

# **Boundary observability inequalities for the 3D Oseen-Stokes system and applications**

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# BOUNDARY OBSERVABILITY INEQUALITIES FOR THE 3D OSEEN–STOKES SYSTEM AND APPLICATIONS

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ABSTRACT. Controllability properties for the Navier–Stokes system are closely related to observability properties for the adjoint Oseen–Stokes system; boundary observability inequalities are derived, for that adjoint system, that will be appropriate to deal with suitable constrained controls, like finite-dimensional controls supported in a given subset of the boundary. As an illustration, a new boundary controllability result for the Oseen–Stokes system is derived. Finally, we discuss some further plausible consequences of the derived inequalities, concerning the Navier–Stokes system.

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## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^3$  be a connected bounded domain located locally on one side of its smooth boundary  $\Gamma = \partial\Omega$ , and let  $I \subseteq \mathbb{R}$  be a nonempty open interval. The Navier–Stokes system, in  $I \times \Omega$ , controlled through the boundary reads

$$(1) \quad \partial_t u + \langle u \cdot \nabla \rangle u - \nu \Delta u + \nabla p + h = 0, \quad \operatorname{div} u = 0, \quad u|_{\Gamma} = \gamma + \zeta$$

where  $\zeta$  is a control taking values in a suitable subspace of square-integrable functions in  $\Gamma$  whose support, in  $x$ , is contained in the closure  $\overline{\Gamma_c}$  of a given open subset  $\Gamma_c \subseteq \Gamma$ . Furthermore, as usual,  $u = (u_1, u_2, u_3)$  and  $p$ , defined for  $(t, x_1, x_2, x_3) \in I \times \Omega$ , are the unknown velocity field and pressure of the fluid,  $\nu > 0$  is the viscosity, the operators  $\nabla$  and  $\Delta$  are respectively the well known gradient and Laplacean in the space variables  $(x_1, x_2, x_3)$ ,  $\langle u \cdot \nabla \rangle v$  stands for  $(u \cdot \nabla v_1, u \cdot \nabla v_2, u \cdot \nabla v_3)$ ,  $\operatorname{div} u := \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3$ , and  $h$  and  $\gamma$  are fixed functions.

It turns out that (local) controllability properties to trajectories for system (1) are often related with observability inequalities for the time-backward “adjoint” Oseen–Stokes system

$$(2) \quad -\partial_t q + \mathcal{B}^*(\hat{u})q - \nu \Delta q + \nabla p + f = 0, \quad \operatorname{div} q = 0, \quad q|_{\Gamma} = 0,$$

where  $\hat{u}$  is a given reference (desired) trajectory of (1) (with  $\zeta = 0$ ),  $f$  is a suitable force, and  $\mathcal{B}^*(\hat{u})$  is the formal adjoint to  $\mathcal{B}(\hat{u}) : v \mapsto \langle \hat{u} \cdot \nabla \rangle v + \langle v \cdot \nabla \rangle \hat{u}$ . We refer the reader to the works [Ima01, BRS11] for the case of internal controls, and for [FI99] for a procedure to obtain boundary controllability results from internal ones. See also [FCGIP04, FI98, GBGP09, Fur04] and references therein.

We are particularly interested in the case where the reference solution  $\hat{u}$  is nonstationary (i.e.,  $\hat{u} = \hat{u}(t)$  depends on time), a situation that often can occur in real world applications, as in the case suitable (say non-gradient) external forces ( $h$  and  $\gamma$ ) depend on time. Moreover, for applications purposes it is often required that the control obeys some general constraints like, for example, to be feedback, finite-dimensional and supported in a given (small) open subset. It turns out that with these constraints on the boundary control, the procedure in [FI99] is (or may be) no longer sufficient to derive the wanted boundary controllability results.

In [BRS11], an internal stabilizing finite-dimensional feedback controller was found for the case of nonstationary reference solutions. Then, one question arises: can we find a similar boundary controller? We can, for example, see that from the internal result and from the procedure in [FI99] we cannot guarantee that the obtained boundary control is finite-dimensional. Also, the methods used in the particular case of a stationary reference solution, in [RT10, BT11, Bar12, BLT06, BT04], use some (spectral-like) properties of the (time-independent) Oseen–Stokes operator  $u \mapsto \nu \Delta u - \mathcal{B}(\hat{u})u + \nabla p_u$  and/or of its “adjoint”  $v \mapsto \nu \Delta v - \mathcal{B}^*(\hat{u})v + \nabla p_v$ , which seem to give us no hint for the nonstationary case. A more promising idea to obtain a positive answer is to adapt the procedure in [BRS11] to the boundary control case, even if we can realize that the adaptation is not straightforward because of new difficulties we will encounter, namely some regularity issues and the “tighter” compatibility conditions relating the solution and the control. In other words, we need to develop first some tools in order to be able to adapt the procedure to the boundary control case.

One of the main ingredients in [BRS11] is a suitable internal truncated observability inequality for system (2), where the truncation is closely related with the finite-dimensional control space; this inequality was derived by truncating the “observed space” in a well known observability inequality we find in [Ima01].

The work [BRS11] and the relation between observability inequalities for the adjoint Oseen–Stokes system (2) and controllability properties for the Navier–Stokes system (1) are the main motivations of this paper. We establish appropriate observability inequalities for (2) to deal with boundary control problems for (1), in particular to deal with the constraints on the finite dimension and on the support of the boundary controls  $\zeta$ . To give an idea, from the results we will derive in section 4.2, we can conclude that the solution of (2), in the case  $f = 0$  and  $I \times \Omega = (a, b) \times \Omega$ , satisfies

$$|q(a)|_{L^2(\Omega, \mathbb{R}^3)}^2 \leq \overline{C}_{\|\hat{u}\|_{\mathcal{W}}} |P_M^\Gamma \chi(p\mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q)|_{L^2((a, b), L^2(\Gamma, \mathbb{R}^3))}^2,$$

where  $\overline{C}_{\|\hat{u}\|_{\mathcal{W}}}$  is some constant depending on the norm of the reference solution  $\hat{u}$  in an appropriate Banach space  $\mathcal{W}$ ,  $P_M^\Gamma : L^2(\Gamma, \mathbb{R}^3) \rightarrow L_M^2(\Gamma, \mathbb{R}^3)$  is a projection onto a  $M$ -dimensional space  $L_M^2(\Gamma, \mathbb{R}^3)$ , and  $\chi : \Gamma \rightarrow \mathbb{R}$  is an a priori given smooth function. This inequality can be related to control problems for system (1) where the controls take their values in the “adjoint” finite dimensional space  $\chi L_M^2(\Gamma, \mathbb{R}^3) = \chi P_M^\Gamma L^2(\Gamma, \mathbb{R}^3)$ , that is,  $\zeta : (a, b) \rightarrow \chi L_M^2(\Gamma, \mathbb{R}^3)$ ; the support of the controls is necessarily contained in the support of  $\chi$ .

As an illustration, we use the derived observability inequalities, to obtain a new controllability result: let  $\{e_i \mid i \in \mathbb{N}_0\}$  be the eigenvector fields of the Stokes operator, forming an orthogonal basis for the subspace  $H \subset L^2(\Omega, \mathbb{R}^3)$  of solenoidal vector fields. Then, we can construct a family  $\{\chi \Psi_n \mid n \in \mathbb{N}_0\} \subset C^1([a, b], C^2(\Gamma, \mathbb{R}^3))$  such that for any given  $N \in \mathbb{N}_0$ , there is a positive integer  $M_{N, |\hat{u}|_{\mathcal{W}}}$  depending on the pair  $(N, |\hat{u}|_{\mathcal{W}})$  with the following property: for any given  $v_0 \in H$ , there is  $\kappa(v_0) \in \mathbb{R}^{M_{N, |\hat{u}|_{\mathcal{W}}}}$ , such that the control  $\zeta = \chi \sum_{n=1}^{M_{N, |\hat{u}|_{\mathcal{W}}}} \kappa_n \Psi_n$ , drives the Oseen–Stokes system

$$\partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v + \nabla p = 0, \quad \operatorname{div} v = 0, \quad v|_\Gamma = \zeta,$$

from  $v(a) = v_0 \in H$ , at time  $t = a$ , to a vector field  $v(b) \in H$ , at time  $t = b$ , with  $(v(b), e_i)_{L^2(\Omega, \mathbb{R}^3)} = 0$  for all  $i \leq N$ . Roughly speaking, there is a control  $\zeta$ , that can be “realized” by  $M_{N, |\hat{u}|_{\mathcal{W}}}$  constants, driving the (first  $N$ ) less stable Stokes modes to zero. Further, the mapping  $v_0 \mapsto \zeta(v_0)$  is linear and continuous.

The rest of the paper is organized as follows. In section 2, we introduce the functional spaces arising in the theory of the Navier–Stokes equations, set up our problem, and recall some well-known facts. In sections 3 and 4 we derive some boundary observability inequalities including some appropriate to deal with finite-dimensional controls supported in a given subset of the boundary. In section 5 we illustrate/recall how the observability inequalities can be used to obtain controllability results, deriving two new controllability results. In section 6 we give some remarks and discuss some further plausible consequences of the derived inequalities and of the controllability results derived in section 5; namely, the boundary versions of the internal results in [BRS11] and [Shi11] concerning, respectively, the stabilization to a nonstationary solution of the Navier–Stokes equations and a property of the stochastic version of the same equations. Finally, the appendix gathers some auxiliary results used in the main text.

**Notation.** We write  $\mathbb{R}$  and  $\mathbb{N}$  for the sets of real numbers and nonnegative integers, respectively, and we define  $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$ . We denote by  $\Omega \subset \mathbb{R}^3$  a bounded domain with a smooth boundary  $\Gamma = \partial\Omega$ . Given a vector function  $u : (t, x_1, x_2, x_3) \mapsto u(t, x_1, x_2, x_3) \in \mathbb{R}^k$ , defined in an open subset of  $\mathbb{R} \times \Omega$ , its partial time derivative  $\partial u / \partial t$  will be denoted by  $\partial_t u$ . Also the partial spatial derivatives  $\partial u / \partial x_i$  will be denoted by  $\partial_{x_i} u$ .

Given a Banach space  $X$  and an open subset  $O \subset \mathbb{R}^n$ , let us denote by  $L^p(O, X)$ , with either  $p \in [1, +\infty)$  or  $p = \infty$ , the Bochner space of measurable functions  $f : O \rightarrow X$ , and such that  $|f|_X^p$  is integrable over  $O$ , for  $p \in [1, +\infty)$ , and such that

ess  $\sup_{x \in O} |f(x)|_X < +\infty$ , for  $p = \infty$ . In the case  $X = \mathbb{R}$  we recover the usual Lebesgue spaces, and  $L^p(O, \mathbb{R}^k) \sim L^p(O, \mathbb{R})^k$ . By  $W^{s,p}(O, \mathbb{R}^k)$ , for  $s \in \mathbb{R}$ , denote the usual Sobolev space of order  $s$ . In the case  $p = 2$ , as usual, we denote  $H^s(O, \mathbb{R}^k) := W^{s,2}(O, \mathbb{R}^k)$ . Recall that  $H^0(O, \mathbb{R}^k) = L^2(O, \mathbb{R}^k)$ . For each  $s > 0$ , we recall also that  $H^{-s}(O, \mathbb{R}^k)$  stands for the dual space of  $H_0^s(O, \mathbb{R}^k) = \text{closure of } \{f \in C^\infty(O, \mathbb{R}^k) \mid \text{supp } f \subset O\}$  in  $H^s(O, \mathbb{R}^k)$ . Notice that  $H^{-s}(O, \mathbb{R}^k)$  is a space of distributions.

For a normed space  $X$ , we denote by  $|\cdot|_X$  the corresponding norm, by  $X'$  its dual, and by  $\langle \cdot, \cdot \rangle_{X', X}$  the duality between  $X'$  and  $X$ . The dual space is endowed with the usual dual norm:  $|f|_{X'} := \sup\{\langle f, x \rangle_{X', X} \mid x \in X \text{ and } |x|_X = 1\}$ .

Let  $X$  and  $Y$  be normed spaces, and let  $Z$  be a Hausdorff topological space. Suppose that both inclusions  $X \subseteq Z$  and  $Y \subseteq Z$  are continuous; then the Cartesian product  $X \times Y$ , the intersection  $X \cap Y$  and the sum  $X + Y$  are supposed to be endowed with the norms  $|(a, b)|_{X \times Y} := (|a|_X^2 + |b|_Y^2)^{1/2}$ ;  $|a|_{X \cap Y} := |(a, a)|_{X \times Y}$ ; and  $|a|_{X+Y} := \inf_{(a^X, a^Y) \in X \times Y} \{|(a^X, a^Y)|_{X \times Y} \mid a = a^X + a^Y\}$ , respectively. We can show that, if  $X$  and  $Y$  are endowed with a scalar product, then also  $X \times Y$ ,  $X \cap Y$ , and  $X + Y$  are. In the case we know that  $X \cap Y = \{0\}$ , we say that  $X + Y$  is a direct sum and we write  $X \oplus Y$  instead.

Given an open interval  $I \subseteq \mathbb{R}$ , then we write  $W(I, X, Y) := \{f \in L^2(I, X) \mid \partial_t f \in L^2(I, Y)\}$ , where the derivative  $\partial_t f$  is taken in the sense of distributions. This space is endowed with the natural norm  $|f|_{W(I, X, Y)} := (|f|_{L^2(I, X)}^2 + |\partial_t f|_{L^2(I, Y)}^2)^{1/2}$ . In the case  $X = Y$  we write  $H^1(I, X) := W(I, X, X)$ . Again, if  $X$  and  $Y$  are endowed with a scalar product, then also  $W(I, X, Y)$  is. The space of continuous linear mappings from  $X$  into  $Y$  will be denoted by  $\mathcal{L}(X \rightarrow Y)$ .

If  $\bar{I} \subset \mathbb{R}$  is a closed bounded interval,  $C(\bar{I}, X)$  stands for the space of continuous functions  $f : \bar{I} \rightarrow X$  with the norm  $|f|_{C(\bar{I}, X)} = \max_{t \in \bar{I}} |f(t)|_X$ .

$\bar{C}_{[a_1, \dots, a_k]}$  denotes a nonnegative function of nonnegative variables  $a_j$  that increases in each of its arguments.

$C, C_i, i = 1, 2, \dots$ , stand for unessential positive constants.

## 2. PRELIMINARIES

**2.1. Functional spaces.** Let  $\Omega \subset \mathbb{R}^3$  be a connected bounded domain of class  $C^\infty$  located locally on one side of its boundary  $\Gamma = \partial\Omega$ . More precisely we suppose that each point  $p \in \Gamma$  has a tubular neighborhood  $\mathcal{T}_p \subset \mathbb{R}^3$  that is diffeomorphic to a cylinder  $\mathbb{C}_p := \{(w_1, w_2, w_3) \in \mathbb{R}^3 \mid w_1^2 + w_2^2 < \rho_p \text{ and } |w_3|_{\mathbb{R}} < \varepsilon_p\}$ , for suitable  $\rho_p, \varepsilon_p > 0$ : there exists a bijective mapping

$$(3) \quad \begin{aligned} \Phi_p : \mathbb{C}_p &\rightarrow \mathcal{T}_p \\ (w_1, w_2, w_3) &\mapsto (w_1, w_2, \Phi_p^0(w_1, w_2)) + w_3 \mathbf{n}_{\Phi_p^0(w_1, w_2)}, \end{aligned}$$

see Figure 1 as an illustration, where for  $\mathbb{C}_p^0 := \{(w_1, w_2, w_3) \in \mathbb{C}_p \mid w_3 = 0\}$  and  $\mathbb{C}_p^- := \{(w_1, w_2, w_3) \in \mathbb{C}_p \mid w_3 < 0\}$  we have

- both  $\Phi_p$  and its inverse  $\Phi_p^{-1} : \mathcal{T}_p \rightarrow \mathbb{C}_p$  are of class  $C^\infty$ ,
- $\Phi_p(\mathbb{C}_p^0) = \mathcal{T}_p \cap \Gamma$  and  $\Phi_p(\mathbb{C}_p^-) = \mathcal{T}_p \cap \Omega$ ,
- $\Phi_p^0$  is of class  $C^\infty$  and  $\mathbf{n}_{\Phi_p^0(w_1, w_2)}$  is the unit outward normal vector to  $\Gamma$  at the point  $(w_1, w_2, \Phi_p^0(w_1, w_2)) \in \Gamma$ .

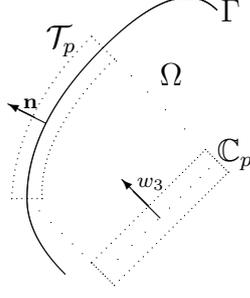


FIGURE 1. Tubular neighborhood.

Due to the incompressibility condition,  $\operatorname{div} u = 0$ , some important subspaces in the study of the systems (1) and (2) are the Lebesgue and Sobolev subspaces

$$\begin{aligned} L_{\operatorname{div}}^r(\Omega, \mathbb{R}^3) &:= \{u \in L^r(\Omega, \mathbb{R}^3) \mid \operatorname{div} u = 0 \text{ in } \Omega\}, \quad 1 \leq r \leq +\infty, \\ H_{\operatorname{div}}^s(\Omega, \mathbb{R}^3) &:= \{u \in H^s(\Omega, \mathbb{R}^3) \mid \operatorname{div} u = 0 \text{ in } \Omega\}, \quad s \geq 0. \end{aligned}$$

The incompressibility condition allows us to define the trace of  $u \cdot \mathbf{n}$  on the boundary  $\Gamma = \partial\Omega$ , where  $\mathbf{n}$  is the outward normal vector to the boundary  $\Gamma$ , and then to write

$$H := \{u \in L_{\operatorname{div}}^2(\Omega, \mathbb{R}^3) \mid u \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \quad H_c := \{u \in L_{\operatorname{div}}^2(\Omega, \mathbb{R}^3) \mid u \cdot \mathbf{n} = 0 \text{ on } \Gamma \setminus \overline{\Gamma_c}\},$$

where  $\Gamma_c$  is an open subset of  $\Gamma$ . Some spaces of more regular vector fields we find throughout the paper are

$$(4) \quad \begin{aligned} V &:= \{u \in H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3) \mid u = 0 \text{ on } \Gamma\}, \quad V_c := \{u \in H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3) \mid u = 0 \text{ on } \Gamma \setminus \overline{\Gamma_c}\}, \\ D(L) &:= V \cap H^2(\Omega, \mathbb{R}^3). \end{aligned}$$

The spaces  $H_{\operatorname{div}}^s(\Omega, \mathbb{R}^3)$  are endowed with the scalar product inherited from  $H^s(\Omega, \mathbb{R}^3)$ ; the spaces  $H$  and  $H_c$  with that inherited from  $L^2(\Omega, \mathbb{R}^3)$ ; the spaces  $V$  and  $V_c$  with that inherited from  $H^1(\Omega, \mathbb{R}^3)$ ; and  $D(L)$  with that inherited from  $H^2(\Omega, \mathbb{R}^3)$ . Notice that if  $\Pi$  is the orthogonal projection in  $L^2(\Omega, \mathbb{R}^3)$  onto  $H$ , it is well known that  $D(L)$  coincides with the domain  $\{u \in V \mid Lu \in H\}$  of the Stokes operator  $L := -\nu\Pi\Delta$ . That is the reason for the notation.

Next, fix a constant  $\sigma > 6/5$ . For any pair of real numbers  $a, b$ , with  $a < b$ , we introduce the Banach spaces  $\mathcal{W}^{(a,b)|\operatorname{wk}}$  and  $\mathcal{W}^{(a,b)|\operatorname{st}}$  of the measurable vector functions  $u = (u_1, u_2, u_3)$ , defined in  $(a, b) \times \Omega$ , satisfying

$$(5) \quad \begin{aligned} |u|_{\mathcal{W}^{(a,b)|\operatorname{wk}}} &:= \left( |u|_{L^\infty((a,b), L_{\operatorname{div}}^\infty(\Omega, \mathbb{R}^3))}^2 + |\partial_t u|_{L^2((a,b), L^\sigma(\Omega, \mathbb{R}^3))}^2 \right)^{1/2} < \infty \\ |u|_{\mathcal{W}^{(a,b)|\operatorname{st}}} &:= \left( |u|_{\mathcal{W}^{(a,b)|\operatorname{wk}}}^2 + |\nabla u|_{L^2((a,b), L^3(\Omega, \mathbb{R}^9))}^2 \right)^{1/2} < \infty. \end{aligned}$$

*Remark 2.1.* Notice that  $\mathcal{W}^{(a,b)|\operatorname{wk}} \subset W((a, b), L^\sigma(\Omega, \mathbb{R}^3), L^\sigma(\Omega, \mathbb{R}^3))$  and, then we have  $\mathcal{W}^{(a,b)|\operatorname{wk}} \subset C([a, b], L^\sigma(\Omega, \mathbb{R}^3))$  (cf. [Tem01, chapter 3, section 1, Lemma 1.1]). Also from,  $\int_\Omega f\phi \, dx \leq |f|_{L^\sigma(\Omega, \mathbb{R}^3)} |\phi|_{L^{\sigma'}(\Omega, \mathbb{R}^3)}$  with  $(1/\sigma) + (1/\sigma') = 1$ , we obtain  $1/\sigma' > 1 - (1/6/5) = 1/6$ , and  $\int_\Omega f\phi \, dx \leq C|f|_{L^\sigma(\Omega, \mathbb{R}^3)} |\phi|_{L^6(\Omega, \mathbb{R}^3)}$ . Since the inclusion  $H^1(\Omega, \mathbb{R}^3) \subset L^6(\Omega, \mathbb{R}^3)$  is continuous (see, e.g., [Neč67, chapter 2, Theorem 3.4]) we can conclude that  $L^\sigma(\Omega, \mathbb{R}^3) \subset H^{-1}(\Omega, \mathbb{R}^3)$ :  $\langle f, \phi \rangle_{H^{-1}(\Omega, \mathbb{R}^3), H_0^1(\Omega, \mathbb{R}^3)} = \int_\Omega f\phi \, dx \leq C_1|f|_{L^\sigma(\Omega, \mathbb{R}^3)} |\phi|_{H^1(\Omega, \mathbb{R}^3)}$ . Thus we have  $\mathcal{W}^{(a,b)|\operatorname{st}} \subset W((a, b), H^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$  and, from classical interpolation results (see, e.g., [LM72]), it follows  $\mathcal{W}^{(a,b)|\operatorname{st}} \subset C([a, b], L^2(\Omega, \mathbb{R}^3))$ . Finally, the lower bound  $6/5$  for  $\sigma$  is motivated from the results in [FCGIP04, Rod14].

Now, we recall that, in [FGH02], the set of traces  $u|_\Gamma$  at the boundary  $\Gamma$  of the elements  $u$  in the space  $W((a, b), H_{\operatorname{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))$  is completely characterized, for each

$s > 1/2$ , with  $s \notin \{3/2, 5/2\}$ . Denoting that trace space by  $G_{\text{av}}^s((a, b), \Gamma)$ , we have that  $v \mapsto v|_{\Gamma}$  is continuous:

$$|v|_{\Gamma}|_{G_{\text{av}}^s((a, b), \Gamma)} \leq C_1 |w|_{W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))}$$

and, there is a continuous extension

$$E_s : G_{\text{av}}^s((a, b), \Gamma) \rightarrow W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))$$

such that  $(E_s w)|_{\Gamma} = w$  and

$$|E_s w|_{W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))} \leq C_2 |w|_{G_{\text{av}}^s((a, b), \Gamma)}.$$

Moreover, from [FGH02, section 2.2] we know that

$$G_{\text{av}}^s((a, b), \Gamma) = G_{\mathbf{t}}^s((a, b), \Gamma) \oplus \mathbf{n}G_{\mathbf{n}, \text{av}}^s((a, b), \Gamma),$$

with  $\begin{cases} G_{\mathbf{t}}^s((a, b), \Gamma) = L^2((a, b), H^{s-(1/2)}(\Gamma, T\Gamma)) \cap H^{r_{\mathbf{t},1}(s)}((a, b), H^{r_{\mathbf{t},2}(s)}(\Gamma, T\Gamma)) \\ G_{\mathbf{n}, \text{av}}^s((a, b), \Gamma) = L^2((a, b), H_{\text{av}}^{s-(1/2)}(\Gamma, \mathbb{R})) \cap H^{r_{\mathbf{n},1}(s)}((a, b), H_{\text{av}}^{r_{\mathbf{n},2}(s)}(\Gamma, \mathbb{R})) \end{cases}$ ; and where  $H_{\text{av}}^r(\Gamma, \mathbb{R}) := \{u \in H^r(\Gamma, \mathbb{R}) \mid \int_{\Gamma} u \, d\Gamma = 0\}$  and  $r_{\mathbf{t},1}(s), r_{\mathbf{t},2}(s), r_{\mathbf{n},1}(s), r_{\mathbf{n},2}(s)$  are constants, in  $\mathbb{R}$ , given by

$$(r_{\mathbf{t},1}(s), r_{\mathbf{t},2}(s)) = \begin{cases} (1, s - (5/2)) & \text{if } 5/2 < s \\ (2s-1/4, 0) & \text{if } 2 \leq s < 5/2 \\ (2s-1/2s, (s-2)(2s-1)/2s) & \text{if } 1/2 < s \leq 2, s \neq 3/2 \end{cases},$$

and

$$(r_{\mathbf{n},1}(s), r_{\mathbf{n},2}(s)) = \begin{cases} (1, s - (5/2)) & \text{if } 3/2 < s, s \neq 5/2 \\ (2s+1/4, -1) & \text{if } 1 \leq s < 3/2 \\ (2s+1/2s+2, 2s^2-3(s+1)/2s+2) & \text{if } 1/2 < s \leq 1 \end{cases}.$$

The space  $G_{\text{av}}^s((a, b), \Gamma)$ , if nothing is said in contrary, is supposed to be endowed with the scalar product  $(u, v)_{G_{\text{av}}^s((a, b), \Gamma)} = (u^{\mathbf{t}} + \mathbf{n}u^{\mathbf{n}}, v^{\mathbf{t}} + \mathbf{n}v^{\mathbf{n}})_{G_{\text{av}}^s((a, b), \Gamma)} := (u^{\mathbf{t}}, v^{\mathbf{t}})_{G_{\mathbf{t}}^s((a, b), \Gamma)} + (u^{\mathbf{n}}, v^{\mathbf{n}})_{G_{\mathbf{n}, \text{av}}^s((a, b), \Gamma)}$ .

*Remark 2.2.* The notation  $T\Gamma$ , in the definition of  $G_{\mathbf{t}}^s((a, b), \Gamma)$ , stands for the tangent bundle of  $\Gamma$ ; the notation underlines that, for each instant of time, the elements of  $G_{\mathbf{t}}^s((a, b), \Gamma)$  are vector functions tangent to  $\Gamma$ , that is, vector fields in  $\Gamma$ .

*Remark 2.3.* Notice that the integral  $\int_{\Gamma} u \, d\Gamma = 0$  is well defined, in the sense of distributions, for  $u \in H^r(\Gamma, \mathbb{R})$  and all  $r \in \mathbb{R}$ : for  $r \geq 0$ , we have  $v \in L^2(\Gamma, \mathbb{R}^3)$  and the integral is well defined; on the other hand for  $r < 0$ , we have that  $H^r(\Gamma, \mathbb{R})$  coincides with the dual space of  $H^{-r}(\Gamma, \mathbb{R})$  (because  $\partial\Gamma = \emptyset$ ), then since the constant function  $1_{\Gamma}(x) := 1, x \in \Gamma$ , is in  $H^{-r}(\Gamma, \mathbb{R})$  (because  $\Gamma$  is bounded), the integral  $\int_{\Gamma} u \, d\Gamma := \langle u, 1_{\Gamma} \rangle_{H^r(\Gamma, \mathbb{R}), H^{-r}(\Gamma, \mathbb{R})}$  is well defined (considering, as usual,  $L^2(\Gamma, \mathbb{R})$  as a pivot space).

For technical reasons we relax a little the trace spaces: we define the superspace  $G^s((a, b), \Gamma)$  of  $G_{\text{av}}^s((a, b), \Gamma)$  by just omitting the average constraint:

$$(6) \quad G^s((a, b), \Gamma) := G_{\mathbf{t}}^s((a, b), \Gamma) \oplus \mathbf{n}G_{\mathbf{n}}^s((a, b), \Gamma)$$

with  $G_{\mathbf{n}}^s((a, b), \Gamma) := L^2((a, b), H^{s-(1/2)}(\Gamma, \mathbb{R})) \cap H^{r_{\mathbf{n},1}(s)}((a, b), H^{r_{\mathbf{n},2}(s)}(\Gamma, \mathbb{R}))$ . The space  $G^s((a, b), \Gamma)$  is endowed with the scalar product  $(u, v)_{G^s((a, b), \Gamma)} = (u^{\mathbf{t}} + \mathbf{n}u^{\mathbf{n}}, v^{\mathbf{t}} + \mathbf{n}v^{\mathbf{n}})_{G^s((a, b), \Gamma)} := (u^{\mathbf{t}}, v^{\mathbf{t}})_{G_{\mathbf{t}}^s((a, b), \Gamma)} + (u^{\mathbf{n}}, v^{\mathbf{n}})_{G_{\mathbf{n}}^s((a, b), \Gamma)}$ .

**Proposition 2.1.** *We have that  $G^s((a, b), \Gamma) = G_{\text{av}}^s((a, b), \Gamma) \oplus H^{r_{\mathbf{n},1}(s)}((a, b), \mathbb{R})\mathbf{n}$ . Moreover, for  $\pi_u := (1/\int_{\Gamma} d\Gamma) \int_{\Gamma} u \cdot \mathbf{n} \, d\Gamma$ , the projections*

$$\begin{aligned} \pi^s : G^s((a, b), \Gamma) &\rightarrow H^{r_{\mathbf{n},1}(s)}((a, b), \mathbb{R})\mathbf{n} \\ u &\mapsto \pi_u \mathbf{n} \end{aligned},$$

and  $1 - \pi^s : G^s((a, b), \Gamma) \rightarrow G_{\text{av}}^s((a, b), \Gamma)$ ,  $u \mapsto u_{\text{av}} := u - \pi^s u$  are continuous.

