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Approximation of the solution of an ill-posed spherical pseudo-differential equation at a given point

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Abstract

This paper presents a method for approximating the solution of an ill-posed spherical pseudo-differential equation at a given point. The approximation is based on the regularized least-squares method of An et. al., SIAM J. Numer. Anal. 50 (2012), 1513–1534. We discuss an a posteriori parameter choice rule and illustrate our theoretical findings by numerical results for the reconstruction of the solution at a given point.

1 Introduction

In many real-life applications, one needs to recover the quantities of interest from indirect observations blurred by noise. In geoscience, for example, mathematical models have the form of ill-posed spherical pseudo-differential equations

$$Ax = y, \tag{1}$$

where A is a pseudo-differential operator that relates continuous functions $x \in C(\Omega_R)$ and $y \in C(\Omega_\rho)$ defined on centered spheres $\Omega_R, \Omega_\rho \in \mathbb{R}^3$ of radii $R \leq \rho$. In satellite geodesy this approach has been introduced in [4, 13] where the spheres Ω_R, Ω_ρ are taken as the models of the surfaces of the Earth and the satellite orbit respectively.

Recall that a spherical pseudo-differential operator A is a linear operator that assigns to any $x \in C(\Omega_R)$ a function

$$Ax := \sum_{k=0}^{\infty} a_k \sum_{j=1}^{2k+1} \hat{x}_{k,j} \frac{1}{\rho} Y_{k,j}\left(\frac{\cdot}{\rho}\right), \tag{2}$$

where

$$\hat{x}_{k,j} = \left\langle \frac{1}{R} Y_{k,j}\left(\frac{\cdot}{R}\right), x(\cdot) \right\rangle_{L_2(\Omega_R)} := \frac{1}{R} \int_{\Omega_R} x(\tau) Y_{k,j}\left(\frac{\tau}{R}\right) d\Omega_R(\tau)$$

are the spherical Fourier coefficients, and $Y_{k,j}(\cdot), j = 1, 2, ..., 2k+1$, are the spherical harmonics [9] of degree k, which are L_2-orthonormalized with respect to the unit sphere $\Omega_1 \in \mathbb{R}^3$. The sequence of real numbers $\{a_k\}_{k=0}^{\infty}$ is referred to as the spherical symbol of A.

One can see that if $a_k \to 0$ as $k \to \infty$ then the operator A is compact. Therefore, it is not continuously invertible (e.g. [3]), and we face an ill-posed problem.

Examples of the ill-posed problem (1), (2) include satellite-to-satellite tracking problem (SST-problem) with $a_k = \frac{k+1}{\rho}\left(\frac{R}{\rho}\right)^k$, satellite gravity gradiometry problem (SGG-problem) with $a_k = \frac{(k+1)(k+2)}{\rho^2}\left(\frac{R}{\rho}\right)^k$ etc. (for more details on these and other examples we can refer an interested reader to [4]).

At this point it is worth to mention that in practice one is usually interested in discrete approximations of the solution of (1), (2). For example, Earth Gravity Model (EGM2008)[11] is parameterized by the spherical Fourier coefficients of the gravitational potential x up to the prescribed degree M.

Note that in applications, such as EGM2008, the function y is assumed to be continuous. However, in practice we are given just a finite amount of points $\{t_i\}_{i=1}^N \subset \Omega_\rho$ at which an information about the values of y is provided. It should be noted also that the satellite data contain measurement errors, which can be modeled, for example, in the following way

$$|y^\epsilon(t_i) - y(t_i)| \leq \epsilon_i, \ i = 1, \dots, N,$$

where we assume that there exists a function $y^\epsilon \in C(\Omega_\rho)$ staying for the noisy version of the original function y, such that

$$\|y^\epsilon - y\|_{C(\Omega_\rho)} \leq \epsilon := \max_{1\leq i\leq N}\{\epsilon_i\},$$

where ϵ_i are measurement errors.

