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A ROBUST MULTIGRID METHOD FOR DISCONTINUOUS GALERKIN DISCRETIZATIONS OF STOKES AND LINEAR ELASTICITY EQUATIONS

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Abstract. We consider multigrid methods for discontinuous Galerkin (DG) $H(\mathrm{div}, \Omega)$-conforming discretizations of the Stokes equation. We first describe a simple Uzawa iteration for the solution of the Stokes problem, which requires a solution of a nearly incompressible linear elasticity problem on every iteration. Then, based on special subspace decompositions of $H(\mathrm{div}, \Omega)$, as introduced in [J. Schöberl. Multigrid methods for a parameter dependent problem in primal variables. *Numerische Mathematik*, 84(1):97–119, 1999], we analyze variable V-cycle and W-cycle multigrid methods with nonnested bilinear forms. We prove that these algorithms are robust, and their convergence rates are independent of the material parameters such as Poisson ratio and of the mesh size.

1. Introduction

In this paper we present a multigrid method for a family of discontinuous Galerkin $H(\mathrm{div}; \Omega)$-conforming discretizations of the Stokes problem and the linear elasticity problem. The discretization for the Stokes problem preserves divergence-free velocity fields and was first introduced in [1]. The same method was also used in [2].

In general, the numerical discretization of the Stokes problem produces systems of linear algebraic equations of saddle-point type. Solving such systems has been the subject of extensive research work and at present several different approaches can be used to solve them efficiently (see [3] and references cited therein).

One widely used approach is to construct a block diagonal preconditioner with two blocks: one containing the inverse or a preconditioner of the stiffness matrix of a vector Laplacian and one containing the inverse of a lumped mass matrix for the pressure. The resulting preconditioned system can then be solved by means of the preconditioned MINRES (minimum residual) method.

Recently, an auxiliary space preconditioner for an $H(\mathrm{div})$-conforming DG discretization of the Stokes problem was proposed in [4]. The auxiliary space preconditioning techniques were introduced in [5] as generalizations of the fictitious space methods (see [6]). Since the solution of the Stokes system has divergence-free velocity component, the problem can easily be reduced to a "second-order" problem in the space Range(curl). In order to apply the preconditioner one needs to solve four elliptic problems, for details, see [4].

There are other multigrid methods which can roughly be classified into two categories: coupled and decoupled methods, cf. [7]. A well-known coupled approach is based on solving small saddle point systems at every grid point or on appropriate patches, cf. [8]. The Schur complement of each small saddle point system can be formed explicitly, and hence it is easy to solve the local problems. However, it is not straightforward to choose appropriate patches when the pressure is discretized

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by continuous elements. Further, when used as a smoothing iteration, this so-called Vanka method needs a proper damping parameter.

One classical decoupled approach is the Uzawa method [9]. A crucial point in applying this method is the right choice of a damping parameter for solving the arising linear elasticity system. As proved in [9] the Uzawa method is very efficient for solving the Stokes problem when the damping parameter is very large. In this case it is important to have a robust solver for the linear elasticity problem, that is, an iterative method that converges uniformly with respect to the Lamé parameters, or equivalently with respect to the Poisson ratio.

In [10], the author proposed and analyzed robust and optimal multigrid methods for the parameter dependent problem of nearly incompressible materials for the $P_2 - P_0$ finite element scheme for the mixed system and for the corresponding non-conforming finite element scheme in primal variables. This approach relies on constructing a locally supported basis for the weakly divergence-free functions. In the present paper we construct suitable subspace decompositions of $H(\text{div}, \Omega)$, as suggested in [10], in order to design and analyze robust multigrid algorithms with nonnested (non-inherited) bilinear forms related to $H(\text{div}; \Omega)$-conforming DG discretizations. Similar ideas were used to build a robust subspace correction method for the system of linear algebraic equations arising from non-conforming finite element discretization based on reduced integration in [11]. An alternative approach is based on using augmented Lagrangian formulations for nearly singular systems, cf. [12].

The computationally most expensive part of the Uzawa algorithm for the Stokes problem is the solution of a nearly incompressible linear elasticity problem. The Uzawa iteration converges rapidly if the damping parameter is very large. However, in this case an efficient multilevel solver is needed for the linear elasticity problem which is uniform with respect to the Lamé parameters. A key component of such a solver is an overlapping block-smoother which corresponds to an appropriate splitting of the space of divergence-free functions, cf. [13]. At the same time, noting that a truly divergence-free function on the coarse grid is also divergence-free on the fine grid, the transfer operator prolongating coarse-grid divergence-free functions to fine grid divergence-free functions is as simple as an inclusion operator. In this paper, we first show that the discretization of the linear elasticity system and the corresponding Stokes problem is locking-free and then establish the approximation and smoothing properties necessary for the multigrid analysis [14, 15].

The layout of the paper is as follows. In Section 2 we state the Stokes problem and the linear elasticity problem. The discontinuous Galerkin discretization for the Stokes and the corresponding linear elasticity problem is given in Section 3. We further prove the stability and approximation of the discretization and show that for the linear elasticity problem it is locking-free. In Section 4, we propose the multigrid method and prove its robust and optimal convergence. Finally, we give some concluding remarks in Section 5.

2. Problem formulation

In this section, we give the formulation of the Stokes and the linear elasticity problem. Let $\Omega \subset R^d$ $(d = 2, 3)$ be a polygonal domain with boundary $\partial\Omega$, $\boldsymbol{f} \in L^2(\Omega)^d$, and $H_0^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega), u|_{\partial\Omega} = 0\}$. We also need the standard Sobolev spaces $H^1(\Omega), H^2(\Omega)$, and the corresponding norms

$$\|u\|_2 = \Big(\sum_{|\boldsymbol{\alpha}|\le 2} \int_\Omega |\frac{\partial^{\boldsymbol{\alpha}} u}{\partial x^{\boldsymbol{\alpha}}}|^2 dx\Big)^{1/2}, \|u\|_1 = \Big(\sum_{|\boldsymbol{\alpha}|\le 1} \int_\Omega |\frac{\partial^{\boldsymbol{\alpha}} u}{\partial x^{\boldsymbol{\alpha}}}|^2 dx\Big)^{1/2}, \|u\| = \Big(\int_\Omega u^2 dx\Big)^{1/2}.$$

The variational formulation of the Stokes and the mixed formulation of the elasticity problem can be written as: Find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), & \text{for all} \quad \boldsymbol{v} \in H_0^1(\Omega)^d, \\ b(\boldsymbol{u}, q) - (\rho p, q) = 0, & \text{for all} \quad q \in L_0^2(\Omega), \end{cases} \tag{2.1}$$

Here, with the usual notation, $\boldsymbol{u}$ is the velocity field (displacement in the case of elasticity), p is the pressure, and $\varepsilon(\boldsymbol{u}) \in L^2(\Omega)^{d\times d}_{sym}$ is the symmetric (linearized) strain rate tensor defined by $\varepsilon(\boldsymbol{u}) = \frac{\nabla \boldsymbol{u}+\nabla \boldsymbol{u}^T}{2}$. For the Stokes equation, one takes $\rho = 0$, and for elasticity equation, we have $\rho = \lambda^{-1}$, with λ being the Lamè parameter defined as $\lambda = \frac{\nu}{1-2\nu}$, $0 \le \nu < \frac{1}{2}$ and ν is the Poisson ratio.

The bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $(\cdot,\cdot)$ are defined by

$$
\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v}) &:= \int_\Omega \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) dx, , \quad \text{for all} \quad \boldsymbol{u},\boldsymbol{v} \in H^1_0(\Omega)^d,\\
b(\boldsymbol{u},q) &:= \int_\Omega q \operatorname{div} \boldsymbol{u} dx, \quad \text{for all} \quad \boldsymbol{u} \in H^1_0(\Omega)^d, q \in L^2_0(\Omega).\\
(p,q) &:= \int_\Omega pq \, dx, \quad \text{for all} \quad p,q \in L^2_0(\Omega).
\end{aligned}
\tag{2.2}
$$

For the linear elasticity problem, we also have the corresponding primal formulation, which is: Find $\boldsymbol{u}$ in $H^1_0(\Omega)^d$ such that

$$
(\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})) + \lambda(\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}), \quad \text{for all} \quad \boldsymbol{v} \in H^1_0(\Omega)^d. \tag{2.3}
$$

The conditions for the existence and uniqueness of the solution $(\boldsymbol{u}, p)$ of (2.1) are well known and understood, see, e.g. [16]. For the relationship between the inf-sup condition for the Stokes problem and the Korn's inequality which guarantees the solvability of the elasticity equations see also [17]. For convenience, in this paper, we assume that the domain Ω is such that the following regularity estimate holds (see e.g. [18] for the limiting case of the Stokes equation and [19, Lemma 2.2] for the corresponding result in linear elasticity):

$$
\|\boldsymbol{u}\|_2 + \|p\|_1 \lesssim \|\boldsymbol{f}\|. \tag{2.4}
$$

Here, in Equation (2.4) and throughout the presentation that follows, the hidden constants in $\lesssim$, $\gtrsim$ and $\eqsim$ are independent of λ and the mesh size h.

3. Discontinuous Galerkin discretization

In this section, we first give some preliminaries and notation for the DG formulations. Next, we derive the DG discretization of the Stokes problem and the equations of linear elasticity and show the relationship between the two problems by the Uzawa method. Finally, we analyze the stability and approximation properties of the discretization.

3.1. **Preliminaries and notation.** We denote by T_h a shape-regular triangulation of mesh-size h of the domain Ω into triangles $\{K\}$. We further denote by E_h^I the set of all interior edges (or faces) of T_h and by E_h^B the set of all boundary edges (or faces); we set $E_h = E_h^I \cup E_h^B$.

For $s \ge 1$, we define

$$
H^s(T_h) = \{\phi \in L^2(\Omega), \text{ such that } \phi|_K \in H^s(K) \text{ for all } K \in T_h\}.
$$

The vector functions are represented column-wise. And, we recall the definitions of the spaces to be used herein:

$$
H(\operatorname{div};\Omega) := \{\boldsymbol{v} \in L^2(\Omega) : \operatorname{div} \boldsymbol{v} \in L^2(\Omega)\},
$$

with the norm

$$
\|\boldsymbol{v}\|^2_{H(\operatorname{div};\Omega)} := \|\boldsymbol{v}\|^2 + \|\operatorname{div} \boldsymbol{v}\|^2.
$$

As is usual in the DG approach, we now define some trace operators. Let $e = \partial K_1 \cap \partial K_2$ be the common boundary (interface) of two subdomains K_1 and K_2 in T_h , and $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be unit normal vectors to e pointing to the exterior of K_1 and K_2, respectively. For any edge (or face) $e \in E_h^I$ and a scalar $q \in H^1(T_h)$, vector $\boldsymbol{v} \in H^1(T_h)^d$ and tensor $\boldsymbol{\tau} \in H^1(T_h)^{d\times d}$, we define the averages

$$\{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}|_{\partial K_1\cap e}\cdot \boldsymbol{n}_1 - \boldsymbol{v}|_{\partial K_2\cap e}\cdot \boldsymbol{n}_2), \quad \{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}|_{\partial K_1\cap e}\boldsymbol{n}_1 - \boldsymbol{\tau}|_{\partial K_2\cap e}\boldsymbol{n}_2),$$
$$\langle\boldsymbol{\tau}\rangle = \frac{1}{2}(\boldsymbol{\tau}|_{\partial K_1\cap e} + \boldsymbol{\tau}|_{\partial K_2\cap e}),$$

and jumps

$$[q] = q|_{\partial K_1\cap e} - q|_{\partial K_2\cap e}, \quad [\boldsymbol{v}] = \boldsymbol{v}|_{\partial K_1\cap e} - \boldsymbol{v}|_{\partial K_2\cap e}, \quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}|_{\partial K_1\cap e}\odot \boldsymbol{n}_1 + \boldsymbol{v}|_{\partial K_2\cap e}\odot \boldsymbol{n}_2,$$

where $\boldsymbol{v}\odot\boldsymbol{n} = \frac{1}{2}(\boldsymbol{v}\boldsymbol{n}^T + \boldsymbol{n}\boldsymbol{v}^T)$ is the symmetric part of the tensor product of $\boldsymbol{v}$ and $\boldsymbol{n}$.

