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Reconstructing obstacles by the enclosure method using in one step the farfield measurements

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Abstract

In this work, we are concerned with the reconstruction of the obstacles by the enclosure method using the farfield measurements in one step. To justify this, first we state the indicator function of the enclosure method linking directly the farfield pattern to the reflected solutions corresponding to the used complex geometrical optics solutions. Second, we use layer potential techniques to derive the needed estimates of the reflected solutions. No condition on the geometry of the obstacle or on the used frequency is needed.

1 Introduction

We assume $D \subset \mathbb{R}^3$ to be a bounded and Lipschitz domain such that $\mathbb{R}^3 \setminus \bar{D}$ is connected. Let $u^i(x,d) := e^{ikx\cdot d}$ be the incident plane wave, where $x \in \mathbb{R}^3$, k is the frequency and $d \in \mathbb{S}^2 := \{x \in \mathbb{R}^3; |x| = 1\}$ is the incident direction. As a model, we consider the scattering problem for the Helmholtz equation with Neumann boundary conditions:

$$\begin{cases} (\Delta + k^2)u^s = 0, \text{ in } \mathbb{R}^3 \setminus \bar{D} \\ \partial_\nu u^s = -\partial_\nu e^{ikx\cdot d}, \text{ on } \partial D \\ \lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \end{cases} \tag{1.1}$$

where $r = |x|$ and the limit is assumed to hold uniformly in all directions $\frac{x}{|x|}$ and the normal derivative $\partial_\nu u^s$ on the boundary is defined in the sense that the limit

$$\partial_\nu u^s = \lim_{h\to 0+} \nu(x)\cdot \operatorname{grad} u^s(x + h\nu(x)),\ x \in \partial D,$$

exists uniformly on ∂D. The unit normal vector ν is directed into the exterior of D. Here we use the notation $\partial_\nu := \frac{\partial}{\partial \nu}$.
It is known that the problem is well-posed in appropriate Hölder or Sobolev spaces and the scattering solution u^s has the asymptotic behavior of the form

$$u^s(x,d) = \frac{e^{ik|x|}}{|x|}\left\{u^\infty(\hat{x},d) + O\left(\frac{1}{|x|}\right)\right\},\ |x| \to \infty,$$

uniformly in all directions $\hat{x} = \frac{x}{|x|}$ where the function u^∞ defined on $\mathbb{S}^2$ is known as the farfield pattern of u^s, see [1]. We are interested in the following inverse problem.

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Inverse Problem: Reconstruct the unknown obstacle D from the farfield measurement $u^\infty(\hat{x}, d)$, for $\hat{x}, d \in \mathbb{S}^2$.

The enclosure method was introduced by Ikehata to reconstruct an obstacle using far-field measurements, see for instance [2] and [3]. The method is divided into two steps. The first step is to recover the Dirichlet to Neumann map, defined on the surface of a known domain containing the obstacle, from the farfield data and the second step is to identify the unknown obstacle from this Dirichlet to Neumann map. In this work, we show that the method can be stated in one step only. For this, we proceed as follows. First, we use the identity (2.1) and the superposition argument to connect the farfield pattern to the used complex geometrical optics solutions (CGO solutions, in short). Second, we use the layer potential theory to derive the natural estimates needed to justify the enclosure method. We show the detailed analysis for CGOs having a linear phase. However, we can also use CGOs having other phases, see [6] where the corresponding estimates have been derived. In the recent work [5], by Nakamura and Potthast, this method was also stated in one step assuming the obstacle to be C^2-regular and strictly convex. Compared to this work, where pseudo-differential methods are used, we propose a shorter proof but most importantly we avoid the strict convexity condition of the obstacle. Let us mention, in addition, that this strict convexity condition of the obstacle was stated as an open question by Ikehata in [3]. Finally, our work is completing [6] where the analysis was based on the Dirichlet to Neumann map.
The paper is organized as follows. In section 2, the indicator function is defined via the farfield map and then the relation between the farfields and the CGO solutions is discussed. In section 3, our main theorem is stated and in section 4, its proof is given.

