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On the differentiability of fluid-structure interaction problems with respect to the problem data

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Abstract. A coupled system of stationary fluid-structure equations in an arbitrary Lagrangian-Eulerian framework is considered in this work. Existence results presented in the literature are extended to show differentiability of the solutions to a stationary fluid-structure interaction problem with respect to the given data, volume forces and boundary values, provided a small data assumption holds. Numerical experiments are used to substantiate the theoretical findings.

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Introduction

Fluid-structure interaction remains one of the most challenging topics to date, although many publications have appeared with specific emphasize on applications, coupling algorithms, and theory. Typical examples are found in industrial processes, aero-elasticity, and biomechanics. Specifically, fluid-structure interactions (FSI) are important to describe flows around elastic structures as for instance in the flutter analysis of airplanes, parachute FSI, or blood flow in the cardiovascular system, possibly with hyperelastic structure models, and heart valve dynamics, see, e.g, [34, 42, 31, 18, 39, 35, 28].

Often, either implicitly or explicitly, the quantity of interest is associated with calculating derivatives of solutions to FSI problems with respect to the given data. Such derivatives with respect to given data such as volume forces or boundary values have been used excessively in numerical papers concerned

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This work is an extension of our previous report *On the differentiability of fluid-structure interaction problems with respect to the problem data; Hamburger Beiträge zur Angewandten Mathematik 2013-06 (2013)* with sensitivity calculations or derivative based minimization problems, see, e.g., [29, 1, 44, 40, 38, 33]. In addition, sensitivity calculations are required for certain a posteriori error estimation techniques, such as the DWR-method [7, 8, 5]. Studies with a particular emphasize on FSI problems are carried out, for instance in [16, 37, 45, 23, 47]. Optimization-related problems subject to FSI have not been tackled, yet. Some first steps have been done in [9] where the case of a nonstationary problem with fixed interface was considered. Further, stabilization of such problems with a lower dimensional structure equations are considered [13, 36].

In recent years, several attempts have been made to prove existence of fluid-structure interaction problems. First results were derived for structures that were modeled as a rigid body, e.g., [14], or given by a finite number of modal functions [15]. Existence results for three dimensional fluid-structure interaction, where the structure was described as an elastic material, have been shown in [20, 21]. The extension to nonstationary problems is far from being trivial. The well-known problem is the regularity gap of the fluid and the structure velocity on the interface. Proofs of well-posedness and existence of nonstationary fluid-structure interaction have been derived in [11, 12, 22]. Moreover, regularity properties of the FSI solution have been studied in [6, 2]. We emphasize that the important dilemma of non-matching coordinate systems of fluids and structures has to be overcome. Consequently, in this work, the well-known arbitrary Lagrangian-Eulerian (ALE) frame of reference (see, e.g., [25, 27, 32]) is used. Using this method, the fluid equations are reformulated on a fixed (but arbitrary) reference configuration.

The goal of this paper is to address a specific problem of theoretical fluid-structure interaction, namely the proof that there is at least one locally unique solution that is Fréchet-differentiable with respect to the boundary data and volume forces; supplemented with its numerical verification, to asses the necessity of the imposed small data assumption. To the best of the authors knowledge, neither results on the differentiability of solutions to FSI problems with respect to the data have been shown, nor detailed numerical studies were conducted. Some work in this direction is contained in [30] where differentiability of the solution-map for a lower-dimensional structure was considered. The purpose of this work is to extend the existence theory for a stationary FSI problem given in [21] to include local uniqueness of at least one solution under a small data assumption. Then, the main contribution of this article, differentiability of the solution with respect to the problem data is proven. These theoretical results are substantiated with some numerical tests.

The paper is organized as follows. In Section 1, we will describe the considered setting for the FSI problem under consideration. Then, some results on the fluid and structure problem will be collected in Section 2. Here, special emphasis is placed on the fluid problem, which is posed in the ALE framework, while the structure equation is standard linear elasticity. The paper culminates in Section 3; here the main result, namely differentiability of solutions to the FSI problem, will be shown. This will be done using a fixed point argument in combination with several applications of the implicit function theorem. Finally, numerical experiments illustrating our findings, in particular the necessity of the small data assumption, are conducted in Section 4.

1. Problem Setting

In the sequel, let $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ be a given domain with a $C^{1,1}$ boundary. We decompose Ω into a fluid part Ω_f and a structure part Ω_s. In Ω_f the incompressible Stokes equations and on Ω_s a linear elastic structure equation is valid. We denote the interface between the two subdomains by $\Gamma_i := \overline{\Omega_f} \cap \overline{\Omega_s}$. Boundary parts where we prescribe Dirichlet conditions, are denoted by Γ_f and Γ_s (for fluid or structure, respectively). These boundary parts are chosen such that $\partial\Omega = \Gamma_f \cup \Gamma_s$. In order to avoid problems due to singularities arising from the change of the boundary conditions, we assume that the interface Γ_i has positive distance to the boundary $\partial\Omega$ of the domain.

We use standard notation for the usual Lebesgue and Sobolev spaces. On the Hilbert space $L^2(\Omega)$, we denote the scalar product by $(\cdot,\cdot)$ and the corresponding norm by $\|\cdot\|$. The space $W^{m,p}(\Omega)$ contains those functions whose weak derivatives up to order m are in $L^p(\Omega)$, and we write $H^m(\Omega) := W^{m,2}(\Omega)$. For any $d-1$ dimensional set $\Gamma \subset \overline{\Omega}$, we define $W^{m,p}_{0,\Gamma}(\Omega) \subset W^{m,p}(\Omega)$ by zero Dirichlet conditions on Γ, assuming $|\Gamma| \neq 0$ in the $d-1$ dimensional sense. For vector valued function spaces, we indicate this by adding the image space to the definition, e.g., $H^m(\Omega;\mathbb{R}^d)$ for H^m functions with values in $\mathbb{R}^d$. Throughout, we assume $p > d$ and hence $W^{2,p}(\Omega) \subset W^{1,\infty}(\Omega)$ by standard embeddings.

Due to the coupling of fluid and structure equations, the domains Ω_f and Ω_s are unknown a priori. Under the assumption that the overall domain Ω is fixed, we will reformulate the coupled system on a fixed domain $\widehat{\Omega} = \Omega$ with boundary $\partial\Omega = \Gamma_f \cup \Gamma_s$. To do so, we introduce a reference configuration by denoting $\widehat{\Omega}_f \subset \Omega$ and $\widehat{\Omega}_s = \Omega \setminus \overline{\widehat{\Omega}_f} \subset \Omega$. The interface $\widehat{\Gamma}_i = \partial\widehat{\Omega}_f \cap \partial\widehat{\Omega}_s$ is assumed to be of class $C^{1,1}$ with positive distance to $\partial\widehat{\Omega}$. Then, the problem of finding the domains $\Omega_{f,s}$ is equivalent to finding a transformation $\widehat{\mathcal{A}}\colon \widehat{\Omega} \to \widehat{\Omega}$ such that $\widehat{\mathcal{A}}(\widehat{\Omega}_f) = \Omega_f$, $\widehat{\mathcal{A}}(\widehat{\Omega}_s) = \Omega_s$, and $\widehat{\mathcal{A}}(\widehat{\Gamma}_i) = \Gamma_i$. This transformation is called the ALE map [25, 27, 32]. A natural choice for $\widehat{\Omega}_s$ is the domain initially occupied by the structure, and hence $\widehat{\mathcal{A}}\colon \widehat{\Omega}_s \to \Omega_s \subset \widehat{\Omega}$ is given by the displacement $\hat{u}_s$, e.g., $\widehat{\mathcal{A}}\Big|_{\widehat{\Omega}_s} = I + \hat{u}_s$, where I denotes the identity on $\mathbb{R}^d$. To obtain its values on the fluid domain, one has to choose an arbitrary continuation, e.g., of harmonic type [49, 27], to obtain a displacement $\hat{u}_f = \widehat{\mathcal{A}} - I$ on the fluid reference domain $\widehat{\Omega}_f$ that satisfies the following geometric coupling condition:

$$I + \hat{u}_f(\hat{x}) = \widehat{\mathcal{A}}(\hat{x}) = I + \hat{u}_s(\hat{x}) \quad \text{on } \widehat{\Gamma}_i.$$

For the extension of boundary values, we introduce the following spaces

$$W_{i,s} := W^{s-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d), \quad s = 1,2,$$
$$W_E = W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f}(\widehat{\Omega}_f;\mathbb{R}^d),$$

together with their respective norms $\|\cdot\|_{i,s}$ and $\|\cdot\|_E$.
Let $\mathcal{E} : W_{i,2} \to W_E$ be a continuous (linear) extension operator. Then, the ALE map on $\widehat{\Omega}_f$ is determined as a function of $\hat{u}_s$ by

$$\widehat{\mathcal{A}}\Big|_{\widehat{\Omega}_f} = \widehat{\mathcal{A}}(\hat{u}_s)\Big|_{\widehat{\Omega}_f} = I + \mathcal{E}(\gamma_i(\hat{u}_s)),$$

where γ_i denotes the trace operator over $\widehat{\Gamma}_i$. This leads to $\hat{u}_f = \hat{u}_s$ on $\widehat{\Gamma}_i$. For reasons that will become clear later, it is desirable to choose the continuation $\mathcal{E}$ such that $\|\widehat{\mathcal{A}}\|_{W^{1,\infty}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}$ is small. For more details on the ALE regularity condition, we also refer the reader to [17].

For later purposes, we define the deformation gradient $\widehat{F}$ and its determinant $\hat{J}$, related to the ALE map, that we will use quite frequently, by

$$\widehat{F} := \widehat{F}(\hat{u}) = \widehat{\nabla}\widehat{\mathcal{A}}, \quad \hat{J} := \hat{J}(\hat{u}) = \det(\widehat{F}). \tag{1.1}$$

The existence proof is based on a fixed point argument. Hence, we begin by stating some properties of the fluid and structure problem.

Remark 1.1. Note that, by Sobolev embeddings and the assumptions on $\mathcal{E}$, we have for any $\hat{u}_s \in W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)$ and $\hat{u}_f = \widehat{\mathcal{A}}(\hat{u}_s)$ that

$$\|\hat{u}_f\|_{W^{1,\infty}(\widehat{\Omega}_f;\mathbb{R}^d)} \lesssim \|\hat{u}_f\|_E \lesssim \|\hat{u}_s\|_{i,2} \lesssim \|\hat{u}_s\|_{W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)},$$

where $\lesssim$ indicates that the inequality holds up to a constant independent of the relevant quantities, in this case $\hat{u}_s$ and $\hat{u}_f$.

In particular, $\hat{J} > 0$ if $\|\hat{u}_s\|_{W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)}$ is sufficiently small and we see that this implies $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}^{-1} \in W^{1,\infty}(\widehat{\Omega};\mathbb{R}^{d\times d})$.