When $e \in E_h^B$ then the above quantities are defined as

$$\{\boldsymbol{v}\} = \boldsymbol{v}|_e\cdot\boldsymbol{n},\ \{\boldsymbol{\tau}\} = \boldsymbol{\tau}|_e\boldsymbol{n},\ \langle\boldsymbol{\tau}\rangle = \boldsymbol{\tau}|_e,$$

and

$$[q] = q|_e,\ [\boldsymbol{v}] = \boldsymbol{v}|_e,\ [\![\boldsymbol{v}]\!] = \boldsymbol{v}|_e\odot\boldsymbol{n}.$$

Since $\boldsymbol{n}_1 = -\boldsymbol{n}_2, \{\varepsilon(\boldsymbol{u})\} = \langle\varepsilon(\boldsymbol{u})\rangle\boldsymbol{n}_1$ and $[\![\boldsymbol{v}]\!] = [\boldsymbol{v}]\odot\boldsymbol{n}_1$, it follows that

$$\begin{aligned}\langle\varepsilon(\boldsymbol{u})\rangle : [\![\boldsymbol{v}]\!] &= \mathrm{trace}([\![\boldsymbol{v}]\!]^T\langle\varepsilon(\boldsymbol{u})\rangle) = \mathrm{trace}([\boldsymbol{v}]\{\varepsilon(\boldsymbol{u})\}^T)\\ &= \{\varepsilon(\boldsymbol{u})\}\cdot[\boldsymbol{v}], \quad \text{for all} \quad \boldsymbol{u},\boldsymbol{v}\in H^1(T_h)^d.\end{aligned} \tag{3.1}$$

If we denote by $\boldsymbol{n}_K$ the outward unit normal to ∂K, it is easy to check that

$$\sum_{K\in T_h}\int_{\partial K}\boldsymbol{v}\cdot\boldsymbol{n}_K q ds = \sum_{e\in E_h}\int_e\{\boldsymbol{v}\}[q]ds, \quad \text{for all} \quad \boldsymbol{v}\in H(\mathrm{div};\Omega), \text{for all} \quad q\in H^1(T_h). \tag{3.2}$$

Also for $\boldsymbol{\tau}\in H^1(\Omega)^{d\times d}$ and for all $\boldsymbol{v}\in H^1(T_h)^d$, we have

$$\sum_{K\in T_h}\int_{\partial K}(\boldsymbol{\tau}\boldsymbol{n}_K)\cdot\boldsymbol{v}ds = \sum_{e\in E_h}\int_e\{\boldsymbol{\tau}\}\cdot[\boldsymbol{v}]ds. \tag{3.3}$$

The finite element spaces are denoted by

$$\boldsymbol{V}_h = \{\boldsymbol{v}\in H(\mathrm{div};\Omega) : \boldsymbol{v}|_K\in\boldsymbol{V}(K),\ K\in T_h;\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$
$$S_h = \{q\in L^2(\Omega) : q|_K\in Q(K),\ K\in T_h;\ \int_\Omega q dx = 0\}.$$

For the DG method, we use the RT pair $RT_l(K)/P_l(K)$ or the BDM pair $BDM_l(K)/P_{l-1}(K)$ or the BDFM pair $BDFM_l(K)/P_{l-1}(K)$ as $\boldsymbol{V}(K)/Q(K)$ which satisfy $\mathrm{div}\,\boldsymbol{V}(K) = Q(K)$ and preserve the divergence-free velocity fields, (see [4]).

We recall the basic approximation properties of these spaces: for all $K\in T_h$ and for all $\boldsymbol{v}\in H^s(K)^d$, there exists $\boldsymbol{v}_I\in\boldsymbol{V}(K)$ such that

$$\|\boldsymbol{v}-\boldsymbol{v}_I\|_{0,K} + h_K|\boldsymbol{v}-\boldsymbol{v}_I|_{1,K} + h_K^2|\boldsymbol{v}-\boldsymbol{v}_I|_{2,K} \lesssim h_K^s|\boldsymbol{v}|_{s,K},\ 2\le s\le l+1. \tag{3.4}$$

3.2. **DG formulations.** We note that according to the definition of $\boldsymbol{V}_h$, the normal component of any $\boldsymbol{v}\in\boldsymbol{V}_h$ is continuous on the internal edges and vanishes on the boundary edges. Therfore, by splitting a vector $\boldsymbol{v}\in\boldsymbol{V}_h$ into its normal and tangential components $\boldsymbol{v}_n$ and $\boldsymbol{v}_t$

$$\boldsymbol{v}_n := (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}, \quad \boldsymbol{v}_t := \boldsymbol{v}-\boldsymbol{v}_n, \tag{3.5}$$

we have

$$\text{for all} \quad e\in E_h \int_e[\boldsymbol{v}_n]\cdot\boldsymbol{\tau}ds = 0, \quad \text{for all} \quad \boldsymbol{\tau}\in H^1(T_h)^d, \boldsymbol{v}\in\boldsymbol{V}_h, \tag{3.6}$$

implying that

$$\text{for all}\quad e\in E_h\quad \int_e [\boldsymbol{v}]\cdot\boldsymbol{\tau}ds=\int_e [\boldsymbol{v}_t]\cdot\boldsymbol{\tau}ds=0,\quad \text{for all}\quad \boldsymbol{\tau}\in H^1(T_h)^d, \boldsymbol{v}\in V_h. \tag{3.7}$$

A direct computation, similar to the one given in (3.1), shows that

$$\begin{aligned}[\boldsymbol{u}_t]\odot\boldsymbol{n}:[\boldsymbol{v}_t]\odot\boldsymbol{n} &= (([\boldsymbol{u}_t]\odot\boldsymbol{n})\boldsymbol{n})\cdot[\boldsymbol{v}_t]=\frac{1}{2}\left(([\boldsymbol{u}_t]\boldsymbol{n}^T+\boldsymbol{n}[\boldsymbol{u}_t]^T)\cdot\boldsymbol{n}\right)\cdot[\boldsymbol{v}_t]\\ &=\frac{1}{2}\left([\boldsymbol{u}_t]+([\boldsymbol{u}_t]\cdot\boldsymbol{n})\boldsymbol{n}\right)\cdot[\boldsymbol{v}_t]=\frac{1}{2}[\boldsymbol{u}_t]\cdot[\boldsymbol{v}_t],\end{aligned}$$

implying that

$$[\![\boldsymbol{u}_t]\!]:[\![\boldsymbol{v}_t]\!]=\frac{1}{2}[\boldsymbol{u}_t]\cdot[\boldsymbol{v}_t]. \tag{3.8}$$

Therefore, the discretization of the Stokes problem (2.1) is given by: Find $(\boldsymbol{u}_h,p_h)\in\boldsymbol{V}_h\times S_h$ such that

$$\left\{\begin{array}{l} a_h(\boldsymbol{u}_h,\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,p_h)=(\boldsymbol{f},\boldsymbol{v}_h)\ ,\quad \text{for all}\quad \boldsymbol{v}_h\in\boldsymbol{V}_h,\\ b_h(\boldsymbol{u}_h,q_h)=0\ ,\quad \text{for all}\quad q_h\in S_h,\end{array}\right. \tag{3.9}$$

where

$$\begin{aligned} a_h(\boldsymbol{u},\boldsymbol{v}) &= \sum_{K\in T_h}\int_K \varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{v})dx-\sum_{e\in E_h}\int_e\{\varepsilon(\boldsymbol{u})\}\cdot[\boldsymbol{v}_t]ds\\ &\quad -\sum_{e\in E_h}\int_e\{\varepsilon(\boldsymbol{v})\}\cdot[\boldsymbol{u}_t]ds+\sum_{e\in E_h}\int_e\eta h_e^{-1}[\boldsymbol{u}_t]\cdot[\boldsymbol{v}_t]ds, \end{aligned}\tag{3.10}$$

$$b_h(\boldsymbol{u},q) = \int_\Omega \nabla\cdot\boldsymbol{u}q dx, \tag{3.11}$$

and η is a properly chosen penalty parameter independent of the mesh size h and so that $a_h(\cdot,\cdot)$ is positive definite.

Remark 3.1. *Noting the identities* (3.1) *and* (3.8)*, we can rewrite* $a_h(\cdot,\cdot)$ *as*

$$\begin{aligned} a_h(\boldsymbol{u},\boldsymbol{v}) &= \sum_{K\in T_h}\int_K \varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{v})dx-\sum_{e\in E_h}\int_e\langle\varepsilon(\boldsymbol{u})\rangle:[\![\boldsymbol{v}_{\boldsymbol{t}}]\!]ds\\ &\quad -\sum_{e\in E_h}\int_e\langle\varepsilon(\boldsymbol{v})\rangle:[\![\boldsymbol{u}_{\boldsymbol{t}}]\!]ds+\sum_{e\in E_h}\int_e 2\eta h_e^{-1}[\![\boldsymbol{u}_t]\!]:[\![\boldsymbol{v}_t]\!]ds, \end{aligned}$$

which matches the bilinear form in [1] *by noting that the normal component of* $\boldsymbol{u}\in\boldsymbol{V}_h$ *is continuous. When compared to the bilinear form in* [4]*, we can see that the jumps of* $\boldsymbol{u}_t$ *on the boundary edges are included for the Dirichlet condition based on the fact that the condition* $\boldsymbol{u}\cdot\boldsymbol{n}=0$ *is included in the definition of the space* $\boldsymbol{V}_h$*.*

Note that $\operatorname{div}\boldsymbol{V}_h=S_h$ means that the approximate velocity $\boldsymbol{u}_h$ of the discrete problem (3.9) satisfies $\operatorname{div}\boldsymbol{u}_h=0$. Hence, we can rewrite the above system as the following equivalent system: Find $(\boldsymbol{u}_h,p_h)\in\boldsymbol{V}_h\times S_h$ such that

$$\left\{\begin{array}{l} a_h(\boldsymbol{u}_h,\boldsymbol{v}_h)+\lambda b_h(\boldsymbol{u}_h,\operatorname{div}\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,p_h)=(\boldsymbol{f},\boldsymbol{v}_h)\ ,\quad \text{for all}\quad \boldsymbol{v}_h\in\boldsymbol{V}_h,\\ b_h(\boldsymbol{u}_h,q_h)=0\ ,\quad \text{for all}\quad q_h\in S_h.\end{array}\right. \tag{3.12}$$

Application of the Uzawa method to (3.12) with damping parameter λ reads: Given $(\boldsymbol{u}_h^l,p^l)$, the new iterate $(\boldsymbol{u}_h^{l+1},p^{l+1})$ is obtained by solving the following system:

$$\left\{\begin{array}{l} a_h(\boldsymbol{u}_h^{l+1},\boldsymbol{v}_h)+\lambda b_h(\boldsymbol{u}_h^{l+1},\operatorname{div}\boldsymbol{v}_h)=(\boldsymbol{f},\boldsymbol{v}_h)-b_h(\boldsymbol{v}_h,p_h^l)\ ,\quad \text{for all}\quad \boldsymbol{v}_h\in\boldsymbol{V}_h,\\ p_h^{l+1}=p_h^l-\lambda\operatorname{div}\boldsymbol{u}_h^{l+1}.\end{array}\right.$$

By the definition of $b_h(\cdot,\cdot)$, namely (3.11), the Uzawa method is just the following: Given $(\boldsymbol{u}_h^l, p^l)$, the new iterate $(\boldsymbol{u}_h^{l+1}, p^{l+1})$ is obtained by solving the following system:

$$\left\{\begin{array}{l} a_h(\boldsymbol{u}_h^{l+1}, \boldsymbol{v}_h) + \lambda(\operatorname{div} \boldsymbol{u}_h^{l+1}, \operatorname{div} \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) - b_h(\boldsymbol{v}_h, p_h^l)\ , \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ p_h^{l+1} = p_h^l - \lambda \operatorname{div} \boldsymbol{u}_h^{l+1}. \end{array}\right. \tag{3.13}$$

Convergence of the this method has been discussed in several works, see, e.g., [9, 12, 20, 21] indicating that for large λ, the iterates converge rapidly to the solution of problem (3.9). As a consequence, the major computational cost lies in solving the discrete problem of the linear elasticity equations (2.3) which reads: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.14}$$

where $A_h(\cdot,\cdot)$ reads

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \lambda(\operatorname{div} \boldsymbol{u}_h, \operatorname{div} \boldsymbol{v}_h), \tag{3.15}$$

and $a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$ is defined by (3.10). Noting that $\operatorname{div} \boldsymbol{V}_h = S_h$, it is immediately seen that the problem (3.15) is equivalent to the discretization of the problem (2.1) as follows: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times S_h$ such that

$$\left\{\begin{array}{l} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h)\ , \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b_h(\boldsymbol{u}_h, q_h) - (\lambda^{-1} p_h, q_h) = 0\ , \quad \text{for all} \quad q_h \in S_h, \end{array}\right. \tag{3.16}$$

where $a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$ is defined by (3.10) and $b_h(\boldsymbol{u}_h, q_h)$ is defined by (3.11).