In such a set up the problem (1), (2) is reduced to the following spherical pseudo-differential operator equation

$$A_M x := \sum_{k=0}^{M} a_k \sum_{j=1}^{2k+1} \hat{x}_{k,j} \frac{1}{\rho} Y_{k,j}\left(\frac{\cdot}{\rho}\right) = y^\epsilon.$$

In some cases, one is not interested in completely knowing x, but is instead in only some quantities derived from it. For example, one is interested in the gravitational potential at some fixed point $\xi \in \Omega_R$ [2] to estimate or predict local gravity anomalies, for example. To do this one may potentially reconstruct the whole solution x using the regularized collocation method and then calculate the value $x(\xi)$. However, the usual methods of regularization of the problem (1), (2) operate in the Hilbert space L_2. Then the issue is that for any particular point $\xi \in \Omega_R$, the value $x(\xi)$ is not well-defined for the $L_2(\Omega_R)$-function x.

The recently published paper [1] is concerned with the polynomial approximation of noisy continuous functions on the sphere. In that paper the authors suggest to approximate the function y from its noisy measurements $y^\epsilon(t_i)$ by means of regularized least-squares method in the space $\mathbb{P}_M$ of all spherical polynomials of degree less or equal to M. The crucial point is that the space $\mathbb{P}_M$ can be seen also as a reproducing kernel Hilbert space $\mathcal{H}$ (RKHS) in Ω_ρ. By the Riesz representation theorem, to every RKHS $\mathcal{H}$ there corresponds a unique symmetric positive definite function $K : \Omega_\rho \times \Omega_\rho \to \mathbb{R}$, called the reproducing kernel of $\mathcal{H}$, that has the following reproducing property

$\forall f \in \mathcal{H}\ \forall t \in \Omega_\rho : f(t) = \langle f(\cdot), K(\cdot,t)\rangle_{\mathcal{H}}$. This means that for $x \in \mathcal{H}$ the evaluation $x(\xi)$ at any fixed point ξ is well-defined.

Therefore, in this paper we are going to present the following method for the reconstruction of $x(\xi)$. In the first stage we apply the approach [1] to preprocess the original noisy data. Then the approximation of the solution x can be obtained by the formal inversion of A_M, and we will be able to approximate $x(\xi)$.

The paper is organized as follows: in the next section we discuss the details of the above mentioned approximation. Section 3 is devoted to theoretical error bounds. Finally, in the last section we discuss an a posteriori regularization paremeter choice rule and present some numerical experiments showing the good performance of our method.

2 Description of the method

We shall assume, in a slight generalization of [1], that the set $\{t_i\}_{i=1}^N \subset \Omega_\rho$ consists of the points of a cubature rule which is exact for all polynomials $p \in \mathbb{P}_{2M}$, i. e.

$$\forall p \in \mathbb{P}_{2M}(\Omega_\rho),\ \ \sum_{i=1}^{N} \omega_i p(t_i) = \int_{\Omega_\rho} p(\zeta) d\Omega_\rho(\zeta), \tag{3}$$

where $d\Omega_\rho(\zeta)$ denotes the area measure on Ω_ρ, and $\omega_i, i = 1,\dots,N$ are positive cubature weights associated with the pointset $\{t_i\}_{i=1}^N \subset \Omega_\rho$. For sufficiently large N one can find in the literature a variety of suitable cubature formulas (see, e.g., [6, 7, 17])

The kernel generating the corresponding RKHS $\mathcal{H}$ can be written in the following form

$$K(t,\tau) = \sum_{k=0}^{M} \beta_k^{-2} \sum_{j=1}^{2k+1} \frac{1}{\rho^2} Y_{k,j}\left(\frac{t}{\rho}\right) Y_{k,j}\left(\frac{\tau}{\rho}\right),\ t,\tau \in \Omega_\rho, \tag{4}$$

where $\beta = (\beta_1, \beta_2, \dots, \beta_M, \dots)$ is an increasing sequence. Note that the inner product of $\mathcal{H}$ associated with this kernel can be defined as follows

$$\langle f,g\rangle_{\mathcal{H}} = \sum_{k=0}^{M} \frac{\beta_k^2}{\rho^2} \sum_{j=1}^{2k+1} \left\langle Y_{k,j}\left(\frac{\cdot}{\rho}\right), f\right\rangle_{L_2(\Omega_\rho)} \left\langle Y_{k,j}\left(\frac{\cdot}{\rho}\right), g\right\rangle_{L_2(\Omega_\rho)}.$$