From [Tem01, chapter 1, Proposition 2.3], there exists a unique vector function  $\Theta \in H^2(\Omega, \mathbb{R}^3)$  solving the Stokes system

$$-\nu \Delta \Theta + \nabla p = 0, \operatorname{div} \Theta = \int_{\Gamma} d\Gamma / \int_{\Omega} d\Omega \text{ in } \Omega, \text{ and } \Theta|_{\Gamma} = \mathbf{n} \text{ on } \Gamma.$$

Now, we can extend the extension  $E_s$  above, defined in  $G_{\text{av}}^s((a, b), \Gamma)$ , to  $G^s((a, b), \Gamma)$ :

**Proposition 2.2.** *Writing each  $u \in G^s((a, b), \Gamma)$  as  $u = u_{\text{av}} + \pi_u \mathbf{n}$ , we define*

$$\begin{aligned} E_s^e : G^s((a, b), \Gamma) &\rightarrow W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3)) \oplus H^{r_{\mathbf{n}}, 1^{(s)}}((a, b), \mathbb{R})\Theta \\ u &\mapsto E_s u_{\text{av}} + \pi_u \Theta \end{aligned}$$

and we endow the space  $H^{r_{\mathbf{n}}, 1^{(s)}}((a, b), \mathbb{R})\Theta$  with the scalar product

$$(\phi\Theta, \psi\Theta)_{H^{r_{\mathbf{n}}, 1^{(s)}}((a, b), \mathbb{R})\Theta} := (\phi, \psi)_{H^{r_{\mathbf{n}}, 1^{(s)}}((a, b), \mathbb{R})}.$$

Then,  $E_s^e$  extends  $E_s$  and is linear and continuous. Moreover, the trace mapping  $v \mapsto v|_{\Gamma}$  from  $W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3)) \oplus H^{r_{\mathbf{n}}, 1^{(s)}}((a, b), \mathbb{R})\Theta$  onto  $G^s((a, b), \Gamma)$  is also linear and continuous.

The proofs of Propositions 2.1 and 2.2 will be given in the appendix, section A.3.

**2.2. The (illustrating) control space.** Let us write  $L^2(\Omega, \mathbb{R}^3) = H \oplus H^{\perp}$ , where  $H^{\perp} = \{\nabla \xi \mid \xi \in H^1(\Omega, \mathbb{R})\}$  denotes the orthogonal complement of  $H$  in  $L^2(\Omega, \mathbb{R}^3)$ , and denote by  $\Pi$  the orthogonal projection  $\Pi : L^2(\Omega, \mathbb{R}^3) \rightarrow H$  in  $L^2(\Omega, \mathbb{R}^3)$  onto  $H$ . For each positive integer  $N$ , we now define the  $N$ -dimensional space  $H_N \subset H$  as follows: let  $\{e_i \mid i \in \mathbb{N}_0\}$  be the orthonormal basis in  $H$  formed by the eigenfunctions of the Stokes operator  $L$ , which domain is defined by (4), and let  $0 < \alpha_1 \leq \alpha_2 \leq \dots$  be the corresponding eigenvalues:  $Le_i = \alpha_i e_i$ , then put

$$(7) \quad H_N := \operatorname{span}\{e_i \mid i \leq N\} \subset D(L) \subset H,$$

and denote by  $\Pi_N$  the orthogonal projection  $\Pi_N : H \rightarrow H_N$  in  $H$  onto  $H_N$ .

Let  $\mathcal{O} \subseteq \Gamma$  be a connected open subset of the boundary  $\Gamma$ , localized on one side of its boundary. We suppose that  $\mathcal{O}$  is a  $C^\infty$ -smooth manifold either boundaryless or with  $C^\infty$ -smooth boundary  $\partial\mathcal{O}$ . Let  $\{\pi_i \mid i \in \mathbb{N}_0\}$  be an orthonormal basis in  $L^2(\mathcal{O}, \mathbb{R})$  formed by the eigenfunctions of the Laplace–de Rham (or Laplace–Beltrami) operator  $\Delta_{\mathcal{O}}$  on the smooth manifold  $\mathcal{O}$ , under Dirichlet boundary conditions,  $\pi_i(p) = 0$  for all  $p \in \partial\mathcal{O}$ . Analogously let  $\{\tau_i \mid i \in \mathbb{N}_0\}$  be an orthonormal basis in  $L^2(\mathcal{O}, T\mathcal{O})$  formed by the vector fields that are eigenfunctions of  $\Delta_{\mathcal{O}}$  on  $T\mathcal{O}$ , also under Dirichlet boundary conditions in the case  $\partial\mathcal{O} \neq \emptyset$ ,  $\tau_i(p) = 0 \in T_p\Gamma$  for all  $p \in \partial\mathcal{O}$ .

Let  $0 \leq \beta_1 \leq \beta_2 \leq \dots$ , and  $0 \leq \gamma_1 \leq \gamma_2 \leq \dots$  be the eigenvalues associated with the systems  $\{\pi_i \mid i \in \mathbb{N}_0\}$  and  $\{\tau_i \mid i \in \mathbb{N}_0\}$ , respectively.

We may write  $L^2(\mathcal{O}, \mathbb{R}^3)$  as an orthogonal sum

$$L^2(\mathcal{O}, \mathbb{R}^3) = L^2(\mathcal{O}, \mathbb{R})\mathbf{n} \oplus L^2(\mathcal{O}, T\mathcal{O}).$$

Notice that  $\{\pi_i \mathbf{n} \mid i \in \mathbb{N}_0\}$  is an orthonormal basis for  $L^2(\mathcal{O}, \mathbb{R})\mathbf{n} = \{f\mathbf{n} \mid f \in L^2(\mathcal{O}, \mathbb{R})\}$ , and the system  $\{\pi_i \mathbf{n} \mid i \in \mathbb{N}_0\} \cup \{\tau_i \mid i \in \mathbb{N}_0\}$  is an orthonormal basis in the space  $L^2(\mathcal{O}, \mathbb{R}^3)$ .

For some more details, and references, concerning the Laplace–de Rham operator see section A.1 in the appendix.

Define, for each  $M \in \mathbb{N}_0$ , the space

$$(8) \quad L_M^2(\mathcal{O}, \mathbb{R}^3) := \operatorname{span}\{\pi_i \mathbf{n}, \tau_i \mid i \in \mathbb{N}_0, i \leq M\}.$$

Denote by  $P_M^\mathcal{O}$  the orthogonal projection  $P_M^\mathcal{O} : L^2(\mathcal{O}, \mathbb{R}^3) \rightarrow L_M^2(\mathcal{O}, \mathbb{R}^3)$  in  $L^2(\mathcal{O}, \mathbb{R}^3)$  onto  $L_M^2(\mathcal{O}, \mathbb{R}^3)$ .

We suppose we are able to apply a control through a subset  $\Gamma_c \subseteq \overline{\Gamma_c} \subseteq \mathcal{O} \subseteq \Gamma$ , where  $\overline{\Gamma_c} = \text{supp } \chi$  is the support of a function  $\chi \in C^\infty(\Gamma, \mathbb{R})$ . Further let  $\epsilon > 0$  and  $\vartheta \in C^2(\Gamma, \mathbb{R})$  be a function such that for all  $x \in \Gamma_c$ ,  $\vartheta(x) \geq \epsilon$  and with  $\text{supp } \vartheta \subseteq \overline{\mathcal{O}}$ . For an illustration purpose, we will give particular attention to the case where the boundary control  $\zeta$  is in the space

$$(9) \quad \begin{aligned} \mathcal{E}_M^1 &:= \chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O} \vartheta G^1((a, b), \Gamma)|_\mathcal{O} \\ &:= \{\zeta \mid \zeta(t) = \chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O} (\vartheta \eta(t)|_\mathcal{O}) \text{ and } \eta \in G^1((a, b), \Gamma)\} \end{aligned}$$

where  $\mathbb{E}_0^\mathcal{O} : L^2(\mathcal{O}, \mathbb{R}) \rightarrow L^2(\Gamma, \mathbb{R})$  stands for the extension by zero outside  $\mathcal{O}$ , and  $P_{\chi^\perp}^\mathcal{O} : L^2(\mathcal{O}, \mathbb{R}^3) \rightarrow \{f \in L^2(\mathcal{O}, \mathbb{R}^3) \mid (f, \chi \mathbf{n})_{L^2(\mathcal{O}, \mathbb{R}^3)} = 0\}$  is the orthogonal projection in  $L^2(\mathcal{O}, \mathbb{R}^3)$  onto  $\{\chi \mathbf{n}|_\mathcal{O}\}^\perp$ . In other words,

$$\mathbb{E}_0^\mathcal{O} \xi(x) := \begin{cases} \xi(x) & \text{if } x \in \mathcal{O} \\ 0 & \text{if } x \in \Gamma \setminus \overline{\mathcal{O}}, \end{cases} \text{ and } P_{\chi^\perp}^\mathcal{O} v := v - ((v, \chi \mathbf{n})_{L^2(\mathcal{O}, \mathbb{R}^3)} / \int_\mathcal{O} \chi^2 d\mathcal{O}) \chi \mathbf{n}.$$

In particular the controls take their values  $\zeta(t)$  in the finite-dimensional space spanned by  $\{\chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} \pi_i \mathbf{n}, \chi \mathbb{E}_0^\mathcal{O} \tau_i \mid i \in \mathbb{N}_0, i \leq M\}$ , for a.e.  $t \in (a, b)$ .

*Remark 2.4.* Notice that  $\zeta(t) \in \chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O} \vartheta L^2(\mathcal{O}, \mathbb{R}^3)$  satisfy the zero-average compatibility condition:  $\int_\Gamma \zeta(t) \cdot \mathbf{n} d\Gamma = \int_\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O} \vartheta \eta(t) \cdot \chi \mathbf{n} d\mathcal{O} = 0$ . The function  $\chi$  guarantees that the controls are supported in  $\overline{\Gamma_c}$ ; the function  $\vartheta$  is needed because we will need suitable continuity properties (cf. Propositions 2.7 and 5.1, needed in the proofs of Theorems 5.2 and 5.3, respectively). Further, as we said, we propose the space (9) mainly as an example guideline; the arguments that will follow may work for other (admissible) control spaces (cf. section 5 where we consider a variation of this control space).

**2.3. The addressed problem.** Consider the following time-forward Oseen–Stokes system, in  $(a, b) \times \Omega$ ,

$$(10) \quad \begin{aligned} \partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v + \nabla p + g &= 0, & \text{div } v &= 0, \\ v|_\Gamma &= \zeta, & v(a) &= v_0, \end{aligned}$$

where  $\hat{u} \in \mathcal{W}^{(a,b)|\text{wk}}$ ,  $g \in L^2((a, b), H^{-1}(\Omega, \mathbb{R}^3))$ ,  $\zeta \in G_{\text{av}}^1((a, b), \Gamma)$  is a control,  $v_0 \in L_{\text{div}}^2(\Omega, \mathbb{R}^3)$ , and  $\mathcal{B}(\hat{u}) : v \mapsto \langle \hat{u} \cdot \nabla \rangle v + \langle v \cdot \nabla \rangle \hat{u}$ .

We will start by the derivation of some observability inequalities concerning the “adjoint” Oseen–Stokes time-backward system, in  $(a, b) \times \Omega$ ,

$$(11) \quad \begin{aligned} -\partial_t q + \mathcal{B}^*(\hat{u})q - \nu \Delta q + \nabla p + f &= 0, & \text{div } q &= 0, \\ q|_\Gamma &= 0, & q(b) &= q_1 \in H, \end{aligned}$$

where  $f \in L^2((a, b), H^{-1}(\Omega, \mathbb{R}^3))$  and  $\mathcal{B}^*(\hat{u})$  is the formal adjoint to  $\mathcal{B}(\hat{u})$ , that is,

$$(12) \quad (\mathcal{B}^*(\hat{u})q, v)_{L^2(\Omega, \mathbb{R}^3)} := \langle q, \mathcal{B}(\hat{u})v \rangle_{H_0^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)}, \quad q \in V, v \in H_{\text{div}}^1(\Omega, \mathbb{R}^3).$$

*Remark 2.5.* Notice that for  $D_s q := (\nabla q + (\nabla q)^\top)$ , where  $A^\top$  denotes the transpose matrix of  $A$ , we have  $\mathcal{B}^*(\hat{u})q = \langle \hat{u} \cdot D_s \rangle q$ , with  $w = (w_1, w_2, w_3) := \langle \hat{u} \cdot D_s \rangle q$  given by  $w_j = \sum_{i=1}^3 \hat{u}_i (\partial_{x_i} q_j + \partial_{x_j} q_i)$  for all  $j \in \{1, 2, 3\}$ . In particular we have  $(\langle \hat{u} \cdot D_s \rangle q) \cdot v = (\langle v \cdot D_s \rangle q) \cdot u$ , for any given pair of vectors  $u, v$  in  $\mathbb{R}^3$ .

Then, one of the derived observability inequalities will be used to obtain a controllability result to the system (10), where (a subspace of)  $\mathcal{E}_M^1$  is taken as the space of the controls.

**2.4. Existence and uniqueness of weak and strong solutions.** Here we present some remarks concerning the solutions of the considered systems (10) and (11). Among the spaces  $G_{\text{av}}^s((a, b), \Gamma)$ , the most interesting for us will be the ones corresponding to  $s \in \{1, 2\}$ , that will be related to so-called weak and strong solutions. Recall the extensions  $E_s$  in section 2.1.

**Definition 2.1.** Given  $\hat{u} \in \mathcal{W}^{(a,b)|\text{wk}}$ ,  $v_0 \in L_{\text{div}}^2(\Omega, \mathbb{R}^3)$ ,  $g \in L^2((a, b), H^{-1}(\Omega, \mathbb{R}^3))$ , and  $\zeta \in G_{\text{av}}^1((a, b), \Gamma)$ ; we say that  $v$ , in the space  $W((a, b), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ , is a weak solution for system (10), if  $v - E_1\zeta \in L^2((a, b), V, V')$  is a weak solution for system

$$(13) \quad \begin{aligned} \partial_t y + \mathcal{B}(\hat{u})y - \nu \Delta y + \nabla p + f &= 0, & \operatorname{div} y &= 0, \\ y|_{\Gamma} &= 0, & y(a) &= y_0 \end{aligned}$$

with  $f = g + \partial_t E_1\zeta + \mathcal{B}(\hat{u})E_1\zeta - \nu \Delta E_1\zeta$ , and  $y_0 = v_0 - E_1\zeta(a) \in H$ . Here weak solution for (13) is understood in the classical sense (cf. [Lio69, Tem95, Tem01]).

**Definition 2.2.** Given  $\hat{u} \in \mathcal{W}^{(a,b)|\text{st}}$ ,  $v_0 \in H_{\text{div}}^1(\Omega, \mathbb{R}^3)$ ,  $g \in L^2((a, b), L^2(\Omega, \mathbb{R}^3))$ , and  $\zeta \in G_{\text{av}}^2((a, b), \Gamma)$ ; we say that  $v$ , in the space  $W((a, b), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$ , is a strong solution for system (10), if  $v - E_2\zeta \in L^2((a, b), D(L), H)$  is a strong solution for system (13) with  $f = g + \partial_t E_2\zeta + \mathcal{B}(\hat{u})E_2\zeta - \nu \Delta E_2\zeta$ , and  $y_0 = v_0 - E_2\zeta(a) \in V$ . Again, strong solution for (13) is understood in the classical sense (cf. [Tem95]).

*Remark 2.6.* The existence and uniqueness of a weak solution in  $W((a, b), V, V')$  for (13), can be proved by standard arguments as in [Tem01] taking into account that, formally

$$\langle \mathcal{B}(\hat{u})y, w \rangle_{H^{-1}(\Omega, \mathbb{R}^3), H_0^1(\Omega, \mathbb{R}^3)} = - \sum_{i,j=1}^3 \int_{\Omega} \hat{u}_i (\partial_{x_i} w_j) y_j \, dx - \sum_{i,j=1}^3 \int_{\Omega} y_i (\partial_{x_i} w_j) \hat{u}_j \, dx,$$

which leads to the estimate  $|\mathcal{B}(\hat{u})y|_{H^{-1}(\Omega, \mathbb{R}^3)} \leq C|\hat{u}|_{L_{\text{div}}^{\infty}(\Omega, \mathbb{R}^3)}|y|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)}$  (cf. [Rod14, Remark 3.1]). For the existence and uniqueness of strong solution for (13) we can use, in addition,

$$\begin{aligned} |\mathcal{B}(\hat{u})y|_{L^2(\Omega, \mathbb{R}^3)} &\leq C_1(|\hat{u}|_{L_{\text{div}}^{\infty}(\Omega, \mathbb{R}^3)}|\nabla y|_{L^2(\Omega, \mathbb{R}^9)} + |\nabla \hat{u}|_{L^3(\Omega, \mathbb{R}^9)}|y|_{L_{\text{div}}^6(\Omega, \mathbb{R}^3)}) \\ &\leq C_2|\hat{u}|_{\mathcal{W}^{(a,b)|\text{st}}}|y|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}. \end{aligned}$$

In the case our control take its values in the space  $\mathcal{E}_M^1$ , a natural question is: what are the admissible initial vector fields  $v_0$  for this type of controls, if we want to guarantee the existence of a weak solution? The answer is not difficult if we give it in a general way, see [Rod14, section 3.1]: let  $\mathcal{Z}$  be a Hilbert space, and  $K_1 : \mathcal{Z} \rightarrow G_{\text{av}}^1((a, b), \Gamma)$  a continuous linear mapping; then the set of admissible weak initial conditions for system (10), with  $\zeta \in K_1\mathcal{Z}$ , is given by  $\mathcal{A}_{K_1} = H + \mathcal{H}_{K_1}$  where  $\mathcal{H}_{K_1} := E_1K_1\mathcal{Z}(a) = \{\gamma(a) \mid \gamma = E_1K_1\eta \text{ and } \eta \in \mathcal{Z}\}$ . Moreover  $\mathcal{H}_{K_1}$  and  $\mathcal{A}_{K_1}$  are Hilbert spaces, with associated range norms

$$\begin{aligned} |u|_{\mathcal{H}_{K_1}} &:= \inf \{|\eta|_{\mathcal{Z}} \mid u = E_1K_1\eta(a), \eta \in \mathcal{Z}\}, \\ |u|_{\mathcal{A}_{K_1}} &:= \inf \{ |(w, z)|_{H \times \mathcal{H}_{K_1}} \mid u = w + z \text{ and } (w, z) \in H \times \mathcal{H}_{K_1} \}, \end{aligned}$$

and there are constants  $C_1, C_2, C_3 > 0$  such that

$$(14a) \quad |u|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)} \leq C_1|u|_{\mathcal{H}_{K_1}}, \quad \text{for all } u \in \mathcal{H}_{K_1};$$

$$(14b) \quad |u|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)} \leq C_2|u|_{\mathcal{A}_{K_1}}, \quad \text{for all } u \in \mathcal{A}_{K_1};$$

$$(14c) \quad |u|_{L_{\text{div}}^2(\Omega, \mathbb{R}^3)} \geq C_3|u|_{\mathcal{A}_{K_1}}, \quad \text{for all } u \in H.$$

From [Rod14, section 3.1] we also have the following existence result:

**Theorem 2.3.** *Given  $\hat{u} \in \mathcal{W}^{(a,b)|\text{wk}}$ ,  $g \in L^2((a,b), H^{-1}(\Omega, \mathbb{R}^3))$ , a Hilbert space  $\mathcal{Z}$ , a continuous linear mapping  $K_1 : \mathcal{Z} \rightarrow G_{\text{av}}^1((a,b), \Gamma)$ ,  $v_0 \in \mathcal{A}_{K_1}$  and  $\eta \in \mathcal{Z}$ , then there exists a weak solution  $v \in W((a,b), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$  for system (10), with  $\zeta = K_1\eta$ . Moreover  $v$  is unique and depends continuously on the given data  $(v_0, g, \eta)$ :*

$$|v|_{W((a,b), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{(a,b)|\text{wk}}}} \left( |v_0|_{L^2_{\text{div}}(\Omega, \mathbb{R}^3)}^2 + |g|_{L^2((a,b), H^{-1}(\Omega, \mathbb{R}^3))}^2 + |\eta|_{\mathcal{Z}}^2 \right).$$

*Remark 2.7.* Notice that in [Rod14] we find the Lebesgue-like notation  $L^\infty((a,b) \times \Omega, \mathbb{R}^3)$  instead of the Bochner-like notation  $L^\infty((a,b), L^\infty(\Omega, \mathbb{R}^3))$ . Here we use the latter because it will be more convenient below. To see that the spaces coincide, first we observe that the inclusion  $L^1((a,b) \times \Omega, \mathbb{R}^3) \subseteq L^1((a,b), L^1(\Omega, \mathbb{R}^3))$  follows from Fubini's Theorem, see e.g. [DS67, section III.11, Theorem 9]; and  $L^1((a,b) \times \Omega, \mathbb{R}^3) \supseteq L^1((a,b), L^1(\Omega, \mathbb{R}^3))$  can be derived from Theorem 17 in [DS67, section III.11] (recalling that functions in  $L^1((a,b) \times \Omega, \mathbb{R}^3)$  are defined up to sets of measure zero). Then, we can write  $L^\infty((a,b) \times \Omega, \mathbb{R}^3) = L^1((a,b) \times \Omega, \mathbb{R}^3)' = L^1((a,b), L^1(\Omega, \mathbb{R}^3))' = L^\infty((a,b), L^\infty(\Omega, \mathbb{R}^3))$ .

Following the same idea in [Rod14, section 3.1], we can also prove the analogous results for strong solutions, where now, we consider a continuous linear mapping  $K_2 : \mathcal{Z} \rightarrow G_{\text{av}}^2((a,b), \Gamma)$ , the set of admissible strong initial conditions is  $\mathcal{A}_{K_2} := V + \mathcal{H}_{K_2}$  with  $\mathcal{H}_{K_2} := E_2 K_2 \mathcal{Z}(a)$ , and

$$\begin{aligned} |u|_{\mathcal{H}_{K_2}} &:= \inf \{ |\eta|_{\mathcal{Z}} \mid u = E_2 K_2 \eta(a), \eta \in \mathcal{Z} \}, \\ |u|_{\mathcal{A}_{K_2}} &:= \inf \{ |(w, z)|_{V \times \mathcal{H}_{K_2}} \mid u = w + z \text{ and } (w, z) \in V \times \mathcal{H}_{K_2} \}, \end{aligned}$$

and there are constants  $C_1, C_2, C_3 > 0$  such that

$$(15a) \quad |u|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)} \leq C_1 |u|_{\mathcal{H}_{K_2}}, \quad \text{for all } u \in \mathcal{H}_{K_2};$$

$$(15b) \quad |u|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)} \leq C_2 |u|_{\mathcal{A}_{K_2}}, \quad \text{for all } u \in \mathcal{A}_{K_2};$$

$$(15c) \quad |u|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)} \geq C_3 |u|_{\mathcal{A}_{K_2}}, \quad \text{for all } u \in V.$$

**Theorem 2.4.** *Given  $\hat{u} \in \mathcal{W}^{(a,b)|\text{st}}$ ,  $g \in L^2((a,b), L^2(\Omega, \mathbb{R}^3))$ , a Hilbert space  $\mathcal{Z}$ , a continuous linear mapping  $K_2 : \mathcal{Z} \rightarrow G_{\text{av}}^2((a,b), \Gamma)$ ,  $v_0 \in \mathcal{A}_{K_2}$  and  $\eta \in \mathcal{Z}$ , then there exists a strong solution  $v \in W((a,b), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$  for system (10), with  $\zeta = K_2\eta$ . Moreover  $v$  is unique and depends continuously on the given data  $(v_0, g, \eta)$ :*

$$|v|_{W((a,b), H_{\text{div}}^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{(a,b)|\text{st}}}} \left( |v_0|_{H_{\text{div}}^1(\Omega, \mathbb{R}^3)}^2 + |g|_{L^2((a,b), L^2(\Omega, \mathbb{R}^3))}^2 + |\eta|_{\mathcal{Z}}^2 \right).$$

Analogously, weak and strong solutions for system (11) can be defined in the classical sense (see [Tem95, section 2.4]), just reversing time. We can derive that

**Theorem 2.5.** *Given  $\hat{u} \in \mathcal{W}^{(a,b)|\text{wk}}$ ,  $f \in L^2((a,b), H^{-1}(\Omega, \mathbb{R}^3))$ , and  $q_1 \in H$ , then there exists a weak solution  $q \in W((a,b), V, V')$  for system (11). Moreover  $q$  is unique and depends continuously on the given data  $(q_1, f)$ :*

$$|q|_{W((a,b), V, V')}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{(a,b)|\text{wk}}}} \left( |q_1|_H^2 + |f|_{L^2((a,b), H^{-1}(\Omega, \mathbb{R}^3))}^2 \right).$$

**Theorem 2.6.** *Given  $\hat{u} \in \mathcal{W}^{(a,b)|\text{wk}}$ ,  $f \in L^2((a,b), L^2(\Omega, \mathbb{R}^3))$ , and  $q_1 \in V$ , then there exists a strong solution  $q \in W((a,b), D(L), H)$  for system (11). Moreover  $q$  is unique and depends continuously on the given data  $(q_1, f)$ :*

$$|q|_{W((a,b), D(L), H)}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{(a,b)|\text{wk}}}} \left( |q_1|_V^2 + |f|_{L^2((a,b), L^2(\Omega, \mathbb{R}^3))}^2 \right).$$

*Remark 2.8.* Notice that, although we have taken the reference solution  $\hat{u}$  in the spaces (5), for the previous results concerning the existence, uniqueness and continuity of the solutions we do not need the condition on the time-derivative:  $\partial_t \hat{u} \in L^2((a, b), L^\sigma(\Omega, \mathbb{R}^3))$ . This condition is needed only for the observability and controllability results that follow.

2.4.1. *Solutions and the illustrating control space.* Notice that,  $P_M^\mathcal{O} \vartheta L^2(\mathcal{O}, \mathbb{R}^3)$  is a subset of  $C^2(\mathcal{O}, \mathbb{R}^3)$ ; then it follows that  $(\chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O} \vartheta L^2(\mathcal{O}, \mathbb{R}^3))|_{\Gamma_c}$  is a subset of  $H_0^2(\Gamma_c, \mathbb{R}^3)$ , and  $\chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O} \vartheta L^2(\mathcal{O}, \mathbb{R}^3) = \mathbb{E}_0^{\Gamma_c}(\chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O} \vartheta L^2(\mathcal{O}, \mathbb{R}^3))|_{\Gamma_c}$  is a subset of  $H^2(\Gamma, \mathbb{R}^3)$ ; here  $\mathbb{E}_0^{\Gamma_c} : H_0^2(\Gamma_c, \mathbb{R}^3) \rightarrow H^2(\Gamma, \mathbb{R}^3)$  stands for the extension by 0 outside  $\Gamma_c$ .

**Proposition 2.7.** *The operator  $\eta \mapsto K_i \eta = K^\mathcal{O} \eta := \chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O}(\vartheta \eta|_{\mathcal{O}})$  maps the space  $G^i((a, b), \Gamma)$  into  $G_{\text{av}}^i((a, b), \Gamma)$ , and is linear and continuous, for  $i \in \{1, 2\}$ .*

We give the proof in the appendix, section A.5. We can find the set of admissible weak conditions  $\mathcal{A}_{K_i}$  as above, and apply Theorem 2.3 to guarantee the existence of weak solutions, in the case the control functions are in  $\mathcal{E}_M^1$ . Analogously we derive the existence of strong solutions for controls in  $\mathcal{E}_M^2 := \chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O}(\vartheta G^2((a, b), \Gamma)|_{\mathcal{O}}) \subset \mathcal{E}_M^1$ .

### 3. LOCALIZED OBSERVABILITY INEQUALITIES

In this section the main goal is to derive some observability inequalities that are somehow appropriate to deal with boundary controls problems with controls supported in a given subset of the boundary. The starting point will be a result on null boundary controllability from [Rod14, section 4]: let us denote by  $G_{\text{av},c}^1((a, b), \Gamma) \subseteq G_{\text{av}}^1((a, b), \Gamma)$  the space of the traces at  $\Gamma$  of the functions in  $W((a, b), V_c, H^{-1}(\Omega, \mathbb{R}^3))$ . We endow  $G_{\text{av},c}^1((a, b), \Gamma)$  with the norm inherited from  $G_{\text{av}}^1((a, b), \Gamma)$ . We observe that

$$G_{\text{av},c}^1((a, b), \Gamma) = \{u \in G_{\text{av}}^1((a, b), \Gamma) \mid u(t) = 0 \text{ in } \Gamma \setminus \overline{\Gamma_c} \text{ for a.e. } t \in (a, b)\}$$

and that  $G_{\text{av},c}^1((a, b), \Gamma)$  is a closed subspace of  $G_{\text{av}}^1((a, b), \Gamma)$ . Notice that if  $(u^n)_{n \in \mathbb{N}}$  is a sequence on  $G_{\text{av},c}^1((a, b), \Gamma)$  and  $u^n$  converges to  $u^\infty$  in  $G_{\text{av}}^1((a, b), \Gamma)$ , then in particular  $u^n$  converges to  $u^\infty$  in  $L^2((a, b), L^2(\Gamma, \mathbb{R}))$ , and so  $u^n|_{\Gamma \setminus \overline{\Gamma_c}}$  converges to  $u^\infty|_{\Gamma \setminus \overline{\Gamma_c}}$  in  $L^2((a, b), L^2(\Gamma \setminus \overline{\Gamma_c}, \mathbb{R}))$ . Necessarily,  $u^\infty|_{\Gamma \setminus \overline{\Gamma_c}} = 0$ . It follows that  $G_{\text{av},c}^1((a, b), \Gamma)$  is complete and, we can consider the system (10) with  $\zeta \in G_{\text{av},c}^1((a, b), \Gamma)$  and  $v_0 \in \mathcal{A}_{K_1}$ , where  $\mathcal{A}_{K_1} = H + \mathcal{H}_{K_1}$  is the space of admissible weak initial conditions for that system, with  $\mathcal{Z} = G_{\text{av},c}^1((a, b), \Gamma)$ ,  $K_1 : G_{\text{av},c}^1((a, b), \Gamma) \rightarrow G_{\text{av}}^1((a, b), \Gamma)$  the inclusion mapping  $\eta \mapsto \eta$ , and  $\mathcal{H}_{K_1} := \{E_1 \zeta(a) \mid \zeta \in G_{\text{av},c}^1((a, b), \Gamma)\}$ .  $\mathcal{H}_{K_1}$  and  $\mathcal{A}_{K_1}$  are supposed to be endowed with the respective range scalar products, and range norms. From [Rod14, section 4], since  $\overline{\Gamma_c} = \text{supp } \chi$  is the support of a function  $\chi \in C^\infty(\Gamma, \mathbb{R})$ , we have the following null controllability property:

**Lemma 3.1.** *Given  $\hat{u} \in \mathcal{W}^{(a,b)|\text{wk}}$  and  $v_0 \in \mathcal{A}_{K_1}$ , there exists a control  $\zeta = \zeta(v_0) \in G_{\text{av},c}^1((a, b), \Gamma)$  such that, for the corresponding solution  $v$  to system (10) with  $g = 0$ , we have  $v(b) = 0$ . Moreover the control may be chosen so that the mapping  $v_0 \mapsto \zeta(v_0)$  is linear and continuous:  $|\zeta(v_0)|_{G_{\text{av},c}^1((a,b),\Gamma)} \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{(a,b)|\text{wk}}}} |v_0|_{\mathcal{A}_{K_1}}$ .*

3.1. **Some simple observability inequalities.** Consider the system (11) with data

$$(16) \quad \hat{u} \in \mathcal{W}^{(a,b)|\text{wk}}, \quad f \in L^2((a, b), H), \quad \text{and } q_1 \in V.$$

By Theorem 2.6 there exists a strong solution  $q \in W((a, b), D(L), H)$  for that system. Consider also, the weak solution  $v$  of (10) with  $g = 0$  and  $\zeta = \zeta(v_0)$ , the control given by

Lemma 3.1. We find

$$\begin{aligned}
& (q_1, v(b))_{L^2(\Omega, \mathbb{R}^3)} - (q(a), v_0)_{L^2(\Omega, \mathbb{R}^3)} = \int_a^b \frac{d}{dt} (q, v)_{L^2(\Omega, \mathbb{R}^3)}(\tau) d\tau \\
& = \int_a^b \left( (\mathcal{B}^*(\hat{u})q - \nu \Delta q + \nabla p_q + f, v)_{L^2(\Omega, \mathbb{R}^3)} \right. \\
& \quad \left. + \langle q, -\mathcal{B}(\hat{u})v + \nu \Delta v - \nabla p_v \rangle_{H_0^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)} \right) dt \\
& = \int_a^b (f, v)_{L^2(\Omega, \mathbb{R}^3)} dt + \int_a^b (-\nu \langle \mathbf{n} \cdot \nabla \rangle q + p_q \mathbf{n}, v)_{L^2(\Gamma, \mathbb{R}^3)} dt,
\end{aligned}$$

from which, we obtain

$$\begin{aligned}
(17) \quad & - (q(a), v_0)_{L^2(\Omega, \mathbb{R}^3)} \\
& = \int_a^b (f, v)_{L^2(\Omega, \mathbb{R}^3)} dt + \int_a^b (-\nu \langle \mathbf{n} \cdot \nabla \rangle q + p_q \mathbf{n}, v)_{L^2(\Gamma, \mathbb{R}^3)} dt.
\end{aligned}$$

Now, considering  $L^2((a, b), L^2(\Gamma, \mathbb{R}^3))$  as a pivot space at the boundary, we consider the dual space  $G^1((a, b), \Gamma)'$  of  $G^1((a, b), \Gamma) \subset L^2((a, b), L^2(\Gamma, \mathbb{R}^3))$ . We have the dense inclusions  $G^1((a, b), \Gamma) \subset L^2((a, b), L^2(\Gamma, \mathbb{R}^3)) \subset G^1((a, b), \Gamma)'$ .