3.3. **Approximation and stability properties.** In this subsection, we analyze the approximation and stability properties of the discrete problems (3.9) and (3.15).

For any $\boldsymbol{u} \in H^1(T_h)^d$, we now define the mesh dependent norms:

$$\begin{aligned} \|\boldsymbol{u}\|_h^2 &= \sum_{K\in T_h} \|\varepsilon(\boldsymbol{u})\|_{0,K}^2 + \sum_{e\in E_h} h_e^{-1} \|[\boldsymbol{u}_t]\|_{0,e}^2, \\ \|\boldsymbol{u}\|_{1,h}^2 &= \sum_{K\in T_h} \|\nabla \boldsymbol{u}\|_{0,K}^2 + \sum_{e\in E_h} h_e^{-1} \|[\boldsymbol{u}_t]\|_{0,e}^2, \end{aligned}$$

Next, for $\boldsymbol{u} \in H^2(T_h)^d$, we define the "DG"-norm

$$\|\boldsymbol{u}\|_{DG}^2 = \sum_{K\in T_h} \|\nabla \boldsymbol{u}\|_{0,K}^2 + \sum_{e\in E_h} h_e^{-1} \|[\boldsymbol{u}_t]\|_{0,e}^2 + \sum_{K\in T_h} h_K^2 |\boldsymbol{u}|_{2,K}^2. \tag{3.17}$$

From the discrete version of the Korn's inequality (see [22, Equation (1.12)]) we have the following norm equivalence result.

Lemma 3.1. *The norms* $\|\cdot\|_{DG}$, $\|\cdot\|_h$, *and* $\|\cdot\|_{1,h}$ *are equivalent in* $\boldsymbol{V}_h$, *namely*

$$\|\boldsymbol{u}\|_{DG} \eqsim \|\boldsymbol{u}\|_h \eqsim \|\boldsymbol{u}\|_{1,h}, \quad \textit{for all } \boldsymbol{u} \in \boldsymbol{V}_h. \tag{3.18}$$

Proof. By the inverse inequality, we clearly have that $\|\boldsymbol{u}\|_{DG} \eqsim \|\boldsymbol{u}\|_{1,h}$. We now focus on proving that $\|\boldsymbol{u}\|_h \eqsim \|\boldsymbol{u}\|_{1,h}$. From the definitions we immediately get $\|\boldsymbol{u}\|_h \le \|\boldsymbol{u}\|_{1,h}$.

To prove the inequality in the other direction, we use [22, Equation (1.12)], namely,

$$\begin{aligned} \sum_{K\in T_h} \|\nabla \boldsymbol{u}\|_{0,K}^2 \;\lesssim\; & \Big(\sum_{K\in T_h} \|\varepsilon(\boldsymbol{u})\|_{0,K}^2 + \sup_{\substack{\boldsymbol{m}\in RM(\Omega)\\ \|\boldsymbol{m}\|_{L^2(\partial\Omega)=1}}} \big(\int_{\partial\Omega} \boldsymbol{u}\cdot \boldsymbol{m} ds\big)^2 \\ & + \sum_{e\in E_h^I} h_e^{-1} \|\pi_e [\boldsymbol{u}]_e\|_{0,e}^2 \Big). \end{aligned} \tag{3.19}$$

Here RM(Ω) denotes the space of rigid body motions,

$$\mathrm{RM}(\Omega) = \left\{ \boldsymbol{a} + \boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{a} \in I\!R^d,\ \boldsymbol{A} \in I\!R^{d\times d},\ \boldsymbol{A} = -\boldsymbol{A}^T \right\},$$

and the operator π_e is the $L^2(e)$-orthogonal projection operator onto $(P_1(e))^d$, the space of vector valued linear polynomial functions on e.

For the second term on the right hand side of (3.19) we have

$$\sup_{\substack{\boldsymbol{m}\in RM(\Omega)\\ \|\boldsymbol{m}\|_{L^2(\partial\Omega)=1}}} (\int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{m} ds)^2 \leq \int_{\partial\Omega} \boldsymbol{u}^2 ds = \sum_{e\in E_h^B} \|\boldsymbol{u}\|_{0,e}^2 \leq \sum_{e\in E_h^B} h_e^{-1} \|[\boldsymbol{u}]_e\|_{0,e}^2.$$

Since π_e is an orthogonal projection for the third term on the right hand side of (3.19) we obtain

$$\sum_{e\in E_h^I} h_e^{-1}\|\pi_e[\boldsymbol{u}]_e\|_{0,e}^2 \leq \sum_{e\in E_h^I} h_e^{-1}\|[\boldsymbol{u}]_e\|_{0,e}^2.$$

Finally, combining the two inequalities above completes the proof. □

Both bilinear forms, $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$, introduced above are continuous and we have

$$\begin{aligned} |a_h(\boldsymbol{u},\boldsymbol{v})| &\lesssim \|\boldsymbol{u}\|_{DG}\|\boldsymbol{v}\|_{DG}, \quad \text{for all} \quad \boldsymbol{u},\ \boldsymbol{v} \in H^2(T_h)^d, \\ |b_h(\boldsymbol{u},q)| &\leq \|\boldsymbol{u}\|_{1,h}\|q\|, \quad \text{for all } \boldsymbol{u} \in H^1(T_h)^d, q \in L_0^2(\Omega). \end{aligned}$$

For our choice of the finite element spaces $\boldsymbol{V}_h$ and S_h we have the following inf-sup condition for $b_h(\cdot,\cdot)$ (see, e.g., [4, 23])

Lemma 3.2. *There exits a constant $\beta > 0$ independent of the mesh size h, such that*

$$\inf_{q_h\in S_h} \sup_{\boldsymbol{u}_h\in\boldsymbol{V}_h} \frac{(\operatorname{div}\boldsymbol{u}_h, q_h)}{\|\boldsymbol{u}_h\|_{1,h}\|q_h\|} \geq \beta. \tag{3.20}$$

As expected, we can also show that $a_h(\cdot,\cdot)$ is coercive, namely, we have the following Lemma whose proof is based on similar proofs in [4, 24].

Lemma 3.3. *For sufficiently large η, independent of the mesh size h, we have*

$$a_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \gtrsim \|\boldsymbol{u}_h\|_h^2, \quad \textit{for all } \boldsymbol{u}_h \in \boldsymbol{V}_h. \tag{3.21}$$

By the equivalence of the norms shown in (3.18) and also by the standard theory for solvability of mixed problems [25], we obtain the following theorem.

Theorem 3.1. *The discrete problem* (3.9) *has a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times S_h$ that satisfies*

$$\operatorname{div}\ \boldsymbol{u}_h = 0 \textit{ in } \Omega. \tag{3.22}$$

Moreover, for every $\boldsymbol{v}_h \in \boldsymbol{V}_h$ with $\operatorname{div}\ \boldsymbol{v}_h = 0$ *and for every $q_h \in S_h$ the following estimates hold:*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{DG} \lesssim \|\boldsymbol{u}-\boldsymbol{v}_h\|_{DG}, \quad \|p-p_h\| \lesssim (\|p-q_h\| + \|\boldsymbol{u}-\boldsymbol{v}_h\|_{DG}) \tag{3.23}$$

with $(\boldsymbol{u},p)$ being the solution of (2.1).

The bilinear forms $a_h(\cdot,\cdot)$ and $A_h(\cdot,\cdot)$ are coercive and also define norms on $\boldsymbol{V}_h$, i.e.,

$$\|\boldsymbol{u}\|_{a_h}^2 = a_h(\boldsymbol{u},\boldsymbol{u}), \quad \|\boldsymbol{u}\|_{A_h}^2 = A_h(\boldsymbol{u},\boldsymbol{u}).$$

We now introduce the canonical interpolation operators $\Pi_h^{\mathrm{div}} : H^1(\Omega)^d \mapsto \boldsymbol{V}_h$. We also denote the L^2-projection on S_h by Q_h. The following Lemma summarizes some of the properties of Π_h^{div} and Q_h needed later.

Lemma 3.4. *For all $\boldsymbol{w} \in H^1(K)^d$ we have*

$$\operatorname{div} \Pi_h^{\operatorname{div}} = Q_h \operatorname{div}\,; \quad |\Pi_h^{\operatorname{div}} \boldsymbol{w}|_{1,K} \lesssim |\boldsymbol{w}|_{1,K};$$
$$\|\boldsymbol{w} - \Pi_h^{\operatorname{div}} \boldsymbol{w}\|_{0,\partial K}^2 \lesssim h_K |\boldsymbol{w}|_{1,K}^2; \quad \|\operatorname{div}(\boldsymbol{w} - \Pi_h^{\operatorname{div}} \boldsymbol{w})\|_{-1} \lesssim h_K \|\operatorname{div} \boldsymbol{w}\|,$$

where $\|r\|_{-1} = \sup_{\chi \in H^1} \frac{(\chi, r)}{\|\chi\|_1}$.

Proof. The proof of the commutativity of $\Pi_h^{\operatorname{div}}$ and div and the first two inequalities are well known and we refer the reader to [26] for the details.

The last inequality follows from the approximation properties of the L^2-orthogonal projection

$$\begin{aligned} \|\operatorname{div} \boldsymbol{w} - \operatorname{div} \Pi_h^{\operatorname{div}} \boldsymbol{w}\|_{-1} &= \sup_{\chi \in H^1} \frac{((I - Q_h) \operatorname{div} \boldsymbol{w}, \chi)}{\|\chi\|_1} = \sup_{\chi \in H^1} \frac{(\operatorname{div} \boldsymbol{w}, (I - Q_h)\chi)}{\|\chi\|_1} \\ &\lesssim \sup_{\chi \in H^1} \frac{\|\operatorname{div} \boldsymbol{w}\| \|(I - Q_h)\chi)\|}{\|\chi\|_1} \lesssim h_K \|\operatorname{div} \boldsymbol{w}\|. \end{aligned}$$

□

The following approximation result shows that the discretization we consider is locking-free.

Theorem 3.2. *Let $(\boldsymbol{u}, p)$ be the solution of (2.1) and $(\boldsymbol{u}_h, p_h)$ be the solution of (3.16). Then we have the following estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{DG}^2 + \lambda^{-1}\|p - p_h\|^2 \lesssim \inf_{\boldsymbol{v} \in \boldsymbol{V}_h, q \in S_h} \Big(\|\boldsymbol{u} - \boldsymbol{v}\|_{DG}^2 + \lambda^{-1}\|p - q\|^2 \Big).$$

Proof. If $(\boldsymbol{u}, p)$ is the solution of the continuous problem (2.1) and $(\boldsymbol{u}_h, p_h)$ is the solution of the discrete problem (3.16) we have that $p = \lambda \operatorname{div} \boldsymbol{u}$, and, since $\operatorname{div} \boldsymbol{V}_h = S_h$ we also have that $p_h = \lambda \operatorname{div} \boldsymbol{u}_h$. The left hand side of the first equation in (3.16) then is given by the bilinear form (3.15), and, since this discrete problem is consistent, we have

$$A_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) = 0, \text{ for all } \boldsymbol{v} \in \boldsymbol{V}_h.$$

Consider now the interpolation $\Pi_h^{\operatorname{div}} \boldsymbol{u} \in \boldsymbol{V}_h$ of $\boldsymbol{u}$ and we set $q = \lambda \operatorname{div} \Pi_h^{\operatorname{div}} \boldsymbol{u}$. Recall that $p = \lambda \operatorname{div} \boldsymbol{u}$, and $p_h = \lambda \operatorname{div} \boldsymbol{u}_h$ and hence (by Lemma 3.4) $q = \lambda Q_h \operatorname{div} \boldsymbol{u} = Q_h p$. We set $\boldsymbol{e}_h = (\boldsymbol{u}_h - \Pi_h^{\operatorname{div}} \boldsymbol{u})$ and from the coercivity of $a_h(\cdot, \cdot)$ we have

$$\begin{aligned} &\|\boldsymbol{e}_h\|_{1,h}^2 + \lambda^{-1}\|p_h - q\|^2 = \|\boldsymbol{e}_h\|_{1,h}^2 + \lambda \|\operatorname{div} \boldsymbol{e}_h\|^2 \\ &\lesssim A_h(\boldsymbol{e}_h, \boldsymbol{e}_h) = A_h(\boldsymbol{u} - \Pi_h^{\operatorname{div}} \boldsymbol{u}, \boldsymbol{e}_h) \\ &\lesssim \|\boldsymbol{u} - \Pi_h^{\operatorname{div}} \boldsymbol{u}\|_{DG} \|\boldsymbol{e}_h\|_{1,h} + \lambda(\operatorname{div}(\boldsymbol{u} - \Pi_h^{\operatorname{div}} \boldsymbol{u}), \operatorname{div} \boldsymbol{e}_h) \\ &= \|\boldsymbol{u} - \Pi_h^{\operatorname{div}} \boldsymbol{u}\|_{DG} \|\boldsymbol{e}_h\|_{1,h}. \end{aligned} \tag{3.24}$$