2. Properties of the subproblems

The fluid problem reads (in Eulerian coordinates):

Problem 2.1. Given a domain Ω_f with a $C^{1,1}$ boundary, and boundary values $g_f \in W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)$ with the compatibility condition, see, e.g, [43, Theorem 2.4],

$$\int_{\Gamma_f} g_f \cdot n_f \, ds = 0. \tag{2.1}$$

To abbreviate, we write $g_f \in W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)/\mathbb{R}$. Denote, again by g_f the continuation of g_f in $W^{2,p}(\Omega_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_i}(\Omega_f;\mathbb{R}^d)$. Then we need to find

$(v_f, p_f) \in H^1_{0,\Gamma_f \cup \Gamma_i}(\Omega_f; \mathbb{R}^d) \times L^2(\Omega_f)/\mathbb{R} + (g_f, 0)$ such that

$$\begin{aligned} -\widehat{\operatorname{div}}\,(\sigma_f) &= 0 \quad \text{in } \Omega_f, \\ \widehat{\operatorname{div}}\, v_f &= 0 \quad \text{in } \Omega_f, \\ v_f &= 0 \quad \text{on } \Gamma_i, \\ v_f &= g_f \quad \text{on } \Gamma_f. \end{aligned}$$

Here, the constitutive tensor σ_f is given by

$$\sigma_f := \sigma_f(v_f, p_f) = -p_f I + \nu_f(\nabla v_f + \nabla v_f^T),$$

where ν_f describes the kinematic viscosity of the fluid.

The existence and regularity of solutions to Problem 2.1 is studied in [43] under the assumption of a regular domain Ω_f. Unfortunately, since the domain Ω_f is given by the unknown mapping $\hat{u}_s$, i.e., $\Omega_f = \widehat{\mathcal{A}}(\widehat{\Omega}_f)$, this can not be asserted a priori.

In order to proceed in the reference domain, we transform the fluid equations to a fixed arbitrary reference configuration:

Problem 2.2. Given a displacement $\hat{u}_s \in W_{i,2}$, and $g_f \in W^{2-1/p,p}(\Gamma_f; \mathbb{R}^d)$ satisfying (2.1). Define $\widehat{F}$ and $\hat{J}$ by (1.1), and the transformed constitutive tensor as

$$\widehat{\sigma}_f := \widehat{\sigma}_f(\hat{v}_f, \hat{p}_f) = -\hat{p}_f I + \nu_f(\widehat{\nabla}\hat{v}_f \widehat{F}^{-1} + \widehat{F}^{-T}\widehat{\nabla}\hat{v}_f^T).$$

Find $(\hat{v}_f, \hat{p}_f) \in H^1_{0,\Gamma_f \cup \Gamma_i}(\widehat{\Omega}_f; \mathbb{R}^d) \times L^2(\widehat{\Omega}_f)/\mathbb{R} + (g_f, 0)$ such that

$$\begin{aligned} -\widehat{\operatorname{div}}\,(\widehat{\sigma}_f \hat{J}\widehat{F}^{-T}) &= 0 \quad \text{in } \widehat{\Omega}_f, \\ \widehat{\operatorname{div}}\,(\hat{J}\widehat{F}^{-1}\hat{v}_f) &= 0 \quad \text{in } \widehat{\Omega}_f, \\ \hat{v}_f &= 0 \quad \text{on } \widehat{\Gamma}_i, \\ \hat{v}_f &= g_f \quad \text{on } \Gamma_f. \end{aligned} \tag{2.2}$$

Under the assumption of $W^{2,p}$ regularity for $\hat{u}_f = \mathcal{E}(\hat{u}_s)$, with sufficiently small norm, the solutions of the Problems 2.1 and 2.2 coincide by setting $\hat{v}_f = v_f \circ \widehat{\mathcal{A}}^{-1}$ and $\hat{p}_f = p_f \circ \widehat{\mathcal{A}}^{-1}$. Since $\Gamma_f = \widehat{\Gamma}_f$ the boundary data g_f do not need to be transformed.

The structure problem (in Lagrangian coordinates) is defined by:

Problem 2.3. Given forces $\hat{f}_s \in L^p(\widehat{\Omega}_s; \mathbb{R}^d)$ and $\hat{g}_s \in W_{i,1}$. Find vector-valued displacements $\hat{u}_s \in W^{2,p}(\widehat{\Omega}_s; \mathbb{R}^d)$, such that

$$\begin{aligned} -\widehat{\operatorname{div}}\,(\widehat{\Sigma}_s) &= \hat{f}_s \quad \text{in } \widehat{\Omega}_s, \\ \hat{u}_f &= 0 \quad \text{on } \widehat{\Gamma}_s, \\ \widehat{\Sigma}_s \hat{n}_s &= \hat{g}_s \quad \text{on } \widehat{\Gamma}_i, \end{aligned} \tag{S}$$

with the Piola-Kirchhoff stress tensor $\widehat{\Sigma}_s$ and the (linear) Green-Lagrange tensor $\widehat{E}$:

$$\widehat{\Sigma}_s := \widehat{\Sigma}_s(\hat{u}_s) = \lambda \mathrm{tr}(\widehat{E})I + 2\mu\widehat{E}, \quad \widehat{E} := \widehat{E}(\hat{u}_s) = \frac{1}{2}(\widehat{\nabla}\hat{u}_s + \widehat{\nabla}\hat{u}_s^T),$$

where the constants λ and μ denote the Lamé parameters, and I the identity matrix $\mathbb{R}^{d\times d}$.

For better readability, we introduce the following spaces for the data of the fluid and structure problem as well as the respective solutions:

$$\begin{aligned}
W_{D,S} &:= L^p(\widehat{\Omega}_s;\mathbb{R}^d),\\
W_{D,F} &:= W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)/\mathbb{R},\\
W_{D,F}^+ &:= L^p(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)/\mathbb{R} \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R},\\
W_S &:= W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d) \cap H^1_{0,\Gamma_s}(\widehat{\Omega}_f;\mathbb{R}^d),\\
W_F &:= W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R},\\
W_F^0 &:= W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f\cup\widehat{\Gamma}_i}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R},
\end{aligned}$$

together with their norms $\|\cdot\|_{D,S}$,$\|\cdot\|_{D,F}$, $\|\cdot\|_{D,F^+}$, $\|\cdot\|_S$, and $\|\cdot\|_F$. Here, the notation $g_f \in W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)/\mathbb{R}$ means the compatibility condition $\int_{\Gamma_f} g_f n_f\,dx = 0$ while $\widehat{P} \in W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}$ stands for $\int_{\widehat{\Omega}_f} \widehat{P}\,dx = 0$.

Remark 2.4. We note that in the above setting other boundary conditions can be considered as well. The only strong requirement we have to enforce is that the solutions of the problems are sufficiently regular on $\widehat{\Gamma}_i$, i.e., $\hat{u}_f$ and $\hat{v}_f$ need to be in $W^{2-1/p,p}(\widehat{\Gamma}_i)$. The choice of the particular boundary values here is guided by the aim to have as little technicalities as possible. For the same reason it is possible to replace the equations with appropriate nonlinear versions, e.g., Navier-Stokes, nonlinear elasticity, etc. However, we refrain from such generalizations as not much value is added by an additional fixed point argument to obtain solvability of the (non-coupled) fluid and structure equations. Similar arguments can be made to incorporate additional volume forces or boundary values.

We will derive some properties of solutions to the Problems 2.2 and 2.3. Before these are stated, we recall two assumptions from [21, Page 83]. They are dealing with the regularity of the transformation $\widehat{F}$ and its determinant $\hat{J}$.

Remark 2.5. We simplify again notation by setting $\widehat{A} := \widehat{F}^{-1}\hat{J}\widehat{F}^{-T}$ and $\widehat{B} := \hat{J}\widehat{F}^{-T}$ in the following section.

Following [21], the following assumptions are made on the $\widehat{A}$ and $\widehat{B}$ to proof existence of the transformed Stokes problem. Assume:

a) $\widehat{A}$ is a symmetric positive definite matrix whose coefficients are elements of $W^{1,p}(\widehat{\Omega}_f)$. There exists some positive constant α such that $\widehat{A} \succcurlyeq \alpha\, I$.
b) $\widehat{B}$ is invertible, with both $\widehat{B}$ and $\widehat{B}^{-1}$ in $W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})$.

c) There is $C_u > 0$ such that $\widehat{A}$ and $\widehat{B}$ verify

$$\|I - \widehat{A}\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \leq C_u,$$
$$\|I - \widehat{B}\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \leq C_u,$$
$$\|I - \widehat{B}^T\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \leq C_u.$$

As both $\widehat{A}$ and $\widehat{B}$ are given by $\widehat{F}$, we remark that the assumptions hold if $\widehat{\mathcal{A}}$ is regular enough, e.g., for small deformations of the fluid mesh meaning $\|\hat{u}_f\|_E \ll 1$, see also our Remark 1.1. Further, we note that the mappings

$$\widehat{A}, \widehat{B}, \widehat{B}^{-1} \colon W_E \to W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})$$

are Fréchet-differentiable with

$$\|D_u\widehat{A}(u)\|_{\mathcal{L}(W_E,W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d}))} \leq C_u,$$
$$\|D_u\widehat{B}(u)\|_{\mathcal{L}(W_E,W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d}))} \leq C_u,$$
$$\|D_u\widehat{B}^{-1}(u)\|_{\mathcal{L}(W_E,W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \leq C_u,$$

with a constant C_u depending on $\|u\|_{W_E}$. We note that, in fact, the constant satisfies $C_u \leq C_{\rho,M} < \infty$ as long as $0 < \rho \leq \hat{J}$ and $\|u\|_{W_E} \leq M$.

Before we come to the existence and regularity of solutions to Problem 2.2 we need to introduce a slightly generalized fluid problem that we need frequently in the following proofs.

Problem 2.6. Given a displacement $\hat{u}_s \in W_{i,2}$, and problem data $(\hat{f}_f, g_f, \widehat{P}) \in W^+_{D,F}$. Define $\widehat{F}$ and $\hat{J}$ by (1.1).
Find $(\hat{v}_f, \hat{p}_f) \in W^0_F + (g_f, 0)$ such that

$$\begin{aligned} -\widehat{\operatorname{div}}\,(\hat{\sigma}_f\widehat{B}) &= \hat{f}_f \quad \text{in } \widehat{\Omega}_f,\\ \widehat{\operatorname{div}}\,(\widehat{B}^T\hat{v}_f) &= \widehat{P} \quad \text{in } \widehat{\Omega}_f,\\ \hat{v}_f &= 0 \quad \text{on } \widehat{\Gamma}_i,\\ \hat{v}_f &= g_f \quad \text{on } \Gamma_f, \end{aligned} \tag{F}$$

with $\widehat{B}$ given by Remark 2.5.