2 The indicator function connecting the farfields with the CGOs

We start with the following identity linking the farfield to the near field

$$u^\infty(\hat{x}, d) = \frac{1}{4\pi} \int_{\partial D} \left\{ u^s(y, d) \partial_\nu e^{-ik\hat{x}\cdot y} - \partial_\nu u^s(y, d) e^{-ik\hat{x}\cdot y} \right\} ds(y), \ \hat{x} \in \mathbb{S}^2, \tag{2.1}$$

see [[1], Theorem 2.5]. For $\tau > 0$ and $t \in \mathbb{R}$, the explicit form of the CGO solution with the linear phase for Helmholtz equation, we shall use, is as follows:

$$v := v(y; \rho, \tau, t) = e^{\tau(y\cdot\rho - t) + i\sqrt{\tau^2+k^2}\, y\cdot\rho^\perp},$$

where $\rho, \rho^\perp \in \mathbb{S}^2$ with $\rho \cdot \rho^\perp = 0$, see [2].
Let us recall the Herglotz wave function

$$v_g(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d) ds(d), \ x \in \mathbb{R}^3,$$

where $g \in L^2(\mathbb{S}^2)$. Let Ω be a C^2-domain containing $\bar{D}$. The Herglotz operator H, defined by $Hg := v_g|_{\partial\Omega}$ is bounded from $L^2(\mathbb{S}^2)$ to $L^2(\partial\Omega)$. We know that H is injective and has a dense range if k^2 is not an eigenvalue of the Dirichlet-Laplacian on Ω, see [1]. Since these eigenvalues are monotonic in terms of Ω then we can change, if necessary, Ω so that k^2 is not an eigenvalue. Hence, we can find a sequence $g_n \in L^2(\mathbb{S}^2)$ such that $Hg_n \to v$ in $L^2(\partial\Omega)$. Recall that both v_{g_n} and v satisfy the Helmholtz equation in Ω. By the well-posedness of the interior problem and the interior estimates, we deduce that $v_{g_n} \to v$ in $C^\infty(\Omega)$. Then by the principle of superposition, from (2.1) we obtain

$$\int_{\mathbb{S}^2} \int_{\mathbb{S}^2} u^\infty(\hat{x}, d) g_n(d) \overline{g_n(\hat{x})} ds(\hat{x}) ds(d) = \frac{1}{4\pi} \int_{\partial D} \left\{ \overline{\partial_\nu v_{g_n}}\, u^s(v_{g_n}) - \partial_\nu u^s(v_{g_n})\, \overline{v_{g_n}} \right\} ds(y), \tag{2.2}$$

where $u^s(v_{g_n})$ is the solution of (1.1) replacing the plane wave $e^{ikx\cdot d}$ by the Herglotz wave v_{g_n}. It has the following form

$$u^s(v_g)(x) = \int_{\mathbb{S}^2} u^s(x,d)g(d)ds(d), \ x \in \mathbb{R}^3 \setminus \bar{D},$$

see for instance [[1], Lemma 3.16]. We define the indicator function of the enclosure method as follows:

$$I(v) := \tau \lim_{n\to\infty} \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} u^\infty(\hat{x},d)g_n(d)\overline{g_n(\hat{x})}ds(\hat{x})ds(d). \tag{2.3}$$

Therefore based on (2.2) and using the facts that $v_{g_n} \to v$ in $C^\infty(\Omega)$, the trace theorem and well-posedness of the scattering problem, we derive the following relation

$$\lim_{n\to\infty} \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} u^\infty(\hat{x},d)g_n(d)\overline{g_n(\hat{x})}ds(\hat{x})ds(d) = \frac{1}{4\pi}\int_{\partial D} \left\{\overline{\partial_\nu v}\ u^s(v) - \partial_\nu u^s(v)\ \overline{v}\right\} ds(y), \tag{2.4}$$

where $u^s(v)$ is the solution of (1.1) replacing the plane wave $e^{ikx\cdot d}$ by the CGO solution v. Remark that

$$\begin{aligned}\int_{\partial D} \overline{\partial_\nu v}\ u^s(v)ds(y) &= -\int_{\partial D} \overline{\partial_\nu u^s(v)}\ u^s(v)ds(y)\\ &= -\int_{\partial\Omega} \overline{\partial_\nu u^s(v)}\ u^s(v)ds(y) - \int_{\Omega\setminus\bar{D}} |\nabla u^s(y)|^2dy + k^2\int_{\Omega\setminus\bar{D}} |u^s(v)|^2dy\end{aligned}$$

and

$$\int_{\partial D} \partial_\nu u^s(v)\ \overline{v}ds(y) = -\int_D \partial_\nu v\ \overline{v}ds(y) = \int_D |\nabla v|^2dy - k^2\int_D |v|^2dy.$$