The last identity above follows from $\operatorname{div}(\boldsymbol{u} - \Pi_h^{\operatorname{div}} \boldsymbol{u}) = (I - Q_h) \operatorname{div} \boldsymbol{u}$ and $\operatorname{div} \boldsymbol{e}_h \in S_h$. By Lemma 3.4, we get

$$\|\boldsymbol{u}_h - \Pi_h^{\operatorname{div}} \boldsymbol{u}\|_{1,h} = \|\Pi_h^{\operatorname{div}} (\boldsymbol{u}_h - \boldsymbol{u})\|_{1,h} \lesssim \|\boldsymbol{u}_h - \boldsymbol{u}\|_{1,h}, \tag{3.25}$$

and hence, the right hand side of (3.24) is bounded by a multiple of $\|\boldsymbol{u} - \Pi_h^{\operatorname{div}} \boldsymbol{u}\|_{DG} \|\boldsymbol{u}_h - \boldsymbol{u}\|_{DG}$. As for any $\epsilon > 0$ we have $ab \leq \epsilon a^2 + \epsilon^{-1} b^2$ and using Lemma 3.4 we have for any $\boldsymbol{v} \in \boldsymbol{V}_h$ and any $q \in S_h$,

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_h\|_{DG}^2 + \lambda^{-1}\|p - p_h\|^2 \lesssim \|\boldsymbol{u} - \Pi_h^{\operatorname{div}} \boldsymbol{u}\|_{DG}^2 + \lambda^{-1}\|p - Q_h p\|^2 \\ &= \|\boldsymbol{u} - \boldsymbol{v} - \Pi_h^{\operatorname{div}} (\boldsymbol{u} - \boldsymbol{v})\|_{DG}^2 + \lambda^{-1}\|p - q - Q_h(p - q)\|^2. \end{aligned} \tag{3.26}$$

Using Lemma 3.4 and taking the infimum over $\boldsymbol{v}$ and $\boldsymbol{q}$ then gives the desired result. □

Remark 3.2. *Let $\boldsymbol{u}$ be the solution of* (2.3) *and $\boldsymbol{u}_h$ be the solution of* (3.15). *From theorem 3.2 and the regularity estimate* (2.4), *we obtain the following estimate*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{DG} \lesssim h\|\boldsymbol{f}\|. \tag{3.27}$$

Remark 3.3. *Let us set*

$$B_\lambda((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h)) = a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) - (\operatorname{div}\boldsymbol{u}_h,q_h) - (\operatorname{div}\boldsymbol{v}_h,p_h) - \lambda^{-1}(p_h,q_h).$$

Then for any given $(\boldsymbol{u}_h,p_h)$, choosing $(\boldsymbol{v}_h,q_h)=(\boldsymbol{u}_h,-p_h)$, by the coercivity of $a_h(\cdot,\cdot)$, it is straightforward to show that the inf-sup condition for $B_\lambda(\cdot,\cdot)$ holds, namely, for any $(\boldsymbol{u}_h,p_h)\in\boldsymbol{V}_h\times S_h$ we have

$$\sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h\times S_h}\frac{B_\lambda((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h))}{\|\boldsymbol{v}_h\|_{1,h}+\lambda^{-1/2}\|q_h\|} \gtrsim (\|\boldsymbol{u}_h\|_{1,h}+\lambda^{-1/2}\|p_h\|). \tag{3.28}$$

For the Stokes equation, we have from [25, Theorem 8.2.1] *and* [27, 28] *that*

$$\sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h\times S_h}\frac{B_\infty((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h))}{\|\boldsymbol{v}_h\|_{1,h}+\|q_h\|} \gtrsim \|\boldsymbol{u}_h\|_{1,h}+\|p_h\|. \tag{3.29}$$

3.4. **An *a priori* estimate for the discrete problem.** Next lemma is an *a priori* estimate on the L_2 norm of the solution of a discrete problem which is later used to prove the so called "smoothing property" – an essential part of the multigrid convergence analysis. We state and prove this estimate here (before any multigrid analysis), since it could be of independent interest.

We consider the finite element spaces introduced earlier: $\boldsymbol{V}_h\subset H(\text{div};\Omega)$ and $S_h\subset L^2_0(\Omega)$. Let $\boldsymbol{w}_1\in\boldsymbol{V}_h$ and $\boldsymbol{w}_2\in\boldsymbol{V}_h$ be given and let $\widetilde{\boldsymbol{u}}\in\boldsymbol{V}_h, \widetilde{p}\in S_h$ solve the discrete problem

$$\begin{aligned} a_h(\widetilde{\boldsymbol{u}},\boldsymbol{v}) - (\operatorname{div}\boldsymbol{v},\widetilde{p}) &= a_h(\boldsymbol{w}_1,\boldsymbol{v}), &&\text{for all}\quad \boldsymbol{v}\in\boldsymbol{V}_h,\\ (\operatorname{div}\widetilde{\boldsymbol{u}},q) &= (\operatorname{div}\boldsymbol{w}_2,q), &&\text{for all}\quad q\in S_h.\end{aligned} \tag{3.30}$$

We note that the inf-sup condition given in (3.29) implies that

$$\begin{aligned}\|\widetilde{\boldsymbol{u}}\|_{1,h}+\|\widetilde{p}\| &\lesssim \sup_{(\boldsymbol{v},q)\in\boldsymbol{V}_h\times S_h}\frac{a(\widetilde{\boldsymbol{u}},\boldsymbol{v})-(\operatorname{div}\boldsymbol{v},\widetilde{p})-(\operatorname{div}\widetilde{\boldsymbol{u}},q)}{\|\boldsymbol{v}\|_{1,h}+\|q\|}\\ &= \sup_{(\boldsymbol{v},q)\in\boldsymbol{V}_h\times S_h}\frac{a_h(\boldsymbol{w}_1,\boldsymbol{v})-(\operatorname{div}\boldsymbol{w}_2,q)}{\|\boldsymbol{v}\|_{1,h}+\|q\|}\lesssim \|\boldsymbol{w}_1\|_{1,h}+\|\operatorname{div}\boldsymbol{w}_2\|.\end{aligned} \tag{3.31}$$

Lemma 3.5. *For the solution of* (3.30) *we have the following estimate:*

$$\|\widetilde{\boldsymbol{u}}\| \lesssim \|\boldsymbol{w}_1\| + \|\operatorname{div}\boldsymbol{w}_2\|_{-1}. \tag{3.32}$$

Proof. We consider the following dual problem: Find $\boldsymbol{\phi}\in(H^1_0(\Omega))^d$ and $\theta\in L^2_0(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{v},\boldsymbol{\phi}) - (\operatorname{div}\boldsymbol{v},\theta) &= (\widetilde{\boldsymbol{u}},\boldsymbol{v}), &&\text{for all}\quad \boldsymbol{v}\in(H^1_0(\Omega))^d,\\ (\operatorname{div}\boldsymbol{\phi},q) &= 0, &&\text{for all}\quad q\in S_h.\end{aligned} \tag{3.33}$$

Let Π_h^{div} be the interpolation operator introduced earlier in Section 3.3. Recall that $\operatorname{div}\boldsymbol{\phi}=0$ and hence $(\operatorname{div}\Pi_h^{\text{div}}\boldsymbol{\phi},\widetilde{p})=0$. From equations (3.30) we then have

$$\begin{aligned} 0 &= a_h(\boldsymbol{w}_1,\Pi_h^{\text{div}}\boldsymbol{\phi}) - a_h(\widetilde{\boldsymbol{u}},\Pi_h^{\text{div}}\boldsymbol{\phi}) + (\operatorname{div}\Pi_h^{\text{div}}\boldsymbol{\phi},\widetilde{p})\\ &= a_h(\boldsymbol{w}_1,\boldsymbol{\phi}) - a_h(\boldsymbol{w}_1,\boldsymbol{\phi}-\Pi_h^{\text{div}}\boldsymbol{\phi}) - a_h(\widetilde{\boldsymbol{u}},\Pi_h^{\text{div}}\boldsymbol{\phi}).\end{aligned} \tag{3.34}$$

Observing that $a(\boldsymbol{\phi},\boldsymbol{v})=a_h(\boldsymbol{\phi},\boldsymbol{v})$ for all $\boldsymbol{v}\in\boldsymbol{V}_h$, from (3.33) and (3.34) we obtain

$$\begin{aligned}\|\widetilde{\boldsymbol{u}}\|^2 &= a_h(\boldsymbol{\phi},\widetilde{\boldsymbol{u}}) - (\operatorname{div}\widetilde{\boldsymbol{u}},\theta)\\ &\quad + a_h(\boldsymbol{w}_1,\boldsymbol{\phi}) - a_h(\boldsymbol{w}_1,\boldsymbol{\phi}-\Pi_h^{\text{div}}\boldsymbol{\phi}) - a_h(\widetilde{\boldsymbol{u}},\Pi_h^{\text{div}}\boldsymbol{\phi})\end{aligned} \tag{3.35}$$

Combining the first and the last term, using the triangle inequality and the continuity of $a_h(\cdot,\cdot)$ then shows that

$$\begin{aligned}\|\widetilde{\boldsymbol{u}}\|^2 &\leq |(\operatorname{div}\widetilde{\boldsymbol{u}},\theta)| + |a_h(\boldsymbol{w}_1,\boldsymbol{\phi})| \\ &\quad +|a_h(\boldsymbol{w}_1,\boldsymbol{\phi}-\Pi_h^{\operatorname{div}}\boldsymbol{\phi})| + |a_h(\widetilde{\boldsymbol{u}},\boldsymbol{\phi}-\Pi_h^{\operatorname{div}}\boldsymbol{\phi})| \\ &\lesssim |(\operatorname{div}\widetilde{\boldsymbol{u}},\theta)| + |a_h(\boldsymbol{w}_1,\boldsymbol{\phi})| + (\|\boldsymbol{w}_1\|_{1,h} + \|\widetilde{\boldsymbol{u}}\|_{1,h})\|\boldsymbol{\phi}-\Pi_h^{\operatorname{div}}\boldsymbol{\phi}\|_{1,h}.\end{aligned}$$

As we have that $\operatorname{div}\widetilde{\boldsymbol{u}} = \operatorname{div}\boldsymbol{w}_2$ for the first term on the right side we get

$$|(\operatorname{div}\widetilde{\boldsymbol{u}},\theta)| = |(\operatorname{div}\boldsymbol{w}_2,\theta)| \leq \|\theta\|_1 \sup_{\chi\in H^1}\frac{(\operatorname{div}\boldsymbol{w}_2,\chi)}{\|\chi\|_1} = \|\operatorname{div}\boldsymbol{w}_2\|_{-1}\|\theta\|_1.$$

For the second term, by the regularity estimate (2.4) we have that $\boldsymbol{\phi}\in(H^2(\Omega))^d$, and, thus, $\boldsymbol{\phi}$ is continuous and $[\boldsymbol{\phi}]=0$. Now, integrating by parts and combining the interface terms from neighboring elements then shows that

$$\begin{aligned}a_h(\boldsymbol{\phi},\boldsymbol{w}_1) &= \sum_{K\in T_h}\int_K \varepsilon(\boldsymbol{\phi}):\varepsilon(\boldsymbol{w}_1)dx - \sum_{e\in E_h}\int_e\{\varepsilon(\boldsymbol{\phi})\}\cdot[(\boldsymbol{w}_1)_t]ds \\ &\quad -\sum_{e\in E_h}\int_e\{\varepsilon(\boldsymbol{w}_1)\}\cdot[\boldsymbol{\phi}_t]ds + \sum_{e\in E_h}\int_e \eta h_e^{-1}[\boldsymbol{\phi}_t]\cdot[(\boldsymbol{w}_1)_t]ds, \\ &= \sum_{K\in T_h}\int_K \varepsilon(\boldsymbol{\phi}):\varepsilon(\boldsymbol{w}_1)dx - \sum_{e\in E_h}\int_e\{\varepsilon(\boldsymbol{\phi})\}\cdot[(\boldsymbol{w}_1)_t]ds \\ &= -\sum_{T\in\mathcal{T}_h}\int_T \operatorname{div}\varepsilon(\boldsymbol{\phi})\cdot\boldsymbol{w}_1 \leq \|\boldsymbol{\phi}\|_2\|\boldsymbol{w}_1\|.\end{aligned}$$

Finally, the desired result follows from the interpolation estimates in Lemma 4.3, the regularity estimate $\|\boldsymbol{\phi}\|_2 + \|\theta\|_1 \lesssim \|\widetilde{\boldsymbol{u}}\|$, inequality (3.31) and the inverse inequalities $\|\boldsymbol{w}_1\|_{1,h} \lesssim h^{-1}\|\boldsymbol{w}_1\|$ and $\|\operatorname{div}\boldsymbol{w}_2\| \lesssim h^{-1}\|\operatorname{div}\boldsymbol{w}_2\|_{-1}$. □

4. Multigrid method

In this section, we design a multigrid algorithm to solve the discrete system (3.15) arising from the DG discretization of the linear elasticity problem. We will show that the algorithm is robust with respect to the parameter λ. Hence, by combining it with the Uzawa method and choosing λ to be large enough, we can also solve the discrete system (3.9) arising from the DG discretization of the Stokes problem very efficiently.