It is immediately clear that, Problem 2.2 can be obtained from (F), by setting $\hat{f}_f = 0$ and $\widehat{P} = 0$.

Theorem 2.7 (Existence of Stokes in the ALE framework). *Let $\widehat{\Omega}_f \subset \mathbb{R}^d$ be any open domain, with a $C^{1,1}$ boundary $\partial\widehat{\Omega}_f = \widehat{\Gamma}_f \cup \widehat{\Gamma}_i$ with positive distance between the two boundary parts. Let $\hat{u}_s \in W_{i,2}$ and $(\hat{f}_f, g_f, \widehat{P}) \in W^+_{D,F}$ be given. Define $\widehat{F}$ and $\hat{J}$ by* (1.1). *In addition, let the assumptions a) - c) hold true, i.e., $\|\hat{u}_s\|_{i,2}$ is small enough. Then, there exists a unique solution $(\hat{v}_f, \hat{p}_f) \in W^0_F + (g_f, 0)$ to* (F) *and it holds the estimate*

$$\|(\hat{v}_f, \hat{p}_f)\|_F \lesssim \|(\hat{f}_f, g_f, \widehat{P})\|_{D,F^+}. \tag{2.3}$$

The hidden constant depends on the assumptions a)−c) and hence on $\hat{u}_s$; it remains bounded in terms of $C_{\rho,M}$ as long as $0 < \rho \le \hat{J}$ and $\|\hat{u}_s\|_{W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)} \le M$. In particular, it holds

$$\|\widehat{\sigma}_f \widehat{B}\hat{n}_f\|_{i,1} \le C\|(\hat{f}_f, g_f, \widehat{P})\|_{D,F^+}.$$

Further, the mapping $\mathcal{F} : W_{i,2} \times W^+_{D,F} \to W_{i,1}$ defined by

$$(\hat{u}_s, \hat{f}_f, g_f, \widehat{P}) \mapsto \widehat{\sigma}_f \widehat{B}\hat{n}_f$$

is continuously differentiable.

Proof. The existence of a unique solution

$$(\hat{v}_f, \hat{p}_f) \in H^1_0(\widehat{\Omega}_f; \mathbb{R}^d) \times L^2(\widehat{\Omega}_f)/\mathbb{R} + (g_f, 0)$$

to (F) can be seen by transformation to the physical domain Ω_f, noting that the mean value of the back transformed $\widehat{P}$ is preserved, see for instance [43, Prop. 2.2] (compare Problem 2.1).

The regularity follows as in [21, Lemma 4] with the obvious modifications for non-homogeneous Dirichlet data and inhomogeneity $\widehat{P}$. To show differentiability, it is sufficient to consider the map $(\hat{u}_s, \hat{f}_f, g_f, \widehat{P}) \mapsto \widehat{\sigma}_f \widehat{B}\hat{n}_f$.

In order to show differentiability, we employ an implicit function type argument. To this end, for given values $\hat{u}_s, \hat{f}_f, g_f, \widehat{P}$ the solution $(\hat{v}_f, \hat{p}_f)$ is given by the equation (F), which we abbreviate as

$$a(\hat{u}_s, \hat{f}_f, g_f, \widehat{P}, \hat{v}_f, \hat{p}_f) = 0,$$

where

$$a\colon W_{i,2} \times W^+_{D,F} \times W_F \to L^p(\widehat{\Omega}_f; \mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}.$$

To show differentiability of $(\hat{v}_f, \hat{p}_f)$ with respect to the given data, we employ the implicit function theorem. To this end, we note a is linear in $(\hat{v}_f, \hat{p}_f)$, and thus

$$D_{v,p} a(\hat{u}_s, \hat{f}_f, g_f, \widehat{P}, \hat{v}_f, \hat{p}_f)\colon W_F \to L^p(\widehat{\Omega}_f; \mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R},$$

corresponds to the transformed Stokes operator on the left of F. Thus, by what we have seen above, the operator
$D_{v,p} a(\hat{u}_s, \hat{f}_f, g_f, \hat{v}_f, \hat{p}_f)$ is invertible. This shows the assertion. □

For later purposes, we intend to derive some properties of the derivative $D_u\mathcal{F}$. To do so, we start with a theorem calculating the derivative of $(\hat{v}_f, \hat{p}_f)$ with respect to the domain transformation $\hat{u}_f = \mathcal{E}\hat{u}_s$.

Theorem 2.8. *The mapping*

$$\begin{aligned} \mathcal{G}\colon W_E &\to W^0_F + (g_f, 0) \\ \hat{u}_f &\mapsto (\hat{v}_f, \hat{p}_f) \end{aligned}$$

given by (F) *with fixed $(\hat{f}_f, g_f, \widehat{P}) \in W^+_{D,F}$ is Fréchet-differentiable and the derivative*

$$\mathcal{G}'(\hat{u}_f)\colon W_E \to W^0_F,$$

is given as follows. For any given $\delta u \in W_E$*, the pair*

$$(\delta v, \delta p) = \mathcal{G}'(\hat{u}_f)\delta u \in W_F^0$$

is given as the unique solution to the problem

$$\begin{aligned}
-\widehat{div}\,((-\delta p I + \nu_f(\nabla \delta v \widehat{F}^{-1} + \widehat{F}^{-T}\nabla \delta v^T))\widehat{B}) \\
= \widehat{div}\big(-\hat{p}_f D_u \widehat{B}\delta u \qquad && \text{in } \widehat{\Omega}_f, \\
+\nu_f\big(\nabla \hat{v}_f D_u \widehat{A}\delta u \\
+ D_u \widehat{F}^{-T}\delta u \nabla \hat{v}_f^T \widehat{B} && \quad (2.4) \\
+ \widehat{F}^{-T}\nabla \hat{v}_f^T D_u \widehat{B}\delta u\big)\big) \\
\widehat{div}\,(\widehat{B}^T \delta v) = -\widehat{div}\,((D_u \widehat{B}^T \delta u)\hat{v}_f) && \text{in } \widehat{\Omega}_f, \\
\delta v = 0 && \text{on } \widehat{\Gamma}_i \cup \Gamma_f,
\end{aligned}$$

where $(\hat{v}_f, \hat{p}_f) = \mathcal{G}(\hat{u}_f)$ *and* $\widehat{A}$ *and* $\widehat{B}$ *are given in Remark* 2.5*. In particular, if* $\|\hat{u}_f\|_E$ *is sufficiently small (compare Remark* 2.5*) then*

$$\|\mathcal{G}'(\hat{u}_f)\|_{\mathcal{L}(W_E, W_F)} \leq C(\hat{u}_f),$$

where $C(\hat{u}_f)$ *depends on* $\|\hat{u}_f\|_E$*, and the data* $(\hat{f}_f, g_f, \widehat{P})$ *only. Further, it holds*

$$C(\hat{u}_f) \to 0 \quad \text{as} \quad \|\hat{u}_f\|_E + \|(\hat{f}_f, g_f, \widehat{P})\|_{D,F^+} \to 0.$$

Proof. To see that problem (2.4) has a unique solution, we show that the prerequisites of (F) are fulfilled, which shows the claimed existence using Theorem 2.7. The regularity requirements for the right hand side are given by definition of $\widehat{F}$ and $(\hat{v}_f, \hat{p}_f)$ (using that $W^{1,p}$ and $W^{2,p}$ are algebras). Further to see that $\widehat{\mathrm{div}}\,((D_u \widehat{B}^T \delta u)\hat{v}_f) \in W^{1,p}(\widehat{\Omega}_f)$ we note that due to the Piola identity, see, e.g., [10, Chapter I, p 39], it holds $\widehat{\mathrm{div}}\,(\hat{J}(u)\widehat{F}^{-T}(u)) = 0$ for all u and thus also $\widehat{\mathrm{div}}\,((D_u \widehat{B}^T \delta u)) = 0$. This implies

$$\widehat{\mathrm{div}}\,((D_u \widehat{B}^T \delta u)\hat{v}_f) = D_u \widehat{B}^T \delta u : \nabla \hat{v}_f \in W^{1,p}(\widehat{\Omega}_f).$$

Thus, the only question remaining is the compatibility condition

$$\int_{\widehat{\Omega}_f} \widehat{\mathrm{div}}\,((D_u \widehat{B}^T \delta u)\hat{v}_f)\, dx = 0.$$

To this end, one applies the Gauss divergence theorem, which shows the compatibility condition using $\hat{v}_f = 0$ on $\widehat{\Gamma}_i$ and $D_u \widehat{B}^T \delta u = 0$ on Γ_f; by definition that $\widehat{\mathcal{A}} = I$ on Γ_f for all $u \in W_E$.

The claimed bound on $\|\mathcal{G}'(\hat{u}_f)\|_{\mathcal{L}(W_E, W_F)}$ follows from the stability estimate (2.3) in Theorem 2.7 and the bounds on $D_u \widehat{F}^{-1}$, $D_u \hat{J}$, $\hat{v}_f$, and the data. For the bounds on $D_u \widehat{F}^{-1}$ and $D_u \hat{J}$ the smallness assumption in Remark 2.5 is required. Moreover, because $(\hat{v}_f, \hat{p}_f)$ in the right hand side of the defining equations of $(\delta v, \delta p)$ tends to zero if $(\hat{u}_f, \hat{f}_f, g_f, \widehat{P}) \to 0$ it holds $C(\hat{u}_f) \to 0$ if all data tends to zero. Note that C_u in Remark 2.5 c), and thus C in (2.3), is non increasing as $\hat{u}_f \to 0$.

Now, to show differentiability we define $(\hat{v}_f, \hat{p}_f) = \mathcal{G}(\hat{u}_f)$ and $(v, p) = \mathcal{G}(\hat{u}_f + \delta u)$ where δu is sufficiently small to assert the smallness assumption for $\hat{u}_f + \delta u$ following Remark 2.5.

Then, from strong convergence of the coefficients in (F), i.e.,

$$\widehat{F}(\hat{u}_f + \delta u)^{-1} \to \widehat{F}(\hat{u}_f)^{-1} \quad \text{as} \quad \|\delta u\|_E \to 0,$$

it follows easily, by comparing the defining equations and application of the stability estimate in Theorem 2.7, that $(v, p) \to (\hat{v}_f, \hat{v}_p)$ in W_F as $\|\delta u\|_E \to 0$.