Hence, by (2.3), (2.4) we obtain the following identity

$$\begin{aligned}4\pi\tau^{-1}I(v) = &-\int_{\partial\Omega} u^s(v)\ \overline{\partial_\nu u^s(v)}ds(y) - \int_{\Omega\setminus\bar{D}} |\nabla u^s(v)|^2dy + k^2\int_{\Omega\setminus\bar{D}} |u^s(v)|^2dy\\ &-\int_D |\nabla v|^2dy + k^2\int_D |v|^2dy.\end{aligned} \tag{2.5}$$

Finally, we have the following key inequality

$$-4\pi\tau^{-1}I(v) \geq \int_{\partial\Omega} u^s(v)\ \overline{\partial_\nu u^s(v)}ds(y) - k^2\int_{\Omega\setminus\bar{D}} |u^s(v)|^2dy - k^2\int_D |v|^2dy + \int_D |\nabla v|^2dy. \tag{2.6}$$

3 Main theorem

As the CGO solution v depends on ρ, τ, t, we denote $I(v)$ by $I_\rho(v,\tau,t)$. For a fixed direction ρ, we recall the support function defined as

$$h_D(\rho) := \sup_{x\in D} x\cdot\rho, \ (\rho \in \mathbb{S}^2).$$

The following is the main theorem of this work.

Theorem 3.1. *Let $\rho \in \mathbb{S}^2$. We have the following characterization of $h_D(\rho)$.*

$$\lim_{\tau\to\infty} |I_\rho(v,\tau,t)| = 0 \quad (t > h_D(\rho)), \ \textit{and precisely, } |I_\rho(v,\tau,t)| \leq Ce^{-c\tau}, \ \tau >> 1, \ c, C > 0, \tag{3.1}$$

$$\liminf_{\tau\to\infty} |I_\rho(v,\tau,h_D(\rho))| > 0, \quad \textit{and precisely, } c \leq |I_\rho(v,\tau,h_D(\rho))| \leq C\tau^2, \ \tau >> 1, \ c, C > 0, \tag{3.2}$$

$$\lim_{\tau\to\infty} |I_\rho(v,\tau,t)| = \infty \quad (t < h_D(\rho)), \ \textit{and precisely, } |I_\rho(v,\tau,t)| \geq Ce^{c\tau}, \ \tau >> 1, \ c, C > 0. \tag{3.3}$$

The properties (3.1)*,* (3.2) *and* (3.3) *are justified if ∂D is of Lipschitz class assuming that k^2 is not an eigenvalue of the Dirichlet-Laplacian on D. If ∂D is of the class $C^{1,1}$ then the eigenvalue condition is not needed.*

From this theorem, we see that, for a fixed direction ρ, the behavior of the indicator function $I_\rho(v,\tau,t)$ changes drastically in terms of τ. Precisely, for $t > h_D(\rho)$ it is decaying exponentially, for $t < h_D(\rho)$ it is growing exponentially and for $t = h_D(\rho)$ it has a polynomial behavior. Using this property of the indicator function we can reconstruct the support function $h_D(\rho)$, $\rho \in \mathbb{S}^2$ from the farfield measurement. Finally, from this support function we can estimate the convex hull of D.

4 Proof of the Main theorem

To prove Theorem 3.1, we need only to show (3.2) as the other properties (3.1) and (3.3) will follow from (3.2) and the identity $I_\rho(v,\tau,t) = e^{2\tau(h_D(\rho)-t)} I_\rho(v,\tau,h_D(\rho))$. In addition, using (2.5), the well-posedness of the scattering problem and the trace theorem, we see that

$$-4\pi\tau^{-1} I_\rho(v,\tau,h_D(\rho)) \leq C\|v\|^2_{H^1(D)}, \text{ where } C > 0.$$