Following [1, 12] consider the functional

$$G_\alpha(p_M) = \sum_{i=1}^{N} \omega_i (p_M(t_i) - y^\epsilon(t_i))^2 + \alpha \|p_M\|_{\mathcal{H}}^2. \tag{5}$$

Denoting the vector space $\mathbb{R}^N$ equipped with the inner product

$$\langle \eta,\gamma\rangle_\omega := \sum_{i=1}^{N} \omega_i \eta_i \gamma_i,\ \ \eta,\gamma \in \mathbb{R}^N,$$

by $\mathbb{R}^N_\omega$, and introducing the sampling operator $S_N : C(\Omega_\rho) \to \mathbb{R}^N_\omega$ for which

$$S_N y^\epsilon := (y^\epsilon(t_1), y^\epsilon(t_2), \dots, y^\epsilon(t_N)),$$

the expression (5) can be presented in the following form

$$G_\alpha(p_M) = \|S_N p_M - S_N y^\epsilon\|_\omega^2 + \alpha \|p_M\|_{\mathcal{H}_K}^2 . \tag{6}$$

In the sequel we will deal with $p_M^* = \arg\min \{G_\alpha(p),\ p \in \mathbb{P}_M\}$.

Theorem 2.1. *Assume that the points $\{t_i\}$ and weights $\{\omega_i\}$ are such that (3) holds true. Then the minimizer p_M^* of (6) has the form*

$$p_M^*(\cdot) = \sum_{k=0}^{M} \frac{1}{1+\alpha\beta_k^2} \sum_{j=1}^{2k+1} \frac{1}{\rho} Y_{k,j}\left(\frac{\cdot}{\rho}\right) \sum_{i=1}^{N} \omega_i \frac{1}{\rho} Y_{k,j}\left(\frac{t_i}{\rho}\right) y^\epsilon(t_i). \tag{7}$$

Note that (7) can be seen as the definition of the operator $T_{\alpha,M} : C(\Omega_\rho) \to \mathbb{P}_M(\Omega_\rho)$ such that the minimizer p_M^* can be written as $p_M^* = T_{\alpha,M} y^\epsilon$.

The theorem can be proven by the same argument as in [1, 12]. For the sake of completeness we present the proof here.

Proof. It is known that the minimizer of the functional (6) can be written as

$$T_{\alpha,M} y^\epsilon = (\alpha I + S_N^* S_N)^{-1} S_N^* S_N y^\epsilon, \tag{8}$$

where $S_N^* : \mathbb{R}_\omega^N \to C(\Omega_\rho)$ is the adjoint of S_N, and I is the identity operator in $\mathcal{H}$.

By the definition

$$\begin{aligned} \langle S_N f, \eta \rangle_\omega &= \sum_{i=1}^{N} \omega_i f(t_i) \eta_i = \sum_{i=1}^{N} \omega_i \langle K(t_i,\cdot), f(\cdot) \rangle_{\mathcal{H}} \eta_i \\ &= \left\langle \sum_{i=1}^{N} \omega_i K(t_i,\cdot)\eta_i, f(\cdot) \right\rangle_{\mathcal{H}} \end{aligned}$$

there holds that

$$(S_N^* \eta)(\cdot) = \sum_{i=1}^{N} \omega_i K(t_i,\cdot)\eta_i,\ \forall \eta \in \mathbb{R}_\omega^N,$$