Now, we show that in fact $(\delta v, \delta p)$ given in the statement of the Theorem is the claimed derivative $\mathcal{G}'(\hat{u}_f)\delta u$. We define the linearization error

$$(e_v, e_p) = \mathcal{G}(\hat{u}_f + \delta u) - \mathcal{G}(\hat{u}_f) - \mathcal{G}'(\hat{u}_f)\delta u = (v - \hat{v}_f - \delta v, p - \hat{p}_f - \delta p).$$

By the following Lemma 2.9 we obtain

$$\|(e_v, e_p)\|_F \leq o(\|\delta u\|_E).$$

This shows the asserted differentiability. □

Lemma 2.9. *Let $\hat{u}_f \in W_E$ be small enough such that the mapping $\mathcal{G}$ from Theorem* 2.8 *is well defined in $\hat{u}_f$. Further, let ϵ be such that $\mathcal{G}$ is well defined for all $\hat{u}_f + \delta u$ where $\|\delta u\|_E \leq \epsilon$. For any such δu, define $(\hat{v}_f, \hat{p}_f) = \mathcal{G}(\hat{u}_f)$, $(v, p) = \mathcal{G}(\hat{u}_f + \delta u)$, and $(\delta v, \delta p)$ by* (2.4).

Then the linearization error $(e_v, e_p) = (v - \hat{v}_f - \delta v, p - \hat{p}_f - \delta p)$ satisfies:

$$\begin{aligned}
\|(e_v, e_p)\|_F &\lesssim \|(v - \hat{v}_f, p - \hat{p}_f)\|_F \|\delta u\|_E + o\Big(\|\delta u\|_E\Big) \\
&= o\Big(\|\delta u\|_E\Big),
\end{aligned}$$

with hidden constant depending on $C_{\rho,M}$ as discussed in Remark 2.5.

Proof. We compare the equation (F) for $(\hat{v}_f, \hat{p}_f)$

$$\begin{aligned}
-\widehat{\operatorname{div}}\,((-\hat{p}_f I + \nu_f(\widehat{\nabla}\hat{v}_f \widehat{F}^{-1} + \widehat{F}^{-T}\widehat{\nabla}\hat{v}_f^T))\widehat{B}) &= \hat{f}_f \quad \text{in } \widehat{\Omega}_f, \\
\widehat{\operatorname{div}}\,(\widehat{B}^T \hat{v}_f) &= \widehat{P} \quad \text{in } \widehat{\Omega}_f, \\
\hat{v}_f &= 0 \quad \text{on } \widehat{\Gamma}_i, \\
\hat{v}_f &= g_f \quad \text{on } \Gamma_f,
\end{aligned}$$

where $\widehat{B} = \hat{J}\widehat{F}^{-T}$, and (v, p) where $F = \widehat{\nabla}\widehat{\mathcal{A}}(\hat{u}_f + \delta u)$, $J = \det F$ and $B = JF^{-T}$

$$\begin{aligned}
-\widehat{\operatorname{div}}\,((-pI + \nu_f(\widehat{\nabla}v F^{-1} + F^{-T}\widehat{\nabla}v^T))B) &= \hat{f}_f \quad \text{in } \widehat{\Omega}_f, \\
\widehat{\operatorname{div}}\,(B^T v) &= \widehat{P} \quad \text{in } \widehat{\Omega}_f, \\
v &= 0 \quad \text{on } \widehat{\Gamma}_i, \\
v &= g_f \quad \text{on } \Gamma_f,
\end{aligned}$$

with the equation (2.4) for $(\delta v, \delta p)$

$$
\begin{aligned}
-\widehat{\operatorname{div}}\,((-\delta p I + \nu_f(\nabla \delta v \widehat{F}^{-1} + \widehat{F}^{-T}\nabla \delta v^T))\widehat{B}) & \\
= \widehat{\operatorname{div}}\big(-\hat{p}_f D_u \widehat{B}\delta u & \quad \text{in } \widehat{\Omega}_f,\\
+\nu_f\big(\nabla \hat{v}_f D_u \widehat{A}\delta u & \\
+ D_u \widehat{F}^{-T}\delta u \nabla \hat{v}_f^T \widehat{B} & \\
+ \widehat{F}^{-T}\nabla \hat{v}_f^T D_u \widehat{B}\delta u\big)\big) & \\
\widehat{\operatorname{div}}\,(\widehat{B}^T\delta v) = -\widehat{\operatorname{div}}\,((D_u \widehat{B}^T\delta u)\hat{v}_f) & \quad \text{in } \widehat{\Omega}_f,\\
\delta v = 0 & \quad \text{on } \widehat{\Gamma}_i \cup \Gamma_f.
\end{aligned}
$$

Comparing the first of each of the three equations yields, using $\widehat{A} = \widehat{F}^{-1}\widehat{B}$ and $A = F^{-1}B$,

$$
\begin{aligned}
-\widehat{\operatorname{div}}\,&((-e_p I + \nu_f(\nabla e_v \widehat{F}^{-1} + \widehat{F}^{-T}\nabla e_v^T))\widehat{B})\\
&= -\widehat{\operatorname{div}}\,((-pI + \nu_f(\widehat{\nabla} v \widehat{F}^{-1} + \widehat{F}^{-T}\widehat{\nabla} v^T)\widehat{B}) - \hat{f}_f\\
&\quad - \widehat{\operatorname{div}}\big(-\hat{p}_f D_u \widehat{B}\delta u + \nu_f \nabla \hat{v}_f D_u \widehat{A}\delta u\\
&\qquad + \nu_f(D_u \widehat{F}^{-T}\delta u \nabla \hat{v}_f^T \widehat{B} + \widehat{F}^{-T}\nabla \hat{v}_f^T D_u \widehat{B}\delta u)\big)\\
&= -\widehat{\operatorname{div}}\,((-pI + \nu_f(\widehat{\nabla} v \widehat{F}^{-1} + \widehat{F}^{-T}\widehat{\nabla} v^T)\widehat{B})\\
&\quad + \widehat{\operatorname{div}}\,((-pI + \nu_f(\widehat{\nabla} v F^{-1} + F^{-T}\widehat{\nabla} v^T)B)\\
&\quad - \widehat{\operatorname{div}}\,((-pI + \nu_f(\widehat{\nabla} v F^{-1} + F^{-T}\widehat{\nabla} v^T)B) - \hat{f}_f\\
&\quad - \widehat{\operatorname{div}}\big(-\hat{p}_f D_u \widehat{B}\delta u + \nu_f \nabla \hat{v}_f D_u \widehat{A}\delta u\\
&\qquad + \nu_f(D_u \widehat{F}^{-T}\delta u \nabla \hat{v}_f^T \widehat{B} + \widehat{F}^{-T}\nabla \hat{v}_f^T D_u \widehat{B}\delta u)\big)\\
&= -\widehat{\operatorname{div}}\,(-p(\widehat{B} - B) - \hat{p}_f D_u \widehat{B}\delta u)\\
&\quad - \nu_f \widehat{\operatorname{div}}\big(\widehat{\nabla} v(\widehat{A} - A) + \nabla \hat{v}_f D_u \widehat{A}\delta u\big)\\
&\quad - \nu_f \widehat{\operatorname{div}}\big((\widehat{F}^{-T} - F^{-T})\widehat{\nabla} v^T \widehat{B} + D_u \widehat{F}^{-T}\delta u \nabla \hat{v}_f^T \widehat{B}\big)\\
&\quad - \nu_f \widehat{\operatorname{div}}\big(F^{-T}\widehat{\nabla} v^T(\widehat{B} - B) + \widehat{F}^{-T}\nabla \hat{v}_f^T D_u \widehat{B}\delta u\big)\\
&=: R_1.
\end{aligned}
\tag{2.5}
$$

Noting that $W^{1,p}$ is an algebra, we see that $R_1 \in L^p(\widehat{\Omega}_f;\mathbb{R}^d)$ and by Frechét-differentiability of

$$
\widehat{A}, \widehat{B}, \widehat{F}\colon W_E \to W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d}); \quad \text{and} \quad \hat{J}\colon W_E \to W^{1,p}(\widehat{\Omega}_f),
$$

in a neighborhood of $\hat{u}_f$; we can estimate the L^p-norm of R_1 as follows, with hidden constants depending on C_u given in Remark 2.5:

$$
\begin{aligned}
\|R_1\|_{L^p(\widehat{\Omega}_f;\mathbb{R}^d)} &\leq \|p\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}} \|B - \widehat{B} - D_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \|p - \hat{p}_f\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}} \|D_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \nu_f \|\widehat{\nabla} v\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \|A - \widehat{A} - D_u\widehat{A}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \nu_f \|\widehat{\nabla}(v - \hat{v}_f)\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \|D_u\widehat{A}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \nu_f \|\widehat{\nabla} v\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \|\widehat{B}\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\qquad \|F^{-T} - \widehat{F}^{-T} - D_u\widehat{F}^{-T}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \nu_f \|\widehat{\nabla}(v - \hat{v}_f)\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \|\widehat{B}\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\qquad \|D_u\widehat{F}^{-T}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \nu_f \|F^{-T}\widehat{\nabla} v^T\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\qquad \|B - \widehat{B} - D_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \nu_f \|F^{-T}\widehat{\nabla} v^T - \widehat{F}^{-T}\widehat{\nabla}\hat{v}_f^T\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\qquad \|D_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\lesssim \|(v - \hat{v}_f, p - \hat{p}_f)\|_F \|\delta u\|_E + o(\|\delta u\|_E).
\end{aligned}
\tag{2.6}
$$

Now, we compare the second of each of the three equations; with the Piola identity, as in the proof of Theorem 2.8, we obtain

$$
\begin{aligned}
\widehat{\operatorname{div}}\,(\widehat{B}^T e_v) &= \widehat{\operatorname{div}}\,(\widehat{B}^T v) - \widehat{\operatorname{div}}\,(B^T v) + \widehat{\operatorname{div}}\,(B^T v) - \widehat{P} \\
&\quad + \widehat{\operatorname{div}}\,((D_u\widehat{B}^T\delta u)\hat{v}_f) \\
&= \widehat{\operatorname{div}}\,((\widehat{B}^T - B^T)v + (D_u\widehat{B}^T\delta u)\hat{v}_f) \\
&= (\widehat{B}^T - B^T) : \widehat{\nabla} v + D_u\widehat{B}^T\delta u : \widehat{\nabla}\hat{v}_f \\
&=: R_2.
\end{aligned}
\tag{2.7}
$$

Further, we have the compatibility condition

$$
\begin{aligned}
\int_{\widehat{\Omega}_f} R_2\,dx &= \int_{\widehat{\Omega}_f} \widehat{\operatorname{div}}\,((\widehat{B}^T - B^T)v + (D_u\widehat{B}^T\delta u)\hat{v}_f)\,dx \\
&= \int_{\widehat{\Gamma}_i} (\widehat{B}^T - B^T)v\hat{n}_f + (D_u\widehat{B}^T\delta u)\hat{v}_f\hat{n}_f\,ds \\
&\quad + \int_{\Gamma_f} (\widehat{B}^T - B^T)v\hat{n}_f + (D_u\widehat{B}^T\delta u)\hat{v}_f\hat{n}_f\,ds = 0
\end{aligned}
$$

since $v, \hat{v}_f = 0$ on $\widehat{\Gamma}_i$ and $\widehat{B}^T = B^T$ on Γ_f and thus $D_u\widehat{B}^T\delta u = 0$ on Γ_f.