Hence, the upper bound of $I_\rho(v,\tau,h_D(\rho))$ in (3.2) follows from the upper bound $\|v\|^2_{H^1(D)} \leq O(\tau)$ given in Lemma 4.6. The proof of the lower bound in (3.2) is more difficult. We divide it into two subsections. In the first subsection, we estimate the boundary integral $\int_{\partial\Omega} u^s(v)\partial_\nu u^s(v)ds(y)$ by $\|v\|_{H^{-t+\frac{3}{2}}(D)}$, $\frac{1}{2} \leq t < 1$, and in second subsection, we derive a similar estimate for the volume integral $\int_{\Omega\setminus\bar{D}} |u^s(v)|^2 dy$. Then using the key inequality (2.6) we will deduce the following estimate for the indicator function:

$$-4\pi\tau^{-1} I_\rho(v,\tau,h_D(\rho)) \geq c\|\nabla v\|^2_{L^2(D)} - C\|v\|^2_{L^2(D)}, \text{ where } c, C > 0, \tag{4.1}$$

from which we obtain the lower bound of (3.2) using the estimates of $\|\nabla v\|_{L^2(D)}$ and $\|v\|_{L^2(D)}$ given in Lemma 4.6.

4.1 Estimate of the boundary integral $\int_{\partial\Omega} u^s(v)\overline{\partial_\nu u^s(v)}ds(y)$

Proposition 4.1. *There exists a positive constant $C := C_t$ independent on v such that*

$$|\int_{\partial\Omega} u^s(v)\overline{\partial_\nu u^s(v)}ds(y)| \leq C\|v\|^2_{H^{-t+\frac{3}{2}}(D)}$$

for $\frac{1}{2} \leq t < 1$.

Proof.

Case 1. *(Assume that k^2 is not an eigenvalue of the Dirichlet-Laplacian in D.)*

We represent u^s as a single layer potential

$$u^s(v) = \int_{\partial D} \Phi(y,z) f(z) ds(z) =: Sf, \tag{4.2}$$

where f satisfies

$$-\frac{1}{2}f + \mathcal{K}^* f = -\partial_\nu v|_{\partial D} \tag{4.3}$$

and $\mathcal{K}^*$ is the adjoint of the double layer potential $\mathcal{K}$ for the Holmholtz operator. Here $\Phi(y,z) := \frac{e^{ik|y-z|}}{4\pi|y-z|}$ is the fundamental solution of the Helmholtz equation. The following lemma, see for instance ([6], Lemma 4.3) and the references therein, shows that the integral equation (4.3) is solvable in the space $H^{-s}(\partial D)$ for $s \in [0,1]$ if we assume that k^2 is not an eigenvalue of the Dirichlet-Laplace operator stated in D.

Lemma 4.2. *Let D be of Lipschitz class and assume that k^2 is not an eigenvalue of the Dirichlet-Laplacian in D. Then the operator*

$$-\frac{1}{2}I + \mathcal{K}^* : H^{-t}(\partial D) \to H^{-t}(\partial D),\ 0 \le t \le 1,$$

is invertible.

Now, we use the estimate

$$\begin{aligned} |\int_{\partial\Omega} u^s(v)\overline{\partial_\nu u^s(v)}ds(y)| &\le \|u^s(v)\|_{H^{\frac{1}{2}}(\partial\Omega)}\|\partial_\nu u^s(v)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \\ &\text{(by the trace theorem)} \\ &\le C\|u^s(v)\|^2_{H^1(B\setminus\bar{\Omega})}, \end{aligned} \tag{4.4}$$

where B is a measurable and bounded set containing Ω.
Using the representation (4.2), we obtain

$$\|u^s(v)\|_{H^1(B\setminus\bar{\Omega})} \le \|f\|_{H^{-t}(\partial D)}\|F\|_{H^1(B\setminus\bar{\Omega})} \tag{4.5}$$

for $0 \le t \le 1$, where $F(y) := \|\Phi(y,\cdot)\|_{H^t(\partial D)}$. Due to the form of $\Phi(\cdot,\cdot)$, $F(y)$ is in $H^1(B\setminus\bar{\Omega})$ since $\bar{D} \subset \Omega$. We recall here the following lemma proved in [[6], Proposition 4.4].