and thereby

$$(S_N^* S_N f)(\cdot) = \sum_{i=1}^{N} \omega_i K(t_i,\cdot) f(t_i).$$

Inserting this expression into (8), we observe that p_M^* solves the equation

$$\alpha p_M^*(\cdot) + \sum_{i=1}^{N} \omega_i K(t_i,\cdot) p_M^*(t_i) = \sum_{i=1}^{N} \omega_i K(t_i,\cdot) y^\epsilon(t_i),$$

which in view of (4) can be rewritten as follows:

$$
\begin{aligned}
\alpha \quad & \sum_{k=0}^{M}\sum_{j=1}^{2k+1}\left\langle Y_{k,j}\left(\frac{\cdot}{\rho}\right), p_M^*(\cdot)\right\rangle_{L_2(\Omega_\rho)} \frac{1}{\rho^2} Y_{k,j}\left(\frac{t}{\rho}\right) \\
+ \quad & \sum_{i=1}^{N}\omega_i \sum_{k=0}^{M}\beta_k^{-2}\sum_{j=1}^{2k+1}\frac{1}{\rho^2}Y_{k,j}\left(\frac{t}{\rho}\right)Y_{k,j}\left(\frac{t_i}{\rho}\right) \\
\times \quad & \sum_{\kappa=0}^{M}\sum_{\iota=1}^{2\kappa+1}\left\langle Y_{\kappa,\iota}\left(\frac{\cdot}{\rho}\right), p_M^*(\cdot)\right\rangle_{L_2(\Omega_\rho)} \frac{1}{\rho^2} Y_{\kappa,\iota}\left(\frac{t_i}{\rho}\right) \\
= \quad & \sum_{i=1}^{N}\omega_i \sum_{k=0}^{M}\beta_k^{-2}\sum_{j=1}^{2k+1}\frac{1}{\rho^2}Y_{k,j}\left(\frac{t}{\rho}\right)Y_{k,j}\left(\frac{t_i}{\rho}\right) y^\epsilon(t_i).
\end{aligned}
$$

Then using (3), one can see that

$$
\begin{aligned}
& \sum_{i=1}^{N}\omega_i \sum_{k=0}^{M}\beta_k^{-2}\sum_{j=1}^{2k+1}\frac{1}{\rho^2}Y_{k,j}\left(\frac{t}{\rho}\right)Y_{k,j}\left(\frac{t_i}{\rho}\right) \\
\times \quad & \sum_{\kappa=0}^{M}\sum_{\iota=1}^{2\kappa+1}\left\langle Y_{\kappa,\iota}\left(\frac{\cdot}{\rho}\right), p_M^*(\cdot)\right\rangle_{L_2(\Omega_\rho)} \frac{1}{\rho^2} Y_{\kappa,\iota}\left(\frac{t_i}{\rho}\right) \\
= \quad & \sum_{k=0}^{M}\beta_k^{-2}\sum_{j=1}^{2k+1}\left\langle Y_{k,j}\left(\frac{\cdot}{\rho}\right), p_M^*(\cdot)\right\rangle_{L_2(\Omega_\rho)} \frac{1}{\rho^2} Y_{k,j}\left(\frac{t}{\rho}\right)
\end{aligned}
$$

Therefore,

$$
\left\langle \frac{1}{\rho}Y_{k,j}\left(\frac{\cdot}{\rho}\right), p_M^*(\cdot)\right\rangle_{L_2(\Omega_\rho)} = \frac{1}{1+\alpha\beta_k^2}\sum_{i=1}^{N}\omega_i\frac{1}{\rho}Y_{k,j}\left(\frac{t_i}{\rho}\right)y^\epsilon(t_i),
$$

for $j = 1, \dots, 2k+1$, $k = 0, 1, \dots, M$, which implies the explicit form of $p_M^* = T_{\alpha,M}y^\epsilon$ given in (7). □