Again, using the algebra properties of $W^{1,p}$ and the properties of $\widehat{B}$, we assert $R_2 \in W^{1,p}(\widehat{\Omega}_f)$ and we calculate

$$\begin{aligned}
\|R_2\|_{W^{1,p}(\widehat{\Omega}_f)} &\leq \|B^T - \widehat{B}^T - D_u\widehat{B}^T\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}\|v\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \\
&\quad + \|v - \hat{v}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}\|D_u\widehat{B}^T\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\lesssim \|v - \hat{v}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \\
&\quad + o(\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}).
\end{aligned} \tag{2.8}$$

Combining the above calculations asserts that (e_v, e_p) solves the problem

$$\begin{aligned}
-\widehat{\operatorname{div}}\,((-e_pI + \nabla e_v\widehat{F}^{-1} + (\nabla e_v\widehat{F}^{-1})^T)\widehat{B}) &= R_1 && \text{in } \widehat{\Omega}_f, \\
\widehat{\operatorname{div}}\,(\widehat{B}^T e_v) &= R_2 && \text{in } \widehat{\Omega}_f, \\
e_v &= 0 && \text{on } \widehat{\Gamma}_i \cup \Gamma_f,
\end{aligned}$$

with bounds on R_1 and R_2 given by (2.5) and (2.7). From the estimates (2.6) and (2.8) the assertion follows by Theorem 2.7. □

Now, we can state the next preliminary result concerning the derivative of the 'Dirichlet-to-Neumann' map given in Theorem 2.7:

Theorem 2.10. *For the mapping $\mathcal{F}$ given by Theorem 2.7 it holds that for any given constant $\varepsilon > 0$, one can assert that*

$$\|D_u\mathcal{F}(\hat{u}_s, \hat{f}_f, g_f, \widehat{P})\| \leq \varepsilon,$$

where the norm is taken with respect to

$$D_u\mathcal{F}(\hat{u}_s, \hat{f}_f, g_f, \widehat{P})\colon W_{i,2} \to W_{i,1},$$

provided that $(\hat{f}_f, g_f, \widehat{P}) \in W^+_{D,F}$, and $\hat{u}_s \in W_{i,2}$ are sufficiently small.

Proof. We note that $\mathcal{F}$ is given as a composition of the following three maps:

$$\begin{aligned}
\mathcal{E}\colon \quad & W_{i,2} \to W_E \\
& \hat{u}_s \mapsto \mathcal{E}(\hat{u}_s) = \hat{u}_f, \\
\mathcal{G}\colon \quad & W_E \to W_F, \\
& \hat{u}_f \mapsto (\hat{v}_f, \hat{p}_f) \text{ solving (F) with fixed data } (\hat{f}_f, g_f, \widehat{P}), \\
\tau\colon \quad & W_E \times W_F \to W_{i,1}, \\
& (\hat{u}_f, \hat{v}_f, \hat{p}_f) \mapsto \hat{J}\hat{\sigma}_f\widehat{F}^{-T}\hat{n}_f.
\end{aligned}$$

Here, for fixed $\hat{f}_f$ and g_f the operator $\mathcal{G}$ is defined as in Theorem 2.8.

This means

$$\mathcal{F}(\hat{u}_s, \hat{f}_f, g_f, \widehat{P}) = \tau\Big(\mathcal{E}(\hat{u}_s), \mathcal{G}(\mathcal{E}(\hat{u}_s))\Big).$$

Noting that $\mathcal{E}$ is linear and bounded, and thus has no influence on the differentiability, we neglect the influence of $\mathcal{E}$ and write $\hat{u}_f \in W_E$ instead of $\mathcal{E}(\hat{u}_s)$. By the chain rule, we get for any direction $\delta u \in W_E$ that

$$D_u\mathcal{F}(\hat{u}_f, \hat{f}_f, g_f, \widehat{P})\delta u = D_u\tau(\hat{u}_f, \mathcal{G}(\hat{u}_f))\delta u + D_{v,p}\tau(\hat{u}_f, \mathcal{G}(\hat{u}_f))\mathcal{G}'(\hat{u}_f)\delta u.$$

Because τ is linear in v and p, it follows

$$D_u\mathcal{F}(\hat{u}_f,\hat{f}_f,g_f,\widehat{P})\delta u = \underbrace{D_u\tau(\hat{u}_f,\mathcal{G}(\hat{u}_f))\delta u}_{(i)} + \underbrace{\tau(\hat{u}_f,\mathcal{G}'(\hat{u}_f)\delta u)}_{(ii)}.$$

We take a closer look on the summands on the right hand side.

(i) For the first term $D_u\tau$, we note

$$\tau(\hat{u}_f,\mathcal{G}(\hat{u}_f)) = \tau(\hat{u}_f,\hat{v}_f,\hat{p}_f) = (-\hat{p}_f I + \nu_f(\widehat{\nabla}\hat{v}_f\widehat{F}^{-1} + (\widehat{\nabla}\hat{v}_f\widehat{F}^{-1})^T))\widehat{B}\hat{n}_f.$$

Consequently, its derivative with respect to the first argument is given as

$$\begin{aligned} D_u\tau(\hat{u}_f,\mathcal{G}(\hat{u}_f))\delta u &= (-\hat{p}_f I + \nu_f(\widehat{\nabla}\hat{v}_f\widehat{F}^{-1} + (\widehat{\nabla}\hat{v}_f\widehat{F}^{-1})^T))(D_u\widehat{B}\delta u)\hat{n}_f \\ &\quad + (\nu_f(\widehat{\nabla}\hat{v}_f D_u\widehat{F}^{-1}\delta u + (\widehat{\nabla}\hat{v}_f D_u\widehat{F}^{-1}\delta u)^T))\widehat{B}\hat{n}_f. \end{aligned}$$

From this, using that $W_{i,1}$ is the trace space of $W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^d)$, and an algebra for $p > d$, we calculate:

$$\begin{aligned} &\|D_u\tau(\hat{u}_f,\mathcal{G}(\hat{u}_f))\delta u\|_{W_{i,1}} \\ &\qquad \le \|\widehat{\sigma}_f(D_u\widehat{B}\delta u)\hat{n}_f\|_{W_{i,1}} \\ &\qquad\quad + \|(\nu_f(\widehat{\nabla}\hat{v}_f D_u\widehat{F}^{-1}\delta u + (\widehat{\nabla}\hat{v}_f D_u\widehat{F}^{-1}\delta u)^T))\widehat{B}\hat{n}_f\|_{W_{i,1}} \\ &\qquad \lesssim \|\widehat{\sigma}_f\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}\|D_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\ &\qquad\quad + \|\hat{v}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}\|\widehat{B}\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}\|D_u\widehat{F}^{-1}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}. \end{aligned}$$

Using the definitions of $\hat{J}$ and $\widehat{F}^{-T}$, we get that for $\|\hat{u}_f\|_E$ sufficiently small, there is a constant, depending on the size of this norm only, such that

$$\|D_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} + \|D_u\widehat{F}^{-1}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \le C\|\delta u\|_E.$$

Hence, for $(\hat{f}_f,g_f,\widehat{P})$ sufficiently small,

$$\|D_u\tau(\hat{u}_f,\mathcal{G}(\hat{u}_f))\|_{\mathcal{L}(W_E,W_{i,1})} \to 0 \quad (\hat{u}_f \to 0)$$

uniformly in $\|\delta u\|_E \le 1$ since $\|\widehat{\sigma}_f\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^d)}, \|\hat{v}_f\|_E \to 0$.

(ii) Now, we come to the second term

$$\tau(\hat{u}_f,\mathcal{G}'(\hat{u}_f)\delta u).$$

By the definition of τ it holds with $(\delta v,\delta p) = \mathcal{G}'(\hat{u}_f)\delta u$

$$\|\tau(\hat{u}_f,\delta v,\delta p)\|_{i,1} \lesssim \|\mathcal{G}'(\hat{u}_f)\|\|\delta u\|_E.$$

Thus, by Theorem 2.8, we get

$$\frac{\|\tau(\hat{u}_f,\mathcal{G}'(\hat{u}_f)\delta u)\|_{i,1}}{\|\delta u\|_E} \to 0 \quad ((\hat{u}_f,\hat{f}_f,g_f,\widehat{P}) \to 0).$$

This yields the assertion.

□

Theorem 2.11 (Existence of the structure problem). *Let $\hat{f}_s \in W_{D,S}$ and $\hat{g}_s \in W_{i,1}$. Then, there exists a unique solution $\hat{u}_s \in W_S$ to* (S). *Further, it holds the estimate:*

$$\|\hat{u}_s\|_S \lesssim \|\hat{f}_s\|_{D,S} + \|\hat{g}_s\|_{i,1}.$$

In addition, the mapping $\mathcal{S} : W_{D,S} \times W_{i,1} \to W_S$ defined by

$$(\hat{f}_s, \hat{g}_s) \mapsto \hat{u}_s$$

is continuously differentiable.

Proof. The proof can be found in Ciarlet [10, Theorem 6.3.6] noting that $\widehat{\Gamma}_i$ has positive distance to $\widehat{\Gamma}_s$; and thus no singularities from non-matching boundary conditions may arise. □

3. Existence of a solution to the fluid-structure problem

In this section, the coupled FSI problem is formulated and results regarding a locally unique solution and differentiability of the solution map are established. Due to the employed fixed-point arguments it is not surprising that we obtain existence, local uniqueness and regularity of a solution under a small data assumption only.

We are prepared to show that there exists a (unique) solution to the coupling of the Problems 2.2 and 2.3. Here the principal unknowns are the fluid velocity $\hat{v}_f$, the fluid pressure $\hat{p}_f$, the structure displacement $\hat{u}_s$, and the fluid domain displacement (mesh motion) $\hat{u}_f$. For the convenience of the reader, we recall the coupling conditions on $\widehat{\Gamma}_i$. The first one is the geometric coupling condition (to specify the ALE map $\widehat{\mathcal{A}}$) such that the unknown fluid domain follows the interface. The next is a velocity condition for the fluid problem and finally, the stress balance on the interface. Thus, we have

$$\begin{aligned} \hat{u}_f &= \mathcal{E}(\gamma_i(\hat{u}_s)) = \gamma_i(\hat{u}_s) && \text{on } \widehat{\Gamma}_i, \\ \hat{v}_f &= 0 && \text{on } \widehat{\Gamma}_i, \\ \widehat{\Sigma}_s \hat{n}_s - \hat{J}\widehat{\sigma}_f \widehat{F}^{-T}\hat{n}_f &= 0 && \text{on } \widehat{\Gamma}_i. \end{aligned} \tag{3.1}$$

It can be inferred from the third condition that $\hat{g}_s = \hat{J}\widehat{\sigma}_f \widehat{F}^{-T}\hat{n}_f$ in the structure Problem 2.3.