Since  $v|_\Gamma = \zeta(v_0) \in G_{\text{av},c}^1((a, b), \Gamma)$ , and the norm of  $G_{\text{av},c}^1((a, b), \Gamma)$  is the one inherited from  $G^1((a, b), \Gamma)$ , we have that  $|(q(a), v_0)_{L^2(\Omega, \mathbb{R}^3)}|_{\mathbb{R}}$  is bounded by

$$\begin{aligned}
C_1 \left( |f|_{L^2((a,b),H)} + |\mathcal{I}|_{\Gamma_c} (p_q \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q) \Big|_{G^1((a,b),\Gamma)'} \right) \\
\times \left( |v|_{L^2((a,b),L^2(\Omega,\mathbb{R}^3))} + |\zeta(v_0)|_{G_{\text{av},c}^1((a,b),\Gamma)} \right),
\end{aligned}$$

where  $\mathcal{I}|_{\Gamma_c}$  the indicator operator:  $\mathcal{I}|_{\Gamma_c} f(t, x) := \begin{cases} f(t, x) & \text{if } x \in \Gamma_c \\ 0 & \text{if } x \in \Gamma \setminus \overline{\Gamma_c} \end{cases}$ , mapping  $L^2((a, b), L^2(\Gamma, \mathbb{R}^3))$  into itself. Thus, from the continuity of  $v_0 \mapsto \zeta(v_0)$ , together with (14b) and Theorem 2.3, we can find that

$$(18) \quad |v|_{L^2((a,b),L^2(\Omega,\mathbb{R}^3))} \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{(a,b)}|\text{wk}}} |v_0|_{\mathcal{A}_{K_1}}$$

and, from (14c), we arrive to the boundary observability inequality

$$(19) \quad |q(a)|_H^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{(a,b)}|\text{wk}}} \left( |f|_{L^2((a,b),H)}^2 + |\mathcal{I}|_{\Gamma_c} (p \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q) \Big|_{G^1((a,b),\Gamma)'}^2 \right)$$

for the solution  $q$  of system (11), with  $\hat{u} \in \mathcal{W}^{(a,b)}|\text{wk}$ , and “the” corresponding pressure function  $p = p_q$ .

Now, let  $G_{\text{av},c}^2((a, b), \Gamma) = \{\gamma|_\Gamma \mid \gamma \in W((a, b), V_c \cap H^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))\} = \{u \in G_{\text{av}}^2((a, b), \Gamma) \mid u(t) = 0 \text{ in } \Gamma \setminus \overline{\Gamma_c}, \text{ for a.e. } t \in (a, b)\}$ .

**Lemma 3.2.** *Given  $\hat{u} \in \mathcal{W}^{(a,b)}|\text{st}$  and  $v_0 \in \mathcal{A}_{K_2}$ , there exists a control  $\zeta = \zeta(v_0) \in G_{\text{av},c}^2((a, b), \Gamma)$  such that, for the corresponding solution  $v$  to system (10) with  $g = 0$ , we have  $v(b) = 0$ . Moreover the control may be chosen so that the mapping  $v_0 \mapsto \zeta(v_0)$  is linear and continuous:  $|\zeta(v_0)|_{G_{\text{av},c}^2((a,b),\Gamma)} \leq \overline{C}_{[\hat{u}]_{\mathcal{W}^{(a,b)}|\text{st}}} |v_0|_{\mathcal{A}_{K_2}}$ .*

*Proof.* We follow the proof of Theorem 4.1 in [Rod14]. Write  $v_0 = v_{0V} + v_{0\mathcal{H}}$  where  $(v_{0V}, v_{0\mathcal{H}}) \in V \times \mathcal{H}_{K_2}$ . Let  $\gamma_{v_{0\mathcal{H}}} \in W((a, b), V_c \cap H^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$  be defined by  $\gamma_{v_{0\mathcal{H}}} \in \{\gamma \in W((a, b), V_c \cap H^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3)) \mid \gamma(a) = 0\}^\perp$  and  $\gamma_{v_{0\mathcal{H}}}(a) = v_{0\mathcal{H}}$ . We have that the mapping  $v_{0\mathcal{H}} \mapsto \gamma_{v_{0\mathcal{H}}}$  is linear and continuous. Now, put  $l = b - a/2$  and let  $\xi$  be a smooth real function, defined in  $[a, a + l]$ , taking the value 1 in a neighborhood of  $t = a$ , and vanishing in a neighborhood of  $t = a + l$ . On the interval of time  $(a, a + l)$  we

apply the control  $\xi\gamma_{v_{0\mathcal{H}}}|_{\Gamma}$ ; in this way we arrive to a point  $v(a+l) \in V$  at time  $t = a+l$ . Moreover  $|v(a+l)|_V \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,a+l)}|_{\text{st}}]}}|v_0|_{\mathcal{A}_{K_2}}$ . Then, in the interval of time  $(a+l, b)$  proceeding as in the proof of Theorem 4.1 in [Rod14], we can conclude that there exists a control  $\gamma_1$  driving the system to zero at time  $t = b$ ; moreover  $\gamma_1 = \bar{v}|_{\Gamma}$  is the restriction to  $\Gamma$  of a suitable  $\bar{v} \in W((a+l, b), V_c \cap H^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$ , and the mapping  $v(a+l) \mapsto \gamma_1$  is linear and continuous,  $|\gamma_1|_{G_{\text{av},c}^2((a+l,b),\Gamma)} \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a+l,b)}|_{\text{st}}]}}|v(a+l)|_V$ .

Therefore the concatenation  $\zeta(v_0) := \gamma_1 \circ \xi\gamma_{v_{0\mathcal{H}}}|_{\Gamma}$ , of the controls  $\xi\gamma_{v_{0\mathcal{H}}}|_{\Gamma}$  and  $\gamma_1$  drive the system from  $v_0$ , at time  $t = a$ , to 0 at time  $t = b$ . Moreover  $v_0 \mapsto \zeta(v_0)$  is linear and continuous,  $|\zeta(v_0)|_{G_{\text{av},c}^2((a,b),\Gamma)} \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)}|_{\text{st}}]}}|v_0|_{\mathcal{A}_{K_2}}$   $\square$

*Remark 3.1.* Notice that in the previous proof if  $v_0 \in V$ , then the first control  $\xi\gamma_{v_{0\mathcal{H}}}|_{\Gamma}$  vanishes and, by standard arguments taking into account the smoothing property of the system (cf. [BRS11, eq. (2.6)]), we can conclude that  $|v(a+l)|_V \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,a+l)}|_{\text{st}}]}}|v_0|_H$ . Thus we obtain that the concatenation satisfies  $|\zeta(v_0)|_{G_{\text{av},c}^2((a,b),\Gamma)} \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)}|_{\text{st}}]}}|v_0|_H$ .

Now, from (17), we can also obtain that  $|(q(a), v_0)_{L^2(\Omega, \mathbb{R}^3)}|_{\mathbb{R}}$  is bounded by

$$C_1 \left( |f|_{L^2((a,b),H)} + |\mathcal{I}|_{\Gamma_c} (p_q \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q) \Big|_{G^2((a,b),\Gamma)'} \right) \times \left( |v|_{L^2((a,b),L^2(\Omega,\mathbb{R}^3))} + |\zeta(v_0)|_{G_{\text{av},c}^2((a,b),\Gamma)} \right)$$

and, using (18), (14c), and Remark 3.1, we derive that for all  $v_0 \in V$  we have

$$|(q(a), v_0)_H|_{\mathbb{R}} \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)}|_{\text{st}}]}} \left( |f|_{L^2((a,b),H)}^2 + |\mathcal{I}|_{\Gamma_c} (p_q \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q) \Big|_{G^2((a,b),\Gamma)'}^2 \right) |v_0|_H.$$

Then from the density of  $V$  in  $H$ , it follows that

$$(20) \quad |q(a)|_H^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)}|_{\text{st}}]}} \left( |f|_{L^2((a,b),H)}^2 + |\mathcal{I}|_{\Gamma_c} (p_q \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q) \Big|_{G^2((a,b),\Gamma)'}^2 \right)$$

for the solution  $q$  of system (11), with  $\hat{u} \in \mathcal{W}^{(a,b)|_{\text{st}}}$ , and “the” corresponding pressure function  $p = p_q$ .

*Remark 3.2.* In (19) and (20), we suppose we have fixed a well defined choice of  $p = p_q$ , that is known to be unique up to an additive constant. The constants  $\overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)}|_{\text{wk}}]}}$  and  $\overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)}|_{\text{st}}]}}$  depend also on  $\Gamma_c$ , on the length of  $(a, b)$ , and on the choice of  $p$ . However notice that (17) holds independently of the choice of  $p_q$ ; indeed if we replace  $p_q$  by  $\tilde{p} = p_q + c$ , with  $c \in \mathbb{R}$ , then from  $\text{div } v = 0$  we derive that  $(\tilde{p}\mathbf{n}, v)_{L^2(\Gamma, \mathbb{R}^3)} = (\nabla \tilde{p}, v)_{L^2(\Omega, \mathbb{R}^3)} = (\nabla p_q, v)_{L^2(\Omega, \mathbb{R}^3)} = (p_q \mathbf{n}, v)_{L^2(\Gamma, \mathbb{R}^3)}$ .

**3.2. Choice of the pressure function.** Often the pressure function  $p$  is chosen to have zero average in  $\Omega$  but, in the study of specific problems, as we will see later in section 5, it may be convenient to set another choice.

**Definition 3.1.** We say that the linear mapping  $p \mapsto c_\sigma p := p - \frac{\sigma p}{\sigma 1_\Omega}$  is an appropriate choice of the pressure function if  $\sigma : H^1(\Omega, \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous linear function, with  $\sigma 1_\Omega \neq 0$ . Here  $1_\Omega$  stands for the function  $1_\Omega(x) := 1$ , for all  $x \in \Omega$ .

Given an appropriate choice  $c_\sigma$ , the pressure function  $p$  in (11) may be supposed, or chosen, to satisfy  $\sigma p = 0$ .

*Remark 3.3.* Notice that if  $\sigma$  defines an appropriate choice, then  $p - c_\sigma p = \frac{\sigma p}{\sigma 1_\Omega}$  is a constant,  $c_\sigma 1_\Omega = 0$ ,  $\sigma c_\sigma p = 0$ , and  $c_\sigma c_\sigma p = c_\sigma p$ . Moreover  $|\sigma \cdot|_{\mathbb{R}}$  is a seminorm in  $H^1(\Omega, \mathbb{R})$  and, since  $\sigma 1_\Omega \neq 0$ , the norms  $|\cdot|_{H^1(\Omega, \mathbb{R})}$  and  $|\nabla \cdot|_{L^2(\Omega, \mathbb{R}^3)} + |\sigma \cdot|_{\mathbb{R}}$  are equivalent

in  $H^1(\Omega, \mathbb{R})$ , see e.g. [Tem97, section II.1.4]. With the above terminology, the “usual” choice of zero-averaged  $p$  in  $\Omega$  corresponds to  $\sigma = \sigma_\Omega$  with  $\sigma_\Omega p := \int_\Omega p \, d\Omega$ .

For strong solutions, we know that the choice  $p = c_{\sigma_\Omega} p$  is in  $L^2((a, b), H^1(\Omega, \mathbb{R}))$ . Now, if  $c_{\sigma_1}$  and  $c_{\sigma_2}$  are two appropriate choices, then we have that

$$\begin{aligned} c_{\sigma_1} c_{\sigma_2} p &= c_{\sigma_2} p - \sigma_1 c_{\sigma_2} p / \sigma_1 1_\Omega = p - \sigma_2 p / \sigma_2 1_\Omega - \sigma_1 \left( p - \frac{\sigma_2 p}{\sigma_2 1_\Omega} \right) / \sigma_1 1_\Omega \\ &= c_{\sigma_1} p - \sigma_2 p / \sigma_2 1_\Omega + \sigma_1 \frac{\sigma_2 p}{\sigma_2 1_\Omega} / \sigma_1 1_\Omega = c_{\sigma_1} p. \end{aligned}$$

That is,  $c_{\sigma_1} c_{\sigma_2} p$  coincides with the appropriate choice  $c_{\sigma_1} p$ , which means that we may choose  $p$  having zero average on  $\Gamma_c$ , which corresponds to  $\sigma = \sigma_c$ , with  $\sigma_c p := \int_{\Gamma_c} p \, d\Gamma_c = \sigma_c \sigma_\Omega p$ .

*Remark 3.4.* Here we will consider only solutions of system (11) with data (16); since these solutions are strong we can guarantee (choosing, e.g.,  $p = c_{\sigma_\Omega} p$ ) that the corresponding pressure function  $p$  is in  $H^1(\Omega, \mathbb{R})$ ; this is why we have defined “appropriate choice” ( $p = c_\sigma p = c_\sigma c_{\sigma_\Omega} p$ ) for this regularity. Of course for weak regularity, i.e., to the case  $p \in L^2(\Omega, \mathbb{R})$ , and not necessarily in  $H^1(\Omega, \mathbb{R})$ , we should consider continuous linear functions  $\sigma : L^2(\Omega, \mathbb{R}) \rightarrow \mathbb{R}$ , with  $\sigma 1_\Omega \neq 0$ .

**3.3. Smoother observability inequalities.** We see that the boundary term in inequalities (19) and (20) vanishes if the “observed” trace  $p\mathbf{n} + \nu \langle \mathbf{n} \cdot \nabla \rangle q$  vanishes in  $\Gamma_c$ ; in this sense we may understand those inequalities as inequalities localized on  $\Gamma_c$ . However, the indicator operator  $\mathcal{I}|_{\Gamma_c}$  would, roughly speaking, suit the case in which we take controls like  $\zeta = \mathcal{I}|_{\Gamma_c} \eta$  in system (10), and it would destroy all regularity of  $\eta$  we may be interested to (or need to) preserve for  $\zeta$  across the boundary of  $\Gamma_c$  (cf. section 2.4.1, where the operator  $\eta \mapsto K^\mathcal{O} \eta$  returns us a control with enough regularity to guarantee the existence of a weak solution for (10)).

Here we present a class of observability inequalities localized on open subsets of  $(a, b) \times \Gamma$ . In particular we will see that  $\mathcal{I}|_{\Gamma_c}$  can be replaced by a general smoother operator. We start by some straightforward corollaries of Lemmas 3.1 and 3.2:

**Corollary 3.3.** *Given  $\hat{u} \in \mathcal{W}^{(a,b)|\text{wk}}$ ,  $v_0 \in H$ , and  $\emptyset \neq (c, d) \subseteq (a, b)$ , there exists a control  $\zeta = \zeta(v_0) \in G_{\text{av},c}^1((a, b), \Gamma)$  such that, for the corresponding solution  $v$  to system (10) with  $g = 0$ , we have  $v(b) = 0$ . Moreover the support of the control is contained in  $[c, d] \times \overline{\Gamma_c}$ , and the mapping  $v_0 \mapsto \zeta(v_0)$  is linear and continuous:  $|\zeta(v_0)|_{G_{\text{av},c}^1((a,b),\Gamma)} \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)|\text{wk}}}] |v_0|_H$ .*

*Proof.* If  $a < c$  we apply zero boundary control for time  $t \in (a, c)$ . Then we apply the control given in Lemma 3.1 (with  $(c, d)$  in the role of  $(a, b)$ ) driving the system to 0 at time  $t = d$ . Finally, if  $d < b$  we apply zero control for time  $t \in (d, b)$ . Now using Theorem 2.3, it is straightforward to check that the proposed concatenated control satisfy the required properties.  $\square$

**Corollary 3.4.** *Given  $\hat{u} \in \mathcal{W}^{(a,b)|\text{st}}$ ,  $v_0 \in V$ , and  $\emptyset \neq (c, d) \subseteq (a, b)$ , there exists a control  $\zeta = \zeta(v_0) \in G_{\text{av},c}^2((a, b), \Gamma)$  such that, for the corresponding solution  $v$  to system (10) with  $g = 0$ , we have  $v(b) = 0$ . Moreover the support of the control is contained in  $[c, d] \times \overline{\Gamma_c}$ , and the mapping  $v_0 \mapsto \zeta(v_0)$  is linear and continuous:  $|\zeta(v_0)|_{G_{\text{av},c}^2((a,b),\Gamma)} \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)|\text{st}}}] |v_0|_V$ .*

*Proof.* The proof is similar to that of Corollary 3.3; we have just to take the control given in Lemma 3.2 in the interval  $(c, d)$ , and use Theorem 2.4 instead.  $\square$

Now, proceeding as in section 3.1, using (17) and the controls given by Corollaries 3.3 and 3.4, we can arrive to the following observability inequalities for the solution  $q$  of

system (11) and the corresponding pressure function  $p = p_q$ : if  $\hat{u} \in \mathcal{W}^{(a,b)|\text{wk}}$ ,

$$(21) \quad |q(a)|_H^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)|\text{wk}}}] \left( |f|_{L^2((a,b),H)}^2 + |\psi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1((a,b),\Gamma)'}^2 \right)$$

and, if  $\hat{u} \in \mathcal{W}^{(a,b)|\text{st}}$ ,

$$(22) \quad |q(a)|_H^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)|\text{st}}}] \left( |f|_{L^2((a,b),H)}^2 + |\psi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^2((a,b),\Gamma)'}^2 \right),$$

where  $\psi \in L^\infty((a,b), L^\infty(\Gamma, \mathbb{R}))$  is any function taking the value 1 in  $(c,d) \times \Gamma_c$  (recall that the support of the control is contained in  $[c,d] \times \overline{\Gamma_c}$ ).

Next we relax a little the observability inequalities (21) and (22). We will need the following auxiliary result, which proof is given in the appendix, section A.4.

**Proposition 3.5.** *Given  $u \in G^i((a,b), \Gamma)$  and  $\varphi \in C^1([a,b], C^2(\Gamma, \mathbb{R}))$ , then  $\varphi u \in G^i((a,b), \Gamma)$  and  $|\varphi u|_{G^i((a,b),\Gamma)} \leq C|\varphi|_{C^1([a,b], C^2(\Gamma, \mathbb{R}))}|u|_{G^i((a,b),\Gamma)}$ , for  $i \in \{1, 2\}$ .*

Now, let  $\phi$  be a function satisfying

$$(23) \quad \begin{aligned} &\phi \in L^\infty((a,b), L^\infty(\Gamma, \mathbb{R})), \text{ and for some } (t_0, x_0) \in [a,b] \times \Gamma \\ &\phi(t_0, x_0) \neq 0 \text{ and } \phi \in C^1([t_0 - \delta, t_0 + \delta] \cap [a,b], C^2(\overline{\mathcal{N}_{x_0}}, \mathbb{R})), \end{aligned}$$

for some  $\delta > 0$  and some neighborhood  $\mathcal{N}_{x_0} \subseteq \Gamma$  of  $x_0$ .

**Theorem 3.6.** *Let  $\phi$  satisfy (23), and let  $(q, p)$  solve system (11), for a fixed appropriate choice of the pressure function  $p$ . Then, if  $\hat{u} \in \mathcal{W}^{(a,b)|\text{wk}}$*

$$(24) \quad |q(a)|_H^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)|\text{wk}}}] \left( |f|_{L^2((a,b),H)}^2 + |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1((a,b),\Gamma)'}^2 \right)$$

and, if  $\hat{u} \in \mathcal{W}^{(a,b)|\text{st}}$

$$(25) \quad |q(a)|_H^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)|\text{st}}}] \left( |f|_{L^2((a,b),H)}^2 + |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^2((a,b),\Gamma)'}^2 \right),$$

where now the constants  $\overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)|\text{wk}}}]}$  and  $\overline{C}_{[\hat{u}|_{\mathcal{W}^{(a,b)|\text{st}}}]}$  depend also on  $\phi$ .

*Proof.* We prove (24); the proof of (25) is completely analogous. First of all, for any  $h \in G^1((a,b), \Gamma)'$  and  $\varphi \in C^1([a,b], C^2(\Gamma, \mathbb{R}))$ , from the definitions

$$|\varphi h|_{G^1((a,b),\Gamma)'} := \sup_{\substack{v \in G^1((a,b),\Gamma) \\ |v|_{G^1((a,b),\Gamma)} = 1}} \langle \varphi h, v \rangle_{G^1((a,b),\Gamma)'}_{G^1((a,b),\Gamma)},$$

$$\langle \varphi h, v \rangle_{G^1((a,b),\Gamma)'}_{G^1((a,b),\Gamma)} := \langle h, \varphi v \rangle_{G^1((a,b),\Gamma)'}_{G^1((a,b),\Gamma)},$$

and from Proposition 3.5, we obtain

$$(26) \quad |\varphi h|_{G^1((a,b),\Gamma)'} \leq C|\varphi|_{C^1([a,b], C^2(\Gamma, \mathbb{R}))}|h|_{G^1((a,b),\Gamma)'}$$

Next, since  $\phi(t_0, x_0) \neq 0$ , and  $\phi$  is regular enough in  $\mathcal{N}^\times := [t_0 - \delta, t_0 + \delta] \cap [a,b] \times \overline{\mathcal{N}_{x_0}}$ , we can set two open subsets  $(c,d) \times \mathcal{O}_\phi$  and  $(c^1, d^1) \times \mathcal{O}_\phi^1$  such that

$$\begin{cases} \overline{\mathcal{O}_\phi} = \text{supp } \tilde{\chi}_\phi \text{ and } \overline{\mathcal{O}_\phi^1} = \text{supp } \tilde{\chi}_\phi^1, \text{ for smooth functions } \tilde{\chi}_\phi \text{ and } \tilde{\chi}_\phi^1; \\ (c,d) \times \mathcal{O}_\phi \subset [c,d] \times \overline{\mathcal{O}_\phi} \subset (c^1, d^1) \times \mathcal{O}_\phi^1 \subset [c^1, d^1] \times \overline{\mathcal{O}_\phi^1} \subset \mathcal{N}^\times; \\ \left| \phi([c^1, d^1] \times \overline{\mathcal{O}_\phi^1}) \Big|_{\mathbb{R}} \subseteq [\varepsilon, +\infty), \text{ with } \varepsilon > 0. \end{cases}$$

Now, let  $\gamma \in C^\infty([a,b] \times \Gamma, \mathbb{R})$  be a smooth function such that  $\gamma = 1$  in  $[c,d] \times \overline{\mathcal{O}_\phi}$  and  $\gamma = 0$  in  $[a,b] \times \Gamma \setminus (c^1, d^1) \times \mathcal{O}_\phi^1$ . Thus

$$\phi^{-1}\gamma(t, x) := \begin{cases} \gamma(t, x)/\phi(t, x) & \text{if } \phi(t, x) \neq 0 \\ 0 & \text{if } \phi(t, x) = 0 \end{cases}$$

is a differentiable mapping,  $\phi^{-1}\gamma \in C^1([a, b], C^2(\Gamma, \mathbb{R}))$ . Consider also the subspace  $G_\phi^1((a, b), \Gamma) := \{v \in G^1((a, b), \Gamma) \mid v(t) = 0 \text{ in } \Gamma \setminus \overline{\mathcal{O}_\phi} \text{ for a.e. } t \in (a, b)\}$ .

From (21) (with  $\mathcal{O}_\phi$  in the role of  $\Gamma_c$ ), we obtain

$$|q(a)|_H^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}(a,b)}]_{\text{wk}}} \left( |f|_{L^2((a,b),H)}^2 + |\gamma(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1((a,b),\Gamma)'}^2 \right),$$

and since  $\gamma = \phi^{-1}\gamma\phi$ , from (26) it follows that

$$|\gamma(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1((a,b),\Gamma)'}^2 \leq C_1 |\phi^{-1}\gamma|_{C^1([a,b],C^2(\Gamma,\mathbb{R}))} |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1((a,b),\Gamma)'}^2$$

and

$$|q(a)|_H^2 \leq \overline{C}_{[\hat{u}|_{\mathcal{W}(a,b)}]_{\text{wk}}}(\phi) \left( |f|_{L^2((a,b),H)}^2 + |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1((a,b),\Gamma)'}^2 \right),$$

that is, (24) holds.  $\square$

*Remark 3.5.* We notice that (24) is an observability inequality localized on  $\text{supp } \phi$ . In many applications, taking  $\phi \in C^1((a, b), C^2(\Gamma, \mathbb{R}))$  instead of (23) should be sufficient and sometimes necessary (see the discussion in the beginning of this section 3.3). We take (23) in (24) because it does not bring any real additional difficulties to the proof.

#### 4. TRUNCATED OBSERVABILITY INEQUALITIES

In the case of finite-dimensional controls, we need suitably truncated observability inequalities, i.e., we need to focus the observation on a suitable finite-dimensional space, closely related to the control space. Inspired by the work in [BRS11] for the case of internal controls, we show below that under the constraints that  $f = 0$  and  $q(b)$  is finite-dimensional, the ‘‘observed space’’ can be truncated, and we still have a boundary observability inequality.

**4.1. Auxiliary results.** We start by recalling the following results:

**Lemma 4.1.** *Let  $X$  and  $Y$  be two Banach spaces, and let  $L : X \rightarrow Y$  be a linear continuous mapping. If  $(x^n)_{n \in \mathbb{N}}$  is a sequence in  $X$  such that  $x^n \rightharpoonup x$  in  $X$ , then  $Lx^n \rightharpoonup Lx$  in  $Y$ .*

*Proof.* Given  $f \in Y'$ , the composition  $f \circ L$  is in  $X'$ , which implies  $\langle f, Lx^n \rangle_{Y',Y} =: \langle f \circ L, x^n \rangle_{X',X} \rightarrow \langle f \circ L, x \rangle_{X',X} := \langle f, Lx \rangle_{Y',Y}$ .  $\square$

**Lemma 4.2.** *Let  $Z$  be a Banach space such that  $Z = X \oplus Y$ , where  $X$  and  $Y$  are closed subspaces of  $Z$ . Then we can rewrite, in a unique way, each  $z \in Z$  as  $z = z_X + z_Y$  with  $(z_X, z_Y) \in X \times Y$ , and the projections  $z \mapsto z_X$ , and  $z \mapsto z_Y$  are continuous. Moreover the norms  $|\cdot|_Z$  and  $|\cdot|_{X \oplus Y}$  are equivalent in  $Z$ .*

*Proof.* Consider the graph  $G_X = \{(z, w) \in Z \times Z \mid w = z_X\}$  of the projection onto  $X$  and let  $(z^n, w^n)$  be a sequence on  $G_X$  converging to  $(z, w) \in Z \times Z$ . Then we have that  $z^n \rightarrow z$  and  $z_X^n = w^n \rightarrow w$  in  $Z$ . Since  $X$  is closed we have that  $w \in X$ , which implies that  $w_X = w$ . Since  $Y$  is closed we also have that  $z_Y^n = z^n - z_X^n \rightarrow z - w \in Y$ , and then  $0 = (z - w)_X = z_X - w$ , i.e.,  $(z, w) = (z, z_X) \in G_X$ . Hence  $G_X$  is closed, and by the Closed Graph Theorem, see e.g. [Con85, section III.12], it follows the continuity of the projection  $z \mapsto z_X$ . It follows that also the projection  $z \mapsto z_Y = z - z_X$  is continuous. Finally we find  $|z|_Z^2 \leq (|z_X|_Z + |z_Y|_Z)^2 \leq 2(|z_X|_Z^2 + |z_Y|_Z^2) = 2|z|_{X \oplus Y}^2 \leq C|z|_Z^2$ , which shows the equivalence of the norms.  $\square$

We will need also the following result, which proof is given in the appendix, section A.6.

**Proposition 4.3.** *Let  $X$  be a Hilbert space, and  $Y$  be a subspace of  $X$ . Denote by  $Y^\perp$  the orthogonal space to  $Y$ . If we denote  $\widehat{L}^2((a, b), Y) := \{u \in L^2((a, b), X) \mid u(t) \in Y, \text{ for a.e. } t \in (a, b)\}$ , then  $\widehat{L}^2((a, b), Y)^\perp = L^2((a, b), Y^\perp)$ , where  $\widehat{L}^2((a, b), Y)^\perp$  stands for the orthogonal subspace to  $\widehat{L}^2((a, b), Y)$  in  $L^2((a, b), X)$ .*

*Remark 4.1.* The Bochner integral is, usually, defined for functions  $f : I \rightarrow B$  where  $B$  is a Banach space, see e.g. [AB94, section 9.8], or [DS67, chapter III, section 3]. In Proposition 4.3, we do not ask  $Y$  to be closed, and that is the reason we avoid the (inadequate) notation  $L^2((a, b), Y)$ .