In view of (7) one can easily implement the formal inversion of A_M to obtain solution x_α of the equation $A_M x = T_{\alpha,M}y^\epsilon$. Indeed,

$$
x_\alpha(\cdot) = \sum_{k=0}^{M}\frac{1}{a_k}\frac{1}{1+\alpha\beta_k^2}\sum_{j=1}^{2k+1}\frac{1}{R}Y_{k,j}\left(\frac{\cdot}{R}\right)\sum_{i=1}^{N}\omega_i\frac{1}{\rho}Y_{k,j}\left(\frac{t_i}{\rho}\right)y^\epsilon(t_i). \qquad (9)
$$

This also can be seen as the definition of the operator $T^{\downarrow}_{\alpha,M} : C(\Omega_\rho) \to C(\Omega_R)$ such that $x_\alpha = T^{\downarrow}_{\alpha,M}y^\epsilon$.

The approximate solution x_α in (9) depends on the regularization parameter α, and on the weights β_k. The choice of the regularization parameter α will be presented in the next section. A choice of the weights β_k has been recently discussed in [12]. In the present study the weights β_k are assumed to be chosen a priori, and in our numerical experiments we use one of the recipes from [12].

3 Error analysis

As one can see from (9), we approximate the unknown solution x by means of the spherical polynomial of degree M. In this context a question about the best approximation of a continuous function by means of a polynomial of degree M naturally appears. It is clear that the smoothness of x determines the best approximation. One may also say that the smoothness of x is encoded in its Fourier coefficients, which allow a construction of nearly the best polynomial approximation. For example, in [14] it is suggested to approximate a function $x \in C(\Omega_R)$ by a spherical polynomial

$$(V_M x)(t) = \sum_{k=0}^{M} h\left(\frac{k}{M}\right) \sum_{j=1}^{2k+1} \frac{1}{R^2} Y_{k,j}\left(\frac{t}{R}\right) \left\langle Y_{k,j}\left(\frac{\cdot}{R}\right), x(\cdot)\right\rangle_{L_2(\Omega_R)},$$

where h is a continuously differentiable function that is called a filter function and satisfies the relation

$$h(t) = \begin{cases} 1, & t \in [0, 1/2], \\ 0, & t \in [1, \infty). \end{cases}$$

Various examples of filter functions h can be found in [15]. In [14] it has been shown that

$$\|x - V_M x\|_{C(\Omega_R)} \leq c \inf_{p \in \mathbb{P}_{[M/2]}} \|x - p\|_{C(\Omega_R)},$$

where c is a generic constant, which may take different values at different occurrences, and $[\cdot]$ denotes the floor function.

In the same way one can define a spherical polynomial

$$(V_M y)(t) = \sum_{k=0}^{M} h\left(\frac{k}{M}\right) \sum_{j=1}^{2k+1} \frac{1}{\rho^2} Y_{k,j}\left(\frac{t}{\rho}\right) \left\langle Y_{k,j}\left(\frac{\cdot}{\rho}\right), x(\cdot)\right\rangle_{L_2(\Omega_\rho)}, \ t \in \Omega_\rho,$$

to approximate the right-hand side of the equation (1), and it is clear that

$$V_M x = A_M^{-1} V_M y = T_{0,M}^{\downarrow} V_M y.$$

The latter equality follows from (3).

Observe now that

$$\left|(T_{0,M}^{\downarrow} - T_{\alpha,M}^{\downarrow}) V_M y(\xi)\right| \leq \alpha \left| \sum_{k=0}^{M} \beta_k^2 \sum_{j=1}^{2k+1} \frac{1}{R^2} Y_{k,j}\left(\frac{\xi}{R}\right) \left\langle Y_{k,j}\left(\frac{\cdot}{R}\right), x\right\rangle_{L_2(\Omega_R)} \right|,$$

and this means that for any M and $\xi \in \Omega_R$ $\left|(T_{0,M}^{\downarrow} - T_{\alpha,M}^{\downarrow}) V_M y(\xi)\right|$ tends to zero with $\alpha \to 0$.