Lemma 4.3. *Let D be a Lipschitz domain and v be a solution of the Helmholtz equation in a domain containing D. Then, there exists a positive constant $C := C_t$ independent of v such that*

$$\|\partial_\nu v\|_{H^{-t}(\partial D)} \le C\|v\|_{H^{-t+\frac{3}{2}}(D)},\ \frac{1}{2} \le t < 1.$$

From (4.3), Lemma 4.2 and Lemma 4.3, we have

$$\|f\|_{H^{-t}(\partial D)} \le C\|\partial_\nu v\|_{H^{-t}(\partial D)} \le C\|v\|_{H^{-t+\frac{3}{2}}(D)} \tag{4.6}$$

for $\frac{1}{2} \le t < 1$. Combining (4.5) and (4.6), we obtain

$$\|u^s(v)\|_{H^1(B\setminus\bar{\Omega})} \le C\|v\|_{H^{-t+\frac{3}{2}}(D)}$$

and hence

$$|\int_{\partial\Omega} u^s(v)\overline{\partial_\nu u^s(v)}ds(y)| \le C\|v\|^2_{H^{-t+\frac{3}{2}}(D)} \tag{4.7}$$

for $\frac{1}{2} \le t < 1$.

Case 2. *(Assume that ∂D is of class $C^{1,1}$ but k^2 may be an eigenvalue of the Dirichlet-Laplacian in D.)*

The idea is to represent the solution of (1.1) as the sum of double and single layer potentials, i.e,

$$\begin{aligned} u^s(v) &:= \int_{\partial D} \frac{\partial\Phi(y,z)}{\partial\nu(z)}h(z)ds(z) + i\eta\int_{\partial D}\Phi(y,z)h(z)ds(z),\ y \in \mathbb{R}^3\setminus\bar{D} \\ &:= Kh + i\eta Sh, \end{aligned} \tag{4.8}$$

with η a positive real number and h satisfies

$$[T + i\eta(-\frac{1}{2}I + \mathcal{K}^*)]h = -\partial_\nu v, \tag{4.9}$$

where the operator T is defined as

$$(Th)(y) := \frac{\partial}{\partial\nu(y)} \int_{\partial D} \frac{\partial\Phi(y,z)}{\partial\nu(z)} h(z)ds(z).$$

Now, we need the following lemma on the existence and uniqueness of the solutions of (4.9).

Lemma 4.4. *Let D be of class $C^{1,1}$. The operator*

$$T + i\eta(-\frac{1}{2}I + \mathcal{K}^*) : H^{1-t}(\partial D) \to H^{-t}(\partial D)$$

is invertible for $0 \le t \le 1$.

Proof. Since ∂D is of class $C^{1,1}$, then the operator

$$T : H^{s+\frac{1}{2}}(\partial D) \to H^{s-\frac{1}{2}}(\partial D)$$

is Fredholm with zero index,[1] for $-1 \le s \le 1$, see [[4], Theorem 7.17]. In addition, for $-\frac{1}{2} \le s \le \frac{1}{2}$, the operator

$$-\frac{1}{2}I + \mathcal{K}^* : H^{s+\frac{1}{2}}(\partial D) \to H^{s+\frac{1}{2}}(\partial D)$$

is bounded, see also [4] for instance. As the inclusion $i : H^{s+\frac{1}{2}}(\partial D) \to H^{s-\frac{1}{2}}(\partial D)$ is compact, the operator

$$-\frac{1}{2}I + \mathcal{K}^* : H^{s+\frac{1}{2}}(\partial D) \to H^{s-\frac{1}{2}}(\partial D)$$

is compact for $-\frac{1}{2} \le s \le \frac{1}{2}$. Hence from [[4], Theorem 2.26], the operator

$$T + i\eta(-\frac{1}{2}I + \mathcal{K}^*) : H^{s+\frac{1}{2}}(\partial D) \to H^{s-\frac{1}{2}}(\partial D)$$

is Fredholm with index zero for $-\frac{1}{2} \le s \le \frac{1}{2}$. Therefore applying Fredholm alternative, the operator $T + i\eta(-\frac{1}{2}I + \mathcal{K}^*)$ is invertible if we can show that $\ker(T + i\eta(-\frac{1}{2}I + \mathcal{K}^*)) = \{0\}$.
Let $f \in H^{s+\frac{1}{2}}(\partial D)$ be a solution to the homogeneous equation $(T + i\eta(-\frac{1}{2}I + \mathcal{K}^*))f = 0$. Define

$$u(f) := \int_{\partial D} \frac{\partial\Phi(y,z)}{\partial\nu(z)} f(z)ds(z) + i\eta \int_{\partial D} \Phi(y,z)f(z)ds(z),\ y \in \mathbb{R}^3 \setminus \partial D.$$