4.1. **Preliminaries.** Let us denote by $\{T_k\}_{k=0}^J$ the partition on every level and denote the finest partition $T_h = T_J$. The edges (faces) of T_k are denoted by E_k. We assume that all the partitions $\{T_k\}_{k=0}^J$ are quasi-uniform with characteristic mesh size h_k and $h_k=\gamma h_{k-1}$, $\gamma\in(0,1)$ and $h_0=\mathcal{O}(1)$. Noting that the last term (the penalty term) in the bilinear form $a_h(\cdot,\cdot)$ depends on the mesh size of the partition.

Thus, for every partition T_k we have discretized the equation (2.3) and we need to specify the space $\boldsymbol{V}_h$ on level k. A natural choice for $\boldsymbol{V}_h$ on level k is $\boldsymbol{M}_k$ defined as follows:

$$\boldsymbol{M}_k = \{\boldsymbol{v}\in H(\operatorname{div};\Omega) : \boldsymbol{v}|_K\in\boldsymbol{V}(K),\ K\in T_k;\ \boldsymbol{v}\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega\}.$$

Moreover, we denote the pressure space S_h on level k by

$$S_k = \left\{q\in L^2(\Omega) : q|_K\in Q(K),\quad K\in T_k;\ \int_\Omega q dx = 0\right\}.$$

Thus, corresponding to the set of refined triangulations $\{T_k\}_{k=0}^J$, we also have a sequence of nested, $H(\mathrm{div}, \Omega)$-conforming finite element vector spaces

$$\boldsymbol{M}_0 \subseteq \boldsymbol{M}_1 \subseteq \boldsymbol{M}_2 \subseteq \cdots \subseteq \boldsymbol{M}_J \subseteq H(\mathrm{div}, \Omega).$$

With every space we associate a bilinear form $a_k(\cdot,\cdot)$ which discretizes the first term on the left hand side of (2.3) on $\boldsymbol{M}_k$, i.e.,

$$\begin{aligned} a_k(\boldsymbol{u},\boldsymbol{v}) = &\sum_{K\in T_k}\int_K \varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{v})dx - \sum_{e\in E_k}\int_e \{\varepsilon(\boldsymbol{u})\}\cdot[\boldsymbol{v_t}]ds \\ &- \sum_{e\in E_k}\int_e \{\varepsilon(\boldsymbol{v})\}\cdot[\boldsymbol{u_t}]ds + \sum_{e\in E_k}\int_e \eta h_k^{-1}[\boldsymbol{u}_t]\cdot[\boldsymbol{v}_t]ds. \end{aligned}$$

Adding the divergence term then gives the bilinear form used to discretize (2.3) on $\boldsymbol{M}_k$, i.e.,

$$A_k(\boldsymbol{u},\boldsymbol{v}) = a_k(\boldsymbol{u},\boldsymbol{v}) + \lambda(\mathrm{div}\,\boldsymbol{u}, \mathrm{div}\,\boldsymbol{v}), \text{for all} \quad \boldsymbol{u},\boldsymbol{v}\in \boldsymbol{M}_k.$$

Our goal is to analyze the V-cycle and W-cycle multigrid algorithms for the solution of the problem: Given $\boldsymbol{f}\in\boldsymbol{M}_J$, find $\boldsymbol{v}\in\boldsymbol{M}_J$ satisfying

$$A_J(\boldsymbol{v},\boldsymbol{\phi}) = (\boldsymbol{f},\boldsymbol{\phi}), \text{ for all } \boldsymbol{\phi}\in\boldsymbol{M}_J. \tag{4.1}$$

To define the algorithm, we need several auxiliary notions. For $k = 0,\cdots,J$, define the operator $\mathbb{A}_k : \boldsymbol{M}_k \to \boldsymbol{M}_k$ by

$$(\mathbb{A}_k\boldsymbol{w},\boldsymbol{\phi}) = A_k(\boldsymbol{w},\boldsymbol{\phi}), \text{ for all } \boldsymbol{\phi}\in\boldsymbol{M}_k.$$

The norms on $\boldsymbol{M}_k$ induced by the $A_k(\cdot,\cdot)$ and $a_k(\cdot,\cdot)$ are denoted by $\|\cdot\|^2_{A_k}$, and $\|\cdot\|^2_{a_k}$ respectively

$$\|\boldsymbol{u}\|^2_{A_k} = A_k(\boldsymbol{u},\boldsymbol{u}), \text{ for all } \boldsymbol{u}\in\boldsymbol{M}_k.$$

We also need the L^2-orthogonal projections on $\boldsymbol{M}_k$, and S_k, denoted by $\boldsymbol{Q}_k : \boldsymbol{L}^2(\Omega)\mapsto\boldsymbol{M}_k$ and the operators $Q_k : \boldsymbol{L}^2(\Omega)\mapsto S_k$ and the canonical interpolation $\Pi_k : [H_0^1(\Omega)]^2\mapsto\boldsymbol{M}_k$. According to the notation of the previous section, Π_k and Q_k are just a shorthand for $\Pi_{h_k}^{\mathrm{div}}$ and Q_{h_k}, and we recall that $Q_k\,\mathrm{div} = \mathrm{div}\,\Pi_k$. Further, we introduce the operators $P_{k-1}: \ \boldsymbol{M}_k\to\boldsymbol{M}_{k-1}$ defined as

$$A_{k-1}(P_{k-1}\boldsymbol{w},\boldsymbol{\phi}) = A_k(\boldsymbol{w},\boldsymbol{\phi}), \text{ for all } \boldsymbol{\phi}\in\boldsymbol{M}_{k-1}. \tag{4.2}$$

Finally, we denote the norm $\|\cdot\|_{1,h}$ on the level k as $\|\cdot\|_{1,k}$ and give the following lemmas.

To define the smoothing process, we require linear operators $R_k : \boldsymbol{M}_k\to\boldsymbol{M}_k$ for $k = 1,\cdots,J$. These operators may be symmetric or nonsymmetric with respect to the inner product $(\cdot,\cdot)$. If R_k is nonsymmetric, then we define R_k^t to be its adjoint and set

$$R_k^{(l)} = \begin{cases} R_k \text{ if } l \text{ is odd,} \\ R_k^t \text{ if } l \text{ is even.} \end{cases}$$

4.2. **Multigrid algorithm.** The multigrid operator $\mathbb{B}_k : \boldsymbol{M}_k\to\boldsymbol{M}_k$ is defined by induction and is given as follows, see, e.g., [15].

Multigrid algorithm Set $\mathbb{B}_0 = \mathbb{A}_0^{-1}$. Assume that $\mathbb{B}_{k-1}$ has been defined and define $\mathbb{B}_k\boldsymbol{g}$ for $\boldsymbol{g}\in\boldsymbol{M}_k$ as follows:

(1) Set $\boldsymbol{x}^0 = 0$ and $\boldsymbol{q}^0 = 0$.
(2) Define $\boldsymbol{x}^l$ for $l = 1,\cdots,m(k)$ by

$$\boldsymbol{x}^l = \boldsymbol{x}^{l-1} + R_k^{(l+m(k))}(\boldsymbol{g} - \mathbb{A}_k\boldsymbol{x}^{l-1}). \tag{4.3}$$

(3) Define $\boldsymbol{y}^{m(k)} = \boldsymbol{x}^{m(k)} + \boldsymbol{q}^p$, where $\boldsymbol{q}^i$ for $i = 1,\cdots,p$ is defined by

$$\boldsymbol{q}^i = \boldsymbol{q}^{i-1} + \mathbb{B}_{k-1}[\boldsymbol{Q}_{k-1}(\boldsymbol{g} - \mathbb{A}_k\boldsymbol{x}^{m(k)}) - \mathbb{A}_{k-1}\boldsymbol{q}^{i-1}]. \tag{4.4}$$

(4) Define $\boldsymbol{y}^l$ for $l = m(k)+1, \cdots, 2m(k)$ by

$$\boldsymbol{y}^l = \boldsymbol{y}^{l-1} + R_k^{(l+m(k))}(\boldsymbol{g} - \mathbb{A}_k \boldsymbol{y}^{l-1}).$$

(5) Set $\mathbb{B}_k \boldsymbol{g} = \boldsymbol{y}^{2m(k)}$.

In this algorithm, $m(k)$ is a positive integer which may vary from level to level and determines the number of smoothing iterations on that level, p is a positive integer. We shall study the cases $p=1$ and $p=2$, which correspond respectively to the symmetric $\mathcal{V}$ and $\mathcal{W}$ cycles of multigrid.

4.3. **Multigrid convergence.** Set $K_k = I - R_k\mathbb{A}_k$, then $K_k^* = I - R_k^t\mathbb{A}_k$ is the adjoint with respect to $A_k(\cdot,\cdot)$. Further, set

$$\widetilde{K}_k^{(m)} = \begin{cases} (K_k^* K_k)^{m/2} \text{ if } l \text{ is odd}, \\ (K_k^* K_k)^{(m-1)/2} K_k^* \text{ if } l \text{ is even}, \end{cases}$$

and denote by $(\widetilde{K}_k^{(m)})^*$ the adjoint of $\widetilde{K}_k^{(m)}$ with respect to $A_k(\cdot,\cdot)$.

For convergence estimates, we shall make a priori assumptions. First we make the following basic assumption:

- (A0) The spectrum of $K_k^* K_k$ is in the interval $[0,1)$.

In order to analyze the approximation property and the smoothing property of the multigrid algorithm, we need to define a norm on level k as follows (cf. [10]),

$$\|\boldsymbol{u}\|_{k,0}^2 := \|\boldsymbol{u}\|^2 + \lambda h_k^2 \|\operatorname{div} \boldsymbol{u}\|^2 + \lambda^2 h_k^2 \|Q_{k-1} \operatorname{div} \boldsymbol{u}\|^2, \quad \boldsymbol{u} \in \boldsymbol{M}_k. \tag{4.5}$$

The second assumption is an approximation assumption in $\|\cdot\|_{k,0}$ norm (known as approximation and regularity assumption in [15]),

- (A1) $\|(I - P_{k-1})\boldsymbol{u}\|_{k,0} \lesssim h_k \|\boldsymbol{u}\|_{A_k}$, for all $\boldsymbol{u} \in \boldsymbol{M}_k$.

The third assumption is a requirement on the smoother,

- (A2) $\|(\widetilde{K}_k^{(m)})^* \boldsymbol{u}\|_{A_k} \lesssim m^{-1/4} h_k^{-1} \|\boldsymbol{u}\|_{k,0}$, for all $\boldsymbol{u} \in \boldsymbol{M}_k$.

Next Lemma is an analogue of a result given in Bramble, Pasciak, Xu [15, Lemma 4.1].