Therefore, there exists a continuous and monotonically increasing function $\varphi_{h,M}(\cdot) = \varphi_{h,M}(\cdot;\xi)$ of α such that $\varphi_{h,M}(0;\xi) = 0$ and

$$\left|(T_{0,M}^{\downarrow} - T_{\alpha,M}^{\downarrow}) V_M y(\xi)\right| \leq \varphi_{h,M}(\alpha;\xi).$$

Note that the function $\varphi_{h,M}$ depends on the solution of (1) and therefore is not assumed to be known.

Now we are going to estimate the error of the approximation (9) at the point of interest ξ, namely the distance $|x(\xi) - x_\alpha(\xi)|$.

It is clear that

$$\begin{aligned} \left| x(\xi) - (T^{\downarrow}_{\alpha,M} y^{\epsilon})(\xi) \right| &\leq |x(\xi) - (V_M x)(\xi)| \qquad (10) \\ + \quad &\left| (V_M x)(\xi) - (T^{\downarrow}_{\alpha,M} V_M y)(\xi) \right| + \left| (T^{\downarrow}_{\alpha,M}(V_M y - y + y - y^{\epsilon}))(\xi) \right|. \end{aligned}$$

It is natural to assume that $\|V_M y - y\|_{C(\Omega_\rho)} < \epsilon$, since otherwise data noise is dominated by the approximation error and no regularization is required. We also restrict ourselves to the case when $\|x - V_M x\|_{C(\Omega_R)} < c\epsilon$, otherwise the term $\|x - V_M x\|_{C(\Omega_R)}$ corresponding to the ideal approximation, when one can directly use the Fourier coefficients of unknown x, will appear in all error bounds.

Then the bound (10) can be reduced to the following one

$$\left| x(\xi) - (T^{\downarrow}_{\alpha,M} y^{\epsilon})(\xi) \right| \leq \varphi_{h,M}(\alpha;\xi) + c\epsilon \sup \left\{ \left| (T^{\downarrow}_{\alpha,M} f)(\xi) \right|, \ \|f\|_{C(\Omega_\rho)} \leq 1 \right\} \qquad (11)$$

Note that the second term of the bound (11) does not depend on the unknown solution x and can be estimated as follows.

Theorem 3.1. *Under the conditions of Theorem 2.1*

$$\sup \left\{ \left| (T^{\downarrow}_{\alpha,M} f)(\xi) \right|, \ \|f\|_{C(\Omega_\rho)} \leq 1 \right\} \leq \frac{1}{R\rho} \sum_{k=0}^{M} \frac{2k+1}{4\pi a_k (1 + \alpha\beta_k^2)} \sum_{i=1}^{N} \omega_i \left| P_k \left(\frac{\xi \cdot t_i}{R\rho} \right) \right|, \qquad (12)$$

where P_k are the Legendre polynomials of degree k.

Proof.

Using (9) and the well-known addition theorem for spherical harmonics [9] we can write

$$\begin{aligned} \left| (T^{\downarrow}_{\alpha,M} f)(\xi) \right| &= \frac{1}{R\rho} \left| \sum_{k=0}^{M} \frac{1}{a_k} \frac{1}{1 + \alpha\beta_k^2} \sum_{i=1}^{N} \omega_i f(t_i) \sum_{j=1}^{2k+1} Y_{k,j}\left(\frac{\xi}{R}\right) Y_{k,j}\left(\frac{t_i}{\rho}\right) \right| \\ &= \frac{1}{R\rho} \left| \sum_{k=0}^{M} \frac{2k+1}{4\pi a_k (1 + \alpha\beta_k^2)} \sum_{i=1}^{N} \omega_i f(t_i) P_k\left(\frac{\xi \cdot t_i}{R\rho}\right) \right| \\ &\leq \frac{\|f\|_{C(\Omega_\rho)}}{R\rho} \sum_{k=0}^{M} \frac{2k+1}{4\pi a_k (1 + \alpha\beta_k^2)} \sum_{i=1}^{N} \omega_i \left| P_k\left(\frac{\xi \cdot t_i}{R\rho}\right) \right|. \end{aligned}$$