Then $u(f)$ solves the exterior homogeneous Neumann problem and therefore $u(f) = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. From the jump relations, we have

$$(u(f)_+ - u(f)_-) = f,\ x \in \partial D$$

and

$$\frac{\partial u(f)_+}{\partial\nu} - \frac{\partial u(f)_-}{\partial\nu} = -i\eta f \text{ on } x \in \partial D,$$

where $u(f)_\pm$ stands for the Dirichlet and $\frac{\partial u(f)_\pm}{\partial\nu}$ for Neumann traces along the boundary ∂D from $\mathbb{R}^3 \setminus \bar{D}$ and D respectively. Therefore the Green's theorem implies that

$$\begin{aligned} i\eta \int_{\partial D} |f|^2 ds &= -\int_{\partial D} \bar{u}(f)_- \frac{\partial u(f)_-}{\partial\nu} ds \\ &= -\int_D |\nabla u(f)|^2 dx + k^2 \int_D |u(f)|^2 dx. \end{aligned}$$

[1] This is the point where we need the $C^{1,1}$ smoothness assumption of ∂D.

Taking the imaginary parts of this equation, we have

$$\eta \int_{\partial D} |f|^2 ds = 0.$$

So we get $f = 0$, since $\eta > 0$. Hence, the operator

$$T + i\eta(-\frac{1}{2}I + \mathcal{K}^*) : H^{s+\frac{1}{2}}(\partial D) \to H^{s-\frac{1}{2}}(\partial D)$$

is invertible for $-\frac{1}{2} \leq s \leq \frac{1}{2}$. Finally, set $t := \frac{1}{2} - s$.

□

From (4.9) and Lemma 4.4, we obtain

$$\|h\|_{H^{1-t}(\partial D)} \leq C\|\partial_\nu v\|_{H^{-t}(\partial D)}, \text{ for } t \in [0, 1]. \tag{4.10}$$

Using the representation (4.8) and the Hölder inequality, we have

$$\|u^s(v)\|_{H^1(B\setminus\bar{\Omega})} \leq \|h\|_{H^{1-t}(\partial D)}\|G\|_{H^1(B\setminus\bar{\Omega})} \tag{4.11}$$

where $G(y) := \|\Phi(y, \cdot)\|_{H^{-1+t}(\partial D)} + \|\nabla_y\Phi(y, \cdot)\|_{H^{-1+t}(\partial D)}$ and B is again a measurable and bounded set containing Ω. As in (4.5), since $\bar{D} \subset \Omega$ then due to the form of $\Phi(x, z)$, G is in $H^1(B \setminus \bar{\Omega})$. Combining (4.4), (4.10), (4.11) and Lemma 4.3, we obtain

$$|\int_{\partial\Omega} u^s(v)\overline{\partial_\nu u^s(v)}ds(y)| \leq C\|v\|^2_{H^{-t+\frac{3}{2}}(D)}, \text{ for } t \in [\frac{1}{2}, 1)^2. \tag{4.12}$$

□

4.2 Estimate of the body integral $\int_{\Omega\setminus\bar{D}} |u^s(v)|^2 dy$

Proposition 4.5. *For $\frac{1}{2} \leq t < 1$, we have the following estimate*

$$\|u^s(v)\|_{L^2(\Omega\setminus\bar{D})} \leq C\|v\|_{H^{-t+\frac{3}{2}}(D)}, \ C > 0.$$

Proof.

Case 1. *(Assume k^2 is not an eigenvalue of the Dirichlet-Laplacian in D.)*
The single layer operator

$$S : H^{-t}(\partial D) \to H^{-t+\frac{3}{2}}_{loc}(\mathbb{R}^3), \ 0 \leq t \leq 1,$$

defined in (4.2), is bounded, see [[4], Theorem 6.12]. Hence from (4.2), we obtain

$$\|u^s(v)\|_{L^2(\Omega\setminus\bar{D})} \leq C\|f\|_{H^{-t}(\partial D)}, \ 0 \leq t \leq 1. \tag{4.13}$$

Recalling (4.6), we deduce from (4.13) that

$$\|u^s(v)\|_{L^2(\Omega\setminus\bar{D})} \leq C\|v\|_{H^{-t+\frac{3}{2}}(D)}, \ \frac{1}{2} \leq t < 1. \tag{4.14}$$

Case 2. *(Assume that ∂D is of class $C^{1,1}$ but k^2 may be an eigenvalue of the Dirichlet-Laplacian in D.)*

---

[2]Since ∂D is of class $C^{1,1}$ then Lemma 4.3 is valid also for $t = 1$, see [6], and hence we can also take $t = 1$ in (4.12).