Then, the coupled problem reads:

Problem 3.1. Given $g_f \in W_{D,F}$ and $\hat{f}_s \in W_{D,S}$, find $(\hat{v}_f, \hat{p}_f, \hat{u}_s) \in W_F^0 \times W_S + (g_f, 0, 0)$ such that

$$
\begin{aligned}
-\widehat{\operatorname{div}}\,(\widehat{\sigma}_f \widehat{B}) &= 0 && \text{in } \widehat{\Omega}_f,\\
\widehat{\operatorname{div}}\,(\widehat{B}^T \hat{v}_f) &= 0 && \text{in } \widehat{\Omega}_f,\\
\hat{v}_f &= 0 && \text{on } \widehat{\Gamma}_i,\\
\hat{v}_f &= g_f && \text{on } \Gamma_f,\\
\hline
-\widehat{\operatorname{div}}\,(\widehat{\Sigma}_s) &= \hat{f}_s && \text{in } \widehat{\Omega}_s,\\
\hat{u}_f &= 0 && \text{on } \widehat{\Gamma}_s,\\
\hline
\hat{u}_f &= \mathcal{E}(\gamma_i(\hat{u}_s)) && \text{on } \widehat{\Omega}_f,\\
\hat{u}_f &= \gamma_i(\hat{u}_s) && \text{on } \widehat{\Gamma}_i,\\
\hat{v}_f &= 0 && \text{on } \widehat{\Gamma}_i,\\
\widehat{\Sigma}_s \hat{n}_s - \hat{J} \widehat{\sigma}_f \widehat{F}^{-T} \hat{n}_f &= 0 && \text{on } \widehat{\Gamma}_i
\end{aligned}
\tag{FSI}
$$

with $\widehat{\sigma}_f$ given by Problem 2.2, $\widehat{\Sigma}_s$ given by Problem 2.3 and $\widehat{F}$ and $\hat{J}$ are defined by (1.1).

The following proof of existence for the FSI problem is provided in [21]. However, a major extension is the differentiability of the solution map.

Theorem 3.2. *Let $g_f \in W_{D,F}$, and $\hat{f}_s \in W_{D,S}$ be given with $3 < p < \infty$. Assuming that g_f, and $\hat{f}_s$ are small enough, then there exists a solution $\widehat{U} = (\hat{v}_f, \hat{p}_f, \hat{u}_s) \in W_F^0 \times W_S + (g_f, 0, 0)$ to the coupled Problem* (FSI)*. Furthermore, there is a constant M, such that there is a locally unique solution satisfying the additional condition:*

$$
\|(\hat{v}_f, \hat{p}_f)\|_F + \|\hat{u}_s\|_S \le M. \tag{3.2}
$$

In addition, the herewith defined mapping

$$
\begin{aligned}
\mathcal{FSI}\colon W_{D,F} \times W_{D,S} &\to W_F \times W_S + (g_f, 0, 0),\\
(g_f, \hat{f}_s) &\mapsto (\hat{v}_f, \hat{p}_f, \hat{u}_s),
\end{aligned}
$$

is continuously differentiable in a neighborhood of $(0, 0)$*.*

Proof. The existence of a solution satisfying (3.2) for some sufficiently small value of M can be obtained by the arguments in [21], in particular Theorem 1. Hence all we need to show is differentiability of the thus defined mapping $\mathcal{FSI}$.

To do this, we note that

$$
(\hat{v}_f, \hat{p}_f, \hat{u}_s) = \mathcal{FSI}(g_f, \hat{f}_s)
$$

if and only if the displacement $\hat{u}_s$ satisfies

$$\hat{u}_s = \mathcal{S}(\hat{f}_s, \mathcal{F}(\mathcal{E}(\hat{u}_s), 0, g_f, 0)). \tag{3.3}$$

The velocity and pressure $(\hat{v}_f, \hat{p}_f)$ depend continuously differentiable on $(\hat{u}_s, g_f)$ by Theorem 2.7. Thus, it is sufficient to show differentiability of the mapping

$$(g_f, \hat{f}_s) \mapsto \hat{u}_s$$

given by the above fix point relation (3.3). To see that this defines a differentiable mapping, we employ the implicit function theorem. We note that

$$D_g\mathcal{S}(\hat{f}_s, \mathcal{F}(\mathcal{E}(\hat{u}_s), 0, g_f, 0))\colon W_{i,1} \to W_S$$

corresponds to the solution operator for S and is thus bounded, see Theorem 2.11. The second part

$$D_u\mathcal{F}(\mathcal{E}(\hat{u}_s), 0, g_f, 0))\colon W^{2,p}(\widehat{\Omega}_s; \mathbb{R}^d) \to W^{1-1/p,p}(\widehat{\Gamma}_i; \mathbb{R}^d)$$

corresponds to the shape derivative of the flow and from Theorem 2.10, we know that $\|D_u\mathcal{F}\|$ can be made arbitrarily small by choosing the data possibly smaller. Thus, $I + D_g\mathcal{S} \circ D_u\mathcal{F}$ is invertible. The implicit function theorem yields the asserted local uniqueness of $(\hat{v}_f, \hat{p}_f, \hat{u}_s)$ and differentiability of $\mathcal{FSI}$. □

4. Numerical experiments

This final section substantiates our theory with the help of numerical tests. These tests build upon well-known benchmark configurations [26]. This setting is perfect (because of many existing computational results) to illustrate several features of our theoretical work. Specifically, in the first set of tests, it is shown how the fluid mesh deformation depends on the regularity of $\widehat{\mathcal{A}}$ and consequently on the quality of the chosen moving mesh operator. Here the biharmonic operator provides higher regularity than standard Laplace or elasticity based mesh-motion. This has been demonstrated in an earlier work for nonstationary fluid-structure interaction in [46, 48]. An indicator for the mesh quality is $\hat{J}$ (that is related to a sufficient small $\|\hat{u}_s\|_S$ and $C_{\rho,M}$ respectively) which has already been used in that previous work. In the second test, we are interested in the sensitivity analysis of the solution map $\mathcal{FSI}$ and numerical demonstration of its differentiability. As quantities of interest, we observe the deflection and stresses of an elastic beam immersed in an incompressible fluid (see Figures 1 and 7). The tests are computed with the open-source project DOpElib [19] based upon the finite element library [4, 3].

In order to be able to calculate solutions for which the small data assumption fails, we use a nonstationary formulation of the Problem (FSI) and calculate a stationary limit.

Remark 4.1. In order to compute biharmonic mesh motion $\Delta^2 u = 0$, we use the Ciarlet-Raviart mixed formulation and (outer) boundary conditions $u = \partial_n u = 0$. Although solving the biharmonic equation on the linear system level is more expensive than standard Laplace or linear elasticity moving mesh techniques, it is too simple to claim that the method is not worth to be used due its higher computational cost. It is clear from our theory that the biharmonic mesh motion model provides us with higher regularity and continuity of the normal derivatives and therefore, results in smoother meshes, while bounding $\hat{J}$ away from zero over a larger range of interface displacement values. Consequently, the ALE-Stokes equations are less degenerated and therefore the Newton solver needs less steps, in our numerical experience, and converges faster in the biharmonic case.

We perform two different test scenarios:

- Example 4.1 (fluid initially in rest): Comparison of mesh motion techniques with respect to $\hat{J}$.
- Example 4.2 (fluid inflow and gravitational force $\hat{f}_s$): Differentiability of the solution map $\mathcal{FSI}$.

The first test case is split into two different sub-cases CSM 1 and CSM 4. The fluid is set to be initially at rest in $\widehat{\Omega}_f$. An external gravitational force $\hat{f}_s$ (with $\hat{f}_s = 2$ in CSM 1 [26] and $\hat{f}_s = 4$ in CSM 4) is applied only to the elastic beam, producing a visible deformation. As mentioned above, the tests are performed as time-dependent problems (backward Euler), leading to a steady state solution. For all models, we use the time step size $k = 0.02s$.

In the first test case CSM 1, the parameters and reference values are taken from [26]. We validate the code and compare the different mesh motion approaches. In the second example CSM 4, only the gravitational force is increased causing the elastic beam to become much more deformed.

Configuration: The computational domain has length $L = 2.5m$ and height $H = 0.41m$. The circle center is positioned at $C = (0.2m, 0.2m)$ with radius $r = 0.05m$. The elastic beam has length $l = 0.35m$ and height $h = 0.02m$. The right lower end is positioned at $(0.6m, 0.19m)$, and the left end is attached to the circle.

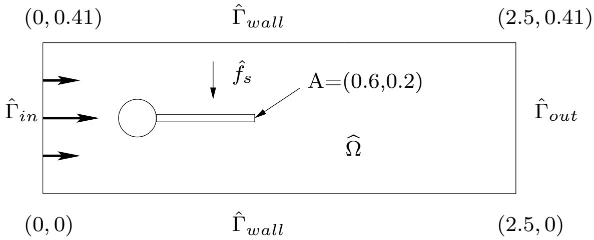

FIGURE 1. Configuration with circle-center $C = (0.2, 0.2)$ and radius $r = 0.05$.

Control points $A(t)$ (with $A(0) = (0.6, 0.2)$) are fixed at the trailing edge of the structure, measuring x- and y-deflections of the beam.

Boundary conditions: For the upper and lower boundaries $\widehat{\Gamma}_{wall}$, the 'no-slip' conditions for velocity are prescribed. Moreover, in Example 4.2, a parabolic inflow velocity profile is given on $\widehat{\Gamma}_{in}$ by

$$v_f(0,y) = \epsilon\, 1.5 \frac{4y(H-y)}{H^2}, \tag{4.1}$$

where ϵ tends to zero, i.e., $\epsilon = \pm 10^{-i}$, $i = 0,1,2,3$. With this, we are able to study the differentiability of the solution map $\mathcal{F}$ in the origin. At the outlet $\widehat{\Gamma}_{out}$ the 'do-nothing' outflow condition is imposed which lead to zero mean value of the pressure at this part of the boundary. The displacements are fixed on all boundaries, i.e, $\hat{u} = 0$ on $\widehat{\Gamma}_{wall} \cup \widehat{\Gamma}_{in} \cup \widehat{\Gamma}_{out}$.