Lemma 4.1. *Assume that (A0), (A1) and (A2) hold and let* $\widetilde{\boldsymbol{u}} = \widetilde{K}_k^{(m)} \boldsymbol{u}$. *Then we have the estimate*

$$-A_k((I - P_{k-1})\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}) \lesssim m^{-1/4} \|\boldsymbol{u}\|_{A_k}^2, \quad \textit{for all} \quad \boldsymbol{u} \in \boldsymbol{M}_k,$$

Proof. By the Cauchy-Schwarz inequality and assumption (A2), we have

$$\begin{aligned} -A_k((I - P_{k-1})\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}) &= -A_k((I - P_{k-1})\widetilde{K}_k^{(m)}\boldsymbol{u}, \widetilde{K}_k^{(m)}\boldsymbol{u}) \\ &= -A_k((\widetilde{K}_k^{(m)})^*(I - P_{k-1})\widetilde{K}_k^{(m)}\boldsymbol{u}, \boldsymbol{u}) \\ &\leq \|(\widetilde{K}_k^{(m)})^*(I - P_{k-1})\widetilde{\boldsymbol{u}}\|_{A_k} \|\boldsymbol{u}\|_{A_k} \\ &\lesssim m^{-1/4} h_k^{-1} \|(I - P_{k-1})\widetilde{\boldsymbol{u}}\|_{k,0} \|\boldsymbol{u}\|_{A_k}. \end{aligned}$$

Next, by assumptions (A1) and (A0) (applied in that order) we have

$$\begin{aligned} -A_k((I - P_{k-1})\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}) &\lesssim m^{-1/4} h_k^{-1} \|(I - P_{k-1})\widetilde{\boldsymbol{u}}\|_{k,0} \|\boldsymbol{u}\|_{A_k} \\ &\lesssim m^{-1/4} \|\widetilde{\boldsymbol{u}}\|_{A_k} \|\boldsymbol{u}\|_{A_k} \lesssim m^{-1/4} \|\boldsymbol{u}\|_{A_k}^2. \end{aligned}$$

□

The estimate in Lemma 4.1 provides the prerequisite to apply the general theory in [15]. Indeed, according to [15], assumptions (A0), (A1) and (A2) and Lemma 4.1 are sufficient to show spectral equivalence for the V-cycle multigrid preconditioner (Theorem 4.1) and uniform convergence result for the W-cycle multigrid method (Theorem 4.2). The first result is just a restatement of [15, Theorem 6] with full regularity.

Theorem 4.1 (Theorem 6 in [15]). *Assume that (A0), (A1) and (A2) hold and define $\mathbb{B}_j$ in Algorithm 4.2 with $p = 1$. Further assume that the number of smoothing steps $m(k)$ satisfy $\beta_0 m(k) \le m(k-1) \le \beta_1 m(k)$ with $\beta_0 \ge 1$ and $\beta_1 > 1$ independent of k. Then the following spectral equivalence holds*

$$\eta_0 A_k(\boldsymbol{u},\boldsymbol{u}) \le A_k(\mathbb{B}_k\mathbb{A}_k\boldsymbol{u},\boldsymbol{u}) \le \eta_1 A_k(\boldsymbol{u},\boldsymbol{u}) \text{ for all } \boldsymbol{u} \in \boldsymbol{M}_k. \tag{4.6}$$

with constants η_0 and η_1 such that

$$\eta_0 \ge \frac{m(k)^\alpha}{M + m(k)^\alpha} \text{ and } \eta_1 \le \frac{M + m(k)^\alpha}{m(k)^\alpha},$$

where M is independent of λ and h where α is the regularity index.

The convergence of the W cycle is also obtained via the analysis in [15].

Theorem 4.2 (Theorem 4 in [15]). *Assume that (A0), (A1) and (A2) hold and that the number of smoothing steps $m(k) = m$ is constant for all k. Then, for sufficiently large m, $\mathbb{B}_k$ defined via the $\mathcal{W}$-cycle algorithm satisfies*

$$|A_k((I - \mathbb{B}_k\mathbb{A}_k)\boldsymbol{u},\boldsymbol{u})| \le \frac{M}{M + m^\alpha}\|\boldsymbol{u}\|_{A_k}^2 \quad \text{for all} \quad \boldsymbol{u} \in \boldsymbol{M}_k.$$

with M, independent of λ and h where α is the regularity index.

We remark here that modifying assumption (A1) one can prove the results above for the case of less than full elliptic regularity. For details we refer to Bramble, Pasciak and Xu [15].

As we have seen, the estimates in Theorems 4.1-4.2 are valid if assumptions (A0), (A1) and (A2) are verified. In the next subsections we show that these assumptions hold in our case.

4.4. **Approximation property.** In this subsection, we verify (A1). One of the difficulties in the analysis is that the bilinear forms $A_k(\cdot,\cdot), k = 1, \cdots, J$ are not nested. We now prove a simple relation between $A_k(\cdot,\cdot)$ and $A_{k-1}(\cdot,\cdot)$.

Lemma 4.2. *If $h_k = \gamma h_{k-1}$, $\gamma \in (0,1)$, then*

$$\|\boldsymbol{u}\|_{A_{k-1}}^2 \le \|\boldsymbol{u}\|_{A_k}^2 \lesssim \|\boldsymbol{u}\|_{A_{k-1}}^2, \quad \text{for all} \quad \boldsymbol{u} \in \boldsymbol{M}_{k-1}. \tag{4.7}$$

Proof. Let $\boldsymbol{u} \in \boldsymbol{M}_{k-1}$. Observe that $[\boldsymbol{u}_t]_e = 0$ for edges $e \in E_k$ which are interior to the elements in T_{k-1}, because $\boldsymbol{u}$ is a continuous, in fact a polynomial, function in each element from T_{k-1}. Hence,

$$\sum_{e\in E_{k-1}} \int_e \eta\gamma^{-1}h_{k-1}^{-1}|[\boldsymbol{u}_t]|^2 ds = \sum_{e\in E_k}\int_e \eta h_k^{-1}|[\boldsymbol{u}_t]|^2 ds, \quad \text{for all} \quad \boldsymbol{u} \in \boldsymbol{M}_{k-1}$$

and we have

$$\begin{aligned} A_k(\boldsymbol{u},\boldsymbol{u}) &= A_{k-1}(\boldsymbol{u},\boldsymbol{u}) + \sum_{e\in E_k}\int_e \eta h_k^{-1}|[\boldsymbol{u}_t]|^2 ds - \sum_{e\in E_{k-1}}\int_e \eta h_{k-1}^{-1}|[\boldsymbol{u}_t]|^2 ds \\ &= A_{k-1}(\boldsymbol{u},\boldsymbol{u}) + (\gamma^{-1} - 1)\sum_{e\in E_{k-1}}\int_e \eta h_{k-1}^{-1}|[\boldsymbol{u}_t]|^2 ds. \end{aligned}$$

The estimates in (4.7) then easily follow from the identity above. □

Remark 4.1. *From Lemma 4.2, for any given $\boldsymbol{u} \in \boldsymbol{M}_k$, we also have*

$$\begin{aligned} \|P_{k-1}\boldsymbol{u}\|_{A_{k-1}}^2 &\le \|P_{k-1}\boldsymbol{u}\|_{A_k}^2 = A_k(\boldsymbol{u}, P_{k-1}\boldsymbol{u}) \le \|\boldsymbol{u}\|_{A_k}\|P_{k-1}\boldsymbol{u}\|_{A_k} \\ &\lesssim \|\boldsymbol{u}\|_{A_k}\|P_{k-1}\boldsymbol{u}\|_{A_{k-1}}, \end{aligned}$$

namely,

$$\|P_{k-1}\boldsymbol{u}\|_{A_{k-1}} \lesssim \|\boldsymbol{u}\|_{A_k}. \tag{4.8}$$

We now introduce the dual problem (which is the same as the primal one in (2.3) because the bilinear form is symmetric): Find $\boldsymbol{w} \in H_0^1(\Omega)^d$ such that

$$(\varepsilon(\boldsymbol{v}) : \varepsilon(\boldsymbol{w})) + \lambda(\operatorname{div} \boldsymbol{v}, \operatorname{div} \boldsymbol{w}) = (\boldsymbol{g}, \boldsymbol{v}), \quad \text{for all} \quad \boldsymbol{v} \in H_0^1(\Omega)^d. \tag{4.9}$$

From the definitions of the bilinear forms $A_{k-1}(\cdot,\cdot)$ and $A_k(\cdot,\cdot)$ we have the following simple identity for the solution $\boldsymbol{w}$ of (4.9):

$$A_k(\boldsymbol{v}, \boldsymbol{w}) = A_{k-1}(\boldsymbol{v}, \boldsymbol{w}), \text{ for all} \quad \boldsymbol{v} \in \boldsymbol{M}_{k-1}. \tag{4.10}$$

This follows immediately, since both $A_{k-1}(\cdot,\cdot)$ and $A_k(\cdot,\cdot)$ are consistent. Indeed, for any $\boldsymbol{v} \in \boldsymbol{M}_{k-1} \subset \boldsymbol{M}_k$ we have $A_k(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{g}, \boldsymbol{v}) = A_{k-1}(\boldsymbol{v}, \boldsymbol{w})$, which proves (4.10).

The next lemma provides estimates on the interpolation error.

Lemma 4.3. *Let $\boldsymbol{w} \in H^{l+1}(\Omega)^d$, $l = 0, 1$, and $\Pi_{k-1}\boldsymbol{w}$ be the interpolant of $\boldsymbol{w}$ in $\boldsymbol{M}_{k-1}$, then*

$$\begin{aligned} \|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|_{A_{k-1}}^2 &\lesssim h_{k-1}^{2l}(|\boldsymbol{w}|_{l+1}^2 + \lambda|\operatorname{div} \boldsymbol{w}|_l^2), \\ \|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|_{A_k}^2 &\lesssim h_{k-1}^{2l}(|\boldsymbol{w}|_{l+1}^2 + \lambda|\operatorname{div} \boldsymbol{w}|_l^2). \end{aligned} \tag{4.11}$$

Proof. By the continuity of $a_k(\cdot,\cdot)$, the trace theorem and the interpolation error estimate (3.4), we have

$$\|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|_{a_{k-1}}^2 \lesssim \|\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}\|_{DG}^2 \lesssim h_{k-1}^{2l}|\boldsymbol{w}|_{l+1}^2.$$

Noting $\operatorname{div} \Pi_{k-1}\boldsymbol{w} = Q_{k-1} \operatorname{div} \boldsymbol{w}$, by the standard approximation error estimate of the projection Q_{k-1}, we have

$$\|\operatorname{div}(\boldsymbol{w} - \Pi_{k-1}\boldsymbol{w})\|^2 = \|\operatorname{div} \boldsymbol{w} - Q_{k-1} \operatorname{div} \boldsymbol{w}\|^2 \lesssim h_{k-1}^{2l}|\operatorname{div} \boldsymbol{w}|_l^2.$$

Combining the above two inequalities and noting the definition of norm $\|\cdot\|_{A_{k-1}}$, we get the first inequality in (4.11). The proof of the second inequality in (4.11) is carried out in a similar fashion. ☐

We now prove a two-level estimate in L^2.

Theorem 4.3. *For all $\boldsymbol{u} \in \boldsymbol{M}_k$ the following estimate holds*

$$\|(I - P_{k-1})\boldsymbol{u}\| \lesssim h_k\|\boldsymbol{u}\|_{A_k}. \tag{4.12}$$

Proof. We estimate $\|(I - P_{k-1})\boldsymbol{u}\|$ using a standard duality argument. Let $\boldsymbol{w} \in H_0^1(\Omega)^d$ be the solution of the dual problem (4.9) with $\boldsymbol{g} = \boldsymbol{u} - P_{k-1}\boldsymbol{u}$. Since, $A_k(\cdot,\cdot)$ is a consistent bilinear form, we have

$$A_k(\boldsymbol{w}, \boldsymbol{v}) = (\boldsymbol{u} - P_{k-1}\boldsymbol{u}, \boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \boldsymbol{M}_k.$$

Now let $\boldsymbol{v} = \boldsymbol{u} - P_{k-1}\boldsymbol{u}$ and $\Pi_{k-1}\boldsymbol{w}$ be the interpolant of $\boldsymbol{w}$ in $\boldsymbol{M}_{k-1}$. Noting that $A_k(\cdot,\cdot), k = 1, \cdots, J$ are symmetric, (4.10) and the definition of operator P_{k-1}, we have

$$\begin{aligned} \|\boldsymbol{u} - P_{k-1}\boldsymbol{u}\|^2 &= A_k(\boldsymbol{w}, \boldsymbol{u} - P_{k-1}\boldsymbol{u}) \\ &= A_k(\boldsymbol{u}, \boldsymbol{w}) - A_k(\boldsymbol{w}, P_{k-1}\boldsymbol{u}) = A_k(\boldsymbol{u}, \boldsymbol{w}) - A_{k-1}(\boldsymbol{w}, P_{k-1}\boldsymbol{u}) \\ &= A_k(\boldsymbol{u}, \boldsymbol{w}) - A_{k-1}(P_{k-1}\boldsymbol{u}, \boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}) - A_{k-1}(P_{k-1}\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w}) \\ &= A_k(\boldsymbol{u}, \boldsymbol{w}) - A_{k-1}(P_{k-1}\boldsymbol{u}, \boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}) - A_k(\boldsymbol{u}, \Pi_{k-1}\boldsymbol{w}) \\ &= A_k(\boldsymbol{u}, \boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}) - A_{k-1}(P_{k-1}\boldsymbol{u}, \boldsymbol{w} - \Pi_{k-1}\boldsymbol{w}). \end{aligned} \tag{4.13}$$

Applying the Cauchy-Schwarz inequality to the right hand side of the identity above and using the approximation estimates given in (4.11), the inequality (4.8) and the regularity estimate (2.4) then lead to

$$\begin{aligned}\|\boldsymbol{u}-P_{k-1}\boldsymbol{u}\|^2 &\le \|\boldsymbol{u}\|_{A_k}\|\boldsymbol{w}-\Pi_{k-1}\boldsymbol{w}\|_{A_k}+\|P_{k-1}\boldsymbol{u}\|_{A_{k-1}}\|\boldsymbol{w}-\Pi_{k-1}\boldsymbol{w}\|_{A_{k-1}}\\ &\lesssim h_{k-1}(\|\boldsymbol{u}\|_{A_k}+\|P_{k-1}\boldsymbol{u}\|_{A_{k-1}})(|\boldsymbol{w}|_2^2+\lambda|\operatorname{div}\boldsymbol{w}|_1^2)^{1/2}\\ &\lesssim h_{k-1}\|\boldsymbol{u}\|_{A_k}(|\boldsymbol{w}|_2^2+\lambda|\operatorname{div}\boldsymbol{w}|_1^2)^{1/2}\lesssim h_{k-1}\|\boldsymbol{u}\|_{A_k}\|\boldsymbol{u}-P_{k-1}\boldsymbol{u}\|.\end{aligned}$$

which completes the proof. □

The next two Lemmas verify the approximation property (A1).