Recalling the space  $H_N \subset H$ , defined in (7), spanned by the first  $N$  eigenfunctions of the Stokes operator, we have the following:

**Lemma 4.4.** *Let  $\phi \in C^1((a, b), C^2(\Gamma, \mathbb{R}))$  be non-identically zero, and let  $(q, p)$  solve system (11), with  $f = 0$  and  $q_1 \in H_N$ , for a fixed appropriate choice  $c_\sigma$  for the pressure function  $p$ , i.e.,  $\sigma p = 0$ . Then, if  $\hat{u} \in \mathcal{W}^{(a, b)|\text{wk}}$*

$$(27) \quad |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{L^2((a, b), H^{1/2}(\Gamma, \mathbb{R}^3))}^2 \leq \overline{C}_{[N, |\hat{u}|_{\mathcal{W}^{(a, b)|\text{wk}}}] } |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1((a, b), \Gamma)'}^2;$$

and, if  $\hat{u} \in \mathcal{W}^{(a, b)|\text{st}}$

$$(28) \quad |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{L^2((a, b), H^{1/2}(\Gamma, \mathbb{R}^3))}^2 \leq \overline{C}_{[N, |\hat{u}|_{\mathcal{W}^{(a, b)|\text{st}}}] } |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^2((a, b), \Gamma)'}^2;$$

where the constants  $\overline{C}_{[N, |\hat{u}|_{\mathcal{W}^{(a, b)|\text{wk}}}]}$  and  $\overline{C}_{[N, |\hat{u}|_{\mathcal{W}^{(a, b)|\text{st}}}]}$  depends only on  $N$ ,  $\Omega$ ,  $\phi$ , on the length of  $(a, b)$ , and on the respective norm of  $\hat{u}$ .

*Proof.* We prove (27). The proof of (28) is completely analogous. We argue by contradiction. Suppose that there exists a sequence of pairs  $((q_1^n, \hat{u}^n))_{n \in \mathbb{N}}$  in  $H_N \times \mathcal{W}^{(a, b)|\text{wk}}$  with  $(|\hat{u}^n|_{\mathcal{W}^{(a, b)|\text{wk}}})_{n \in \mathbb{N}}$  bounded, such that the solution  $(q^n, p^n)$  of the system

$$(29) \quad \begin{aligned} -\partial_t q^n + \mathcal{B}^*(\hat{u}^n)q^n - \nu\Delta q^n + \nabla p^n &= 0, & \operatorname{div} q^n &= 0, \\ q^n|_\Gamma &= 0, & q^n(b) &= q_1^n \in H_N \end{aligned}$$

satisfies the inequality

$$(30) \quad |\phi(p^n\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q^n)|_{L^2((a, b), H^{1/2}(\Gamma, \mathbb{R}^3))}^2 > n|\phi(p^n\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q^n)|_{G^1((a, b), \Gamma)'}^2,$$

where the pressure functions  $p^n$  are supposed to agree with the fixed choice, i.e.,  $\sigma p^n = 0$ , for all  $n \in \mathbb{N}$ . Notice that  $q_1^n = 0$  implies that  $q^n = 0$  and that  $p^n$  is a constant function, and from  $\sigma p^n = 0$ , we obtain that  $p^n = 0$ ; in this case (30) is not satisfied, i.e., necessarily  $q_1^n \neq 0$ , for all  $n \in \mathbb{N}$ . On the other hand, since the mapping sending  $q(b)$  to the corresponding solution  $(q, p)$  is linear, there is no loss of generality in assuming that  $|q_1^n|_V = 1$ . The boundedness of  $(|\hat{u}^n|_{\mathcal{W}^{(a, b)|\text{wk}}})_{n \in \mathbb{N}}$  implies that  $(\hat{u}^n)_{n \in \mathbb{N}}$  and  $(\partial_t \hat{u}^n)_{n \in \mathbb{N}}$  are bounded sequences in  $L^\infty((a, b), L^\infty_{\operatorname{div}}(\Omega, \mathbb{R}^3))$  and  $L^2((a, b), L^\sigma(\Omega, \mathbb{R}^3))$ , respectively. It follows from standard arguments that the sequences  $(q^n)_{n \in \mathbb{N}}$  and  $(\partial_t q^n)_{n \in \mathbb{N}}$  are bounded in  $L^2((a, b), D(L))$  and  $L^2((a, b), H)$ , respectively. Since, by the Kakutani's Theorem, see e.g. [Con85, chapter V, Theorem 4.2], the unit ball in a reflexive Banach space is weakly compact and, by the Alaoglu's Theorem, see e.g. [Con85, chapter V, Theorem 3.1], the unit ball in  $L^\infty((a, b), L^\infty_{\operatorname{div}}(\Omega, \mathbb{R}^3))$  is compact in the weak-\* topology, there exists a subsequence of  $(q_1^n, q^n, \hat{u}^n)$  (for which we preserve the same notation), a  $V$ -unit vector  $q_1^\infty \in H_N$ ,  $q^\infty \in W((a, b), D(L), H)$ , and  $\hat{u}^\infty \in \mathcal{W}^{(a, b)|\text{wk}}$  such that

$$\begin{aligned} q_1^n &\rightharpoonup q_1^\infty && \text{in } H_N; \\ q^n &\rightharpoonup q^\infty && \text{in } L^2((a, b), D(L)); \\ \partial_t q^n &\rightharpoonup \partial_t q^\infty && \text{in } L^2((a, b), H); \\ \hat{u}^n &\rightharpoonup_* \hat{u}^\infty && \text{in } L^\infty((a, b), L^\infty_{\operatorname{div}}(\Omega, \mathbb{R}^3)); \\ \partial_t \hat{u}^n &\rightharpoonup \partial_t \hat{u}^\infty && \text{in } L^2((a, b), L^\sigma(\Omega, \mathbb{R}^3)). \end{aligned}$$

Since  $W((a, b), H^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3)) \subset L^2((a, b), H^1(\Omega, \mathbb{R}^3))$  is a compact inclusion, see e.g. [Tem01, chapter 3, Theorem 2.1], we can suppose (taking a subsequence) that

$$q^n \rightarrow q^\infty \text{ in } L^2((a, b), V).$$

Next, from  $|\nabla p^n|_{L^2(\Omega, \mathbb{R}^3)}^2 = (\nabla p^n, \partial_t q^n - \mathcal{B}^*(\hat{u}^n)q^n + \nu \Delta q^n)_{L^2(\Omega, \mathbb{R}^3)} = (\nabla p^n, -\mathcal{B}^*(\hat{u}^n)q^n + \nu \Delta q^n)_{L^2(\Omega, \mathbb{R}^3)}$ , we derive  $|\nabla p^n|_{L^2(\Omega, \mathbb{R}^3)}^2 \leq \bar{C} \left[ |\hat{u}^n|_{L^\infty((a, b), L^\infty_{\text{div}}(\Omega, \mathbb{R}^3))} \right] |\nabla p^n|_{L^2(\Omega, \mathbb{R}^3)} |q^n|_{H^2_{\text{div}}(\Omega, \mathbb{R}^3)} \leq 1/2 \bar{C}^2 \left[ |\hat{u}^n|_{L^\infty((a, b), L^\infty_{\text{div}}(\Omega, \mathbb{R}^3))} \right] |q^n|_{H^2_{\text{div}}(\Omega, \mathbb{R}^3)}^2 + 1/2 |\nabla p^n|_{L^2(\Omega, \mathbb{R}^3)}^2$ , that is,

$$|\nabla p^n|_{L^2(\Omega, \mathbb{R}^3)}^2 \leq \bar{C}^2 \left[ |\hat{u}^n|_{L^\infty((a, b), L^\infty_{\text{div}}(\Omega, \mathbb{R}^3))} \right] |q^n|_{D(L)}^2;$$

we conclude that  $(\nabla p^n)_{n \in \mathbb{N}}$  is bounded in the space  $L^2((a, b), L^2(\Omega, \mathbb{R}^3))$ . Furthermore since  $\sigma p^n = 0$  we obtain that  $(p^n)_{n \in \mathbb{N}}$  is bounded in the space  $L^2((a, b), H^1(\Omega, \mathbb{R}))$ , see Remark 3.3. Thus we can suppose, using Lemmas 4.1, 4.2 and Proposition 4.3, that

$$(31) \quad \begin{array}{l} p^n \rightharpoonup p^\infty \quad \text{in } L^2((a, b), L^2(\Omega, \mathbb{R})) \\ \nabla p^n \rightharpoonup \nabla p^\infty \quad \text{in } L^2((a, b), L^2(\Omega, \mathbb{R}^3)) \end{array}, \quad p^\infty = c_\sigma p^\infty \in L^2((a, b), H^1(\Omega, \mathbb{R})).$$

Next step is to pass to the limit in (29), for that we will need some preliminary computations. Let us rewrite

$$\mathcal{B}^*(\hat{u}^n)q^n - \mathcal{B}^*(\hat{u}^\infty)q^\infty = \mathcal{B}^*(\hat{u}^n)(q^n - q^\infty) + \mathcal{B}^*(\hat{u}^n - \hat{u}^\infty)q^\infty$$

and let us be given  $v \in L^2((a, b), L^2(\Omega, \mathbb{R}^3))$ ; we find

$$\begin{aligned} & |(\mathcal{B}^*(\hat{u}^n)(q^n - q^\infty), v)_{L^2((a, b), L^2(\Omega, \mathbb{R}^3))}|_{\mathbb{R}} \\ & \leq C |\hat{u}^n|_{L^\infty((a, b), L^\infty_{\text{div}}(\Omega, \mathbb{R}^3))} |q^n - q^\infty|_{L^2((a, b), V)} |v|_{L^2((a, b), L^2(\Omega, \mathbb{R}^3))} \rightarrow 0; \end{aligned}$$

and, from Remark 2.5, we also obtain

$$\begin{aligned} & (\mathcal{B}^*(\hat{u}^n - \hat{u}^\infty)q^\infty, v)_{L^2((a, b), L^2(\Omega, \mathbb{R}^3))} = \int_a^b \left( \int_\Omega \langle (\hat{u}^n - \hat{u}^\infty) \cdot D_s \rangle q^\infty \cdot v \, d\Omega \right) dt \\ & = \int_a^b \left( \int_\Omega \langle v \cdot D_s \rangle q^\infty \cdot (\hat{u}^n - \hat{u}^\infty) \, d\Omega \right) dt \\ & = \langle \hat{u}^n - \hat{u}^\infty, \langle v \cdot D_s \rangle q^\infty \rangle_{(L^1((a, b), L^1(\Omega, \mathbb{R}^3)))', L^1((a, b), L^1(\Omega, \mathbb{R}^3))} \rightarrow 0, \end{aligned}$$

and we can conclude that

$$(32) \quad \mathcal{B}^*(\hat{u}^n)q^n \rightharpoonup \mathcal{B}^*(\hat{u}^\infty)q^\infty \text{ in } L^2((a, b), L^2(\Omega, \mathbb{R}^3)).$$

From Lemma 4.1 it also follows that  $\Delta q^n \rightharpoonup \Delta q^\infty$  in  $L^2((a, b), L^2(\Omega, \mathbb{R}^3))$ , and then we can pass to the weak limit in (29) and obtain

$$(33) \quad \begin{array}{l} -\partial_t q^\infty + \mathcal{B}^*(\hat{u}^\infty)q^\infty - \nu \Delta q^\infty + \nabla p^\infty = 0, \quad \text{div } q^\infty = 0, \\ q^\infty|_\Gamma = 0, \quad q^\infty(b) = q_1^\infty \in H_N. \end{array}$$

Accordingly to Propositions 2.2 and 3.5, given  $v \in G^1((a, b), \Gamma)$ , we can extend  $\phi v \in G^1((a, b), \Gamma)$  to  $E_1^e \phi v \in W((a, b), H^1_{\text{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \oplus H^{3/4}((a, b), \mathbb{R})\Theta$ , with  $\phi v = E_1^e \phi v|_\Gamma$ ; we obtain

$$\begin{aligned} & \langle \phi(p^n \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^n), v \rangle_{G^1((a, b), \Gamma)', G^1((a, b), \Gamma)} \\ & = \langle p^n \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^n, \phi v \rangle_{G^1((a, b), \Gamma)', G^1((a, b), \Gamma)} \\ & = (\nabla p^n - \nu \Delta q^n, E_1^e \phi v)_{L^2((a, b), L^2(\Omega, \mathbb{R}^3))} + (p^n, \text{div}(E_1^e \phi v))_{L^2((a, b), L^2(\Omega, \mathbb{R}^3))} \\ & \quad + (-\nu \nabla q^n, \nabla(E_1^e \phi v))_{L^2((a, b), L^2(\Omega, \mathbb{R}^3))}. \end{aligned}$$

and, taking the limit we obtain

$$\begin{aligned} & \langle \phi(p^n \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^n), v \rangle_{G^1((a,b), \Gamma)', G^1((a,b), \Gamma)} \\ & \rightarrow (\nabla p^\infty - \nu \Delta q^\infty, E_1^e \phi v)_{L^2((a,b), L^2(\Omega, \mathbb{R}^3))} + (p^\infty, \operatorname{div}(E_1^e \phi v))_{L^2((a,b), L^2(\Omega, \mathbb{R}^3))} \\ & \quad + (-\nu \nabla q^\infty, \nabla(E_1^e \phi v))_{L^2((a,b), L^2(\Omega, \mathbb{R}^9))} \\ & = \langle \phi(p^\infty \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^\infty), v \rangle_{G^1((a,b), \Gamma)', G^1((a,b), \Gamma)} \end{aligned}$$

i.e.,  $\phi(p^n \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^n) \rightharpoonup_* \phi(p^\infty \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^\infty)$  in  $G^1((a, b), \Gamma)'$ . In particular we have  $|\phi(p^\infty \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^\infty)|_{G^1((a,b), \Gamma)'} \leq \liminf_{n \rightarrow +\infty} |\phi(p^n \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^n)|_{G^1((a,b), \Gamma)'}$ . Therefore, from (30), we have  $|\phi(p^\infty \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^\infty)|_{G^1((a,b), \Gamma)'}^2 \leq \liminf_{n \rightarrow +\infty} 1/n |\phi(p^n \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^n)|_{L^2((a,b), H^{1/2}(\Omega, \mathbb{R}^3))}$ , i.e.,

$$|\phi(p^\infty \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^\infty)|_{G^1((a,b), \Gamma)'}^2 = 0,$$

because  $\phi(p^n \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^n)$  is bounded in  $L^2((a, b), H^{1/2}(\Gamma, \mathbb{R}^3))$ .

Applying now the observability inequality (24) to system (33) considered on the interval  $(a + r, b)$  with  $0 \leq r < b - a$ , we conclude that  $q^\infty(t) = 0$  for  $a \leq t < b$ . Since  $q^\infty \in C([a, b], V)$ , we obtain  $q_1^\infty = q^\infty(b) = 0$ . This contradicts the fact that  $q_1^\infty \in H_N$  is a  $V$ -unit vector. The contradiction proves that (27) holds.  $\square$

**4.2. Truncation in space variable.** For a given open connected smooth submanifold  $\mathcal{O} \subseteq \Gamma$  of the boundary  $\Gamma$ , recall the space  $L_M^2(\mathcal{O}, \mathbb{R}^3) := \operatorname{span}\{\pi_i \mathbf{n}, \tau_i \mid i \in \mathbb{N}_0, i \leq M\}$  defined in section 2.3, and the orthogonal projection  $P_M^\mathcal{O} : L^2(\mathcal{O}, \mathbb{R}^3) \rightarrow L_M^2(\mathcal{O}, \mathbb{R}^3)$ .

**Theorem 4.5.** *Let  $N \in \mathbb{N}_0$  and let  $(q, p)$  solve system (11) with  $q_1 \in H_N$  and  $f = 0$ . Fix also an appropriate choice for pressure function  $p$ . Let us be given also two differentiable functions  $\phi$  and  $\tilde{\phi}$  in  $C^1([a, b], C^2(\Gamma, \mathbb{R}))$  with nonempty support, an open connected submanifold  $\mathcal{O} \subseteq \Gamma$ , and  $\varepsilon > 0$  such that  $\operatorname{supp} \phi \subseteq [a, b] \times \overline{\mathcal{O}}$  and  $|\tilde{\phi}(t, x)|_{\mathbb{R}} \geq \varepsilon$  for all  $(t, x) \in \operatorname{supp} \phi$ . Then, if  $\hat{u} \in \mathcal{W}^{(a,b)|\operatorname{wk}}$  there exists a positive integer  $M = \overline{C}_{[N, |\hat{u}|_{\mathcal{W}^{(a,b)|\operatorname{wk}}}]}$  such that*

$$(34) \quad |q(a)|_H^2 \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}^{(a,b)|\operatorname{wk}}}] |\tilde{\phi} \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a,b), \Gamma}^2}$$

and, if  $\hat{u} \in \mathcal{W}^{(a,b)|\operatorname{st}}$  there exists a positive integer  $M = \overline{C}_{[N, |\hat{u}|_{\mathcal{W}^{(a,b)|\operatorname{st}}}]}$  such that

$$(35) \quad |q(a)|_H^2 \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}^{(a,b)|\operatorname{st}}}] |\tilde{\phi} \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^2(a,b), \Gamma}^2},$$

where the constants  $\overline{C}_{[|\hat{u}|_{\mathcal{W}^{(a,b)|\operatorname{wk}}}]}$  and  $\overline{C}_{[|\hat{u}|_{\mathcal{W}^{(a,b)|\operatorname{st}}}]}$  depend only on  $\Omega, \mathcal{O}, \phi, \tilde{\phi}$ , and on the respective norm of  $\hat{u}$ .

*Proof.* Again we prove (34), the proof of (35) is completely analogous. Consider the Laplace–de Rham operator, defined by:

$$(36) \quad \begin{aligned} \Delta_\mathcal{O} : H^2(\mathcal{O}, \mathbb{R}^3) \cap H_0^1(\mathcal{O}, \mathbb{R}^3) & \rightarrow L^2(\mathcal{O}, \mathbb{R}^3) \\ u = (u \cdot \mathbf{n})\mathbf{n} + u_t & \mapsto (\Delta_\mathcal{O}(u \cdot \mathbf{n}))\mathbf{n} + \Delta_\mathcal{O}u_t \end{aligned}$$

mapping  $H^2(\mathcal{O}, \mathbb{R}^3) \cap H_0^1(\mathcal{O}, \mathbb{R}^3)$  onto  $L^2(\mathcal{O}, \mathbb{R}^3)$ , see section 2.3.

Denote by  $D(\Delta_\mathcal{O}^s) := \{u \in L^2(\mathcal{O}, \mathbb{R}^3) \mid \Delta_\mathcal{O}^s u \in L^2(\mathcal{O}, \mathbb{R}^3)\}$ , the domain of its fractional power  $\Delta_\mathcal{O}^s$ ,  $s \in [0, 1]$ . Notice that, for  $u = \sum_{i \in \mathbb{N}_0} u_{\mathbf{n}}^i \pi_i \mathbf{n} + \sum_{i \in \mathbb{N}_0} u_t^i \tau_i$ , we may write  $\Delta_\mathcal{O}^s u = \sum_{i \in \mathbb{N}_0} u_{\mathbf{n}}^i \beta_i^s \pi_i \mathbf{n} + \sum_{i \in \mathbb{N}_0} u_t^i \gamma_i^s \tau_i$ , where  $\beta_i$  and  $\gamma_i$  are the eigenvalues associated with  $\pi_i$  and  $\tau_i$ , respectively. Moreover we can endow  $D(\Delta_\mathcal{O}^s)$  with the scalar product  $(u, v)_{D(\Delta_\mathcal{O}^s)} :=$

$(\Delta_{\mathcal{O}}^s u, \Delta_{\mathcal{O}}^s v)_{L^2(\mathcal{O}, \mathbb{R}^3)}$ . Notice that the system  $\{\pi_i \mathbf{n}, \tau_i \mid i \in \mathbb{N}_0\}$  is orthogonal in  $D(\Delta_{\mathcal{O}}^s)$ , for all  $s \in [0, 1]$ . We find

$$\begin{aligned} & |\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a, b, \Gamma)}^2 \\ & \leq C |\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{L^2((a, b), L^2(\Gamma, \mathbb{R}^3))}^2 \\ (37) \quad & = C |(1 - P_M^{\mathcal{O}})(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{L^2((a, b), L^2(\mathcal{O}, \mathbb{R}^3))}^2. \end{aligned}$$

On the other side, let  $0 \leq k \leq 2$ . Since the mapping  $f \mapsto \phi f|_{\mathcal{O}}$  is in

$$\begin{aligned} \mathcal{L}(L^2((a, b), L^2(\Gamma, \mathbb{R}^3)) \rightarrow L^2((a, b), L^2(\mathcal{O}, \mathbb{R}^3))) \\ \cap \mathcal{L}(L^2((a, b), H^2(\Gamma, \mathbb{R}^3)) \rightarrow L^2((a, b), H_0^2(\mathcal{O}, \mathbb{R}^3))); \end{aligned}$$

by an interpolation argument, we can conclude (e.g., using Theorem A.1 and Lemma A.5) that it also maps

$$L^2((a, b), H^k(\Gamma, \mathbb{R}^3)) = [L^2((a, b), H^2(\Gamma, \mathbb{R}^3)), L^2((a, b), L^2(\Gamma, \mathbb{R}^3))]_{1-(k/2)}$$

continuously into

$$\begin{aligned} & [L^2((a, b), H_0^2(\mathcal{O}, \mathbb{R}^3)), L^2((a, b), L^2(\mathcal{O}, \mathbb{R}^3))]_{1-(k/2)} \\ & = L^2((a, b), [H_0^2(\mathcal{O}, \mathbb{R}^3), L^2(\mathcal{O}, \mathbb{R}^3)]_{1-(k/2)}) \\ & \subseteq L^2((a, b), [D(\Delta_{\mathcal{O}}^1), D(\Delta_{\mathcal{O}}^0)]_{1-(k/2)}) = L^2((a, b), D(\Delta_{\mathcal{O}}^{k/2})). \end{aligned}$$

Then, from (37), we can write in particular

$$\begin{aligned} & |\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a, b, \Gamma)}^2 \\ & \leq C_1 \theta_M^{-(1/3)} |(1 - P_M^{\mathcal{O}})(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{L^2((a, b), D(\Delta_{\mathcal{O}}^{1/6}))}^2 \\ & \leq C_1 \theta_M^{-(1/3)} |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}}|_{L^2((a, b), D(\Delta_{\mathcal{O}}^{1/6}))}^2 \end{aligned}$$

where  $\theta_M = \min\{\beta_i, \gamma_i \mid i > M\}$ . Now we find

$$[H_0^2(\mathcal{O}, \mathbb{R}), L^2(\mathcal{O}, \mathbb{R})]_{1-(1/6)} \subseteq D(\Delta_{\mathcal{O}}^{1/6}) \subseteq [H^2(\mathcal{O}, \mathbb{R}), L^2(\mathcal{O}, \mathbb{R})]_{1-(1/6)},$$

and (from [LM72, chapter 1, Theorems 9.6 and 11.6]) we can conclude that  $D(\Delta_{\mathcal{O}}^{1/6}) = H^{1/3}(\mathcal{O}, \mathbb{R}^3)$ , with equivalent norms. Thus

$$\begin{aligned} & |\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a, b, \Gamma)}^2 \\ & \leq C_2 \theta_M^{-(1/3)} |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}}|_{L^2((a, b), H^{1/3}(\mathcal{O}, \mathbb{R}^3))}^2. \end{aligned}$$

From the continuity of the restriction to  $\mathcal{O}$ , from  $H^1(\Gamma, \mathbb{R}^3)$  onto  $H^1(\mathcal{O}, \mathbb{R}^3)$  and from  $L^2(\Gamma, \mathbb{R}^3)$  onto  $L^2(\mathcal{O}, \mathbb{R}^3)$ , again by an interpolation argument, we conclude that it is also continuous from  $H^{1/3}(\Gamma, \mathbb{R}^3)$  onto  $H^{1/3}(\mathcal{O}, \mathbb{R}^3)$  and we obtain

$$\begin{aligned} & |\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a, b, \Gamma)}^2 \\ & \leq C_2 \theta_M^{-(1/3)} |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{L^2((a, b), H^{1/3}(\Gamma, \mathbb{R}^3))}^2. \end{aligned}$$

Using the inequality (27), in Lemma 4.4, we arrive to

$$\begin{aligned} & |\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1(a, b, \Gamma)}^2 \\ (38) \quad & \leq \theta_M^{-(1/3)} \overline{C}_{[N, |\hat{u}|_{W(a, b)}]_{\text{wk}}} |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1(a, b, \Gamma)}^2. \end{aligned}$$

Now let  $\xi \in C^\infty([a, b], C^\infty(\Gamma, \mathbb{R}))$  be a nonnegative function taking the value 1 if  $|\tilde{\phi}|_{\mathbb{R}} \geq \varepsilon$  and vanishing if  $|\tilde{\phi}|_{\mathbb{R}} \leq \varepsilon/2$ . In particular  $\xi\phi = \phi$  and  $\tilde{\phi}^{-1}\xi = \xi/\tilde{\phi} \in C^1([a, b], C^2(\Gamma, \mathbb{R}))$ . Hence

$$\begin{aligned} & |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1(a, b), \Gamma}'^2 = |\tilde{\phi}\tilde{\phi}^{-1}\xi\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1(a, b), \Gamma}'^2 \\ & = |\tilde{\phi}\tilde{\phi}^{-1}\xi\mathbb{E}_0^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a, b), \Gamma}'^2 \\ & \leq 2C_3|\tilde{\phi}^{-1}\xi|_{C^1([a, b], C^2(\Gamma, \mathbb{R}))}^2|\tilde{\phi}\mathbb{E}_0^\mathcal{O}P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a, b), \Gamma}'^2 \\ & \quad + 2C_3|\xi|_{C^1([a, b], C^2(\Gamma, \mathbb{R}))}^2|\mathbb{E}_0^\mathcal{O}(1 - P_M^\mathcal{O})(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a, b), \Gamma}'^2 \\ & \leq C_4|\tilde{\phi}\mathbb{E}_0^\mathcal{O}P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a, b), \Gamma}'^2 + \theta_M^{-(1/3)}C_5|\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1(a, b), \Gamma}'^2 \end{aligned}$$

and, choosing the integer  $M$  so large that

$$\theta_M^{-(1/3)}C_5 = \theta_M^{-(1/3)}\overline{C}_{[N, |\hat{u}|_{\mathbb{W}(a, b)}]_{\text{wk}}}|\xi|_{C^1([a, b], C^2(\Gamma, \mathbb{R}))}^2 \leq \frac{1}{2},$$

we obtain  $|\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1(a, b), \Gamma}'^2 \leq 2C_4|\tilde{\phi}\mathbb{E}_0^\mathcal{O}P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a, b), \Gamma}'^2$ . Combining this with (24) (with  $f = 0$ ), we arrive to the required inequality (34).  $\square$

*Remark 4.2.* Notice that the integer  $M$  in Theorem 4.5, depends on  $N$  but, the constants in the observability inequalities (34) and (35) do not. We can, of course, take  $\tilde{\phi} = 1$  identically; however, as we will see in the example in section 5, it is useful to consider the more general case.

**4.3. Further truncation in time variable.** In the work [Shi11], an observability inequality truncated in both space and time variable was used to derive suitable results for the stochastic Navier–Stokes equations perturbed by an internal random force localized in a subset of the domain  $\Omega$ . Inspired by this results, here we show that we can also truncate the observability inequality in time variable.

We will need the following proposition, which proof is given in the appendix, section A.7.

**Proposition 4.6.** *The inclusion  $G^1((a, b), \Gamma) \subseteq H^{1/4}((a, b), H^{-(1/4)}(\Gamma, \mathbb{R}^3))$  holds and is continuous.*

Let us consider the Laplace–de Rham operator in  $(a, b)$  with, for example, homogeneous Dirichlet boundary conditions:

$$\begin{aligned} \Delta_t : H^2((a, b), \mathbb{R}) \cap H_0^1((a, b), \mathbb{R}) &\rightarrow L^2((a, b), \mathbb{R}) \\ f &\mapsto -\partial_t \partial_t f. \end{aligned}$$

It is well known that the orthonormal system of eigenfunctions, and corresponding eigenvalues, are given by  $\{\sigma_n := (2/b-a)^{1/2} \sin(n\pi(x-a/b-a)) \mid n \in \mathbb{N}_0\}$ , and  $\{\lambda_n = (n\pi/b-a)^2 \mid n \in \mathbb{N}_0\}$ ;  $\Delta_t \sigma_n = \lambda_n \sigma_n$ . Next, given a Hilbert space  $X$ , we define the following mapping  $P_M^t$  in  $L^2((a, b), X)$

$$P_M^t f(t) := \sum_{n=1}^M \left( \int_a^b \sigma_n(\tau) f(\tau) \, d\tau \right) \sigma_n.$$

**Proposition 4.7.** *The mapping  $P_M^t$  is an orthogonal projection in  $L^2((a, b), X)$  onto  $P_M^t L^2((a, b), X) = \sum_{n=1}^M \sigma_n X$ . Moreover we may write*

$$f = \sum_{n \in \mathbb{N}_0} \left( \int_a^b \sigma_n(\tau) f(\tau) d\tau \right) \sigma_n = \lim_{M \rightarrow +\infty} P_M^t f;$$

$$|f|_{L^2((a, b), X)}^2 = \sum_{n \in \mathbb{N}_0} \left| \left( \int_a^b \sigma_n(\tau) f(\tau) d\tau \right) \sigma_n \right|_{L^2((a, b), X)}^2 = \sum_{n \in \mathbb{N}_0} \left| \int_a^b \sigma_n(\tau) f(\tau) d\tau \right|_X^2$$

The proof of this proposition is straightforward, though nontrivial; for the sake of completeness we present it in the appendix, section A.8. Now, given  $f \in H_0^1((a, b), X)$ , we find that

$$\begin{aligned} |f|_{H^1((a, b), X)}^2 &= |f|_{L^2((a, b), X)}^2 + |\partial_t f|_{L^2((a, b), X)}^2 \\ &= \left| \sum_{n \in \mathbb{N}_0} \left( \int_a^b \sigma_n(\tau) f(\tau) d\tau \right) \sigma_n \right|_{L^2((a, b), X)}^2 + \left| \sum_{n \in \mathbb{N}_0} \left( \int_a^b \sigma_n(\tau) f(\tau) d\tau \right) \partial_t \sigma_n \right|_{L^2((a, b), X)}^2 \\ &= \sum_{n \in \mathbb{N}_0} \left| \int_a^b \sigma_n(\tau) f(\tau) d\tau \right|_X^2 + \sum_{n \in \mathbb{N}_0} \lambda_n \left| \int_a^b \sigma_n(\tau) f(\tau) d\tau \right|_X^2 \\ &= \sum_{n \in \mathbb{N}_0} (1 + \lambda_n) \left| \int_a^b \sigma_n(\tau) f(\tau) d\tau \right|_X^2 \end{aligned}$$

and similarly, we can derive that  $|P_M^t f|_{H^1((a, b), X)}^2 = \sum_{n=1}^M (1 + \lambda_n) \left| \int_a^b \sigma_n(\tau) f(\tau) d\tau \right|_X^2$ . Notice that  $(\sigma_m, \sigma_n)_{L^2((a, b), \mathbb{R})} = \delta_n^m$  and  $(\partial_t \sigma_m, \partial_t \sigma_n)_{L^2((a, b), \mathbb{R})} = (-\partial_t \partial_t \sigma_m, \sigma_n)_{L^2((a, b), \mathbb{R})} = \delta_n^m \lambda_m$ , where  $\delta_n^m$  is the Kronecker delta  $\delta_n^m = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$ . In particular we have  $|P_M^t f|_{H_0^1((a, b), X)}^2 \leq |f|_{H_0^1((a, b), X)}^2$ . Further, we can conclude that  $P_M^t$  is also an orthogonal projection in  $H_0^1((a, b), X)$  onto  $P_M^t H_0^1((a, b), X) = P_M^t L^2((a, b), X)$ .