Since ∂D is of class be $C^{1,1}$, then the operator

$$K : H^{1-t}(\partial D) \to H^{-t+\frac{3}{2}}(\Omega \setminus \bar{D})$$

is a bounded, see [[4], Corollary 6.14]. In particular

$$\|Kh\|_{L^2(\Omega\setminus\bar{D})} \leq C\|h\|_{H^{1-t}(\partial D)},\ t \in [0,1], \tag{4.15}$$

where h is the density used in (4.8). Combining (4.10), (4.15) and from Lemma 4.3, we obtain

$$\|Kh\|_{L^2(\Omega\setminus\bar{D})} \leq C\|v\|_{H^{-t+\frac{3}{2}}(D)} \text{ for } t \in [\frac{1}{2}, 1). \tag{4.16}$$

Again using the fact that the single layer potential

$$S : H^{1-t}(\partial D) \to H^{-t+\frac{3}{2}}(\Omega \setminus \bar{D}),\ \text{for } t \in [0,1]$$

is a bounded operator, then from (4.10) and Lemma 4.3, we have the estimates

$$\begin{aligned} \|Sh\|_{L^2(\Omega\setminus\bar{D})} &\leq C\|h\|_{H^{1-t}(\partial D)} \\ &\leq C\|\partial_\nu v\|_{H^{-t}(\partial D)} \\ &\leq C\|v\|_{H^{-t+\frac{3}{2}}(D)},\ \text{for } \frac{1}{2} \leq t < 1. \end{aligned} \tag{4.17}$$

Therefore (4.8), (4.16) and (4.17) imply

$$\|u^s(v)\|^2_{L^2(\Omega\setminus\bar{D})} \leq C\|v\|^2_{H^{-t+\frac{3}{2}}(D)},\ \text{for } \frac{1}{2} \leq t < 1. \tag{4.18}$$

□

4.3 End of the proof of Theorem 3.1

We choose $\frac{1}{2} < t < 1$, then by interpolation and using the Young inequality, we obtain:

$$\|v\|^2_{H^{-t+\frac{3}{2}}(D)} \leq \epsilon\|\nabla v\|^2_{L^2(D)} + \frac{C}{\epsilon}\|v\|^2_{L^2(D)} \tag{4.19}$$

with some $C > 0$ fixed and every $\epsilon > 0$. Combining Proposition 2.6, (4.1), Proposition 4.5 and (4.19) we end up with

$$-4\pi\tau^{-1} I_\rho(v, \tau, h_D(\rho)) \geq c\|\nabla v\|^2_{L^2(D)} - C\|v\|^2_{L^2(D)} \tag{4.20}$$

with positive constants c and C independent on v and τ.
Finally, we recall the following lemma concerning estimates of the CGO solution v shown in Lemma 3.5 and Lemma 3.7 in [6].

Lemma 4.6. *We have the following estimates*

1.

$$\frac{\|\nabla v\|^2_{L^2(D)}}{\|v\|^2_{L^2(D)}} \geq C\tau^2 \quad (\tau \gg 1),$$

2.

$$c\tau^{-3} \leq \int_D |v|^2 dx \leq C\tau^{-1},\ \tau \gg 1,$$

3.
$$c\tau^{-1} \leq \int_D |\nabla v|^2 dx \leq C\tau, \ \tau \gg 1,$$
where $c, C > 0$ *and independent on* τ*.*

From (4.20) and Lemma 4.6 we deduce that
$$-I_\rho(v, \tau, h_D(\rho)) \geq C > 0 \ (\tau >> 1),$$
which ends the proof.

References

[1] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 1998.

[2] M. Ikehata. How to draw a picture of an unknown inclusion from boundary measurements. Two mathematical inversion algorithms. *J. Inverse Ill-Posed Probl.*, 7(3):255–271, 1999.

[3] M. Ikehata. The probe and enclosure methods for inverse obstacle scattering problems. The past and present. *arXiv:1002.0198v4 [math.AP].*

[4] W. McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[5] G. Nakamura and R. Potthast. Multi-waves enclosure method directly using the far-field patterns. *Preprint.*

[6] M. Sini and K. Yoshida. On the reconstruction of interfaces using complex geometrical optics solutions for the acoustic case. *Inverse Problems*, 28(5):055013, 2012.