Lemma 4.4. *For all $\boldsymbol{u}\in\boldsymbol{M}_k$ we have the estimate*

$$\lambda\|Q_{k-1}\operatorname{div}(\boldsymbol{u}-P_{k-1}\boldsymbol{u})\|\lesssim\|\boldsymbol{u}\|_{A_k}. \tag{4.14}$$

Proof. For any given $\boldsymbol{u}\in\boldsymbol{M}_k$ and any $\boldsymbol{v}\in\boldsymbol{M}_{k-1}$, from the definition of P_{k-1} in (4.2), we have

$$a_{k-1}(P_{k-1}\boldsymbol{u},\boldsymbol{v})+\lambda(\operatorname{div}(P_{k-1}\boldsymbol{u}),\operatorname{div}\boldsymbol{v})=a_k(\boldsymbol{u},\boldsymbol{v})+\lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{v}),$$

or, equivalently,

$$\lambda(Q_k\operatorname{div}\boldsymbol{u},Q_{k-1}\operatorname{div}\boldsymbol{v})-\lambda(Q_{k-1}\operatorname{div}(P_{k-1}\boldsymbol{u}),Q_{k-1}\operatorname{div}\boldsymbol{v})=a_k(\boldsymbol{u},\boldsymbol{v})-a_{k-1}(P_{k-1}\boldsymbol{u},\boldsymbol{v}).$$

By the properties of the L^2-projections on S_k and S_{k-1} and the fact that $S_{k-1}\subset S_k$ we have $Q_{k-1}Q_k=Q_{k-1}$ and $Q_{k-1}^2=Q_{k-1}$. Therefore,

$$(Q_{k-1}\operatorname{div}(\boldsymbol{u}-P_{k-1}\boldsymbol{u}),\operatorname{div}\boldsymbol{v})=\lambda^{-1}(a_k(\boldsymbol{u},\boldsymbol{v})-a_{k-1}(P_{k-1}\boldsymbol{u},\boldsymbol{v})). \tag{4.15}$$

Note that the continuity of the bilinear form $a_k(\cdot,\cdot)$ implies that $\|\boldsymbol{v}\|_{a_k}\lesssim\|\boldsymbol{v}\|_{1,k-1}$ and $\|\boldsymbol{v}\|_{a_{k-1}}\lesssim\|\boldsymbol{v}\|_{1,k-1}$. Using now the trivial bound $a_{k-1}(\boldsymbol{w},\boldsymbol{w})\le A_{k-1}(\boldsymbol{w},\boldsymbol{w})$, which holds for all $\boldsymbol{w}\in\boldsymbol{M}_{k-1}$, and the inequality (4.8) for the right hand side of (4.15) we obtain

$$\begin{aligned}a_k(\boldsymbol{u},\boldsymbol{v})-a_{k-1}(P_{k-1}\boldsymbol{u},\boldsymbol{v}) &\lesssim (\|\boldsymbol{u}\|_{a_k}+\|P_{k-1}\boldsymbol{u}\|_{a_{k-1}})\|\boldsymbol{v}\|_{1,k-1}\\ &\lesssim \|\boldsymbol{u}\|_{A_k}\|\boldsymbol{v}\|_{1,k-1}.\end{aligned}$$

The inf-sup condition (3.20) and the inequality above then show that

$$\begin{aligned}\|Q_{k-1}\operatorname{div}(\boldsymbol{u}-P_{k-1}\boldsymbol{u})\| &\lesssim \sup_{\boldsymbol{v}\in\boldsymbol{M}_{k-1}}\frac{(Q_{k-1}\operatorname{div}(\boldsymbol{u}-P_{k-1}\boldsymbol{u}),\operatorname{div}\boldsymbol{v})}{\|\boldsymbol{v}\|_{1,k-1}}\\ &= \lambda^{-1}\sup_{\boldsymbol{v}\in\boldsymbol{M}_{k-1}}\frac{a_k(\boldsymbol{u},\boldsymbol{v})-a_{k-1}(P_{k-1}\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{1,k-1}}\\ &\lesssim \lambda^{-1}\|\boldsymbol{u}\|_{A_k}.\end{aligned}$$

The proof is complete. □

Next lemma estimates the last term in the definition of $\|\boldsymbol{u}-P_{k-1}\boldsymbol{u}\|_{k,0}$.

Lemma 4.5. *If $\lambda\gtrsim 1$, then the following estimate holds for all $\boldsymbol{u}\in\boldsymbol{M}_k$.*

$$\lambda\|\operatorname{div}(\boldsymbol{u}-P_{k-1}\boldsymbol{u})\|^2\lesssim\|\boldsymbol{u}\|_{A_k}^2. \tag{4.16}$$

Proof. We observe that $Q_{k-1}\operatorname{div}P_{k-1}\boldsymbol{u}=\operatorname{div}P_{k-1}\boldsymbol{u}$ and then, by the triangle inequality and Lemma 4.4, we have

$$\begin{aligned}\|\operatorname{div}(\boldsymbol{u}-P_{k-1}\boldsymbol{u})\| &\le \|\operatorname{div}\boldsymbol{u}-Q_{k-1}\operatorname{div}\boldsymbol{u}\|+\|Q_{k-1}\operatorname{div}(\boldsymbol{u}-P_{k-1}\boldsymbol{u})\|\\ &\lesssim \|\operatorname{div}\boldsymbol{u}\|+\lambda^{-1}\|\boldsymbol{u}\|_{A_k}.\end{aligned}$$

The proof is completed by first squaring both sides, then multiplying by λ and finally using the inequality $(a+b)^2\le 2(a^2+b^2)$ and the fact that $\lambda\gtrsim 1$. We have,

$$\lambda\|\operatorname{div}(\boldsymbol{u}-P_{k-1}\boldsymbol{u})\|^2 \lesssim \lambda\|\operatorname{div}\boldsymbol{u}\|^2+\lambda^{-1}\|\boldsymbol{u}\|_{A_k}^2\lesssim\|\boldsymbol{u}\|_{A_k}^2.$$

□

Combining the L^2-estimate (4.12), and the estimates given in Lemma 4.4, and Lemma 4.5, we obtain the following theorem, which verifies (A1).

Theorem 4.4. *The following approximation estimate holds for $\lambda \gtrsim 1$ and for all $\boldsymbol{u} \in \boldsymbol{M}_k$.*

$$\|(I - P_{k-1})\boldsymbol{u}\|_{k,0} \lesssim h_k \|\boldsymbol{u}\|_{A_k}.$$

4.5. **Smoothing property.** In this subsection, we verify the smoothing property (A2). We only consider the 3-dimensional case because the 2-dimensional case is similar and simpler. We denote by $\mathcal{V}_k$, and $\mathcal{E}_k$ the sets of vertices and edges, respectively, of the partition T_k. For $\nu \in \mathcal{V}_k \cup \mathcal{E}_k$ we define

$$T_k^\nu = \{K \in T_k : \nu \subset K\},\ \bar{\Omega}_k^\nu = \cup_{K \in T_k^\nu} \bar{K},\ \Omega_k^\nu = \text{interior}(\ \bar{\Omega}_k^\nu).$$

Thus Ω_k^ν is the subdomain of Ω formed by the patch of elements meeting at ν, and T_k^ν is the restriction of the mesh partition T_k to Ω_k^ν.

We now consider the decomposition of these spaces as sums of spaces supported in small patches of elements. Define

$$\boldsymbol{M}_k^\nu = \{\boldsymbol{r} \in \boldsymbol{M}_k : \operatorname{supp} \boldsymbol{r} \subset \bar{\Omega}_k^\nu\},\ \nu \in \mathcal{V}_k \cup \mathcal{E}_k.$$

Then

$$\boldsymbol{M}_k = \sum_{i \in \mathcal{V}_k} \boldsymbol{M}_k^i = \sum_{e \in \mathcal{E}_k} \boldsymbol{M}_k^e.$$

For each of these decompositions there is a corresponding estimate on the sum of the squares of the L^2-norms of the summands. For example, we can decompose an arbitrary element $\boldsymbol{u} \in \boldsymbol{M}_k$ as $\boldsymbol{u} = \sum_{i \in \mathcal{V}_k} \boldsymbol{u}^i$ with $\boldsymbol{u}^i \in \boldsymbol{M}_k^i$ so that the estimate

$$\sum_{i \in \mathcal{V}_k} \|\boldsymbol{u}^i\|^2 \lesssim \|\boldsymbol{u}\|^2 \tag{4.17}$$

holds with a constant depending only on the shape regularity of the mesh.

Since the kernel basis functions of the operator div are captured by the above subspaces $\boldsymbol{M}_k^i$, we must use a block damped Jacobi smoother or a block Gauss-Seidel smoother where the blocks correspond to one of the above L^2-decompositions in order to preserve the structure of the kernel. For example, we can use a vertex block damped Jacobi smoother, a vertex block Gauss-Seidel smoother, an edge block damped Jacobi smoother, or an edge block Gauss-Seidel smoother.

Remark 4.2. *We should point out that the block Gauss-Seidel smoother satisfies the assumption (A0). But for the block damped Jacobi smoother, we need to choose the damping parameter such that the basic assumption (A0) is satisfied. A damped Richardson smoother $I - \tau A_k$ would need a damping parameter τ proportional to λ^{-1}. Thus the components of the error in the kernel of A_k would be smoothed out very slow as λ is large. We should also point out that in the 2-dimensional case, we can only use vertex block smoothers.*

In the rest of this subsection, we only consider the vertex block damped Jacobi smoother since the others are similar, and define the operator $P_{k,i} : \boldsymbol{M}_k \to \boldsymbol{M}_k^i$ for $i \in \mathcal{V}_k$ by

$$A_k(P_{k,i}\boldsymbol{u}, \boldsymbol{v}_i) = A_k(\boldsymbol{u}, \boldsymbol{v}_i) \text{ for all } \boldsymbol{u} \in \boldsymbol{M}_k, \boldsymbol{v}_i \in \boldsymbol{M}_k^i.$$

We use exact local solvers and hence the block damped Jacobi smoother R_k is given by $R_k = \tau \sum_{i \in \mathcal{V}_k} P_{k,i} A_k^{-1} := \tau D_k^{-1}$, where τ is the damping parameter such that (A0) is satisfied. In this case, $K_k^* = K_k$ and $\widetilde{K}_k^{(m)} = K_k^m$. By the assumption (A0), the estimate

$$\|K_k^m \boldsymbol{u}\|_{A_k}^2 = (D_k^{-1} A_k K_k^{2m} \boldsymbol{u}, \boldsymbol{u})_{D_k} \lesssim m^{-1} \|\boldsymbol{u}\|_{D_k}^2 \tag{4.18}$$

is well known in multigrid theory (see e.g. Hackbusch [14]).