Now, for simplicity, given a finite orthogonal sequence  $\mathcal{S} = \{v_i \mid i = 1, 2, \dots, k\} \subseteq X$  in the Hilbert space  $X$ , let  $\mathcal{F} = \text{span } \mathcal{S}$  and define the operator  $\Delta_{t, \mathcal{F}} : H^2((a, b), \mathcal{F}) \cap H_0^1((a, b), \mathcal{F}) \rightarrow L^2((a, b), \mathcal{F})$ , sending  $f(t) = \sum_{i=1}^k f_i(t) v_i$  to  $\sum_{i=1}^k \Delta_t f_i(t) v_i$ . It turns out that  $\Delta_{t, \mathcal{F}}^s f(t) := \sum_{i=1}^k (\Delta_t^s f_i(t)) v_i$ , and

$$(39) \quad |f|_{D(\Delta_{t, \mathcal{F}}^s)}^2 = \sum_{i=1}^k |f_i|_{D(\Delta_t^s)}^2 |v_i|_X^2; \text{ for } s \in [0, 1].$$

**Theorem 4.8.** *Let  $N \in \mathbb{N}_0$  and let  $(q, p)$  solve system (11) with  $q_1 \in H_N$  and  $f = 0$ , for an appropriate choice for pressure function  $p$ . Let us be given also two differentiable functions  $\phi, \tilde{\phi} \in C^1([a, b], C^2(\Gamma, \mathbb{R}))$ , with nonempty support, an open connected smooth submanifold  $\mathcal{O} \subseteq \Gamma$ , and  $\varepsilon > 0$  such that  $\text{supp } \phi \subseteq [a, b] \times \bar{\mathcal{O}}$  and  $|\tilde{\phi}(t, x)|_{\mathbb{R}} \geq \varepsilon$  for all  $(t, x) \in \text{supp } \phi$ . Then, if  $\hat{u} \in \mathcal{W}^{(a, b)|\text{wk}}$  there exists a positive integer  $M = \bar{C}_{[N, |\hat{u}|_{\mathcal{W}^{(a, b)|\text{wk}}}]}$  such that*

$$(40) \quad |q(a)|_H^2 \leq \bar{C}_{[|\hat{u}|_{\mathcal{W}^{(a, b)|\text{wk}}}] } |\tilde{\phi} P_M^t \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p\mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a, b), \Gamma}^2,$$

and, if  $\hat{u} \in \mathcal{W}^{(a, b)|\text{st}}$  there exists a positive integer  $M = \bar{C}_{[N, |\hat{u}|_{\mathcal{W}^{(a, b)|\text{st}}}]}$  such that

$$(41) \quad |q(a)|_H^2 \leq \bar{C}_{[|\hat{u}|_{\mathcal{W}^{(a, b)|\text{st}}}] } |\tilde{\phi} P_M^t \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p\mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^2(a, b), \Gamma}^2,$$

where the constants  $\bar{C}_{[\hat{u}]_{\mathcal{W}(a,b)}^{\text{wk}}}$  and  $\bar{C}_{[\hat{u}]_{\mathcal{W}(a,b)}^{\text{st}}}$  depend only on  $\Omega$ ,  $\mathcal{O}$ ,  $\phi$ ,  $\tilde{\phi}$ , and on the respective norm of  $\hat{u}$ .

*Proof.* Again we prove (40), the proof of (41) is completely analogous (e.g., starting by using the continuity of the inclusion  $G^1(a, b), \Gamma)' \subset G^2(a, b), \Gamma)'$ ). From Proposition 4.6, we can derive

$$\begin{aligned} & |(1 - P_M^t) \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a,b),\Gamma'}^2 \\ & \leq C |(1 - P_M^t) \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{H^{-(1/4)}((a,b), H^{1/4}(\Gamma, \mathbb{R}^3))}^2 \end{aligned}$$

and, from the continuity of the extension by zero outside  $\mathcal{O}$ , from  $H^s(\mathcal{O}, \mathbb{R}^3)$  into  $H^s(\Gamma, \mathbb{R}^3)$  for  $0 \leq s < 1/2$  (cf. [LM72, chapter 1, section 11.3]), we can write

$$\begin{aligned} & |(1 - P_M^t) \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a,b),\Gamma'}^2 \\ & \leq C |(1 - P_M^t) P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{H^{-(1/4)}((a,b), H^{1/4}(\mathcal{O}, \mathbb{R}^3))}^2 \end{aligned}$$

Now, set  $\mathcal{F} := L_M^2(\mathcal{O}, \mathbb{R}^3) = P_M^\mathcal{O} L^2(\mathcal{O}, \mathbb{R}^3)$ . By an analogous argument as in the proof of Theorem 4.5 we can prove that  $H^{1/4}(\mathcal{O}, \mathbb{R}^3) = \text{D}(\Delta_{\mathcal{O}}^{1/8})$ ,  $H^{1/4}((a, b), \mathcal{F}) = \text{D}(\Delta_{t,\mathcal{F}}^{1/8})$ , and  $H^{-(1/4)}((a, b), \mathcal{F}) = \text{D}(\Delta_{t,\mathcal{F}}^{-(1/8)})$ , with equivalent norms. Thus, using (39), we can derive

$$\begin{aligned} & |(1 - P_M^t) \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a,b),\Gamma'}^2 \\ & \leq C_1 |(1 - P_M^t) P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{\text{D}(\Delta_{t,\mathcal{F}_{1/8}}^{-(1/8)})}^2 \\ & = C_1 |\Delta_{t,\mathcal{F}_{1/8}}^{-(1/8)}(1 - P_M^t) P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{\text{D}(\Delta_{t,\mathcal{F}_{1/8}}^0)}^2 \\ & \leq C_1 \Theta_{t,M}^{-(1/4)} |P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{\text{D}(\Delta_{t,\mathcal{F}_{1/8}}^0)}^2 \\ & \leq C_1 \Theta_{t,M}^{-(1/4)} |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}}|_{L^2((a,b), \text{D}(\Delta_{\mathcal{O}}^{1/8}))}^2 \end{aligned}$$

where  $\theta_{t,M} = \min\{\lambda_i \mid i > M\} = ((M+1)\pi/b-a)^2$ , and  $\mathcal{F}_{1/8}$  means that  $\mathcal{F}$  is endowed with the  $\text{D}(\Delta_{\mathcal{O}}^{1/8})$ -norm. Proceeding as in the proof of Theorem 4.5, and using (27), we obtain

$$\begin{aligned} & |(1 - P_M^t) \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a,b),\Gamma'}^2 \\ & \leq C_2 \Theta_{t,M}^{-(1/4)} |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{L^2((a,b), H^{1/4}(\Gamma, \mathbb{R}^3))}^2 \\ (42) \quad & \leq \Theta_{t,M}^{-(1/4)} \bar{C}_{[N, \hat{u}]_{\mathcal{W}(a,b)}^{\text{wk}}} |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1(a,b),\Gamma'}^2. \end{aligned}$$

Next, again as in the proof of Theorem 4.5, we set  $\xi \in C^\infty([a, b], C^\infty(\Gamma, \mathbb{R}))$  be a non-negative function taking the value 1 if  $|\tilde{\phi}|_{\mathbb{R}} \geq \varepsilon$  and vanishing if  $|\tilde{\phi}|_{\mathbb{R}} \leq \varepsilon/2$ . Writing

$$\begin{aligned} & \tilde{\phi} \tilde{\phi}^{-1} \xi \phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q) = \tilde{\phi} \tilde{\phi}^{-1} \xi \mathbb{E}_0^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}}) \\ & = \tilde{\phi} \tilde{\phi}^{-1} \xi P_M^t \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}}) + \tilde{\phi} \tilde{\phi}^{-1} \xi (1 - P_M^t) \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}}) \\ & \quad + \tilde{\phi} \tilde{\phi}^{-1} \xi \mathbb{E}_0^\mathcal{O} (1 - P_M^\mathcal{O})(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}}) \end{aligned}$$

and using (38), (42), and  $\tilde{\phi} \tilde{\phi}^{-1} \xi \phi = \phi$ , we find

$$\begin{aligned} & |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1(a,b),\Gamma'}^2 \\ & \leq C_3 |\tilde{\phi}^{-1} \xi|_{C^1([a,b], C^2(\Gamma, \mathbb{R}))}^2 |\tilde{\phi} P_M^t \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a,b),\Gamma'}^2 \\ & \quad + (\Theta_{t,M}^{-(1/4)} + \Theta_M^{-(1/3)}) |\xi|_{C^1([a,b], C^2(\Gamma, \mathbb{R}))}^2 \bar{C}_{[N, \hat{u}]_{\mathcal{W}(a,b)}^{\text{wk}}} |\phi(p\mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla \rangle q)|_{G^1(a,b),\Gamma'}^2 \end{aligned}$$

and, choosing  $M \in \mathbb{N}_0$  so large that  $(\Theta_M^{-1/3} + \Theta_{t,M}^{-1/4})|\xi|_{C^1([a,b], C^2(\Gamma, \mathbb{R}))}^2 \overline{C}_{[N, |\hat{u}|_{\mathcal{W}(a,b)}]_{\text{wk}}} \leq \frac{1}{2}$ , we obtain  $|\phi(p\mathbf{n} - \nu \langle \mathbf{n}, \nabla \rangle q)|_{G^1(a,b), \Gamma}^2 \leq C_{4(\phi, \tilde{\phi})} |\tilde{\phi} P_M^t \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p\mathbf{n} - \nu \langle \mathbf{n}, \nabla \rangle q)|_{\mathcal{O}})|_{G^1(a,b), \Gamma}^2$ . Combining this with (24) (with  $f = 0$ ), we arrive to the required inequality (40).  $\square$

## 5. EXAMPLE OF APPLICATION

Recall the space  $\mathcal{E}_M^1 = \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} \vartheta G^1((a, b), \Gamma)|_{\mathcal{O}}$  defined in (9), and its subspace  $\mathcal{E}_M^2 = \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} \vartheta G^2((a, b), \Gamma)|_{\mathcal{O}}$  defined in section 2.4.1. Here we use the truncated observability inequalities (35) and (41) to derive two controllability results for the Oseen–Stokes system (10), where the control  $\zeta$  is taken in (a subspace of)  $\mathcal{E}_M^2$ .

It turns out that, while inequality (35) is appropriate to deal with the control space  $\mathcal{E}_M^2$ , inequality (41) is appropriate to deal with controls in

$$\begin{aligned} \mathcal{G}_M &:= \varphi \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} P_M^t \tilde{\varphi} \vartheta G^2((a, b), \Gamma)|_{\mathcal{O}} \\ &:= \{\zeta \mid \zeta = \varphi \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} P_M^t (\tilde{\varphi} \vartheta \eta)|_{\mathcal{O}} \text{ and } \eta \in G^2((a, b), \Gamma)\}. \end{aligned}$$

Consider the operator

$$(43) \quad \eta \mapsto K_t^{\mathcal{O}} \eta := \varphi \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} P_M^t (\tilde{\varphi} \vartheta \eta)|_{\mathcal{O}}.$$

**Proposition 5.1.** *The operator  $K_t^{\mathcal{O}}$  is linear and continuous from  $G^i((a, b), \Gamma)$  into  $G_{\text{av}}^i((a, b), \Gamma)$ , for  $i \in \{1, 2\}$ .*

The proof of the proposition will be given in the appendix, section A.9; it will follow from Proposition 2.7 and some interpolation arguments.

Next, we recall also the space  $H_N$  and the orthogonal projection  $\Pi_N : H \rightarrow H_N$  (see section 2.3). Let  $\varphi \in C^1((a, b), \mathbb{R})$  be such that  $\text{supp}(\varphi) \neq \emptyset$  and  $\varphi(t)$  vanishes in a neighborhood of  $\{a, b\}$ , say  $\varphi(t) = 0$  for some  $0 < \delta < b-a/2$  and all  $t \in [a, a+\delta] \cup [b-\delta, b]$ .

**Theorem 5.2.** *For each  $N \in \mathbb{N}$  there exists an integer  $M = \overline{C}_{[N, |\hat{u}|_{\mathcal{W}(a,b)}]_{\text{st}}} \in \mathbb{N}_0$  such that, for every  $v_0 \in H$ , we can find  $\eta = \eta(v_0) \in G^2((a, b), \Gamma)$ , depending linearly on  $v_0$ , such that the boundary control  $\zeta = \varphi \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} (\vartheta \eta)|_{\mathcal{O}}$  drives the system (10), with  $g = 0$ , to a vector  $v(b) \in V$  such that  $\Pi_N v(b) = 0$ . Moreover, there exists a constant  $\overline{C}_{[|\hat{u}|_{\mathcal{W}(a,b)}]_{\text{st}}}$ , depending on  $|\hat{u}|_{\mathcal{W}(a,b)}|_{\text{st}}$  and  $\varphi$  but, not on the pair  $(N, v_0)$ , such that*

$$|\eta|_{G^2((a,b), \Gamma)}^2 \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}(a,b)}]_{\text{st}}} |v_0|_H^2.$$

Next, let  $\tilde{\varphi} \in C^1([a, b], \mathbb{R}) \cap H_0^1((a, b), \mathbb{R})$  be a function such that  $|\tilde{\varphi}(t)|_{\mathbb{R}} \geq \varepsilon > 0$  for all  $t \in \text{supp}(\phi)$ .

**Theorem 5.3.** *For each  $N \in \mathbb{N}$  there exists an integer  $M = \overline{C}_{[N, |\hat{u}|_{\mathcal{W}(a,b)}]_{\text{st}}} \in \mathbb{N}_0$  such that, for every  $v_0 \in H$ , we can find  $\eta = \eta(v_0) \in G^2((a, b), \Gamma)$ , depending linearly on  $v_0$ , such that the boundary control  $\zeta = \varphi \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} P_M^t (\tilde{\varphi} \vartheta \eta)|_{\mathcal{O}}$  drives the system (10), with  $g = 0$ , to a vector  $v(b) \in V$  such that  $\Pi_N v(b) = 0$ . Moreover, there exists a constant  $\overline{C}_{[|\hat{u}|_{\mathcal{W}(a,b)}]_{\text{st}}}$ , depending on  $|\hat{u}|_{\mathcal{W}(a,b)}|_{\text{st}}$ ,  $\varphi$ , and  $\tilde{\varphi}$  but, not on the pair  $(N, v_0)$ , such that*

$$(44) \quad |\eta|_{G^2((a,b), \Gamma)}^2 \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}(a,b)}]_{\text{st}}} |v_0|_H^2.$$

The proofs of Theorems 5.2 and 5.3 are completely analogous. So we will prove only Theorem 5.3 where we shall use the observability inequality (41); to prove Theorem 5.2 we can use (35) instead. We start by recalling the following:

**Lemma 5.4.** *If  $q \in D(L)$ , then  $\langle \mathbf{n}, \nabla \rangle q$  is tangent to  $\Gamma$ .*

*Proof.* Since  $q|_{\Gamma} = 0$  we have that  $\nabla q_j = \alpha_j \mathbf{n}$  on  $\Gamma$ , for a suitable function  $\alpha_j$  and for each  $j \in \{1, 2, 3\}$ . Then we can derive that  $\partial_{x_i} q_j = \alpha_j \mathbf{n}_i$  and  $(\langle \mathbf{n} \cdot \nabla \rangle q) \cdot \mathbf{n} = \sum_{j=1}^3 (\sum_{i=1}^3 \mathbf{n}_i \partial_{x_i} q_j) \mathbf{n}_j = \sum_{j=1}^3 (\sum_{i=1}^3 \mathbf{n}_i^2 \alpha_j) \mathbf{n}_j = \sum_{j=1}^3 \alpha_j \mathbf{n}_j$ , on the boundary  $\Gamma$ . On the other hand, from  $0 = \operatorname{div} q = \sum_{j=1}^3 \partial_{x_j} q_j$ , we obtain that  $0 = (\operatorname{div} q)|_{\Gamma} = \sum_{j=1}^3 \alpha_j \mathbf{n}_j$ . Therefore, we have  $(\langle \mathbf{n} \cdot \nabla \rangle q) \cdot \mathbf{n} = 0$  on  $\Gamma$ .  $\square$

*Proof of Theorem 5.3.* We shall follow the idea in the proof of Lemma 3.2 in [BRS11]. First, we extend the orthogonal projection  $\Pi : L^2(\Omega, \mathbb{R}^3) \rightarrow H$  to a projection mapping  $\Pi : H^{-1}(\Omega, \mathbb{R}^3) \rightarrow V'$  by setting  $\langle \Pi f, u \rangle_{V', V} := \langle f, u \rangle_{H^{-1}(\Omega, \mathbb{R}^3), H_0^1(\Omega, \mathbb{R}^3)}$ . Recall that we can write  $H^{-1}(\Omega, \mathbb{R}^3) = V' \oplus \{\nabla p \mid p \in L^2(\Omega, \mathbb{R})\}$  (see [Tem01, chapter 1, section 1.4, Proposition 1.1 and Remark 1.9]). Observe that, given  $p \in L^2(\Omega, \mathbb{R})$ , we have  $\langle \Pi \nabla p, u \rangle_{V', V} = \langle \nabla p, u \rangle_{H^{-1}(\Omega, \mathbb{R}^3), H_0^1(\Omega, \mathbb{R}^3)} = 0$ , that is,  $\Pi \nabla p = 0$ ; in other words, writing  $H^{-1}(\Omega, \mathbb{R}^3) = V' \oplus \{\nabla p \mid p \in L^2(\Omega, \mathbb{R})\}$ ,  $\Pi$  coincides with the projection onto  $V'$ .

Then, we fix  $\epsilon > 0$  and consider the following minimization problem:

PROBLEM 5.1. *Given  $M, N \in \mathbb{N}$  and  $v_0 \in H$ , find the minimum of the quadratic functional*

$$J_{\epsilon}(v, \eta) := |\eta|_{G^2((a, b), \Gamma)}^2 + (1/\epsilon) |\Pi_N v(b)|_H^2,$$

*subject to the constraint  $F(v, \eta) = (0, 0, 0)$ , in the space*

$$\mathcal{X} := W_H((a, b), H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \times G^2((a, b), \Gamma),$$

*where*

$$\begin{aligned} F : \mathcal{X} &\rightarrow \mathcal{Y} := H \times L^2((a, b), V') \times G_{\operatorname{av}, H}^1((a, b), \Gamma) \\ (v, \eta) &\mapsto (v(a) - v_0, \Pi(v_t - \nu \Delta v + \mathcal{B}(\hat{u})v), v|_{\Gamma} - K_t^{\mathcal{O}} \eta) \end{aligned}$$

*with*

$$\begin{aligned} W_H((a, b), H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \\ &:= \{u \in W((a, b), H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \mid u(a) \in H\}; \\ G_{\operatorname{av}, H}^1((a, b), \Gamma) &:= \{u|_{\Gamma} \mid u \in W_H((a, b), H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))\}. \end{aligned}$$

Since the constraint can be rewritten as  $A(v, \eta) = (v_0, 0, 0)$  where  $A$  is the linear mapping  $A(v, \eta) := F(v, \eta) + (v_0, 0, 0)$ , we have that Problem 5.1 has a unique minimizer  $(\bar{v}_{\epsilon}, \bar{\eta}_{\epsilon})$ , which linearly depends on  $v_0 \in H$  (see e.g. Theorem A.2 in [BRS11]).

From Theorem 2.3 we can derive that the derivative of  $F$ ,  $A = dF$ , is surjective. Thus, by the Karush–Kuhn–Tucker Theorem (e.g., see [BRS11, section A.1]), it follows that there exists a Lagrange multiplier  $(\mu^{\epsilon}, q^{\epsilon}, \rho^{\epsilon}) \in \mathcal{Y}' = H \times L^2((a, b), V) \times G_{\operatorname{av}, H}^1((a, b), \Gamma)'$  such that

$$dJ_{\epsilon}(\bar{v}^{\epsilon}, \bar{\eta}^{\epsilon}) + (\mu^{\epsilon}, q^{\epsilon}, \rho^{\epsilon}) \circ dF(\bar{v}^{\epsilon}, \bar{\eta}^{\epsilon}) = 0,$$

where the symbol “ $\circ$ ” stands for the composition of two linear operators. It follows that, for all  $(z, \xi) \in \mathcal{X}$ , we have

$$\begin{aligned} (45) \quad 0 &= 2(1/\epsilon) (\Pi_N \bar{v}^{\epsilon}(b), z(b))_H + (\mu^{\epsilon}, z(a))_H \\ &\quad + \int_a^b \langle z_t + \mathcal{B}(\hat{u})z - \nu \Delta z, q^{\epsilon} \rangle_{H^{-1}(\Omega, \mathbb{R}^3), H_0^1(\Omega, \mathbb{R}^3)} dt \\ &\quad + (\rho^{\epsilon}, z|_{\Gamma})_{G_{\operatorname{av}, H}^1((a, b), \Gamma)', G_{\operatorname{av}, H}^1((a, b), \Gamma)} dt, \end{aligned}$$

$$(46) \quad 0 = 2(\bar{\eta}^{\epsilon}, \xi)_{G^2((a, b), \Gamma)} + (\rho^{\epsilon}, -K_t^{\mathcal{O}} \xi)_{G_{\operatorname{av}, H}^1((a, b), \Gamma)', G_{\operatorname{av}, H}^1((a, b), \Gamma)}.$$

Letting  $z$  run over  $W((a, b), V, V')$  (e.g., proceeding as in the proof of Lemma 3.2 in [BRS11]) we can verify that relation (45) implies that  $q^{\epsilon}$  solves system (11) with  $f = 0$ ,  $q^{\epsilon}(b) = -2(1/\epsilon) \Pi_N \bar{v}^{\epsilon}(b)$ , and a suitable pressure function  $p_c^{\epsilon}$ . Further,  $q^{\epsilon}(a) = \mu^{\epsilon}$ .

Next, we let  $z$  run over  $W_H((a, b), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$  in (45); we can derive that  $\rho^\epsilon = p_q^\epsilon \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q^\epsilon$  and, in particular, we have that  $\rho^\epsilon \in L^2((a, b), L^2(\Gamma, \mathbb{R}^3))$ . Therefore, we can obtain

$$\begin{aligned} & (\rho^\epsilon, K_t^\mathcal{O} \xi)_{G_{\text{av}, H}^1((a, b), \Gamma)', G_{\text{av}, H}^1((a, b), \Gamma)} \\ &= (\rho^\epsilon, \varphi \chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O} P_M^t (\tilde{\varphi} \vartheta \xi |_\mathcal{O}))_{L^2((a, b), L^2(\Gamma, \mathbb{R}^3))} \\ &= (\tilde{\varphi} \vartheta P_M^t \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O} P_{\chi^\perp}^\mathcal{O} (\varphi \chi \rho^\epsilon |_\mathcal{O}), \xi)_{L^2((a, b), L^2(\Gamma, \mathbb{R}^3))} \end{aligned}$$

and, from (46), it follows that necessarily  $2\mathcal{A}\bar{\eta}^\epsilon = \tilde{\varphi} \vartheta P_M^t \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O} P_{\chi^\perp}^\mathcal{O} (\varphi \chi \rho^\epsilon |_\mathcal{O})$ , where  $\mathcal{A}$  is the natural isomorphism

$$(47) \quad \langle \mathcal{A}u, v \rangle_{G^2((a, b), \Gamma)', G^2((a, b), \Gamma)} := (u, v)_{G^2((a, b), \Gamma)}$$

from  $G^2((a, b), \Gamma)$  onto  $G^2((a, b), \Gamma)'$ . Notice that the mapping  $v \mapsto (u, v)_{G^2((a, b), \Gamma)}$  is in  $G^2((a, b), \Gamma)'$ , and (47) just says that we denote this mapping by  $\mathcal{A}u$ . That  $\mathcal{A}$  is, indeed, bijective follows from the Lax–Milgram Lemma (cf. [Tem01, chapter 1, Theorem 2.2], [Neč67, chapter 1, section 3.1]).

Therefore, we obtain

$$(48) \quad 2\mathcal{A}\bar{\eta}^\epsilon = \tilde{\varphi} \vartheta P_M^t \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O} P_{\chi^\perp}^\mathcal{O} (\varphi \chi (p_q^\epsilon \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q^\epsilon) |_\mathcal{O}).$$

Combining the above identities, we can arrive to

$$\begin{aligned} \frac{d}{dt} (q^\epsilon, \bar{v}^\epsilon)_{L^2(\Omega, \mathbb{R}^3)} &= (q_t^\epsilon, \bar{v}^\epsilon)_{L^2(\Omega, \mathbb{R}^3)} + \langle q^\epsilon, \bar{v}_t^\epsilon \rangle_{H_0^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)} \\ &= (p_q^\epsilon \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q^\epsilon, \bar{v}^\epsilon |_\Gamma)_{L^2(\Gamma, \mathbb{R}^3)} \\ &= (p_q^\epsilon \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q^\epsilon, K_t^\mathcal{O} \bar{\eta}^\epsilon)_{L^2(\Gamma, \mathbb{R}^3)} \\ &= (\tilde{\varphi} \vartheta P_M^t \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O} P_{\chi^\perp}^\mathcal{O} (\varphi \chi (p_q^\epsilon \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q^\epsilon) |_\mathcal{O}), \bar{\eta}^\epsilon)_{L^2(\Gamma, \mathbb{R}^3)} \end{aligned}$$

and, integrating in time over the interval  $(a, b)$ ,

$$\begin{aligned} & (q^\epsilon(b), \bar{v}^\epsilon(b))_H - (q^\epsilon(a), \bar{v}^\epsilon(a))_H \\ &= (\tilde{\varphi} \vartheta P_M^t \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O} P_{\chi^\perp}^\mathcal{O} (\varphi \chi (p_q^\epsilon \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q^\epsilon) |_\mathcal{O}), \bar{\eta}^\epsilon)_{L^2((a, b), L^2(\Gamma, \mathbb{R}^3))} \\ &= (2\mathcal{A}\bar{\eta}^\epsilon, \bar{\eta}^\epsilon)_{L^2((a, b), L^2(\Gamma, \mathbb{R}^3))} = 2|\bar{\eta}^\epsilon|_{G^2((a, b), \Gamma)}^2; \end{aligned}$$

so, from  $q^\epsilon(b) = -2(1/\epsilon)\Pi_N \bar{v}^\epsilon(b)$ , we obtain

$$(49) \quad 2|\bar{\eta}^\epsilon|_{G^2((a, b), \Gamma)}^2 + 2(1/\epsilon)|\Pi_N \bar{v}^\epsilon(b)|_H^2 = -(q^\epsilon(a), \bar{v}^\epsilon(a))_H.$$

We wish to use the truncated observability inequality (41) to estimate the right-hand side of (49); to this end, it will be convenient to choose the pressure function  $p_q^\epsilon$  in a suitable way (cf. section 3.2). We choose  $p_q^\epsilon$  such that  $\sigma(p_q^\epsilon) := \int_\Gamma \chi^2 p_q^\epsilon d\Gamma = 0$ . Then, using also Lemma 5.4, we observe that  $P_{\chi^\perp}^\mathcal{O} (\varphi \chi (p_q^\epsilon \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q^\epsilon) |_\mathcal{O}) = \varphi P_{\chi^\perp}^\mathcal{O} (\chi p_q^\epsilon \mathbf{n} |_\mathcal{O}) - \varphi \chi \langle \mathbf{n} \cdot \nabla \rangle q^\epsilon |_\mathcal{O} = \varphi \chi (p_q^\epsilon \mathbf{n} - \langle \mathbf{n} \cdot \nabla \rangle q^\epsilon) |_\mathcal{O}$ , and by the observability inequality (41), with  $\phi(t, x) = \varphi(t) \chi(x)$  and  $\tilde{\phi}(t, x) = \tilde{\varphi}(t) \vartheta(x)$  for  $(t, x) \in [a, b] \times \Gamma$ , there exists an integer  $M$  such that

$$(50) \quad \begin{aligned} |q^\epsilon(a)|_H^2 &\leq \bar{C}_{[\hat{u}|_{\mathcal{W}(a, b)}]_{\text{st}}} |\tilde{\varphi} \vartheta P_M^t \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O} (\varphi \chi (p_q^\epsilon \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q^\epsilon) |_\mathcal{O})|_{G^2(a, b), \Gamma'}^2 \\ &= \bar{C}_{[\hat{u}|_{\mathcal{W}(a, b)}]_{\text{st}}} |2\mathcal{A}\bar{\eta}^\epsilon|_{G^2(a, b), \Gamma'}^2. \end{aligned}$$

Further, from (49), for every  $\alpha > 0$  we can write

$$\begin{aligned} 4|\bar{\eta}^\epsilon|_{G^2((a, b), \Gamma)}^2 + 4(1/\epsilon)|\Pi_N \bar{v}^\epsilon(b)|_H^2 &\leq \alpha |q^\epsilon(a)|_H^2 + (1/\alpha) |\bar{v}^\epsilon(a)|_H^2 \\ &\leq 4\alpha \bar{C}_{[\hat{u}|_{\mathcal{W}(a, b)}]_{\text{st}}} |\mathcal{A}\bar{\eta}^\epsilon|_{G^2(a, b), \Gamma'}^2 + (1/\alpha) |\bar{v}^\epsilon(a)|_H^2 \end{aligned}$$

and, setting  $\alpha = \left(2\overline{C}_{[\hat{u}]_{\mathcal{W}(a,b)|_{\text{st}}}}\right)^{-1}$ , we obtain

$$(51) \quad |\bar{\eta}^\epsilon|_{G^2((a,b),\Gamma)}^2 + 2(1/\epsilon)|\Pi_N \bar{v}^\epsilon(b)|_H^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}(a,b)|_{\text{st}}}} |v_0|_H^2.$$

In particular, the family  $\{\bar{\eta}^\epsilon \mid \epsilon > 0\}$  is bounded in  $G^2((a,b),\Gamma)$ , from which it follows the boundedness of the family  $\{\bar{v}^\epsilon \mid \epsilon > 0\}$  in  $W_H((a,b), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ . Indeed, we notice that the constraint  $F(\bar{v}^\epsilon, \bar{\eta}^\epsilon) = (0, 0, 0)$  means that the triple  $(v, g, \zeta) = (\bar{v}^\epsilon, 0, K_t^\mathcal{O} \bar{\eta}^\epsilon)$  solves (10), with  $v(a) = v_0$  and then the boundedness follows from Proposition 5.1 and Theorem 2.3.