Parameters: We choose for our computation the following parameters. For the fluid we use $\nu_f = 1kgm^{-1}s^{-1}$. The elastic structure is characterized by the Lamé coefficients $\mu_s = 5 * 10^5 kgm^{-1}s^{-2}$ and $\lambda_s = 2 * 10^6 kgm^{-1}s^{-2}$.

4.1. Comparison of mesh motion techniques with respect to $\hat{J}$

We compare three different mesh motion techniques. First, harmonic extension with constant diffusion parameter (Harm) [49], then harmonic extension with Jacobi-based stiffening (Harm(J)) [41], and finally biharmonic mesh motion (Biharm) [24]. The deflection of the elastic beam is computed in the point $A = (0.6, 0.2)$. Comparing fixed deflections of the beam, we observe in Tables 1 and 2 that the biharmonic mesh motion model delivers the most regular meshes since $\hat{J}$ remains larger compared to the other two techniques.

Table 1. Results for CSM 1: We present different mesh motion techniques and their influence on the mesh quality. Here, we compare $\min(\hat{J})$ when the elastic beam has reached certain values of its tip deflection $u_y(A)$. All mesh motion techniques (MMT) are able to compute this test case, however the quality is best when using the biharmonic technique because $\min_{\text{Biharm}}(\hat{J}) > \min_{\text{Harm(J)}}(\hat{J}) > \min_{\text{Harm}}(\hat{J})$.

MMT	$u_y(A)[\times 10^{-2}m]$	$\min(\hat{J})$
Harm	−1.0	0.811
	−2.0	0.635
	−4.0	0.269
Harm(J)	−1.0	0.826
	−2.0	0.673
	−4.0	0.426
Biharm	−1.0	0.926
	−2.0	0.853
	−4.0	0.707

It should be noted, that this demonstrates that the smallness assumptions made in Section 2, to assert $\hat{J} > 0$, depend strongly on the choice of the extension $\mathcal{E}$.

TABLE 2. Results for CSM 4: We present different mesh motion techniques (MMT) and their influence on the mesh quality. Here, we compare $\min(\hat{J})$ when the elastic beam has reached certain values of its tip deflection $u_y(A)$. Here, the harmonic extension fails for a tip-deflection of $-5.8 \times 10^{-2}m$ whereas the other two techniques achieve $u_y(A) = 13.17 \times 10^{-2}m$ and $u_y(A) = -13.56 \times 10^{-2}m$ before they fail.

MMT	$u_y(A)[\times 10^{-2}m]$	$\min(\hat{J})$
Harm	-1.0	0.803
	-5.0	0.082
	-5.8	< 0.0
Harm(J)	-1.0	0.814
	-5.0	0.311
	-10.0	0.047
	-13.17	< 0.0
Biharm	-1.0	0.917
	-5.0	0.629
	-10.0	0.255
	-13.56	< 0.0

4.2. Differentiability of the solution map $\mathcal{FSI}$

Large Data. In this test, we use the simple harmonic (Harm) and the biharmonic (Biharm) mesh motion technique and each with force $\hat{f}_s = -4$ (CSM 4). We study the differentiability of the solution map $\mathcal{FSI}$ while keeping the gravity at the same value for all tests and only vary the magnitude of the parabolic inflow profile as given in Equation (4.1). Specifically, we are interested in the case $\epsilon \to 0$.

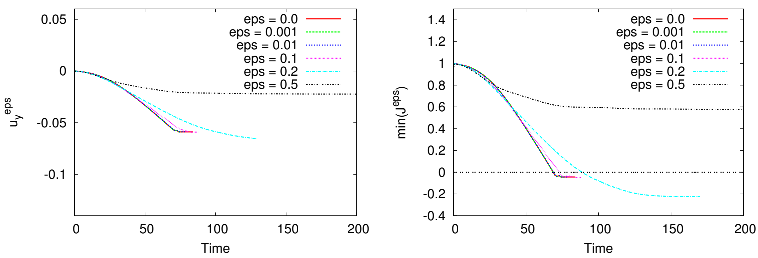

FIGURE 2. Function plots for u_y^ϵ and $\min(\hat{J}^\epsilon)$ for different ϵ using the harmonic mesh motion technique (CSM 4).

Instead of monitoring the respective norms on W_S and W_F, we analyze the behavior of a dependent quantity. Namely, we will compare the deflection of the tip A of the beam.

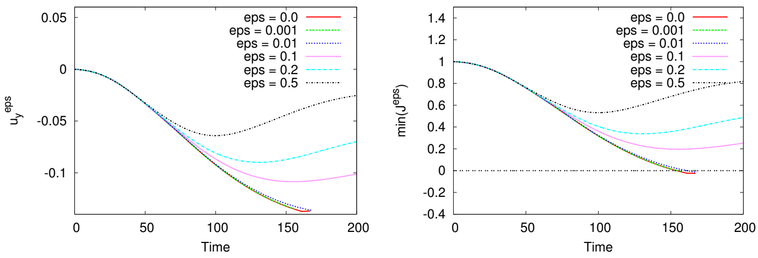

FIGURE 3. Function plots for u_y^ϵ and $\min(\hat{J}^\epsilon)$ for different ϵ using the biharmonic mesh motion technique (CSM 4).

As we monitor in Figure 2, the beam-deflection increases when the inflow magnitude gets smaller. This observation is physically reasonable. Correspondingly, the same observation holds for the $\min(\hat{J})$ because larger beam deflection means larger mesh deformation and therefore, degeneration of the ALE map. Here, the biharmonic mesh motion models yields much better results as monitored in Figure 3. In comparison, the harmonic mesh motion technique fails with $\min(\hat{J}) < 0$ in time-step 70 while for the biharmonic mesh motion technique $\min(\hat{J}) < 0$ is achieved not until time-step 152. In both cases, even for $g_f = 0$, the tip has not reached its stationary value.

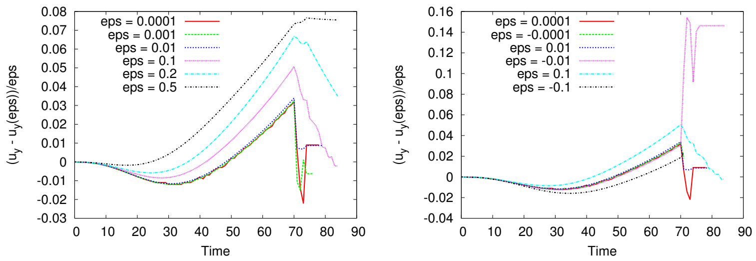

FIGURE 4. Function plots for CSM 4 using the harmonic mesh motion technique for $\frac{u_y - u_y^\epsilon}{\epsilon}$ for different positive ϵ. At time step 70 $\min(\hat{J})$ becomes negative (at left). At right, two corresponding negative and positive ϵ are compared. As $\epsilon_+, \epsilon_- \to 0$, the functional value $\frac{u_y - u_y^\epsilon}{\epsilon}$ converges in at time step 69 (just before the break down).

To check for differentiability of the position of the tip A, we calculate the y-component u_y of the displacement at this point for varying values of the inflow velocity $g_f|_{\widehat{\Gamma}_{in}} = \varepsilon v_f(0, y)$. In Figure 4 and 5, we observe the difference quotients between the corresponding values $u_y = u_y^0$ (the exact value) and the ϵ-related u_y^ε. For decreasing positive ϵ, the solution converges as $\epsilon \to 0$. However, we notice, that there is a big change in the difference quotient once

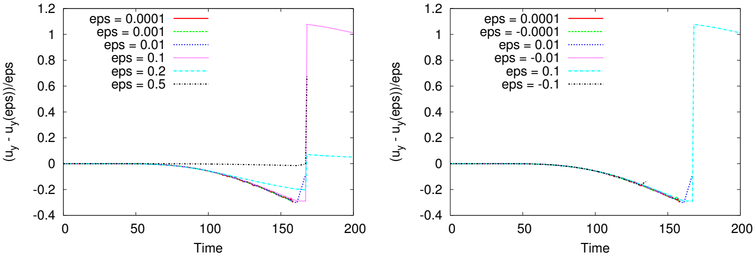

FIGURE 5. Function plots for CSM 4 using the biharmonic mesh motion technique for $\frac{u_y - u_y^\epsilon}{\epsilon}$ for different positive ϵ. At time step 152 $\min(\hat{J})$ becomes negative (at left). At right, two corresponding negative and positive ϵ are compared. As $\epsilon_+, \epsilon_- \to 0$, the functional value $\frac{u_y - u_y^\epsilon}{\epsilon}$ converges in at time step 152 (just before the break down).

$\min(\hat{J}) < 0$. In the second subplot in Figure 4, we see that for negative and positive ϵ the limit is identical up to the point $\min(\hat{J}) < 0$. However, after the value of $\min(\hat{J})$ gets negative the graphs in the two Figures 4 and 5 differs. This illustrates that the assumption of smallness of the data, in particular the thus implied condition $\hat{J} = \hat{J}(\hat{u}_f) > 0$, can not be dropped and moreover that calculated results with $\min(\hat{J})$ are not reliable for the calculation of sensitivities.

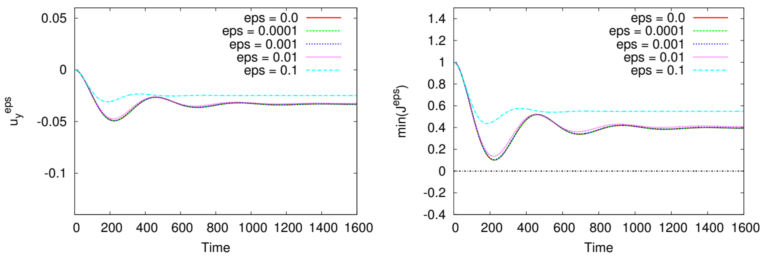

FIGURE 6. Function plots for u_y^ϵ and $\min(\hat{J}^\epsilon)$ for different ϵ using the harmonic mesh motion technique with small force $\hat{f}_s = -1.0$. Here, the right hand side force is chosen small enough such that the method does not break down.

Small Data. Finally, we consider a CSM test case, where $\hat{f}_s = -1.0$ is smaller than in CSM 4. As we observe in Figure 6 the harmonic mesh motion technique is capable of calculating the stationary limit. The effect of $\hat{J} \to 0$ is visualized in Figure 7.

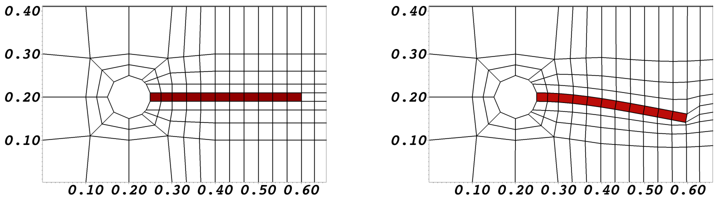

FIGURE 7. Snapshot of test case $\hat{f}_s = -1.0$ at time 220 with the highest deflection of the tip A. The computational domain (i.e., the reference configuration) $\widehat{\Omega}$ is shown left. In the right subfigure, the physical domain $\widehat{\mathcal{A}}(\widehat{\Omega}) = \Omega$ visualizes the impact of $\hat{J} \to 0$, namely the mesh degeneration close to the beam tip.