□

Let us denote by $\psi_{N,M}(\alpha;\xi)$ the right-hand side of the bound (12). It is clear that $\psi_{N,M}(\cdot;\xi)$ is a decreasing function of α. Then in view of the decomposition (11) the desired value of the regularization parameter $\alpha = \alpha_{opt}$ is the one that balances both terms in the bound

$$\left|x(\xi) - (T^{\downarrow}_{\alpha,M} y^{\epsilon})(\xi)\right| \le \varphi_{h,M}(\alpha;\xi) + c\epsilon\psi_{N,M}(\alpha;\xi),$$

i. e. $\varphi_{h,M}(\alpha_{opt};\xi) = c\epsilon\psi_{N,M}(\alpha_{opt};\xi)$.

Note that the value of such $\alpha = \alpha_{opt}$ cannot be found without knowledge of $\varphi_{h,M}$. At the same time, there is a parameter choice rule that does not require this knowledge, but nevertheless it guarantees an error bound which is only by a constant factor worse than $\varphi_{h,M}(\alpha_{opt};\xi) + c\epsilon\psi_{N,M}(\alpha_{opt};\xi)$. This rule is sometimes called the balancing principle (BP) and is well documented in the literature (see, e.g., the recent book [8] and references therein).

In BP the regularization parameter α is selected from some finite set, say $\Delta_L := \left\{\alpha_i = q^i\alpha_0, i = 1, 2, ..., L\right\}$, with $q \in (0, 1)$ and L large enough.

Applying the balancing principle to our problem we start with the smallest parameter α_L and increase stepwise $\alpha_{i-1} = \alpha_i/q, i = L, L-1, ...,$ until $\alpha_* := \alpha_z$ is the parameter for which

$$\left|(T^{\downarrow}_{\alpha_z,M} y^{\epsilon})(\xi) - (T^{\downarrow}_{\alpha_{z+1},M} y^{\epsilon})(\xi)\right| > \vartheta\epsilon\psi_{N,M}(\alpha_{z+1};\xi), \tag{13}$$

for the first time. Here ϑ is a design parameter; in the experiments below it is taken as $\vartheta = 10^{-4}$ and $\vartheta = 10^{-3}$. The regularization parameter α_* is the parameter of our choice. Note that for choosing α_* we need only the knowledge of the upper bound given by (12).

In the next section we present numerical experiments demonstrating the performance of BP and some of its advantages.

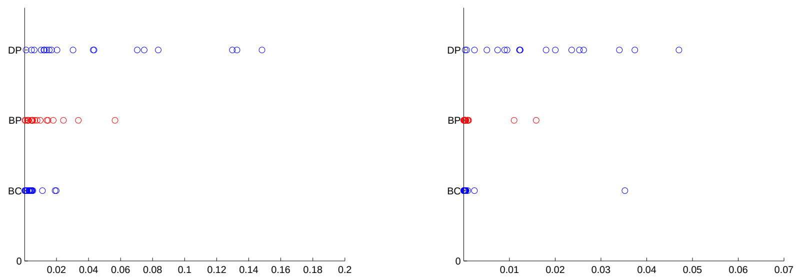
4 Numerical examples

At this point it is worth to mention that the most commonly used a posteriori method for choosing the regularization parameter is the Discrepancy principle (DP) (see, e.g., [3], p. 83). In the context of spherical pseudo-differential equations DP has been discussed recently in [10]. Recall that in accordance with DP the parameter $\alpha = \alpha_{DP}$ is chosen as the largest one from Δ_L such that $\|S_N A_M x_\alpha - S_N y^\epsilon\|_\omega \le \epsilon$. Then $x_{\alpha_{DP}}(\xi)$ can be used as an approximation for $x(\xi)$.