Thus, we can find a decreasing sequence  $\epsilon_n \searrow 0$  such that  $\eta^{\epsilon_n} \rightharpoonup \eta^0$  in  $G^2((a,b),\Gamma)$  and  $\bar{v}^{\epsilon_n} \rightharpoonup v^0$  in  $W_H((a,b), H_{\text{div}}^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ . It follows (e.g. from Lemma 4.1) that  $\bar{v}^{\epsilon_n} \rightharpoonup v^0$  in  $L^2((a,b), H_{\text{div}}^1(\Omega, \mathbb{R}^3))$ ,  $\bar{v}_t^{\epsilon_n} \rightharpoonup v_t^0$  in  $L^2((a,b), H^{-1}(\Omega, \mathbb{R}^3))$ , and  $\bar{v}^{\epsilon_n}(b) \rightharpoonup v^0(b)$  in  $H$ . A standard limiting argument shows that also  $(v, g, \zeta) = (v^0, 0, K_t^\mathcal{O} \eta^0)$  solves (10), with  $v(a) = v_0$ . Furthermore, from (51), we have  $2|\Pi_N \bar{v}^\epsilon(b)|_H^2 \leq \epsilon \overline{C}_{[\hat{u}]_{\mathcal{W}(a,b)|_{\text{st}}}} |v_0|_H^2 \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Now, the fact that  $\Pi_N \bar{v}^{\epsilon_n}(b) \rightharpoonup \Pi_N v^0(b)$  in  $H$ , implies that  $|\Pi_N v^0(b)|_H \leq \liminf_{n \rightarrow +\infty} |\Pi_N \bar{v}^{\epsilon_n}(b)|_H = 0$ . Analogously, it follows from (51) that  $\eta = \eta^0$  satisfies (44):  $|\eta^0|_{G^2((a,b),\Gamma)}^2 \leq \liminf_{n \rightarrow +\infty} |\bar{\eta}^{\epsilon_n}|_{G^2((a,b),\Gamma)}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}(a,b)|_{\text{st}}}} |v_0|_H^2$ .

It remains to show that the control  $\eta$  may be chosen depending linearly on  $v_0$ . For that we follow the idea in the proof of Lemma 3.5 in [BRS11]: let  $N \in \mathbb{N}_0$  and let  $M$  be the integer in (50); consider the following variation of Problem 5.1

**PROBLEM 5.2.** *Given  $v_0 \in H$ , find the minimum of the quadratic functional*

$$J_\infty(v, \eta) := |\eta|_{G^2((a,b),\Gamma)}^2$$

*subject to the constraint  $\tilde{F}(v, \eta) = (0, 0, 0, 0)$ , in the space  $\mathcal{X}$  where*

$$\begin{aligned} \tilde{F} : \mathcal{X} &\rightarrow \tilde{\mathcal{Y}} := \mathcal{Y} \times H_N \\ (v, \eta) &\mapsto (F(v, \eta), \Pi_N v(b)). \end{aligned}$$

Now, reasoning as above (using the Theorem A.2 in [BRS11]), we can conclude that Problem 5.2 has an unique minimizer  $(\bar{v}, \bar{\eta})(v_0)$  depending linearly on  $v_0$ . Notice that necessarily we still have  $|\bar{\eta}|_{G^2((a,b),\Gamma)}^2 \leq |\eta^0|_{G^2((a,b),\Gamma)}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}(a,b)|_{\text{st}}}} |v_0|_H^2$ .  $\square$

**Constancy of the control.** Notice that the control  $\zeta = K_t^\mathcal{O} \eta = K_t^\mathcal{O} \bar{\eta}$ , given in Theorem 5.3 (and minimizing Problem 5.2), can be “realized” by an element  $\kappa \in \mathbb{R}^{2M^2}$ :  $K_t^\mathcal{O} \eta(t, x) = \check{K} \kappa = \sum_{i,j=1}^M \kappa_{i,j} \varphi(t) \sigma_i(t) \chi(x) \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}(\pi_j \mathbf{n})|_x + \sum_{i,j=1}^M \kappa_{i,M+j} \varphi(t) \sigma_i(t) \chi(x) \mathbb{E}_0^\mathcal{O} \tau_j|_x$ , where the  $\sigma_i$ s (see section 4.3), the  $\pi_j$ s, and the  $\tau_j$ s (see section 2.2) are eigenfunctions and eigenvector fields of the Dirichlet Laplacean operator in  $(a,b)$  and in  $\mathcal{O}$ . Notice that if  $\check{K}$  has a nontrivial kernel  $\mathcal{N}(\check{K}) = \{\kappa \in \mathbb{R}^{2M^2} \mid \check{K} \kappa = 0\}$ , then  $\kappa$  is not unique but, for given  $\kappa \in \mathbb{R}^{2M^2}$  we can set the unique  $\check{\kappa} \in \mathbb{R}^{2M^2}$  solving

$$\check{K} \check{\kappa} = \check{K} \kappa \quad \text{and} \quad \check{\kappa} \in \mathcal{N}(\check{K})^\perp$$

where  $\mathcal{N}(\check{K})^\perp$  stands for the orthogonal complement, in  $\mathbb{R}^{2M^2}$ , of the kernel  $\mathcal{N}(\check{K})$ . In this way  $|\check{\kappa}|_{\mathbb{R}^{2M^2}}$  and  $|\check{K} \check{\kappa}|_{G_{\text{av}}^2((a,b),\Gamma)}$  are two norms in the finite-dimensional space  $\mathcal{N}(\check{K})^\perp$ ; it follows that for  $\check{\kappa}(v_0) \in \mathcal{N}(\check{K})^\perp$  with  $\check{K} \check{\kappa}(v_0) = K_t^\mathcal{O} \bar{\eta}(v_0)$ , we have  $|\check{\kappa}(v_0)|_{\mathbb{R}^{2M^2}}^2 \leq \overline{C}_{[M]} |K_t^\mathcal{O} \bar{\eta}(v_0)|_{G_{\text{av}}^2((a,b),\Gamma)}^2 \leq \overline{C}_{[\hat{u}]_{\mathcal{W}(a,b)|_{\text{st}}}} |v_0|_H^2$ .

## 6. FINAL REMARKS

**6.1. On further plausible consequences.** Departing from a theorem analogous to Theorem 5.2, in [BRS11] it was proven the internal feedback stabilization to a nonstationary solution for the Navier–Stokes equations. We can conjecture that the analogous result holds in the boundary control case. Of course, there are details that must be checked that we prefer to address in a future paper; here, we confine the illustration of applications of the observability inequalities to the examples in Theorems 5.2 and 5.3.

Also, Theorem 4.8 is inspired in the work in [Shi11] concerning the randomly forced Navier–Stokes equation with space-time internal localized noise. From a localized internal observability inequality, analogous to the boundary inequalities in Theorem 4.8, and using appropriate controls, it was proven in [Shi11] that the Markov process generated by the restriction of solutions to the instants of time proportional to the period possesses a unique stationary distribution, which is exponentially mixing. Then we can also conjecture that the analogous result holds with space-time boundary localized noise, as a consequence of the observability inequalities in Theorem 4.8. Again, there are details to be checked.

**6.2. On some of the regularity assumptions.** In section 2.3 we suppose we are able to apply a control through a subset  $\Gamma_c \subseteq \Gamma$ , that is the support of a given function  $\chi \in C^\infty(\Gamma, \mathbb{R})$ . Then, we (must) choose an open superset  $\mathcal{O} \supseteq \overline{\Gamma}_c$  for which we have/know the existence of the systems of eigenfunctions and eigenvector fields of the Laplace–de Rham operator. This freedom to choose an auxiliary superset on the support of the controls can be important for applications. Moreover, the asked smoothness of  $\partial\mathcal{O}$  may be not necessary; for example if  $\overline{\Gamma}_c \subset R \subset \Gamma$  where  $R$  is an open flat rectangle, we can find the corresponding systems of smooth eigenfunctions and eigenvector fields. Indeed, identifying  $R \sim [0, s] \times [0, r]$ , we find the system of eigenfunctions  $\mathcal{F} = \{2/(sr)^{1/2} \sin(n_1\pi z_1/s) \sin(n_2\pi z_2/r) \mid n = (n_1, n_2) \in \mathbb{N}_0^2\}$ , and the system of eigenvector fields  $(\mathcal{F}, 0) \cup (0, \mathcal{F})$ , with  $(z_1, z_2) \in [0, s] \times [0, r]$  being “the” coordinates in  $R$ .

In section 2.1, we suppose  $C^\infty$  regularity for the boundary  $\partial\Omega$  because we use some results that have been derived for  $C^\infty$ -smooth Riemannian manifolds, namely results from [FGH02, Aub82, Tay97, Sch95]. The derivation of the necessary results for less regular boundaries is out of the scope of this work. Anyway, concerning the control space in section 5, the  $C^\infty$ -regularity is only needed for the auxiliary subset  $\mathcal{O} \subseteq \Gamma$  containing the support  $\overline{\Gamma}_c$  of the admissible boundary controls; away from  $\overline{\mathcal{O}} \subseteq \Gamma$  the  $C^4$ -regularity is sufficient (to use, in section 3, the results from [Rod14]).

## — APPENDIX —

**A.1. Laplace–de Rham operator.** We assume some familiarity with some basic tools from differential geometry. We refer to [Car67, dC94, Jos05, Tra84].

Let  $p \in \Gamma$  and consider the diffeomorphism  $\Phi_p$ , in (3), mapping the cylinder  $\mathbb{C}_p$  onto the tubular neighborhood  $\mathcal{T}_p$ . First we recall that we may see the open subset  $\mathcal{T}_p \subset \mathbb{R}^3$  with its induced Euclidean metric as the manifold  $(\mathbb{C}_p, g)$  for a suitable Riemannian metric tensor  $g = \sum_{i,j=1}^3 g_{ij} dw^i \otimes dw^j$ . We may suppose that the ordered triple  $(w^1, w^2, w^3)$  preserves the orientation of  $\mathcal{T}_p$  (otherwise we just change  $w^1$  with  $w^2$  in the triple). Let  $\partial/\partial w^i$  be the vector field induced in  $\mathcal{T}_p$  by the new coordinate function  $w^i$ ,  $i = 1, 2, 3$ . If  $(x^1, x^2, x^3)$  are the Euclidean coordinate functions in  $\mathcal{T}_p \subset \mathbb{R}^3$  we find that

$$(A.1) \quad \begin{aligned} \partial/\partial w^i \Big|_{(w^1, w^2, w^3)} &= \partial/\partial x^i + \partial_{w^i} \Phi_p^0 \Big|_{(w^1, w^2, w^3)} \partial/\partial x^3 \text{ for } i \in \{1, 2\}, \text{ and} \\ \partial/\partial w^3 \Big|_{(w^1, w^2, w^3)} &= \mathbf{n}_{\Phi_p^0(w^1, w^2)}. \end{aligned}$$

Since  $(w^1, w^2, \Phi_p^0(w^1, w^2)) \in \Gamma$ , it will follow that both  $\partial/\partial w^1$  and  $\partial/\partial w^2$  are orthogonal to  $\partial/\partial w^3$ . Moreover the length of  $\partial/\partial w^i$ , for  $i \in \{1, 2\}$ , is given by  $\left(1 + (\partial_{w^i}\Phi_p^0)^2\right)^{1/2}$ ; the Euclidean scalar product  $(\partial/\partial w^1, \partial/\partial w^2)$  is equal to  $\partial_{w^1}\Phi_p^0\partial_{w^2}\Phi_p^0$ ; and the length of  $\partial/\partial w^3$  is 1. Thus the metric tensor becomes

$$g = \left(1 + (\partial_{w^1}\Phi_p^0)^2\right) dw^1 \otimes dw^1 + \partial_{w^1}\Phi_p^0\partial_{w^2}\Phi_p^0(dx^1 \otimes dw^2 + dx^2 \otimes dx^1) \\ + \left(1 + (\partial_{w^2}\Phi_p^0)^2\right) dw^2 \otimes dw^2 + dw^3 \otimes dw^3.$$

The Euclidean volume element in  $\mathcal{T}_p$  may then be written as

$$d\mathcal{C}_p = \sqrt{\bar{g}}dw^1 \wedge dw^2 \wedge dw^3, \quad \text{with } \bar{g} := \det[g_{ij}] = 1 + (\partial_{w^1}\Phi_p^0)^2 + (\partial_{w^2}\Phi_p^0)^2.$$

Let  $\mathcal{O} \subseteq \Gamma$  be a smooth connected manifold, either with or without boundary. The Laplace–de Rham operator  $\Delta_{\mathcal{O}}$  on the two-dimensional manifold  $\mathcal{O}$ , is defined locally in  $\mathcal{T}_p \cap \mathcal{O}$ , by means of compositions of the Hodge star  $*$ , exterior derivative  $d$ , sharp  $\sharp$  and flat  $\flat$  mappings: for a given  $k$ -differential form  $\alpha$  we put  $\Delta_{\mathcal{O}}\alpha := -( *d *d + d *d *)\alpha$ .  $\Delta_{\mathcal{O}}$  maps  $k$ -forms into  $k$ -forms. A function  $f$  is a 0-form, and it turns out that for functions we have  $d * f = 0$  so  $\Delta_{\mathcal{O}}f = - * d * df$ . To compute the Laplacean (Laplace–de Rham) of a vector field  $v \in T\mathcal{O}$  we use in addition the sharp  $\sharp$  and flat  $\flat$  mappings:

$$(A.2) \quad \Delta_{\mathcal{O}}v := (\Delta_{\mathcal{O}}v^{\sharp})^{\flat}.$$

We recall that  $\sharp$  maps vector fields into 1-forms, and  $\flat$  maps 1-forms into vector fields: for a vector field  $V = \sum_{i=1}^3 V^i \partial/\partial w^i$  and a 1-form  $\alpha = \sum_{i=1}^3 \alpha_i dw^i$ , we have  $V^{\sharp} := \sum_{i,j=1}^3 g_{ij} V^i dw^j$  and  $\alpha^{\flat} := \sum_{i,j=1}^3 g^{ij} \alpha_i \partial/\partial w^j$ , where  $[g^{ij}]$  stands for the inverse matrix of  $[g_{ij}]$ . It turns out that  $\sharp$  and  $\flat$  are inverse to each other:  $(V^{\sharp})^{\flat} = V$  and  $(\alpha^{\flat})^{\sharp} = \alpha$ .<sup>1</sup>

*Eigenfunctions and eigenvector fields.* We are interested in functions and vector fields vanishing outside a submanifold  $\mathcal{O} \subseteq \Gamma$ . Since we need some regularity for those functions and vector fields, two cases must be considered: the case  $\partial\mathcal{O} \neq \emptyset$  and the case  $\partial\mathcal{O} = \emptyset$ .

- *The case  $\partial\mathcal{O} \neq \emptyset$ .* Consider the Laplace–de Rham operator  $\Delta_{\mathcal{O}}$ :

$$\Delta_{\mathcal{O}} : H^2(\mathcal{O}, Y) \cap H_0^1(\mathcal{O}, Y) \rightarrow L^2(\mathcal{O}, Y) \\ u \mapsto \Delta_{\mathcal{O}}u,$$

where  $H_0^1(\mathcal{O}, Y)$  is the closure of the space of smooth mappings  $C_c^\infty(\mathcal{O}, Y)$ , having a compact support contained in  $\mathcal{O}$ , in the  $H^1(\mathcal{O}, Y)$ -norm.

For the case of functions and 1-forms, respectively  $Y = \mathbb{R}$  and  $Y = T^*\mathcal{O}$ , it follows that  $\Delta_{\mathcal{O}}$  is an isomorphism, see e.g. [Sch95, Theorem 3.4.10]. See also [Tay97, section 5.1] for the particular case of functions.

Notice that we consider that the 1-forms satisfy the (homogeneous) Dirichlet boundary conditions  $w|_{\partial\mathcal{O}} = 0$ , where the restriction has the same meaning as in [Sch95], i.e.,  $w|_{\partial\mathcal{O}} : \cup_{p \in \partial\mathcal{O}} T_p\Gamma \rightarrow \mathbb{R}$ ,  $(w|_{\partial\mathcal{O}})|_p(v) := w|_p(v)$ , for any  $v \in T_p\Gamma$ ,  $p \in \partial\mathcal{O}$ .<sup>2</sup>

For vector fields, i.e., in the case  $Y = T\mathcal{O}$ , from (A.2) follows that  $\Delta_{\mathcal{O}}V = U$  if, and only if,  $\Delta_{\mathcal{O}}V^{\sharp} = U^{\sharp}$  and, then  $\Delta_{\mathcal{O}} : H^2(\mathcal{O}, Y) \cap H_0^1(\mathcal{O}, Y) \rightarrow L^2(\mathcal{O}, Y)$  is also a isomorphism in this case. Notice that from well known properties of the Hodge star,

<sup>1</sup>In some works the roles of  $\sharp$  and  $\flat$  are changed. The Laplace–de Rham operator is defined to have nonnegative eigenvalues; in Euclidean (flat) domains it coincides with the symmetric of the “usual” Laplacean,  $\Delta_{\mathcal{O}} = -\Delta$ .

<sup>2</sup>As we see, to define  $w|_{\partial\mathcal{O}}$ , we essentially need  $w_p$  to be well defined in  $T_p\Gamma$ , for  $p \in \partial\mathcal{O}$ . Notice also that, for some authors, the terminology “Dirichlet boundary conditions”, for 1-forms, stand for different boundary conditions as in [Tay97, section 5.9]; the meaning we use here coincides, in the Euclidean case, to say that of all coordinate components of the 1-form  $w$  must vanish.

wedge product and interior product mappings, see e.g. [Tay97] or [Rod08, section 5.7], we can write  $*(\alpha \wedge * \beta) = -\iota_{\alpha^b} * * \beta = \beta(\alpha^b) = g(\beta^b, \alpha^b)$ , from which we conclude that  $(\alpha, \beta)_{L^2(\mathcal{O}, T^*\mathcal{O})} := \int_{\mathcal{O}} \alpha \wedge * \beta = \int_{\mathcal{O}} *(\alpha \wedge * \beta) d\mathcal{O} = \int_{\mathcal{O}} g(\beta^b, \alpha^b) d\mathcal{O} =: (\alpha^b, \beta^b)_{L^2(\mathcal{O}, T\mathcal{O})}$ .

Moreover  $\Delta_{\mathcal{O}}$  is self-adjoint and have compact inverse. We can deduce the existence of a system of eigenvalues  $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ , with  $\lambda_k \rightarrow +\infty$ , and corresponding eigenforms  $\Delta_{\mathcal{O}}\zeta_k = \lambda_k\zeta_k$ , forming a complete orthonormal system  $\{\zeta_k \mid k \in \mathbb{N}_0\}$  in  $L^2(\mathcal{O}, Y)$ . That the first eigenvalue is nonzero follows from the fact that  $\Delta_{\mathcal{O}}w = 0$  and  $w|_{\partial\mathcal{O}} = 0$  imply that  $0 = (\Delta_{\mathcal{O}}w, w)_{L^2(\mathcal{O}, Y)} = (dw, dw)_{L^2(\Gamma_{\mathcal{O}}, Y)} + (\operatorname{div} w, \operatorname{div} w)_{L^2(\mathcal{O}, Y)}$ , i.e.,  $dw = 0 = \operatorname{div} w$ , where  $\operatorname{div} w := - * d * w$ , and by [Sch95, Theorem 3.4.4] it follows that  $w = 0$ , necessarily. In the case of functions,  $Y = \mathbb{R}$ , we have also that the first eigenfunction do not change sign in  $\mathcal{O}$ , see [Tay97, chapter 5, Proposition 2.4]. The eigenforms are  $C^\infty$ -smooth due to [Sch95, Theorem 3.4.10]

- *The case  $\partial\mathcal{O} = \emptyset$ .* In this case  $\mathcal{O}$  is a connected component of  $\Gamma$ . In the boundaryless case we still have the existence of a system of eigenvalues  $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ , with  $\lambda_k \rightarrow +\infty$ , and corresponding smooth eigenforms  $\zeta_k$ , with  $\Delta_{\mathcal{O}}\zeta_k = \lambda_k\zeta_k$ , forming a complete orthonormal system  $\{\zeta_k \mid k \in \mathbb{N}_0\}$  in  $L^2(\mathcal{O}, Y)$ . The finite-dimensional eigenspace corresponding to the first eigenvalue  $\lambda_1 = 0$ , is the space of harmonic forms  $w$ , defined by  $dw = 0$  if  $w$  is a function, and by  $dw = 0$  and  $\operatorname{div} w = 0$  if  $w$  is a 1-form. For more details see [Tay97, section 5.8].

*Sobolev spaces.* By an interpolation argument, for example, reasoning as in [LM72], we can identify, for any  $s \in \mathbb{R}$ , the Sobolev space  $H^s(\mathcal{O}, \mathbb{R})$  with the domain  $D((1 + \Delta_{\mathcal{O}})^{s/2})$  of the operator  $1 + \Delta_{\mathcal{O}} : H^2(\mathcal{O}, \mathbb{R}) \rightarrow L^2(\mathcal{O}, \mathbb{R})$ :  $H^s(\mathcal{O}, \mathbb{R}) = \{f \mid (1 + \Delta_{\mathcal{O}})^{s/2}f \in L^2(\mathcal{O}, \mathbb{R})\}$ , and endow it with the norm

$$(A.3) \quad |f|_{H^s(\mathcal{O}, \mathbb{R})} := |(1 + \Delta_{\mathcal{O}})^{s/2}f|_{L^2(\mathcal{O}, \mathbb{R})}$$

Alternatively we can use the partition of unity, associated to a given atlas, approach as in [Tay97]. However, though more abstract, the interpolation setting can sometimes lead to simpler expositions. A short comment on the equivalence of these two approaches is given in section A.10.

**A.2. On interpolation and fractional Sobolev-Bochner spaces.** Here we recall some results on interpolation, mainly from [LM72]. We present some proofs just for the sake of completeness.

Given a Banach space  $X$ , the norm of the Sobolev-like space  $H^s((a, b), X)$  can be defined by means of the Fourier transform (see, e.g., [FGH02]). First  $H^s((a, b), X)$  can be defined as  $H^s((a, b), X) := \{\tilde{u}|_{(a, b)} \mid \tilde{u} \in H^s(\mathbb{R}, X)\}$ ; and the Fourier transform, in the (time) variable  $t \in (a, b)$ , of  $\tilde{u}$  is defined by  $\mathcal{F}_t(\tilde{u})(\tau) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i\tau r} \tilde{u}(r) dr$ . Then, the space  $H^s(\mathbb{R}, X)$  is endowed with the norm

$$(A.4) \quad |\tilde{u}|_{H^s(\mathbb{R}, X)} := |(1 + |\tau|^2)^{s/2} \mathcal{F}_t(\tilde{u})|_{L^2(\mathbb{R}, X)},$$

and  $H^s((a, b), X)$  can be seen as the quotient space  $H^s(\mathbb{R}, X)/H_{\mathbb{R} \setminus (a, b)}^s(\mathbb{R}, X)$ , with  $H_{\mathbb{R} \setminus (a, b)}^s(\mathbb{R}, X) := \{u \in H^s(\mathbb{R}, X) \mid \operatorname{supp} u \subset \mathbb{R} \setminus (a, b)\}$ , and is endowed with the quotient norm

$$(A.5) \quad |u|_{H^s((a, b), X)} := \inf_{F_s} \left\{ |F_s u|_{H^s(\mathbb{R}, X)} \mid F_s u|_{(a, b)} = u \right\}.$$

where the infimum is taken over all continuous extensions

$$F_s : H^s(\mathbb{R}, X)/H_{\mathbb{R} \setminus (a, b)}^s(\mathbb{R}, X) \rightarrow H^s(\mathbb{R}, X).$$

*Remark A.1.* Let  $m \in \mathbb{N}_0$ . A continuous extension  $\hat{F}_m: H^m((a, b), X) \rightarrow H^m(\mathbb{R}, X)$ ,  $u \mapsto \hat{F}_m u$  can be constructed from a standard procedure; e.g., for  $l = b - a > 0$ , we define the function 
$$\begin{cases} \bar{u}(a - t) & := \sum_{i=1}^m \lambda_i u(a + (t/i)), & \text{if } t \in (0, l) \\ \bar{u}(t) & := u(t), & \text{if } t \in (a, b). \end{cases}$$
 Similarly we can construct an extension to  $(b, b + l)$ , and we arrive to an extension  $\hat{u}$  to  $(a - l, b + l)$ . Now we may multiply by a  $C^\infty$ -smooth function  $\phi$  supported in  $[a - (l/2), b + (l/2)]$  and taking the value 1 in  $[a, b]$ . The obtained extension,  $u \mapsto \hat{F}_m u := \phi \hat{u}$ , is continuous from  $H^j((a, b), X)$  into  $H^j(\mathbb{R}, X)$  for all  $0 \leq j \leq m$ , i.e.,

$$\hat{F}_m \in \mathcal{L}(H^j((a, b), X) \rightarrow H^j(\mathbb{R}, X)), \text{ for all } j \in \mathbb{N}, j \leq m,$$

if the constants  $\lambda_i$ s solve the system:  $1 = \sum_{i=1}^m (-1)^{k-1} \lambda_i / i^{k-1}$ ,  $k = 1, 2, \dots, m$  (cf. [FI98, section 2.1]).

*Remark A.2.* Notice that the extension  $\bar{u}$  in Remark A.1 is well defined for functions in  $L^2((a, b), X)$ , indeed we can suppose that  $u(a + t)$  is defined for all  $t \in (0, l) \setminus \mathcal{N}$  where  $\mathcal{N}$  is a set of measure zero. Then  $\bar{u}(a - t)$  is defined for all  $t \in (0, l) \setminus \cup_{i=1}^m i\mathcal{N}$ , that is,  $\bar{u}(a - t)$  is well defined for a.e.  $t \in (0, l)$ .

A.2.1. *The space  $H^s((a, b), X)$  as an interpolation space.*

**Definition A.1.** A pair of Hilbert spaces  $(X, Y)$  is said an interpolation pair if  $X \subseteq Y$ , and the inclusion is dense and continuous.<sup>3</sup>

**Theorem A.1** (Interpolation Theorem). *Let us be given two pairs  $(X, Y)$  and  $(X_1, Y_1)$  of interpolation pairs. If  $L$  is a linear and continuous operator both from  $X$  into  $Y$  and from  $X_1$  into  $Y_1$ , then it is also linear and continuous from  $[X, Y]_\theta$  into  $[X_1, Y_1]_\theta$ ,  $0 < \theta < 1$ . Moreover  $|L|_{\mathcal{L}([X, Y]_\theta \rightarrow [X_1, Y_1]_\theta)} \leq C \max\{|L|_{\mathcal{L}(X \rightarrow X_1)}, |L|_{\mathcal{L}(Y \rightarrow Y_1)}\}$ .*

This theorem can be found in [LM72, chapter 1, Theorem 5.1]. The estimate follows from the last equation in the proof of the Theorem 5.1 in [LM72, chapter 1], and from Remark 4.2 in [LM72, chapter 1].

*Remark A.3.* From [LM72, chapter 1, Theorem 4.2], for  $0 < \theta < 1$ , we have the following trace characterization:  $[X, Y]_\theta = \{f(0) \mid \mathcal{F}_t f \in L^2(\mathbb{R}, X) \text{ and } |\tau|^{(2\theta)-1} \mathcal{F}_t f \in L^2(\mathbb{R}, Y)\}$ . This characterization is used in [Gri67] as definition of the interpolation space.

The following Reiteration Theorem can be found in [LM72, chapter 1, section 6.1].

**Theorem A.2** (Reiteration Theorem). *Let  $(X, Y)$  be an interpolation pair, and let  $\theta, \theta_0, \theta_1 \in (0, 1)$  such that  $\theta_1 > \theta_0$ . Then, we have  $[[X, Y]_{\theta_0}, [X, Y]_{\theta_1}]_\theta = [X, Y]_{(1-\theta)\theta_0 + \theta\theta_1}$ , with equivalent norms.*

**Lemma A.3.** *Let us be given a Hilbert space  $X$ , then both  $(H^2(\mathbb{R}, X), L^2(\mathbb{R}, X))$  and  $(H^2((a, b), X), L^2((a, b), X))$  are interpolation pairs.*

*Proof.* The continuity of the inclusion is clear, from the definitions. The densities follow from the density of  $C^\infty([a, b], X) \subset H^2((a, b), X)$ , in  $L^2((a, b), X)$ ; and from  $\{u \in C^\infty(\mathbb{R}, X) \mid \text{supp } u \text{ is compact}\} \subset H^2(\mathbb{R}, X)$ , in  $L^2(\mathbb{R}, X)$ ; see e.g. [LM72, chapter 1, Theorem 2.1].  $\square$

Now, fix  $s \geq 0$ . Inspired by [LM72, chapter 1, section 7.1], see also [Gri67], we define the spaces of measurable functions  $H_{\mathcal{F}}^s(\mathbb{R}, X) = \{f \mid \tau \mapsto (1 + |\tau|^2)^{s/2} f(\tau) \in L^2(\mathbb{R}, X)\}$ , that is,  $H_{\mathcal{F}}^s(\mathbb{R}, X)$  is the domain of the auto-adjoint operator  $f(\cdot) \mapsto \Lambda^{s/2} f(\cdot) := (1 + |\cdot|^2)^{s/2} f(\cdot)$ .

<sup>3</sup>In the Literature we can find more general definitions for interpolation pair.

It turns out that for  $s \in [0, 1]$ , and following [LM72],  $H_{\mathcal{F}}^{2s}(\mathbb{R}, X)$  and  $H_{\mathcal{F}}^s(\mathbb{R}, X)$  can be seen as the interpolation spaces

$$H_{\mathcal{F}}^{2s}(\mathbb{R}, X) = [H_{\mathcal{F}}^2(\mathbb{R}, X), L^2(\mathbb{R}, X)]_{1-s}, \quad H_{\mathcal{F}}^s(\mathbb{R}, X) = [H_{\mathcal{F}}^1(\mathbb{R}, X), L^2(\mathbb{R}, X)]_{1-s}.$$

On the other hand the Fourier transform is an isomorphism from  $H^i(\mathbb{R}, X)$  onto  $H_{\mathcal{F}}^i(\mathbb{R}, X)$ , with  $i \in \{1, 2\}$ , and from  $L^2(\mathbb{R}, X)$  onto itself. By the Interpolation Theorem, we can derive that it also defines an isomorphism from  $[H^i(\mathbb{R}, X), L^2(\mathbb{R}, X)]_{1-s}$  onto  $H_{\mathcal{F}}^{is}(\mathbb{R}, X)$ , that is,

$$[H^i(\mathbb{R}, X), L^2(\mathbb{R}, X)]_{1-s} = \mathcal{F}_t^{-1} H_{\mathcal{F}}^{is}(\mathbb{R}, X) = H^{is}(\mathbb{R}, X), \quad i \in \{1, 2\}.$$

By definition we have  $H^s((a, b), X) := \{\tilde{u}|_{(a,b)} \mid \tilde{u} \in H^s(\mathbb{R}, X)\}$ ; on the other hand, the restriction mapping  $u \mapsto u|_{(a,b)}$  is a surjective mapping in  $\mathcal{L}(H^j(\mathbb{R}, X), H^j((a, b), X))$ , for  $j \in \{0, 1, 2\}$ ; then, by the Interpolation Theorem, we can conclude, in particular, that it maps  $H^{is}(\mathbb{R}, X)$  onto  $[H^i((a, b), X), L^2((a, b), X)]_{1-s}$ . Therefore we also have the characterization  $H^{is}((a, b), X) = [H^i((a, b), X), L^2((a, b), X)]_{1-s}$ , for  $i \in \{1, 2\}$ .

