}$$

with $\Upsilon v_0^*(t) := \frac{\lambda}{2} v_0^*(t) + e^{-\lambda t} \Xi \Xi^\diamond P_\nabla R_{\hat{u}}^{\lambda, t} v_0^*(t) \mathbf{n}$.

Therefore, we can conclude that $p^{L_{O_{\hat{u}}}^t(v_0^*)}(t)$, at time t , depends only on $v_0^*(t)$, for a.e. $t \in \mathbb{R}_0$, which in turn implies that $\partial_s|_{s=t} (v_0^*(s)) = L_{O_{\hat{u}}}^t v_0^* = L_{\hat{u}(t)} v_0^*(t)$ depends also only

on $v_0^*(t)$, for a.e. $t \in \mathbb{R}_0$. Finally, from (A.1) and (A.2), for $\kappa_0^*(t) = e^{-\frac{\lambda}{2}t} \tilde{\kappa}(t)$ we find

$$(A.6) \quad \begin{aligned} \kappa_0^*(t) &= e^{-\lambda t} \Xi^\diamond P_\nabla \left(-\frac{\lambda}{2} R_{\hat{u}}^{\lambda, t} + \partial_s|_{s=t} R_{\hat{u}}^{\lambda, s} + R_{\hat{u}}^{\lambda, t} L_{\hat{u}(t)} \right) v_0^*(t) \mathbf{n} \\ &\quad + 2^{-1} e^{-\lambda t} \Xi^\diamond (p_{q_{s_1}}(t) \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q_{s_1}(t)), \end{aligned}$$

We also have that, since $q_{s_1}(t) \in D(L)$ for a.e. $t > 0$, $\langle \mathbf{n} \cdot \nabla \rangle q_{s_1}(t)$ is tangent to Γ (see, e.g., [Rod14a, Lemma 5.5]) which implies that $\Xi^\diamond \langle \mathbf{n} \cdot \nabla \rangle q_{s_1}(t) = Q_l^M \Xi^\diamond \langle \mathbf{n} \cdot \nabla \rangle q_{s_1}(t)$. On the other hand, since $p_{q_{s_1}}(t) \mathbf{n}$ is normal to the boundary, then $\Xi^\diamond p_{q_{s_1}}(t) \mathbf{n} = Q_f^M \Xi^\diamond (p_{q_{s_1}}(t) \mathbf{n})$. Thus, since $\kappa_0^*(t)$ is normal to the boundary, we have that necessarily $\Xi^\diamond \langle \mathbf{n} \cdot \nabla \rangle q_{s_1}(t)$ vanishes. On the other hand, from (48), $p_{q_{s_1}}(t)$ solves

$$\begin{aligned} \Delta p_{q_{s_1}}(t) &= -\operatorname{div} \mathcal{B}^*(\hat{u}(t)) q_{s_1}(t), \text{ in } \Omega \\ \nabla p_{q_{s_1}}(t) \cdot \mathbf{n} &= (\nu \Delta q_{s_1}(t) - \mathcal{B}^*(\hat{u}(t)) q_{s_1}(t)) \cdot \mathbf{n}, \text{ in } \Gamma \end{aligned}$$

and, since $q_{s_1}(t) = \Pi R_{\hat{u}}^{\lambda, t} v_0^*(t)$, we can conclude that $p_{q_{s_1}}(t)$ depends linearly, continuously, and only on $v_0^*(t)$, that is, we can write $p_{q_{s_1}}(t) =: \mathcal{P} v_0^*(t)$ with \mathcal{P} linear and continuous, for a.e. $t > 0$. It follows that $\kappa_0^*(t)$ and the optimal control $\Xi \kappa_0^*(t)$ depend only on $v_0^*(t)$, for a.e. $t \in \mathbb{R}_0$:

$$(A.7) \quad \kappa_0^*(t) = e^{-\lambda t} \Xi^\diamond \left(P_\nabla \left(\frac{\lambda}{2} R_{\hat{u}}^{\lambda, t} - \partial_s|_{s=t} R_{\hat{u}}^{\lambda, s} - R_{\hat{u}}^{\lambda, t} L_{\hat{u}(t)} \right) v_0^*(t) \mathbf{n} + 2^{-1} \mathcal{P} v_0^*(t) \mathbf{n} \right)$$

Remark A.3. Notice that $\kappa_0^*(t)$ is continuous (in time variable) so we can say that the right-hand side of (A.7) is also continuous, but this does not imply that each term in the right-hand side is well defined for each instant of time. However, we know that $v_0^*(t)$ is defined for every time and that p_q and $L_{O_{\hat{u}}} v_0^*$ are in $L_{\text{loc}}^2(\mathbb{R}_0, L^2(\Omega, \mathbb{R}^3))$; furthermore we have that $L_{\hat{u}(t)} v_0^*(t) = \partial_s|_{s=t} (v_0^*(s)) \in \mathcal{A}_{\Xi_1}$ for a.e. $t \in \mathbb{R}_0$: since the eigenfunctions π_i , $i \in \{1, 2, \dots, M\}$, are smooth we can extend each $\pi_i \mathbf{n}$ to a function in $H_{\text{div}}^1(\Omega, \mathbb{R}^3)$; indeed given a vector function $f \in H_{\text{av}}^{\frac{3}{2}}(\Gamma, \mathbb{R}^3)$ we can extend it to the solution $\widehat{F} f \in H_{\text{div}}^2(\Omega, \mathbb{R}^3)$ of the Stokes system (77). Then, we can define the extension $\widehat{F}: \mathcal{E}_M^{\mathbf{n}} \rightarrow W((0, T), H_{\text{div}}^2(\Omega, \mathbb{R}^3), H_{\text{div}}^2(\Omega, \mathbb{R}^3))$ by

$$\widehat{F} \Xi \kappa(t) = \sum_{i=1}^M \kappa_i(t) \widehat{F} \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} \pi_i \mathbf{n} = 0.$$

From Definitions 2.3 and 2.4, and Remark 2.7, we can write for all $0 < \varepsilon < T$, $v_0^*|_{(0, T)} = y - \widehat{F} \Xi \kappa_0^*|_{(0, T)}$, where $y \in W((0, T), V, V') \cap W((\varepsilon, T), D(L), H)$; which implies that for a.e. $t \in \mathbb{R}_\varepsilon$, $\partial_s|_{s=t} (v_0^*(s)) \in H + \mathcal{A}_{\Xi_1} = \mathcal{A}_{\Xi_1}$.

From the above observations we can derive that all the terms but $\partial_s|_{s=t} R_{\hat{u}}^{\lambda, s} v_0^*(t)$ are defined for a.e. $t \in \mathbb{R}_0$, which implies that necessarily also $\partial_s|_{s=t} R_{\hat{u}}^{\lambda, s} v_0^*(t)$ is. Therefore, the identity (A.7) is “term-wise” meaningful for a.e. $t \in \mathbb{R}_0$.

Acknowledgments. The author acknowledges partial support from the Austrian Science Fund (FWF): P 26034-N25.

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