In contrast to the balancing principle described above, DP is not adjusted to the reconstruction of a solution at a fixed point. To the best of our knowledge, the balancing principle (13) is the first rule that can be used for this purpose in the context of ill-posed spherical pseudo-differential equations. As it can be seen from the numerical tests below, in the problem of pointwise estimation the balancing principle (13) not only outperforms DP, but also performs similar to the best choice of $\alpha = \alpha_{best}$ in the sense of the error measure $\|x - x_\alpha\|_{C(\Omega_R)}$ (the later choice is not practically implementable since it requires the knowledge of x).

In all our experiments we follow [5, 10] and assume that $\{t_i\}_{i=1}^N$ is the set of Gauss-Legendre points, for which the positive quadrature weights are known analytically. In this case $N = 2(M + 1)^2$, and we take $M = 30$.

The data are simulated in the following way. First we generate a spherical function



(a) Relative errors for the case $\vartheta = 10^{-3}$, $a_k = 1.48^k, \upsilon = -7/2, \beta_k^2 = a_k^{-1}(k+1)^{7/2}, k = 1, 2, \ldots, M$ for 20 simulations of the data.

(b) Relative errors for the case $\vartheta = 10^{-4}$, $a_k = 1.48^k, \upsilon = -11/2, \beta_k^2 = a_k^{-1}(k+1)^{11/2}, k = 1, 2, \ldots, M$ for 20 simulations of the data.

Figure 1: Numerical illustrations. The comparison of the performance of the method (9) with the balancing principle (BP), the discrepancy principle (DP), and the best choice of α (BC). The horizontal axes show values of the relatives errors (the vertical axes have no significance)

$$y(t) = \sum_{k=0}^{M} a_k \sum_{j=1}^{2k+1} \hat{x}_{k,j} \frac{1}{\rho} Y_{k,j}\left(\frac{t}{\rho}\right), \quad t \in \Omega_\rho,$$

where $\hat{x}_{k,j} = (k+1)^{\upsilon} g_{k,j}$, $k = 0, ..., M$, $j = 1, ...2k+1$, $\upsilon < 0$, and $g_{k,j}$ are uniformly distributed random values from $[-1, 1]$.The noisy version y^{ϵ} is simulated by adding the noise values to the values of the initial function y at the points $\{t_i\}_{i=1}^{N}$. The simulated noise values are given as the components of a random vector $0.05\eta/\left\|\eta\right\|_{\infty}$, where $\eta = [\eta_1, \eta_2, ..., \eta_N]$, and η_i are uniformly distributed on $[-1, 1]$.

To assess the performance of the considered schemes we measure the relative error

$$\frac{|x(\xi) - x_{\alpha}(\xi)|}{|x(\xi)|},$$

where in spherical coordinate system ξ is given as $\xi = (0, 3.065, R)$,

$$x = \sum_{k=0}^{M} \sum_{j=1}^{2k+1} \hat{x}_{k,j} \frac{1}{R} Y_{k,j}\left(\frac{\cdot}{R}\right),$$

and the approximation x_α is given by (9), where β_k are chosen a priori as $\beta_k = a_k^{-1/2}(k+1)^{\vartheta/4}$ that corresponds to one of the recipes suggested in [12] (see, e.g., the formula (4.4) there).

The results are displayed in Figure 1, where each subfigure corresponds to different values of υ. In Figure 1 each circle exhibits a value of the relative error in solving the problem with one of 20 simulated data, for each of the following cases: approximation of $x(\xi)$ by (9) with BP (13) for choosing α from $\Delta_L, L = 60, q = \alpha_0 = 0.8$, the same regularization method with α chosen by

DP [10], and the best approximation of the whole solution x in the sense of the error $\|x - x_\alpha\|_{C(\Omega_R)}$ with $\alpha \in \Delta_L$.

From Figure 1 we see that the choice of the regularization parameter α according to the proposed balancing principle improves the accuracy of the reconstruction of $x(\xi)$ compared to a standard parameter choice strategy, such as DP.

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