We can conclude that, see also [Gri67, section 3] and [LM72, chapter 1, section 9.4], for  $0 \leq s \leq 1$

$$(A.6) \quad \begin{aligned} H^{2s}(\mathbb{R}, X) &= [H^2(\mathbb{R}, X), L^2(\mathbb{R}, X)]_{1-s}, \\ H^{2s}((a, b), X) &= [H^2((a, b), X), L^2((a, b), X)]_{1-s}, \\ H^s(\mathbb{R}, X) &= [H^1(\mathbb{R}, X), L^2(\mathbb{R}, X)]_{1-s}, \\ H^s((a, b), X) &= [H^1((a, b), X), L^2((a, b), X)]_{1-s}. \end{aligned}$$

A.2.2. *The space  $H^i((a, b), [X, Y]_{\theta})$  as an interpolation space.*

**Lemma A.4.** *Let  $i \in \{0, 1\}$ . If  $(X, Y)$  is an interpolation pair, then also the pairs  $(H^i((a, b), X), H^i((a, b), Y))$  and  $(H^i(\mathbb{R}, X), H^i(\mathbb{R}, Y))$  are.*

*Proof.* First of all, though a bit long, the proof is straightforward. However, we present it for the sake of completeness and because we are going to use one of its arguments again later in section A.6.

By the density remark in the proof of Lemma A.3, given  $u \in L^2((a, b), Y)$  we can find a smooth function  $\phi^n \in C^\infty([a, b], Y)$  such that  $|\phi^n - u|_{L^2((a,b),Y)} \leq 1/n$ . Next we can approximate  $\phi^n$  by a step function, e.g., we set  $N = N(n) \in \mathbb{N}_0$  such that  $N \geq \max\{|\phi^n|_{C^1([a,b],Y)}^2, n^2/3\}$  and set the function  $\psi^n(t) := \phi^n(a + m(b-a)/N)$ , for  $t \in I_N^m := [a + m(b-a)/N, a + (m+1)(b-a)/N)$ , and  $m \in \{0, 1, \dots, N-1\}$ . Notice that  $|\phi^n - \psi^n|_{L^2(I_N^m, Y)}^2 = \int_{I_N^m} |\phi^n(t) - \phi^n(a + m(b-a)/N)|_Y^2 dt \leq |\phi^n|_{C^1(\overline{I_N^m}, Y)}^2 \int_{I_N^m} |t - (a + m(b-a)/N)|_{\mathbb{R}}^2 dt$ , from which we derive  $|\phi^n - \psi^n|_{L^2(I_N^m, Y)}^2 \leq |\phi^n|_{C^1(\overline{I_N^m}, Y)}^2 \int_0^{b-a/N} t^2 dt \leq |\phi^n|_{C^1(\overline{I_N^m}, Y)}^2 (b-a)^3/3N^3 \leq (b-a)^3/3N^2$ . We now approximate each value  $\phi^n(a + m(b-a)/N) \in Y$ , by a vector  $\xi_N^{n,m} \in X$  such that  $|\xi_N^{n,m} - \phi^n(a + m(b-a)/N)|_Y \leq 1/n$ , and define the step function  $\tilde{\psi}^n(t) := \xi_N^{n,m}$ , for  $t \in I_N^m$ . We find that  $|\tilde{\psi}^n - u|_{L^2((a,b),Y)}^2 \leq |\tilde{\psi}^n - \psi^n + \psi^n - \phi^n + \phi^n - u|_{L^2((a,b),Y)}^2 \leq 3((b-a)(1/n^2) + N(b-a)^3/3N^2 + 1/n^2)$ , from which it follows  $|\tilde{\psi}^n - u|_{L^2((a,b),Y)}^2 \leq 3(1+(b-a)^3+(b-a))/n^2$ . We can conclude that  $L^2((a, b), X)$  is dense in  $L^2((a, b), Y)$ . Therefore, given  $u \in H^1((a, b), Y)$  we can find  $w \in L^2((a, b), X)$  such that  $|\partial_t u - w|_{L^2((a,b),Y)}^2 \leq 1/n$ . Since  $u \in C([a, b], Y)$  (see e.g. [Tem01, chapter 3, Lemma 1.1]), we can define  $p^w(t) := u(a) + \int_a^t w(\tau) d\tau$ , for  $t \in (a, b)$ , and we have  $(u - p^w)(t) = \int_a^t \partial_t u - w d\tau$ ; it follows that  $|(u - p^w)(t)|_Y^2 = (\int_a^t \partial_t u - w d\tau, \int_a^t \partial_t u - w ds)_Y = \int_a^t (\partial_t u - w, \int_a^t \partial_t u - w ds)_Y d\tau \leq |\partial_t u - w|_{L^2((a,t),Y)} |(u - p^w)(t)|_Y (t-a)^{1/2}$ , from which we can derive  $|(u - p^w)(t)|_Y^2 \leq |\partial_t u - w|_{L^2((a,t),Y)}^2 (t-a) \leq 1/n(t-a)$ , and then  $|u - p^w|_{H^1((a,b),Y)}^2 \leq 1/n(1 + (b-a)^2/2)$ . Now, set  $\bar{u}_a \in X$  such that  $|\bar{u}_a - u(a)|_Y^2 < 1/n$  and  $q^w := p^w + \bar{u}_a - u(a) = \bar{u}_a + \int_a^t w(\tau) d\tau$ .

Then we have  $q^w \in H^1((a, b), X)$  and  $|u - q^w|_{H^1((a, b), Y)}^2 \leq 2|u - p^w|_{H^1((a, b), Y)}^2 + 2|\bar{u}_a - u(a)|_{H^1((a, b), Y)}^2 \leq \bar{C}_{[b-a]}^{1/n}$ . We can conclude that  $H^1((a, b), X)$  is dense in  $H^1((a, b), Y)$ .

Similarly, given  $u \in L^2(\mathbb{R}, Y)$  we can find a smooth function  $\phi^n \in \{\phi \in C^\infty(\mathbb{R}, Y) \mid \text{supp}(\phi) \text{ is compact}\}$  such that  $|\phi^n - u|_{L^2(\mathbb{R}, Y)} \leq 1/n$ . Let  $T > 0$  be such that  $\text{supp}(\phi_n) \subset (-T, T)$ . Then we can just repeat the proof above replacing  $a$  by  $-T$  and  $b$  by  $T$ ; we conclude that  $L^2(\mathbb{R}, X)$  is dense in  $L^2(\mathbb{R}, Y)$ , and  $H^1(\mathbb{R}, X)$  is dense in  $H^1(\mathbb{R}, Y)$ .

It remains to observe that from the continuity of the inclusion  $X \subseteq Y$ , it easily follows that of the inclusions  $H^i(I, X) \subseteq H^i(I, Y)$ , for any interval  $I \subseteq \mathbb{R}$ , and  $i \in \{0, 1\}$ .  $\square$

**Lemma A.5.** *Let  $r \geq 0$  and  $\theta \in [0, 1]$  be real numbers. Then, for any given open interval  $I \subseteq \mathbb{R}$ , we have that  $[H^r(I, X), H^r(I, Y)]_\theta = H^r(I, [X, Y]_\theta)$ .*

*Proof.* The statement clearly holds for  $\theta \in \{0, 1\}$ . For  $\theta \in (0, 1)$ , from Remark A.3, we have

$$[H^r(\mathbb{R}, X), H^r(\mathbb{R}, Y)]_\theta = \left\{ f(0, \cdot) \mid \begin{array}{l} \mathcal{F}_s f \in L^2(\mathbb{R}, H^r(\mathbb{R}, X)) \\ |\sigma|^{(2\theta)-1} \mathcal{F}_s f \in L^2(\mathbb{R}, H^r(\mathbb{R}, Y)) \end{array} \right\},$$

where  $\mathcal{F}_s f = \mathcal{F}_s f(\sigma, t)$ , is the Fourier transform of  $f = f(s, t)$ , with respect to the variable  $s$ . Denote also by  $\mathcal{F}_t f(s, \tau)$  the Fourier transform with respect to the variable  $t$ ; we have

$$\begin{aligned} & [H^r(\mathbb{R}, X), H^r(\mathbb{R}, Y)]_\theta \\ &= \left\{ f(0, \cdot) \mid \int_{\mathbb{R}} (1 + |\tau|^2)^r d\tau \int_{\mathbb{R}} |\mathcal{F}_t \mathcal{F}_s f|_X^2 + \left| |\sigma|^{(2\theta)-1} \mathcal{F}_t \mathcal{F}_s f \right|_Y^2 d\sigma < +\infty \right\} \\ &= \left\{ f(0, \cdot) \mid \int_{\mathbb{R}} (1 + |\tau|^2)^r d\tau \int_{\mathbb{R}} |\mathcal{F}_s \mathcal{F}_t f|_X^2 + \left| |\sigma|^{(2\theta)-1} \mathcal{F}_s \mathcal{F}_t f \right|_Y^2 d\sigma < +\infty \right\} \\ &= \left\{ f(0, \cdot) \mid \int_{\mathbb{R}} (1 + |\tau|^2)^r |\mathcal{F}_t f(0, \tau)|_{[X, Y]_\theta}^2 d\tau < +\infty \right\} = H^r(\mathbb{R}, [X, Y]_\theta). \end{aligned}$$

Now, since the restriction mapping  $u \mapsto u|_I$ , maps  $H^r(\mathbb{R}, Z)$  onto  $H^r(I, Z)$ , with  $Z \in \{X, Y\}$  and, there exists a continuous extension from  $H^r(I, Z)$  onto  $H^r(\mathbb{R}, Z)$ , by the Interpolation Theorem, we can derive that the restriction mapping maps  $H^r(\mathbb{R}, [X, Y]_\theta)$  onto  $[H^r(I, X), H^r(I, Y)]_\theta$ , i.e., we have that  $H^r(I, [X, Y]_\theta) := H^r(\mathbb{R}, [X, Y]_\theta)|_I = [H^r(I, X), H^r(I, Y)]_\theta$ .  $\square$

*Remark A.4.* The identity  $[H^{s_1}(I, X), H^{s_2}(I, Y)]_\theta = H^{(1-\theta)s_1 + \theta s_2}(I, [X, Y]_\theta)$  can be found in [LM72, chapter 1, section 9.4]. However, to be coherent with our definition of interpolation space (and with the setting in [LM72, chapter 1, section 2.1]), we would need (at least) to impose the condition  $s_1 \geq s_2$ . To give an idea, let  $X \subseteq Y$  continuously and let  $I = (0, 1)$ , then for any  $\bar{x} \in X$ , we have that the function  $\psi(t, x) := t^{1/2} \bar{x}$  is in  $L^2(I, X) \setminus H^1(I, Y)$ . Indeed,  $|\psi|_{L^2(I, X)}^2 = |\bar{x}|_X^2 |t|_{L^1(I, \mathbb{R})} < +\infty$ , and  $|\psi|_{H^1(I, Y)}^2 \geq |\partial_t \psi|_{L^2(I, Y)}^2 = (1/4) |\bar{x}|_Y^2 |t^{-1}|_{L^1(I, \mathbb{R})} = +\infty$ .

*Remark A.5.* Let  $s_1 \geq s_2$  and let  $(X, Y)$  be an interpolation pair. It follows that  $(H^{s_1}(I, X), H^{s_2}(I, Y))$  is also an interpolation pair and that  $[H^{s_1}(I, X), H^{s_2}(I, Y)]_\theta = H^{(1-\theta)s_1 + \theta s_2}(I, [X, Y]_\theta)$ . Indeed, following [LM72, chapter 1, section 2.1] we can identify  $X$  with the domain of a suitable auto-adjoint, positive and unbounded operator  $\Lambda : X \rightarrow Y$ . Then we can make use of the operator  $\mathcal{F}_t u \mapsto \widehat{\Lambda} \mathcal{F}_t u := (1 + |\tau|^2)^{s_1 - s_2} \Lambda \mathcal{F}_t u$  to prove the identity  $[H^{s_1}(\mathbb{R}, X), H^{s_2}(\mathbb{R}, Y)]_\theta = H^{(1-\theta)s_1 + \theta s_2}(\mathbb{R}, [X, Y]_\theta)$ , and then the analogous identity for a given interval  $I$  will follow by a restriction and interpolation argument. Notice that we can identify  $[H^{s_1}(\mathbb{R}, X), H^{s_2}(\mathbb{R}, Y)]_\theta$  with  $\mathcal{F}_t^{-1} D(\widehat{\Lambda}^{1-\theta})$ , and from  $\widehat{\Lambda}^{1-\theta} = (1 + |\tau|^2)^{(s_1 - s_2)(1-\theta)} \Lambda^{1-\theta}$  we have that  $D(\widehat{\Lambda}^{1-\theta}) = \{\mathcal{F}_t u \in L^2(\mathbb{R}, Y) \mid$

$(1 + |\tau|^2)^{s_2 + (1-\theta)(s_1 - s_2)} \Lambda^{1-\theta} \mathcal{F}_t u \in L^2(\mathbb{R}, Y)$ , from which it follows that  $\mathcal{F}_t^{-1} D(\widehat{\Lambda}^{1-\theta}) = H^{(1-\theta)s_1 + \theta s_2}(\mathbb{R}, D(\Lambda^{1-\theta}))$ . Finally, from the identity  $D(\Lambda^{1-\theta}) = [X, Y]_\theta$ , we can conclude that  $[H^{s_1}(\mathbb{R}, X), H^{s_2}(\mathbb{R}, Y)]_\theta = H^{(1-\theta)s_1 + \theta s_2}(\mathbb{R}, [X, Y]_\theta)$  (cf. [LM72, chapter 1, section 9.4]).

**A.3. Proofs of Propositions 2.1 and 2.2.** We start with the following:

**Lemma A.6.** *Given  $v \in G_{\mathbf{n}}^s((a, b), \Gamma)$ , we have that  $\int_{\Gamma} v \, d\Gamma \in H^{r_{\mathbf{n},1}(s)}((a, b), \mathbb{R})$ . Moreover the mapping  $I_{\Gamma} : v \mapsto \int_{\Gamma} v \, d\Gamma$  is in  $\mathcal{L}(G_{\mathbf{n}}^s((a, b), \Gamma) \rightarrow H^{r_{\mathbf{n},1}(s)}((a, b), \mathbb{R}))$ .*

*Proof.* For a given  $u \in G_{\mathbf{n}}^s(\mathbb{R}, \Gamma)$ , we find that

$$\begin{aligned} |I_{\Gamma} u|_{H^{r_{\mathbf{n},1}(s)}(\mathbb{R}, \mathbb{R})}^2 &= \int_{\mathbb{R}} (1 + |\tau|^2)^{r_{\mathbf{n},1}(s)} |\mathcal{F}_t I_{\Gamma} u(\tau)|_{\mathbb{R}}^2 \, d\tau = \int_{\mathbb{R}} (1 + |\tau|^2)^{r_{\mathbf{n},1}(s)} \left| \int_{\Gamma} \mathcal{F}_t u(\tau, x) \, d\Gamma \right|_{\mathbb{R}}^2 \, d\tau \\ &= \int_{\mathbb{R}} (1 + |\tau|^2)^{r_{\mathbf{n},1}(s)} \left| \langle \mathcal{F}_t u(\tau, x), 1_{\Gamma} \rangle_{H^{r_{\mathbf{n},2}(s)}(\Gamma, \mathbb{R}), H^{-r_{\mathbf{n},2}(s)}(\Gamma, \mathbb{R})} \right|_{\mathbb{R}}^2 \, d\tau \\ &\leq \int_{\mathbb{R}} (1 + |\tau|^2)^{r_{\mathbf{n},1}(s)} |\mathcal{F}_t u(\tau, x)|_{H^{r_{\mathbf{n},2}(s)}(\Gamma, \mathbb{R})}^2 |1_{\Gamma}|_{H^{-r_{\mathbf{n},2}(s)}(\Gamma, \mathbb{R})}^2 \, d\tau \\ &= |1_{\Gamma}|_{H^{-r_{\mathbf{n},2}(s)}(\Gamma, \mathbb{R})}^2 |u|_{H^{r_{\mathbf{n},1}(s)}(\mathbb{R}, H^{r_{\mathbf{n},2}(s)}(\Gamma, \mathbb{R}))}^2, \end{aligned}$$

which implies that  $|I_{\Gamma} u|_{H^{r_{\mathbf{n},1}(s)}(\mathbb{R}, \mathbb{R})}^2 \leq |1_{\Gamma}|_{H^{-r_{\mathbf{n},2}(s)}(\Gamma, \mathbb{R})}^2 |u|_{G_{\mathbf{n}}^s(\mathbb{R}, \Gamma)}^2$ . Now, we observe that  $G_{\mathbf{n}}^s((a, b), \Gamma) = G_{\mathbf{n}}^s(\mathbb{R}, \Gamma)|_{(a,b)}$ , and  $\int_{\Gamma} u|_{(a,b)} \, d\Gamma = (\int_{\Gamma} u \, d\Gamma)|_{(a,b)}$ , for each  $u \in G_{\mathbf{n}}^s(\mathbb{R}, \Gamma)$ ; therefore, we have  $\int_{\Gamma} G_{\mathbf{n}}^s((a, b), \Gamma) \, d\Gamma = \int_{\Gamma} G_{\mathbf{n}}^s(\mathbb{R}, \Gamma)|_{(a,b)} \, d\Gamma = (\int_{\Gamma} G_{\mathbf{n}}^s(\mathbb{R}, \Gamma) \, d\Gamma)|_{(a,b)} \subseteq H^{r_{\mathbf{n},1}(s)}((a, b), \mathbb{R})$ . The linearity and continuity of the mapping  $I_{\Gamma}$  are not difficult to check: use the existence of a continuous extension  $F^s : G_{\mathbf{n}}^s((a, b), \Gamma) \rightarrow G_{\mathbf{n}}^s(\mathbb{R}, \Gamma)$  (see section A.2) and the continuity of the restriction  $u \mapsto u|_{(a,b)}$  from  $G_{\mathbf{n}}^s(\mathbb{R}, \Gamma)$  onto  $G_{\mathbf{n}}^s((a, b), \Gamma)$ , together with the Interpolation Theorem.  $\square$

*Proof of Proposition 2.1.* Given  $u \in G_{\mathbf{n}}^s((a, b), \Gamma)$  we write  $u = (u - \phi_u) + \phi_u$ , with  $\phi_u = 1/\int_{\Gamma} \, d\Gamma \int_{\Gamma} u \, d\Gamma$ . By Lemma A.6 we have that  $\phi_u \in H^{r_{\mathbf{n},1}(s)}((a, b), \mathbb{R})$ ; we observe that  $\phi_u$  is independent of  $x \in \Gamma$ ,  $\phi_u(t, x) = \phi_u(t)$  for all  $(t, x) \in \Gamma$ , then we obtain

$$\begin{aligned} |\phi_u|_{H^{r_1}((a,b), H^{r_2}(\Gamma, \mathbb{R}))}^2 &= \inf_{\bar{E}} \int_{\mathbb{R}} (1 + |\tau|^2)^{r_1} |\mathcal{F}_t(\bar{E}\phi_u)(\tau, x)|_{H^{r_2}(\Gamma, \mathbb{R})}^2 \, d\tau \\ &\leq \inf_E \int_{\mathbb{R}} (1 + |\tau|^2)^{r_1} |\mathcal{F}_t(E\phi_u)(\tau)|_{H^{r_2}(\Gamma, \mathbb{R})}^2 \, d\tau \\ &= \inf_E \int_{\mathbb{R}} (1 + |\tau|^2)^{r_1} |\mathcal{F}_t(E\phi_u)(\tau)|_{\mathbb{R}}^2 |1_{\Gamma}|_{H^{r_2}(\Gamma, \mathbb{R})}^2 \, d\tau = |1_{\Gamma}|_{H^{r_2}(\Gamma, \mathbb{R})}^2 |\phi_u|_{H^{r_1}((a,b), \mathbb{R})}^2, \end{aligned}$$

where  $\bar{E}$  runs over all continuous extensions  $H^{r_1}((a, b), H^{r_2}(\Gamma, \mathbb{R})) \rightarrow H^{r_1}(\mathbb{R}, H^{r_2}(\Gamma, \mathbb{R}))$ , and  $E$  runs over all continuous extensions  $H^{r_1}((a, b), \mathbb{R}) \rightarrow H^{r_1}(\mathbb{R}, \mathbb{R})$ . Now, setting  $(r_1, r_2) = (0, s - (1/2))$ , from  $0 < r_{\mathbf{n},1}(s)$  we can conclude that  $\phi_u \in L^2((a, b), H^{s-(1/2)}(\Gamma, \mathbb{R}))$ , and setting  $(r_1, r_2) = (r_{\mathbf{n},1}(s), r_{\mathbf{n},2}(s))$  we obtain  $\phi_u \in H^{r_{\mathbf{n},1}(s)}((a, b), H^{r_{\mathbf{n},2}(s)}(\Gamma, \mathbb{R}))$ ; it follows that  $\phi_u \in G_{\mathbf{n}}^s((a, b), \Gamma)$  and  $u - \phi_u \in G_{\mathbf{n},\text{av}}^s((a, b), \Gamma)$ . Therefore, we have that

$$G^s((a, b), \Gamma) = G_{\text{av}}^s((a, b), \Gamma) \oplus H^{r_{\mathbf{n},1}(s)}((a, b), \mathbb{R})\mathbf{n}.$$

Notice that  $G_{\text{av}}^s((a, b), \Gamma) \cap H^{r_{\mathbf{n},1}(s)}((a, b), \mathbb{R})\mathbf{n} = \{0\}$ , because for each  $u$  in the intersection, we have that  $u = \phi \mathbf{n}$  with  $\phi \in H^{r_{\mathbf{n},1}(s)}((a, b), \mathbb{R})$ , and  $0 = \int_{\Gamma} u \cdot \mathbf{n} \, d\Gamma$ , which implies  $0 = \phi \int_{\Gamma} \, d\Gamma$ , i.e.,  $\phi = 0$ . From the proof of Lemma A.6, we can also conclude the continuity of the projection  $u \mapsto \phi_u$  in  $G_{\mathbf{n}}^s((a, b), \Gamma)$ . Thus, since  $v \mapsto v \cdot \mathbf{n}$  is continuous from  $G^s((a, b), \Gamma)$  onto  $G_{\mathbf{n}}^s((a, b), \Gamma)$ , it follows also the continuity of  $v \mapsto \pi_v \mathbf{n}$  from  $G^s((a, b), \Gamma)$  onto itself, where  $\pi_v := 1/\int_{\Gamma} \, d\Gamma \int_{\Gamma} v \cdot \mathbf{n} \, d\Gamma$ .  $\square$

*Proof of Proposition 2.2.* Clearly  $E_s^e$  extends  $E_s$ :  $E_s^e u = E_s u$  for all  $u \in G_{\text{av}}^s((a, b), \Gamma)$ ; the linearity also follows straightforwardly. Notice that  $W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3)) \cap H^{r_{n,1}(s)}((a, b), \mathbb{R})\Theta = \{0\}$ , because for each  $w$  in the intersection, we have  $w = \psi\Theta$ , with  $\psi \in H^{r_{n,1}(s)}((a, b), \mathbb{R})$  and  $\text{div } w = 0$ , from which we obtain  $0 = \text{div}(\psi\Theta) = \psi \int_{\Gamma} d\Gamma / \int_{\Omega} d\Omega$ , i.e.,  $\psi = 0$ . Next, for simplicity we denote  $S^s := W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3)) \oplus H^{r_{n,1}(s)}((a, b), \mathbb{R})\Theta$ , and we find

$$\begin{aligned} |E_s^e u|_{S^s} &= |E_s u_{\text{av}} + \pi_u \Theta|_{S^s} \leq |E_s u_{\text{av}}|_{S^s} + |\pi_u \Theta|_{S^s} \\ &= |E_s u_{\text{av}}|_{W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))} + |\pi_u \Theta|_{H^{r_{n,1}(s)}((a, b), \mathbb{R})\Theta} \\ &\leq C |u_{\text{av}}|_{G_{\text{av}}^s((a, b), \Gamma)} + |\pi_u|_{H^{r_{n,1}(s)}((a, b), \mathbb{R})} \end{aligned}$$

and, from Proposition 2.1, we obtain  $|E_s^e u|_{S^s} \leq C_1 |u|_{G^s((a, b), \Gamma)}$ .

It remains to prove the continuity of the trace. Since  $S^s$  is a direct sum, from the definition of the norm of the sum, we have that  $|v|_{S^s}^2 = |v_{\text{div}}|_{W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))}^2 + |\psi\Theta|_{H^{r_{n,1}(s)}((a, b), \mathbb{R})\Theta}^2$ , with  $v = v_{\text{div}} + \psi\Theta$ ,  $v_{\text{div}} \in W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))$  and  $\psi \in H^{r_{n,1}(s)}((a, b), \mathbb{R})$ . Then, it follows that

$$\sqrt{2}|v|_{S^s} \geq |v_{\text{div}}|_{W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))} + |\psi\Theta|_{H^{r_{n,1}(s)}((a, b), \mathbb{R})\Theta}.$$

We know that the trace mapping is continuous from  $W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))$  onto  $G_{\text{av}}^s((a, b), \Gamma) \subset G^s((a, b), \Gamma)$ ; on the other hand, we have  $|\psi\Theta|_{\Gamma} |_{G^s((a, b), \Gamma)} = |\psi \mathbf{n}|_{G^s((a, b), \Gamma)} = |\psi|_{G_{\mathbf{n}}^s((a, b), \Gamma)} \leq C |\psi|_{H^{r_{n,1}(s)}((a, b), \mathbb{R})} = C |\psi\Theta|_{H^{r_{n,1}(s)}((a, b), \mathbb{R})\Theta}$ . Therefore,  $|v|_{\Gamma} |_{G^s((a, b), \Gamma)} \leq |v_{\text{div}}|_{\Gamma} |_{G^s((a, b), \Gamma)} + |\psi \mathbf{n}|_{G^s((a, b), \Gamma)} \leq C_1 (|v_{\text{div}}|_{W((a, b), H_{\text{div}}^s(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))} + |\psi\Theta|_{H^{r_{n,1}(s)}((a, b), \mathbb{R})\Theta})$ ; which implies  $|v|_{\Gamma} |_{G^s((a, b), \Gamma)} \leq C_1 \sqrt{2} |v|_{S^s}$ .  $\square$

**A.4. Proof of Proposition 3.5.** The proof will follow from a reiteration-like argument. We start with the following auxiliary result:

**Lemma A.7.** *Let  $v \in H^k((a, b), H^j(\Gamma, Z))$ , with  $\{k, j\} \in \{0, 1\} \times \{-2, -1, 0, 1, 2\}$ , where  $Z$  is either  $\mathbb{R}$  or  $T\Gamma$ . Let also  $\varphi \in C^1([a, b], C^2(\Gamma, \mathbb{R}))$ . Then  $\Phi : v \mapsto \varphi v$  maps  $H^k((a, b), H^j(\Gamma, Z))$  into itself, and we have the estimate  $|\varphi v|_{H^k((a, b), H^j(\Gamma, Z))} \leq C |\varphi|_{C^1([a, b], C^2(\Gamma, \mathbb{R}))} |v|_{H^k((a, b), H^j(\Gamma, Z))}$ .*

*Proof.* It is not hard to check that if  $j$  is nonnegative, then  $\varphi v \in H^k((a, b), H^j(\Gamma, Z))$ , and  $|\varphi v|_{H^k((a, b), H^j(\Gamma, Z))} \leq C |\varphi|_{C^k([a, b], C^j(\Gamma, \mathbb{R}))} |v|_{H^k((a, b), H^j(\Gamma, Z))}$ . In the case  $j < 0$ , we recall that, for each  $t \in (a, b)$  and each  $\xi \in C^{-j}(\Gamma, \mathbb{R})$ ,  $\langle \xi v(t), \psi(t) \rangle_{H^j(\Gamma, Z), H^{-j}(\Gamma, Z)} := \langle v(t), \xi \psi(t) \rangle_{H^j(\Gamma, Z), H^{-j}(\Gamma, Z)}$  and that, for  $k \in \{0, 1\}$  and for simplicity, we may write  $|\varphi v|_{H^k((a, b), H^j(\Gamma, Z))} = |\varphi v|_{L^2((a, b), H^j(\Gamma, Z))} + |k \partial_t(\varphi v)|_{L^2((a, b), H^j(\Gamma, Z))}$ . Then, writing  $\partial_t(\varphi v) = (\partial_t \varphi)v + \varphi \partial_t v$ , we can arrive to  $|\varphi v|_{H^k((a, b), H^j(\Gamma, Z))} \leq C |\varphi|_{C^k([a, b], C^{-j}(\Gamma, \mathbb{R}))} |v|_{L^2((a, b), H^j(\Gamma, Z))} + C |\varphi|_{C^0([a, b], C^{-j}(\Gamma, \mathbb{R}))} |k \partial_t v|_{L^2((a, b), H^j(\Gamma, Z))}$ ; as a corollary, it follows  $|\varphi v|_{H^k((a, b), H^j(\Gamma, Z))} \leq C |\varphi|_{C^1([a, b], C^2(\Gamma, \mathbb{R}))} |v|_{H^k((a, b), H^j(\Gamma, Z))}$ .  $\square$

Now, recalling (A.6) and Lemma A.5, we observe that

$$\begin{aligned} L^2((a, b), H^{3/2}(\Gamma, Z)) &= [L^2((a, b), H^2(\Gamma, Z)), L^2((a, b), L^2(\Gamma, Z))]_{1/4}, \\ H^1((a, b), H^{-(1/2)}(\Gamma, \mathbb{R})) &= [H^1((a, b), L^2(\Gamma, \mathbb{R})), H^1((a, b), H^{-1}(\Gamma, \mathbb{R}))]_{1/2}, \\ H^{3/4}((a, b), L^2(\Gamma, T\Gamma)) &= [H^1((a, b), L^2(\Gamma, T\Gamma)), L^2((a, b), L^2(\Gamma, T\Gamma))]_{1/4}, \end{aligned}$$

from which, using the Interpolation Theorem and Lemma A.7, we can conclude that Proposition 3.5 holds for  $i = 2$ . Indeed, we obtain that

$$\Phi : v \mapsto \varphi v \text{ maps } L^2((a, b), H^{3/2}(\Gamma, Z)) \text{ into itself,}$$

and for a suitable constant  $C > 0$ , the norm  $|\Phi|_{\mathcal{L}(L^2((a,b), H^{3/2}(\Gamma, Z)) \rightarrow L^2((a,b), H^{3/2}(\Gamma, Z)))}$  is bounded by  $C \max_{S \in \{L^2((a,b), H^2(\Gamma, Z)), L^2((a,b), L^2(\Gamma, Z))\}} \left\{ |\Phi|_{\mathcal{L}(S \rightarrow S)} \right\}$ , i.e.,

$$|\varphi v|_{L^2((a,b), H^{3/2}(\Gamma, Z))} \leq C_1 |\varphi|_{C^1([a,b], C^2(\Gamma, \mathbb{R}))} |v|_{L^2((a,b), H^{3/2}(\Gamma, Z))}.$$