By additive Schwarz techniques [29, 30] the induced norm $\|\boldsymbol{u}\|_{D_k} = (D_k\boldsymbol{u}, \boldsymbol{u})^{1/2}$ can be written as

$$\|\boldsymbol{u}\|_{D_k}^2 = \inf_{\boldsymbol{u}=\sum \boldsymbol{u}_k^i} \sum_{i\in\mathcal{V}_k} \|\boldsymbol{u}_k^i\|_{A_k}^2. \tag{4.19}$$

Remark 4.3. *If the estimate $\|\boldsymbol{u}\|_{D_k} \lesssim h_k^{-1}\|\boldsymbol{u}\|_{k,0}$ would be true, the assumption (A2) would be proved. Unfortunately, the proof of Lemma 4.9 suggests that it is not true.*

On the other hand, choosing τ sufficiently small it is obvious that $\|K_k^m\boldsymbol{u}\|_{A_k} \leq \|\boldsymbol{u}\|_{A_k}$ (the assumption (A0) holds). Then an interpolation between this estimate and the estimate (4.18) gives

$$\|K_k^m\boldsymbol{u}\|_{A_k} \lesssim m^{-1/4}\|\boldsymbol{u}\|_{[D_k,A_k]},$$

where $\|\boldsymbol{u}\|_{[D_k,A_k]}$ is the interpolation norm between $\|\cdot\|_{D_k}$ and $\|\cdot\|_{A_k}$ with parameter 1/2. Thus, one way to verify assumption (A2), is to show that

$$\|\boldsymbol{u}\|_{[D_k,A_k]} \lesssim h_k^{-1}\|\boldsymbol{u}\|_{k,0}, \tag{4.20}$$

and the rest of this section is devoted to this. We now define a decomposition of $\boldsymbol{u} \in \boldsymbol{M}_k$ which is stable in $\|\cdot\|_{k,0}$ norm and then show the estimates.

We consider three solutions of problem (3.30) defined as follows:

(4.21) $(\boldsymbol{u}_1, p_1)$ is the solution of (3.30) with $\boldsymbol{w}_1 = \boldsymbol{u}$, $\boldsymbol{w}_2 = 0$.

(4.22) $(\boldsymbol{u}_2, p_2)$ is the solution of (3.30) with $\boldsymbol{w}_1 = 0$, $\boldsymbol{w}_2 = \boldsymbol{u} - \boldsymbol{\Pi}_{k-1}\boldsymbol{u}$.

(4.23) $(\boldsymbol{u}_3, p_3)$ is defined as the solution of (3.30) with $\boldsymbol{w}_1 = 0$, $\boldsymbol{w}_2 = \Pi_{k-1}\boldsymbol{u}$.

It is straightforward to check that $\boldsymbol{u} - \boldsymbol{u}_1 - \boldsymbol{u}_2 - \boldsymbol{u}_3$ and $p_1 + p_2 + p_3$ satisfy the equation (3.30) with $\boldsymbol{w}_1 = 0$ and $\boldsymbol{w}_2 = 0$ and therefore $\boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3 = \boldsymbol{u}$. With these settings in hand, we have the following stability result.

Lemma 4.6. *For the decomposition given in* (4.21)–(4.23) *we have*

$$\|\boldsymbol{u}_1\|_{k,0} + \|\boldsymbol{u}_1\|_{k,0} + \|\boldsymbol{u}_3\|_{k,0} \lesssim \|\boldsymbol{u}\|_{k,0}, \tag{4.24}$$

$$\|\boldsymbol{u}_2\| \lesssim \lambda^{-1}\|\boldsymbol{u}\|_{k,0}. \tag{4.25}$$

Proof. Computing $\|\cdot\|_{k,0}$ for all the components shows that

$$\|\boldsymbol{u}_1\|_{k,0} = \|\boldsymbol{u}_1\|, \tag{4.26}$$

$$\|\boldsymbol{u}_2\|_{k,0} = \|\boldsymbol{u}_2\| + \lambda h_k^{-1}\|\operatorname{div}(\boldsymbol{u} - \Pi_{k-1}\boldsymbol{u})\|, \tag{4.27}$$

$$\|\boldsymbol{u}_3\|_{k,0} = \|\boldsymbol{u}_3\| + \lambda^2 h_k^{-1}\|\operatorname{div}\Pi_{k-1}\boldsymbol{u}\|. \tag{4.28}$$

The rest of the proof is immediate from the definitions of the components (4.21)–(4.22), the definition of the $\|\cdot\|_{k,0}$ norm, Lemma 3.5 and Lemma 3.4. □

4.6. **Smoothing property via interpolation.** Define the $H(\text{curl};\Omega)$-conforming finite element space on level k (see, e.g., [13])

$$\boldsymbol{W}_k = \{\boldsymbol{w} \in H(\text{curl}, \Omega) : \boldsymbol{w}|_K \in \boldsymbol{W}(K), K \in T_k, \boldsymbol{w}\times\boldsymbol{n}|_{\partial\Omega} = 0\},$$

and the three spaces $\boldsymbol{M}_k, S_k$ and $\boldsymbol{W}_k$ are related by the exact sequences ([13])

$$0 \longrightarrow \boldsymbol{W}_k \xrightarrow{\text{curl}} \boldsymbol{M}_k \xrightarrow{\text{div}} S_k \longrightarrow 0.$$

Furthermore, we define

$$\boldsymbol{W}_k^\nu = \{\boldsymbol{r} \in \boldsymbol{W}_k : \operatorname{supp}\boldsymbol{r} \subset \bar{\Omega}_k^\nu\},\ \nu \in \mathcal{V}_k \cup \mathcal{E}_k.$$

Then

$$\boldsymbol{W}_k = \sum_{i\in\mathcal{V}_k} \boldsymbol{W}_k^i = \sum_{e\in\mathcal{E}_k} \boldsymbol{W}_k^e.$$

Note that for any $\boldsymbol{v} \in \boldsymbol{M}_k$, we have that $\|\boldsymbol{v}\|_{A_k} \lesssim \|\boldsymbol{v}\|_{D_k}$ and $\|\boldsymbol{v}\|_{D_k} \leq \|\boldsymbol{v}\|_{D_k}$ and this implies that

$$\|\boldsymbol{v}\|_{[D_k,A_k]} \lesssim \|\boldsymbol{v}\|_{D_k}. \tag{4.29}$$

The next two lemmas bound only the $\|\cdot\|_{D_k}$-norm, which is sufficient in view of (4.29).

Lemma 4.7. *Let $\boldsymbol{u}_1$ defined as in* (4.21)*. Then*

$$\|\boldsymbol{u}_1\|_{D_k} \lesssim h_k^{-1}\|\boldsymbol{u}_1\|_{k,0}. \tag{4.30}$$

Proof. Since $\operatorname{div}\boldsymbol{u}_1 = 0$, we have $\boldsymbol{u}_1 = \operatorname{curl}\boldsymbol{w}_k$ (see [13]), where $\boldsymbol{w}_k \in \boldsymbol{W}_k$.

Noting that $\boldsymbol{w}_k = \sum_{i\in\mathcal{V}_k}\boldsymbol{w}_k^i$, where $\boldsymbol{w}_k^i \in \boldsymbol{W}_k^i$ and $\operatorname{curl}\boldsymbol{w}_k^i \in \boldsymbol{M}_k^i$, by identity (4.19) and inequality (4.17), we have

$$\begin{aligned}\|\boldsymbol{u}_1\|_{D_k}^2 &= \inf_{\boldsymbol{u}_1=\sum\boldsymbol{u}_1^i}\sum_{i\in\mathcal{V}_k}\|\boldsymbol{u}_1^i\|_{A_k}^2 \leq \sum_{i\in\mathcal{V}_k}\|\operatorname{curl}\boldsymbol{w}_k^i\|_{A_k}^2 = \sum_{i\in\mathcal{V}_k}\|\operatorname{curl}\boldsymbol{w}_k^i\|_{a_k}^2\\ &= \sum_{i\in\mathcal{V}_k}\|\boldsymbol{u}_1^i\|_{a_k}^2 \lesssim h_k^{-2}\sum_{i\in\mathcal{V}_k}\|\boldsymbol{u}_1^i\|^2 \lesssim h_k^{-2}\|\boldsymbol{u}_1\|^2 = h_k^{-2}\|\boldsymbol{u}_1\|_{k,0}^2.\end{aligned}$$

The proof of the lemma is complete. □

Lemma 4.8. *Let $\boldsymbol{u}_2$ be defined as in* (4.22)*. Then*

$$\|\boldsymbol{u}_2\|_{D_k} \lesssim h_k^{-1}\|\boldsymbol{u}\|_{k,0}. \tag{4.31}$$

Proof. By the identity (4.19) and Lemma 4.6, we have

$$\|\boldsymbol{u}_2\|_{D_k}^2 = \inf_{\boldsymbol{u}_2=\sum\boldsymbol{u}_2^i}\sum_{i\in\mathcal{V}_k}\|\boldsymbol{u}_2^i\|_{A_k}^2 \lesssim \sum_{i\in\mathcal{V}_k}h_k^{-2}\lambda\|\boldsymbol{u}_2^i\|^2 \lesssim h_k^{-2}\lambda\|\boldsymbol{u}_2\|^2 \lesssim h_k^{-2}\|\boldsymbol{u}\|_{k,0}^2.$$

The proof is complete. □

Corollary 4.1. *From the inequality* (4.29) *and the Lemmas 4.7 and 4.8, we immediately have*

$$\begin{aligned}\|\boldsymbol{u}_1\|_{[D_k,A_k]} &\lesssim h_k^{-1}\|\boldsymbol{u}_1\|_{k,0},\\ \|\boldsymbol{u}_2\|_{[D_k,A_k]} &\lesssim h_k^{-1}\|\boldsymbol{u}\|_{k,0}.\end{aligned} \tag{4.32}$$

Lemma 4.9. *Let $\boldsymbol{u}_3$ be defined as in* (4.23)*. Then*

$$\|\boldsymbol{u}_3\|_{[D_k,A_k]} \lesssim h_k^{-1}\|\boldsymbol{u}\|_{k,0}. \tag{4.33}$$

Proof. By the inf-sup condition (3.31) we have $\|\boldsymbol{u}_3\|_{1,k} + \|p_3\| \lesssim \|Q_{k-1}\operatorname{div}\boldsymbol{u}\|$. Furthermore, $\operatorname{div}\boldsymbol{u}_3 = Q_{k-1}\operatorname{div}\boldsymbol{u}$ by definition. These together with the identity (4.19) give

$$\begin{aligned}\|\boldsymbol{u}_3\|_{A_k}^2 &\lesssim (\|\boldsymbol{u}_3\|_{1,k}^2 + \lambda\|\operatorname{div}\boldsymbol{u}_3\|^2)\\ &\lesssim \|Q_{k-1}\operatorname{div}\boldsymbol{u}\|^2 + \lambda\|Q_{k-1}\operatorname{div}\boldsymbol{u}\|^2 \lesssim \lambda^{-1}h_k^{-2}\|\boldsymbol{u}\|_{k,0}^2.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|\boldsymbol{u}_3\|_{D_k}^2 &= \inf_{\boldsymbol{u}_3=\sum\boldsymbol{u}_3^i}\sum_{i\in\mathcal{V}_k}\|\boldsymbol{u}_3^i\|_{A_k}^2 \lesssim \sum_{i\in\mathcal{V}_k}h_k^{-2}\lambda\|\boldsymbol{u}_3^i\|^2 \lesssim h_k^{-2}\lambda\|\boldsymbol{u}_3\|^2\\ &\lesssim \lambda h_k^{-2}\|\boldsymbol{u}_3\|_{k,0}^2 \lesssim \lambda h_k^{-2}\|\boldsymbol{u}\|_{k,0}^2.\end{aligned}$$

A standard interpolation argument, see, e.g., [31], concludes the proof. □

We close this subsection by the following theorem which verifies (A2).

Theorem 4.5. *The following estimate holds for all $\boldsymbol{u} \in \boldsymbol{M}_k$.*

$$\|(\widetilde{K}_k^{(m)})^*\boldsymbol{u}\|_{A_k} \lesssim m^{-1/4}h_k^{-1}\|\boldsymbol{u}\|_{k,0}. \tag{4.34}$$

Proof. By Lemma 4.6, inequalities (4.32) and (4.33), we obtain the smoothing property (4.34). □

5. Conclusions

We presented a multigrid algorithm for discontinuous Galerkin $H(\mathrm{div};\Omega)$-conforming discretizations of the Stokes and linear elasticity equations. A variable V-cycle and a W-cycle are designed to solve the linear elasticity problem in the present situation of nonnested bilinear forms. The convergence rate of the algorithm is proved to be independent of the Lamè parameters (or, equivalently, the Poisson ratio) and of the mesh size, which shows that the multigrid method is robust and optimal. Combining the multigrid method for the linear elasticity problem together with the Uzawa method, we can also solve the Stokes problem efficiently. The numerical experiments to verify the theoretical results are the future work to do.

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