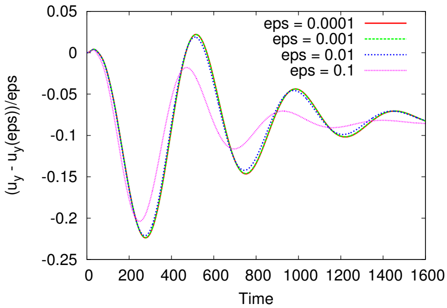

FIGURE 8. Function plots for CSM test case with $\hat{f}_s = -1.0$ using the harmonic mesh motion technique for $\frac{u_y - u_y^\epsilon}{\epsilon}$ for different positive ϵ. The error converges for $\epsilon \to 0$.

In Figure 8, the corresponding difference quotients are calculated. We observe uniform convergence of these difference quotients, highlighting our derived differentiability results.

References

[1] M. Allen and K. Maute, Reliability-based shape optimization of structures undergoing fluid-structure interaction phenomena, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 3472–3495.

[2] G. Avalos, I. Lasiecka, and R. Triggiani, Higher regularity of a coupled parabolic-hyperbolic fluid-structure interactive system, Georgian Mathematical Journal, 15 (2010), pp. 403–437.

[3] W. Bangerth, R. Hartmann, and G. Kanschat, deal.II – a general purpose object oriented finite element library, ACM Trans. Math. Softw., 33 (2007), pp. 24/1–24/27.

[4] W. Bangerth, T. Heister, and G. Kanschat, `deal.II` differential equations analysis library, technical reference. `http://www.dealii.org`.

[5] W. Bangerth and R. Rannacher, Adaptive Finite Element Methods for Differential Equations, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, Boston, Berlin, 1. ed., 2003.

[6] V. Barbu, Z. Grujic, I. Lasiecka, and A. Tuffaha, Smoothness of weak solutions to a nonlinear fluid-structure interaction model, Indiana University Mathematics Journal, 57 (2008), pp. 1173–1207.

[7] R. Becker and R. Rannacher, A feed-back approach to error control in finite element methods: Basic analysis and examples, East-West J. Numer. Math., 4 (1996), pp. 237–264.

[8] ———, An optimal control approach to a posteriori error estimation, in Acta Numerica 2001, A. Iserles, ed., Cambridge University Press, 2001, pp. 1–102.

[9] F. Bucci and I. Lasiecka, Optimal boundary control with critical penalization for a pde model of fluid-solid interactions, Calc. Var. Partial Differential Equations, 37 (2010), pp. 217–235.

[10] P. G. Ciarlet, Mathematical Elasticity. Volume 1: Three Dimensional Elasticity, North-Holland, 1984.

[11] D. Coutand and S. Shkoller, Motion of an elastic solid inside an incompressible viscous fluid, Arch. Ration. Mech. Anal., 176 (2005), pp. 25–102.

[12] ———, The interaction between quasilinear elastodynamics and the navier-stokes equations, Arch. Ration. Mech. Anal., 179 (2006), pp. 303–352.

[13] A. C. Deneuvy, Theoretical study and optimization of a fluid-structure interaction problem, M2AN Math. Model. Numer. Anal., 22 (1988), pp. 75–92.

[14] B. Desjardins and M. J. Esteban, Existence of weak solutions for the motion of rigid bodies in a viscous fluid, Arch. Ration. Mech. Anal., 146 (1999), pp. 59–71.

[15] B. Desjardins, M. J. Esteban, C. Grandmont, and P. Le Tallec, Weak solutions for a fluid-structure interaction problem, Rev. Mat. Complut., 14 (2001), pp. 523–538.

[16] T. Dunne, T. Richter, and R. Rannacher, Numerical simulation of fluid-structure interaction based on monolithic variational formulations, Comtemporary Challenges in Mathematical Fluid Mechanics, Springer, World Scientific, Singapore, 2010, pp. 1–75.

[17] L. Formaggia and F. Nobile, A stability analysis for the arbitrary Lagrangian Eulerian formulation with finite elements, East-West Journal of Numerical Mathematics, 7 (1999), pp. 105–132.

[18] L. Formaggia, A. Quarteroni, and A. Veneziani, eds., Cardiovascular mathematics, vol. 1 of MS&A. Modeling, Simulation and Applications, Springer-Verlag Italia, Milan, 2009. Modeling and simulation of the circulatory system.

[19] C. Goll, T. Wick, and W. Wollner, DOpE: Differential Equations and Optimization Environment, 2014. http://www.dopelib.net.

[20] C. Grandmont, Existence et unicité de solutions d'un problème de couplage fluide-structure bidimensionnel stationnaire, C. R. Acad. Sci. Paris, 326 (1998), pp. 651–656.

[21] C. Grandmont, Existence for a three-dimensional steady state fluid-structure interaction problem, J. Math. Fluid Mech., 4 (2002), pp. 76–94.

[22] ———, Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate, SIAM J. Math. Anal., 40 (2008), pp. 716–737.

[23] T. Grätsch and K.-J. Bathe, Goal-oriented error estimation in the analysis of fluid flows with structural interactions, Comp. Methods Appl. Mech. Engrg., 195 (2006), pp. 5673–5684.

[24] B. Helenbrook, Mesh deformation using the biharmonic operator, Int. J. Numer. Methods Engrg., (2001), pp. 1–30.

[25] C. Hirth, A. Amsden, and J. Cook, An arbitrary Lagrangian-Eulerian computing method for all flow speeds, J. Comput. Phys., 14 (1974), pp. 227–253.

[26] J. Hron and S. Turek, Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow, vol. 53, Springer-Verlag, 2006, pp. 146–170.

[27] T. Hughes, W. Liu, and T. Zimmermann, Lagrangian-eulerian finite element formulation for incompressible viscous flows, Comput. Methods Appl. Mech. Engrg., 29 (1981), pp. 329–349.

[28] U. Langer and H. Yang, Partitioned solution algorithms for fluid–structure interaction problems with hyperelastic models, Journal of Computational and Applied Mathematics, 276 (2015), pp. 47 – 61.

[29] E. Lund, H. Moller, and L. A. Jakobsen, Shape design optimization of stationary fluid-structure interaction problems with large displacements and turbulence, Struct. Multidisc. Optim., 25 (2003), pp. 383–392.

[30] C. M. Murea and C. Vázquez, Sensitivity and approximation of coupled fluid-structure equations by virtual control method, Appl. Math. Optim., 52 (2005), pp. 183–218.

[31] F. Nobile and C. Vergara, An effective fluid-structure interaction formulation for vascular dynamics by generalized robin conditions, SIAM J. Scientific Computing, 30 (2008), pp. 731–763.

[32] W. Noh, A time-dependent two-space-dimensional coupled Eulerian-Lagrangian code, vol. 3 of Methods Comput. Phys., Academic Press, New York, 1964, pp. 117–179.

[33] M. Perego, A. Veneziani, and C. Vergara, A variational approach for estimating the compliance of the cardiovascular tissue: An inverse

fluid-structure interaction problem, SIAM Journal on Scientific Computing, 33 (2011), pp. 1181–1211.

[34] S. Piperno and C. Farhat, Paritioned procedures for the transient solution of coupled aeroelastic problems - part ii: energy transfer analysis and three-dimensional applications, Comput. Methods Appl. Mech. Engrg., 190 (2001), pp. 3147–3170.

[35] A. Quaini, S. Canic, R. Glowinski, S. Igo, C.J.Hartley, W. Zoghbi, and S. Little, Validation of a 3d comoputational fluid-structure interaction model simulating flow through elastic aperature, Journal of Biomechanics, 45 (2012), pp. 310–318.

[36] J.-P. Raymond, Feedback stabilization of a fluid-structure model, SIAM J. Control Optim., 48 (2010), pp. 5398–5443.

[37] T. Richter, Goal-oriented error estimation for fluid-structure interaction problems, Comp. Methods Appl. Mech. Engrg., 223-224 (2012), pp. 38–42.

[38] T. Richter and T. Wick, Optimal control and parameter estimation for stationary fluid-structure interaction problems, SIAM J. Sci. Comp., 35 (2013), pp. B1085–B1104.

[39] N. D. D. Santos, J.-F. Gerbeau, and J. Bourgat, A partitioned fluid-structure algorithm for elastic thin valves with contact, Comp. Methods Appl. Mech. Engrg., 197 (2008), pp. 1750–1761.

[40] M. Schäfer, D. Sternel, G. Becker, and P. Pironkov, Efficient Numerical Simulation and Optimization of Fluid-Structure Interaction, vol. 53, Springer-Verlag, 2010, pp. 133–160.

[41] K. Stein, T. Tezduyar, and R. Benney, Mesh moving techniques for fluid-structure interactions with large displacements, J. Appl. Mech., 70 (2003), pp. 58–63.

[42] K. Takizawa and T. Tezduyar, Computational methods for parachute fluid-structure interactions, Archives of Computational Methods in Engineering, 19 (2012), pp. 125–169.

[43] R. Temam, Navier-Stokes equations. Theory and numerical analysis, North-Holland Publishing Co., Amsterdam, 1977. Studies in Mathematics and its Applications, Vol. 2.

[44] S. Turek, J. Hron, M. Razzaq, H. Wobker, and M. Schäfer, Numerical benchmarking of fluid-structure interaction: a comparison of different discretization and solution approaches, Fluid Structure Interaction II: modelling, simulation, optimization, Springer, Heidelberg, 2010, pp. 413–424.

[45] K. van der Zee, E. van Brummelen, and R. de Borst, Goal-oriented error estimation for Stokes flow interacting with a flexible channel, International Journal of Numerical Methods in Fluids, 56 (2008), pp. 1551–1557.

[46] T. Wick, Fluid-structure interactions using different mesh motion techniques, Comput. Struct., 89 (2011), pp. 1456–1467.

[47] ———, Goal-oriented mesh adaptivity for fluid-structure interaction with application to heart-valve settings, Arch. Mech. Engrg., 59 (2012), pp. 73–99.

[48] ———, Solving monolithic fluid-structure interaction problems in arbitrary Lagrangian Eulerian coordinates with the deal.ii library, Archive of Numerical Software, 1 (2013), pp. 1–19.

[49] A. Winslow, Numerical solution of the quasi-linear poisson equation in a nonuniform triangle mesh, J. Comput. Phys., (1967), pp. 149–172.

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