By a similar reasoning we can obtain similar estimates  $|\varphi v|_S \leq C |\varphi|_{C^1([a,b], C^2(\Gamma, T\Gamma))} |v|_S$ , where  $S$  is either  $H^1((a,b), H^{-(1/2)}(\Gamma, \mathbb{R}))$  or  $H^{3/4}((a,b), L^2(\Gamma, T\Gamma))$ . These estimates imply that

$$(A.7) \quad |\varphi v|_{G^2((a,b), \Gamma)} \leq C |\varphi|_{C^1([a,b], C^2(\Gamma, \mathbb{R}))} |v|_{G^2((a,b), \Gamma)}.$$

Analogously, we can derive that

$$\begin{aligned} L^2((a,b), H^{1/2}(\Gamma, Z)) &= [L^2((a,b), H^1(\Gamma, Z)), L^2((a,b), L^2(\Gamma, Z))]_{1/2}, \\ H^{3/4}((a,b), H^{-1}(\Gamma, \mathbb{R})) &= [H^1((a,b), H^{-1}(\Gamma, \mathbb{R})), L^2((a,b), H^{-1}(\Gamma, \mathbb{R}))]_{1/4}, \\ H^{1/2}((a,b), H^{-(1/2)}(\Gamma, T\Gamma)) &= [H^1((a,b), H^{-(1/2)}(\Gamma, T\Gamma)), L^2((a,b), H^{-(1/2)}(\Gamma, T\Gamma))]_{1/2}, \end{aligned}$$

with

$$\begin{aligned} H^1((a,b), H^{-(1/2)}(\Gamma, T\Gamma)) &= [H^1((a,b), L^2(\Gamma, T\Gamma)), H^1((a,b), H^{-1}(\Gamma, T\Gamma))]_{1/2}, \\ L^2((a,b), H^{-(1/2)}(\Gamma, T\Gamma)) &= [L^2((a,b), L^2(\Gamma, T\Gamma)), L^2((a,b), H^{-1}(\Gamma, T\Gamma))]_{1/2}. \end{aligned}$$

Therefore, arguing as above, we can conclude that  $|\varphi v|_S \leq C |\varphi|_{C^1([a,b], C^2(\Gamma, \mathbb{R}))} |v|_S$ , where  $S$  is either  $L^2((a,b), H^{1/2}(\Gamma, Z))$ ,  $H^{3/4}((a,b), H^{-1}(\Gamma, \mathbb{R}))$ , or  $H^{1/2}((a,b), H^{-(1/2)}(\Gamma, T\Gamma))$ ; which allow us to conclude that

$$(A.8) \quad |\varphi v|_{G^1((a,b), \Gamma)} \leq C |\varphi|_{C^1([a,b], C^2(\Gamma, \mathbb{R}))} |v|_{G^1((a,b), \Gamma)}.$$

From (A.7) and (A.8), it follows the statement of Proposition 3.5.  $\square$

**A.5. Proof of Proposition 2.7.** We start with the following:

**Lemma A.8.** *Let  $r \geq 0$  and  $-1 \leq s \leq 2$ , then  $K^\mathcal{O} : \eta \mapsto K^\mathcal{O}\eta := E_0^\mathcal{O} \chi P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O}(\vartheta\eta|_\mathcal{O})$  maps  $H^r(\mathbb{R}, H^s(\Gamma, \mathbb{R}^3))$  into itself, linearly and continuously.*

*Proof.* We find

$$\begin{aligned} |K^\mathcal{O}\eta|_{H^r(\mathbb{R}, H^s(\Gamma, \mathbb{R}^3))}^2 &= \int_{\mathbb{R}} (1 + |\tau|^2)^r |\mathcal{F}_t K^\mathcal{O}\eta(\tau, \cdot)|_{H^s(\Gamma, \mathbb{R}^3)}^2 d\tau \\ (A.9) \quad &= \int_{\mathbb{R}} (1 + |\tau|^2)^r |\mathcal{F}_t \chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O}(\vartheta\eta|_\mathcal{O})(\tau, \cdot)|_{H^s(\Gamma, \mathbb{R}^3)}^2 d\tau. \end{aligned}$$

Now, let  $0 \leq s \leq 2$ . Using analogous arguments as in section A.4 we can derive that

$$\begin{aligned} &|\mathcal{F}_t \chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O}(\vartheta\eta|_\mathcal{O})(\tau, \cdot)|_{H^s(\Gamma, \mathbb{R}^3)}^2 = |\chi \mathbb{E}_0^\mathcal{O} P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O}(\vartheta\mathcal{F}_t\eta|_\mathcal{O})(\tau, \cdot)|_{H^s(\Gamma, \mathbb{R}^3)}^2 \\ &\leq |\chi|_{C^2(\Gamma, \mathbb{R}^3)}^2 |P_{\chi^\perp}^\mathcal{O} P_M^\mathcal{O}(\vartheta\mathcal{F}_t\eta|_\mathcal{O})(\tau, \cdot)|_{H^s(\Gamma, \mathbb{R}^3)}^2 \\ &\leq 2|\chi|_{C^2(\Gamma, \mathbb{R}^3)}^2 \left( |P_M^\mathcal{O}(\vartheta\mathcal{F}_t\eta|_\mathcal{O})(\tau, \cdot)|_{H^s(\Gamma, \mathbb{R}^3)}^2 + |((P_M^\mathcal{O}(\vartheta\mathcal{F}_t\eta|_\mathcal{O})(\tau, \cdot), \chi\mathbf{n})_{L^2(\Gamma, \mathbb{R}^3)}) / \int_{\mathcal{O}} \chi^2 d\mathcal{O}} \chi\mathbf{n}|_{H^s(\Gamma, \mathbb{R}^3)}^2 \right) \\ &\leq C \left( |P_M^\mathcal{O}(\vartheta\mathcal{F}_t\eta|_\mathcal{O})(\tau, \cdot)|_{H^s(\Gamma, \mathbb{R}^3)}^2 + |P_M^\mathcal{O}(\vartheta\mathcal{F}_t\eta|_\mathcal{O})(\tau, \cdot)|_{L^2(\Gamma, \mathbb{R}^3)}^2 \right) \\ &\leq C_1 |P_M^\mathcal{O}(\vartheta\mathcal{F}_t\eta|_\mathcal{O})(\tau, \cdot)|_{H^s(\Gamma, \mathbb{R}^3)}^2. \end{aligned}$$

Further using some interpolation arguments, we have that  $D(\Delta_{\mathcal{O}}^{s/2}) \subseteq H^s(\mathcal{O}, \mathbb{R}^3)$  is a continuous inclusion, where  $D(\Delta_{\mathcal{O}}^{s/2})$  is the fractional domain of the (Dirichlet) Laplacean,

defined in (36). In particular, using the orthogonality of the eigenfunctions  $\pi_i$  and eigen-vector fields  $\tau_i$  in  $D(\Delta_{\mathcal{O}}^{s/2})$ , we can write

$$\begin{aligned} & |\mathcal{F}_t \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}}(\vartheta \eta|_{\mathcal{O}}(\tau, \cdot))|_{H^s(\Gamma, \mathbb{R}^3)}^2 \leq C_2 |P_M^{\mathcal{O}}(\vartheta \mathcal{F}_t \eta|_{\mathcal{O}}(\tau, \cdot))|_{D(\Delta_{\mathcal{O}}^{s/2})}^2 \\ & \leq C_2 |\vartheta \mathcal{F}_t \eta|_{\mathcal{O}}(\tau, \cdot)|_{D(\Delta_{\mathcal{O}}^{s/2})}^2. \end{aligned}$$

By interpolation arguments, we can also derive that  $\xi \mapsto \vartheta \xi|_{\mathcal{O}}$  maps  $H^s(\Gamma, \mathbb{R}^3)$  into  $D(\Delta_{\mathcal{O}}^{s/2})$  continuously, from which it follows

$$|\mathcal{F}_t \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}}(\vartheta \eta|_{\mathcal{O}}(\tau, \cdot))|_{H^s(\Gamma, \mathbb{R}^3)}^2 \leq C_3 |\mathcal{F}_t \eta(\tau, \cdot)|_{H^s(\Gamma, \mathbb{R}^3)}^2$$

and, from (A.9),

$$(A.10) \quad |K^{\mathcal{O}} \eta|_{H^r(\mathbb{R}, H^s(\Gamma, \mathbb{R}^3))}^2 \leq C_3 |\eta|_{H^r(\mathbb{R}, H^s(\Gamma, \mathbb{R}^3))}^2, \quad \text{for } 0 \leq s \leq 2.$$

In the case  $-1 \leq s < 0$  we just notice that

$$\begin{aligned} & \langle \mathcal{F}_t K^{\mathcal{O}} \eta, \phi \rangle_{H^s(\Gamma, \mathbb{R}^3), H^{-s}(\Gamma, \mathbb{R}^3)} = (\mathcal{F}_t K^{\mathcal{O}} \eta, \phi)_{L^2(\Gamma, \mathbb{R}^3)} \\ & = (\mathcal{F}_t \eta, \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}}(\chi \phi|_{\mathcal{O}}))_{L^2(\Gamma, \mathbb{R}^3)} = \langle \mathcal{F}_t \eta, \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}}(\chi \phi|_{\mathcal{O}}) \rangle_{H^s(\Gamma, \mathbb{R}^3), H^{-s}(\Gamma, \mathbb{R}^3)} \end{aligned}$$

from which, noticing also that the mapping  $\phi \mapsto \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}}(\chi \phi|_{\mathcal{O}})$  is continuous, it follows that  $|\mathcal{F}_t K^{\mathcal{O}} \eta|_{H^s(\Gamma, \mathbb{R}^3)}^2 \leq C |\mathcal{F}_t \eta|_{H^s(\Gamma, \mathbb{R}^3)}^2$ ; hence, from (A.9), we arrive to

$$(A.11) \quad |K^{\mathcal{O}} \eta|_{H^r(\mathbb{R}, H^s(\Gamma, \mathbb{R}^3))}^2 \leq C |\eta|_{H^r(\mathbb{R}, H^s(\Gamma, \mathbb{R}^3))}^2, \quad \text{for } -1 \leq s < 0,$$

which ends the proof.  $\square$

From Lemma A.8 we can derive, in particular, that  $K^{\mathcal{O}} \in \mathcal{L}(G^i(\mathbb{R}, \Gamma) \rightarrow G_{\text{av}}^i(\mathbb{R}, \Gamma))$ . Using the idea in Remark A.1 we can construct an extension  $F \in \mathcal{L}(G^i((a, b), \Gamma) \rightarrow G^i(\mathbb{R}, \Gamma)) \cap \mathcal{L}(G_{\text{av}}^i((a, b), \Gamma) \rightarrow G_{\text{av}}^i(\mathbb{R}, \Gamma))$ . Moreover we will have  $FK^{\mathcal{O}} = K^{\mathcal{O}}F$ , roughly speaking, because  $K^{\mathcal{O}}$  does not depend on the time variable, while  $F$  depends essentially on the time variable. Then we obtain that

$$\begin{aligned} |K^{\mathcal{O}} \eta|_{G_{\text{av}}^i((a, b), \Gamma)}^2 & := \inf_{E \in \mathcal{L}(G_{\text{av}}^i((a, b), \Gamma) \rightarrow G_{\text{av}}^i(\mathbb{R}, \Gamma))} |EK^{\mathcal{O}} \eta|_{G_{\text{av}}^i(\mathbb{R}, \Gamma)}^2 \leq |FK^{\mathcal{O}} \eta|_{G_{\text{av}}^i(\mathbb{R}, \Gamma)}^2 \\ & \leq |K^{\mathcal{O}} F \eta|_{G_{\text{av}}^i(\mathbb{R}, \Gamma)}^2 \leq C |F \eta|_{G^i(\mathbb{R}, \Gamma)}^2 \leq C_1 |\eta|_{G^i((a, b), \Gamma)}^2, \end{aligned}$$

which proves the Proposition 2.7.  $\square$

**A.6. Proof of Proposition 4.3.** Given  $w \in L^2((a, b), Y^\perp)$  and  $u \in \widehat{L}^2((a, b), Y)$  we find that  $(w, u)_{L^2((a, b), X)} = \int_a^b (w(t), u(t))_X dt = 0$ , which implies that  $L^2((a, b), Y^\perp) \subseteq \widehat{L}^2((a, b), Y)^\perp$ . On the other hand, suppose that there exists  $v \in \widehat{L}^2((a, b), Y)^\perp \setminus L^2((a, b), Y^\perp)$ ; then, denoting by  $x_Z \in Z$  the orthogonal projection of  $x \in X$  onto a given closed subspace  $Z \subseteq X$ , we can rewrite  $v = v_{\overline{Y}^\perp} + v_{\overline{Y}}$ , for a.e.  $t \in (a, b)$ . From  $v_{\overline{Y}^\perp} \in L^2((a, b), \overline{Y}^\perp)$  and  $\overline{Y}^\perp = Y^\perp$ , we derive that  $v_{\overline{Y}} \in \left( \widehat{L}^2((a, b), Y)^\perp \setminus L^2((a, b), Y^\perp) \right) - L^2((a, b), Y^\perp) = \widehat{L}^2((a, b), Y)^\perp + L^2((a, b), Y^\perp) = \widehat{L}^2((a, b), Y)^\perp$ , so  $v_{\overline{Y}}$  is in the intersection  $L^2((a, b), \overline{Y}) \cap \widehat{L}^2((a, b), Y)^\perp$ . Proceeding as in the proof of Lemma A.4, we can construct a sequence of piecewise constant (step) functions  $v_{\overline{Y}}^n : (a, b) \rightarrow \overline{Y}$ ,  $n \in \mathbb{N}_0$ , with each  $v_{\overline{Y}}^n$  taking only a finite number of values in  $\overline{Y}$ , such that  $v_{\overline{Y}}^n \rightarrow v_{\overline{Y}}$  in  $L^2((a, b), \overline{Y})$ . Now, for each  $n \in \mathbb{N}_0$ , we can approximate each taken constant by another one in  $Y$  and obtain a sequence  $\tilde{v}_{\overline{Y}}^n \in \widehat{L}^2((a, b), Y)$  such that  $|v_{\overline{Y}}^n - \tilde{v}_{\overline{Y}}^n|_{L^2((a, b), \overline{Y})} < 1/n$ ; so we still have  $\tilde{v}_{\overline{Y}}^n \rightarrow v_{\overline{Y}}$  in  $L^2((a, b), \overline{Y})$  (in particular we see that  $\widehat{L}^2((a, b), Y)$  is dense in  $L^2((a, b), \overline{Y})$ ). Thus  $|v_{\overline{Y}}|_{L^2((a, b), \overline{Y})}^2 = \lim_{n \rightarrow +\infty} (v_{\overline{Y}}, \tilde{v}_{\overline{Y}}^n)_{L^2((a, b), \overline{Y})} = 0$ , and we derive that  $v = v_{\overline{Y}^\perp} + v_{\overline{Y}} =$

$v_{\overline{Y}^\perp} \in L^2((a, b), Y^\perp)$ , which contradicts  $v \in \widehat{L^2}((a, b), Y)^\perp \setminus L^2((a, b), Y^\perp)$ . The contradiction implies that  $\widehat{L^2}((a, b), Y)^\perp \subseteq L^2((a, b), Y^\perp)$ .  $\square$

**A.7. Proof of Proposition 4.6.** From section 2.1 we have that

$$G^1((a, b), \Gamma) \subseteq L^2((a, b), H^{1/2}(\Gamma, \mathbb{R}^3)) \cap H^{1/2}((a, b), H^{-1}(\Gamma, \mathbb{R}^3))$$

continuously. Then, Proposition 4.6 will follow from the following:

**Lemma A.9.** *Let  $r_1, r_2 \geq 0$  and  $s_1, s_2 \in \mathbb{R}$  be real numbers. Let  $I$  be any open interval  $I \subseteq \mathbb{R}$  (either bounded or not), then the inclusion*

$$H^{r_1}(I, H^{s_1}(\Gamma, \mathbb{R})) \cap H^{r_2}(I, H^{s_2}(\Gamma, \mathbb{R})) \subseteq H^{(r_1+r_2)/2}(I, H^{(s_1+s_2)/2}(\Gamma, \mathbb{R}))$$

holds and is continuous.

*Proof.* We can suppose that  $\Gamma$  is a connected manifold; if that is not the case, then it is a disjoint union of such manifolds and the Sobolev spaces in  $\Gamma$  will be just the Cartesian product of the corresponding spaces in each connected component.

We will use the characterization (A.3), with  $\Gamma$  in the role of  $\mathcal{O}$ . Consider the Laplace–de Rham operator (cf. (36)),

$$(A.12) \quad \begin{aligned} \Delta_\Gamma : H^2(\Gamma, \mathbb{R}^3) &\rightarrow L^2(\Gamma, \mathbb{R}^3) \\ u = (u \cdot \mathbf{n})\mathbf{n} + u_t &\mapsto (\Delta_\Gamma(u \cdot \mathbf{n}))\mathbf{n} + \Delta_\Gamma u_t. \end{aligned}$$

Notice that  $1 + \Delta_\Gamma$ , is a symmetric operator, and the same holds for its fractional powers,  $u = \sum_{i \in \mathbb{N}_0} u_{\mathbf{n}}^i \pi_i \mathbf{n} + \sum_{i \in \mathbb{N}_0} u_t^i \tau_i \mapsto (1 + \Delta_\Gamma)^s u = \sum_{i \in \mathbb{N}_0} u_{\mathbf{n}}^i (1 + \beta_i)^s \pi_i \mathbf{n} + \sum_{i \in \mathbb{N}_0} u_t^i (1 + \gamma_i)^s \tau_i$ ,  $s \in [0, 1]$ , where  $\beta_i$  and  $\gamma_i$  are the eigenvalues associated with the eigenfunctions  $\pi_i$  and eigenvector fields  $\tau_i$  of  $\Delta_\Gamma$ , respectively.

Denoting by  $\mathcal{F}_t : f(t, x) \mapsto \mathcal{F}_t f(\tau, x)$  the Fourier transform in the variable  $t$ , we find that

$$\begin{aligned} |f|_{H^{(r_1+r_2)/2}(\mathbb{R}, H^{(s_1+s_2)/2}(\Gamma, \mathbb{R}))}^2 &= \int_{\mathbb{R}} (1 + |\tau|^2)^{(r_1+r_2)/2} |(1 + \Delta_\Gamma)^{(s_1+s_2)/4} \mathcal{F}_t f(\tau, x)|_{L^2(\Gamma, \mathbb{R})}^2 d\tau \\ &\leq \prod_{i=1}^2 \left( \int_{\mathbb{R}} (1 + |\tau|^2)^{r_i} |(1 + \Delta_\Gamma)^{s_i/2} \mathcal{F}_t f(\tau, x)|_{L^2(\Gamma, \mathbb{R})}^2 d\tau \right)^{1/2} \\ &= |f|_{H^{r_1}(\mathbb{R}, H^{s_1}(\Gamma, \mathbb{R}))} |f|_{H^{r_2}(\mathbb{R}, H^{s_2}(\Gamma, \mathbb{R}))} \leq 1/2 (|f|_{H^{r_1}(\mathbb{R}, H^{s_1}(\Gamma, \mathbb{R}))}^2 + |f|_{H^{r_2}(\mathbb{R}, H^{s_2}(\Gamma, \mathbb{R}))}^2) \\ &= 1/2 |f|_{H^{r_1}(\mathbb{R}, H^{s_1}(\Gamma, \mathbb{R})) \cap H^{r_2}(\mathbb{R}, H^{s_2}(\Gamma, \mathbb{R}))}^2. \end{aligned}$$

Hence the Lemma holds with  $I = \mathbb{R}$ . Let now  $I = (a, b) \neq \mathbb{R}$  with  $a < b$ ,  $\{a, b\} \in \mathbb{R} \cup \{-\infty, +\infty\}$ . We derive that

$$\begin{aligned} H^{(r_1+r_2)/2}(I, H^{(s_1+s_2)/2}(\Gamma, \mathbb{R})) &= H^{(r_1+r_2)/2}(\mathbb{R}, H^{(s_1+s_2)/2}(\Gamma, \mathbb{R}))|_I \\ &\supseteq (H^{r_1}(\mathbb{R}, H^{s_1}(\Gamma, \mathbb{R})) \cap H^{r_2}(\mathbb{R}, H^{s_2}(\Gamma, \mathbb{R})))|_I \\ &= H^{r_1}(I, H^{s_1}(\Gamma, \mathbb{R})) \cap H^{r_2}(I, H^{s_2}(\Gamma, \mathbb{R})). \end{aligned}$$

Now (proceeding e.g. as in Remark A.1), setting  $m \in \mathbb{N}_0$  such that  $m \geq \max\{r_1, r_2\}$ , we can find an extension  $F^m$  in the intersection

$$\bigcap_{(\rho, \sigma) \in \{(r_1, s_1), (r_2, s_2), (r_1+r_2, s_1+s_2)/2\}} \mathcal{L}(H^\rho(I, H^\sigma(\Gamma, \mathbb{R})) \rightarrow H^\rho(\mathbb{R}, H^\sigma(\Gamma, \mathbb{R})))$$

(e.g. by a suitable interpolation argument using (A.6); notice that the extension in Remark A.1, depends essentially in the (time) variable  $t \in (a, b)$ , and not really on the

Banach space  $X$ ). Thus, we derive that

$$\begin{aligned} |f|_{H^{(r_1+r_2)/2}(I, H^{(s_1+s_2)/2}(\Gamma, \mathbb{R}))}^2 &:= \inf_E |Ef|_{H^{(r_1+r_2)/2}(\mathbb{R}, H^{(s_1+s_2)/2}(\Gamma, \mathbb{R}))}^2 \\ &\leq |F^m f|_{H^{(r_1+r_2)/2}(\mathbb{R}, H^{(s_1+s_2)/2}(\Gamma, \mathbb{R}))}^2 \leq 1/2 |F^m f|_{H^{r_1}(\mathbb{R}, H^{s_1}(\Gamma, \mathbb{R})) \cap H^{r_2}(\mathbb{R}, H^{s_2}(\Gamma, \mathbb{R}))}^2 \\ &\leq C |f|_{H^{r_1}(I, H^{s_1}(\Gamma, \mathbb{R})) \cap H^{r_2}(I, H^{s_2}(\Gamma, \mathbb{R}))}^2, \end{aligned}$$

with  $E$  running over  $\mathcal{L}(H^{(r_1+r_2)/2}(I, H^{(s_1+s_2)/2}(\Gamma, \mathbb{R})) \rightarrow H^{(r_1+r_2)/2}(\mathbb{R}, H^{(s_1+s_2)/2}(\Gamma, \mathbb{R})))$ . This finishes the proof.  $\square$

*Remark A.6.* Concerning the intersection space in Lemma A.9, we see that our “space” variable lives in  $\Gamma$ . However we would like to refer to [Gri67] where we can find some work on a class of spaces including  $H^{r_1}(I, L^2(O, \mathbb{R})) \cap L^2(I, H^{s_2}(O, \mathbb{R}))$ , where  $O \subset \mathbb{R}^n$  is an open subset.

**A.8. Proof of Proposition 4.7.** Let  $\{x_j \mid j \in \mathbb{N}_0\}$  be an orthonormed basis in  $X$ ; then  $\mathcal{F} := \{\sigma_n x_j \mid (j, n) \in \mathbb{N}_0^2\}$  is an orthonormed basis in  $L^2((a, b), X)$ : that  $\mathcal{F}$  is orthonormed follows from  $\int_a^b (\sigma_n x_j, \sigma_m x_i)_X dt = (x_j, x_i)_X \int_a^b \sigma_n \sigma_m dt = \delta_{(j,n)}^{(i,m)}$ , where

$$\delta_{p_1}^{p_2} \text{ is the Kronecker delta: } \delta_{p_1}^{p_2} = \begin{cases} 1, & \text{if } p_1 = p_2 \\ 0, & \text{if } p_1 \neq p_2 \end{cases}. \text{ On the other side given } f \in$$

$L^2((a, b), X)$  orthogonal to all the elements of  $\mathcal{F}$ , we find that  $0 = \int_a^b \sigma_n (f, x_j)_X dt$ , and that  $\int_a^b |(f, x_j)_X|_{\mathbb{R}}^2 dt \leq \int_a^b |f|_X^2 |x_j|_X^2 dt = \int_a^b |f|_X^2 dt < +\infty$ , which allow us to conclude that for all  $j \in \mathbb{N}_0$ ,  $(f(t), x_j)_X = 0$  for a.e.  $t \in (a, b)$ ; which in turn implies that  $f(t) = 0$  for a.e.  $t \in (a, b)$ , that is,  $f = 0$ . Recall that the union of a sequence of sets with Lebesgue measure zero is still a set of Lebesgue measure zero, see [Smi83, chapter 13, section 2], [Wei73, section 2.2].

Therefore, since  $L^2((a, b), X)$  is complete, and an orthogonal basis is unconditional, we can write  $\left(\int_a^b \sigma_n(t) f(t) dt\right) \sigma_n = \lim_{k \rightarrow +\infty} \sum_{j=1}^k \left(\int_a^b (\sigma_n(t) x_j, f(t))_X dt\right) \sigma_n x_j$ , and  $f = \lim_{M \rightarrow +\infty} \sum_{n=1}^M \left(\int_a^b \sigma_n(t) f(t) dt\right) \sigma_n$ . Furthermore notice that

$$\begin{aligned} |f|_{L^2((a,b), X)}^2 &= \sum_{n, j \in \mathbb{N}_0} (f, \sigma_n x_j)_{L^2((a,b), X)}^2 = \sum_{n \in \mathbb{N}_0} \sum_{j \in \mathbb{N}_0} (f, \sigma_n x_j)_{L^2((a,b), X)}^2 \\ &= \sum_{n \in \mathbb{N}_0} \sum_{j \in \mathbb{N}_0} \left(\int_a^b (\sigma_n(t) x_j, f(t))_X dt\right)^2 = \sum_{n \in \mathbb{N}_0} \left|\left(\int_a^b \sigma_n(t) f(t) dt\right) \sigma_n\right|_{L^2((a,b), X)}^2 \\ &= \sum_{n \in \mathbb{N}_0} \left|\left(\int_a^b \sigma_n(t) f(t) dt\right)\right|_X^2, \text{ and} \\ |P_M^t f|_{L^2((a,b), X)}^2 &= \sum_{n=1}^M \left|\left(\int_a^b \sigma_n(t) f(t) dt\right) \sigma_n\right|_{L^2((a,b), X)}^2 = \sum_{n=1}^M \left|\left(\int_a^b \sigma_n(t) f(t) dt\right)\right|_X^2. \end{aligned}$$

Now, it is straightforward to check that  $P_M^t$  is an orthogonal projection in  $L^2((a, b), X)$  onto  $\sum_{n=1}^M \sigma_n X$ . The proof is completed.  $\square$

**A.9. Proof of Proposition 5.1.** From Propositions 2.7 and 3.5 we have that  $\varphi K^O \in \mathcal{L}(G^i((a, b), \Gamma) \rightarrow G_{\text{av}}^i((a, b), \Gamma))$ , with  $i \in \{1, 2\}$ . Thus, since  $K_t^O = \varphi K^O P_M^t \tilde{\varphi}$ , it follows that it remains to show the continuity of the mapping  $\eta \mapsto P_M^t(\tilde{\varphi} \eta)$ , from  $G^i((a, b), \Gamma)$  into itself.

From section 4.3 we know that  $f \mapsto P_M^t f$  is a an orthogonal projection in  $L^2((a, b), X)$  and in  $H_0^1((a, b), X)$ . It follows that  $P_M^t \tilde{\varphi} \in \mathcal{L}(H^1((a, b), X) \rightarrow H_0^1((a, b), X))$  and

$P_M^t \tilde{\varphi} \in \mathcal{L}(L^2((a, b), X) \rightarrow L^2((a, b), X))$ . By interpolation it will follow that  $P_M^t \tilde{\varphi}$  is a linear and bounded operator from  $H^s((a, b), X)$  into  $[H_0^1((a, b), X), L^2((a, b), X)]_{1-s} \subseteq H^s((a, b), X)$ .

For  $X$  we can take either  $H^r(\Gamma, \mathbb{R})$  or  $H^r(\Gamma, T\Gamma)$ ,  $r \in \mathbb{R}$ . Letting the triple  $(s, r, X)$  run over  $\{(0, i - (1/2), T\Gamma), (r_{\mathbf{t},1}(i), r_{\mathbf{t},1}(i), T\Gamma), (0, i - (1/2), \mathbb{R}), (r_{\mathbf{n},1}(i), r_{\mathbf{n},1}(i), \mathbb{R})\}$  we can conclude that  $P_M^t \tilde{\varphi} \in \mathcal{L}(G^i((a, b), \Gamma) \rightarrow G^1((a, b), \Gamma))$ . Recall that the real numbers  $r_{\mathbf{t},1}(i)$ ,  $r_{\mathbf{t},2}(i)$ ,  $r_{\mathbf{n},1}(i)$ , and  $r_{\mathbf{n},2}(i)$  are defined in section 2.1.  $\square$

#### A.10. A remark on the definitions of the Sobolev spaces on the boundary.

For convenience, throughout the paper we have used several definitions/characterizations of Sobolev spaces on the boundary  $\Gamma = \partial\Omega$ . Indeed, we can see the space  $H^s(\Gamma, \mathbb{R})$  ( $s > 0$ ) as the space of traces on  $\Gamma$  of the functions in  $H^{s+(1/2)}(\Omega, \mathbb{R})$ , see for example [Neč67, chapter 2, section 5]; alternatively looking at  $\Gamma$  as a two-dimensional manifold we can define, for example for  $s \in [0, 2]$ ,  $H^s(\Gamma, \mathbb{R})$  as the domain  $D((1 + \Delta_\Gamma)^{s/2}) = [D((1 + \Delta_\Gamma)^1), D((1 + \Delta_\Gamma)^0)]_{1-(s/2)}$ , see for example [Dod81]. We can also define the Sobolev spaces by means of the Levi-Civita connection (covariant derivative), as in [Aub82, Dod81], or using an atlas of  $\Gamma$  and a partition of unity argument, as in [Tay97].

For compact manifolds, either with or without boundary, all this definitions are equivalent. For the equivalence of the covariant derivative and domains of fractional powers of  $(1 + \Delta_\Gamma)$  approaches we refer to [Dod81]. For the equivalence of the atlas and interpolation (i.e. domains of fractional powers of  $(1 + \Delta_\Gamma)$ ) approaches we refer to [Tay97, chapter 4, section 3].

Notice that in the case of functions, defined in the boundaryless manifold  $\Gamma = \partial\Omega$ , we have  $D((1 + \Delta_\Gamma)^1) = H^2(\Gamma, \mathbb{R})$  and  $D((1 + \Delta_\Gamma)^0) = L^2(\Gamma, \mathbb{R})$ ; furthermore, for a subset  $\mathcal{O} \subseteq \Gamma$ , we can define  $H^s(\mathcal{O}, \mathbb{R}) := H^s(\Gamma, \mathbb{R})|_{\mathcal{O}}$ .

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